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Research Paper

New Double Entire Difference Sequence Spaces Generated by Double Musielak-Orlicz Function

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ABSTRACT: *In this paper we introduce some double entire difference sequence spaces defined by double Musielak-Orlicz function $\mathcal{M} = (M_{kl})$. We also make an effort to study some topological properties and a few inclusion relations between these spaces.*

Keywords: *Double sequence space, Double entire difference sequence space, Musielak-Orlicz function.*
Mathematical Subject Classification: *40A05, 40C05, 40D25.*

I. INTRODUCTION

The initial works on double sequence is found in Bromwich [1]. Later on, it was studied by Hardy [2], Moricz [3], Moricz and Rhoades[4], Tripathy ([6] [5]), Basarir and Sonalcan[7] and many others. Hardy [2] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser[8] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [9] have recently studied the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly cesaro summable double sequences, Mursaleen [10] and Mursaleen and Edely [11] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduce M-Core for double sequences and determined those four dimensional matrices transforming every bounded sequence $x = (x_{mn})$ into one whose core is a subset of the M-Core of x. More recently, Altay and Basar [12] have defined the spaces $BS, BS(t), CS_p, CS_{bp}, CS_r$ and BV of double sequence consisting of all double series whose sequence of partial sums are in the space $\mathcal{M}_u, \mathcal{M}_u(t), C_p, C_{bp}, C_r$ and $\mathcal{L}_u$ respectively and also examined some properties of those sequence spaces as well as the $\alpha-$duals of these spaces BS, BV, CS_{bp} and the $\beta(v)-$ duals of the spaces CS_{bp} and CS_r of double series. Now, recently Basar and Sever [13] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known ℓ_q of single sequences and determined some properties of the $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in Pringsheim sense i.e a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by P-limit x= L) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$, such that $|x_{kl} - L| < \epsilon$, whenever $k, l > n$ (see [15]). We shall write more briefly as P-convergent. The double sequence $x = (x_{kl})$ is bounded if there exists a positive number M such that $|x_{kl}| < M$ for all k, l.

Orlicz function is defined as the function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.
Lindenstrauss and Tzafriri [14] used the concept of Orlicz functions to define the space

$$\ell_M = \left\{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right\}. \tag{1.1}$$

called Orlicz sequence space, and proved that every Orlicz sequence space contains a subspace isomorphic to $\ell_p (1 \leq p < \infty)$. The sequence space ℓ_M defined in (1.1) is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\} \tag{1.2}$$

It is shown in [14] that every Orlicz sequence space l_M contains a subspace isomorphic to $\ell_p (p \geq 1)$

An Orlicz function is said to satisfy the $\Delta_2 -$ condition for all values of u if there exists a constant $k > 0$ such that $M(2u) \leq kM(u), u \geq 0$. The $\Delta_2 -$ condtion is equivalent to $M(nu) \leq knM(u)$, for all values of u and $n > 1$.

A sequence space $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak –Orlicz function. (see [19], [20])
A sequence $\aleph = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|.u - M_k(u): u \geq 0\}\ k = 1,2,...$$

Is called the complimentary function of a Musielak-Orlicz function M. for a given Musielak-Orlicz function M, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follow

$$t_{\mathcal{M}} = \{x \in \omega: I_M(cx) < \infty, \infty \text{for all } c > 0\}$$
$$h_{\mathcal{M}} = \{x \in \omega: I_M(cx) < \infty, \text{for all } c > 0\}$$

Where I_M is a convex modular defined by

$$I_M = \sum_{k=1}^{\infty} M_k(x_k) \text{ and } x = (x_k) \in t_{\mathcal{M}}$$

Consider $t_{\mathcal{M}}$ equipped with Luxemburg norm

$$\|x\| = \inf\left\{k > 0: I_M\left(\frac{x}{k}\right) \leq 1\right\}$$

Or equipped with the Orlicz norm

$$\|x\| = \inf\left\{\frac{1}{k}\left(1 + I_M(kx)\right): k > 0\right\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [16], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. This notion was further generalized by Et and Colak [17] defined the sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Let m, n a non negative integers we have the following spaces:

$$Z(\Delta_n^m) = \{x = (x_k) \in \omega: (\Delta_n^m x_k) \in Z\}$$

For $Z = c, c_0,$ and l_∞ where

$$\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1}), \text{ and } \Delta_n^0 x_k = x_k$$

For all $k \in N$, which is equivalent to binomial representation

$$\Delta_n^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{+ni}$$

It was proved that the generalized sequence space $Z(\Delta_n^m)$, where $Z = \ell_\infty, c$ or c_0, is a Banach space with norm defined by

$$\|x\|_{\Delta_n^m} = \sum_{i=1}^m |x_i| + \sup|\Delta_n^m x_k|.$$

The following inequality will be used throughout the paper. Let $p = (p_{kl})$ be a double sequence of positive real numbers with $0 < p_{kl} \leq \sup_{kl} p_{kl} = H$ and $D = \max[1, 2^{H-1}]$. Then the factorable sequences $\{a_{kl}\}$ and $\{b_{kl}\}$ in the complex plane, we have

$$|a_{kl} + b_{kl}|^{p_{kl}} \leq K(|a_{kl}|^{p_{kl}} + |b_{kl}|^{p_{kl}}) \qquad (1.3)$$

II. BASIC DEFINITIONS

Definition 2.1 A double sequence space E is said to be normal (solid) if
$(y_{kl}) \in E$ whenever $|y_{kl}| \leq |x_{kl}|$ for all $k, l \in \mathbb{N}$, and $(x_{kl}) \in E$.
Definition 2.2 A double sequence space is said to be symmetric if $u = (u_{kl}) \in E$ and $\|u\| = \|x\|$ whenever $x = (x_{kl}) \in E$ and $u \in S(x)$.
Definition 2.3 A BK-Space is a Banach sequence space E in which the coordinate maps are continuous.
Definition 2.4 A sequence $x = (x_{kl})$ is said to be double analytic if $\sup_{kl}|x_{kl}|^{1/k+l} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2.
Definition 2.5 A sequence $x = (x_{kl})$ is said to be double entire if $P - \lim_{kl}|x_{kl}|^{1/k+l} = 0$, for all $k, l \in \mathbb{N}$. The vector space of all double entire sequences will be denoted by Γ^2.
Definition 2.6 Let $\mathcal{M} = (M_{kl})$ be a double sequence of Orlicz functions, then $\mathcal{M}$ is called double Musielak-Orlicz function.

Definition 2.7 Let $\mathcal{M} = (\mathrm{M_{kl}})$ be a double sequence of Orlicz function, X be locally convex Hausdorff topological linear space whose topology is determined by the set of continuous seminorm q. The symbol $\Lambda^2(\mathrm{X})$ and $\Gamma^2(\mathrm{X})$ denote the space of all analytic and entire sequences respectively defined over X. We define a new double sequence space:

$$\Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s) = \Bigg\{ x = (x_{kl}) \in \Gamma^2(X): \frac{1}{mn}\sum_{k,l=1}^{m,n} (kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0,$$
$$m, n \to \infty, \text{uniformly in } m, n > 0, s \geq 0, \text{for some } \rho > 0 \Bigg\}$$

This space is the extension to double sequence of the space defined and studied by Siddiqui and Balili [18].

III. MAIN RESULTS

We shall prove the following theorems in this paper.

Theorem 3.1 Let $\mathcal{M} = (\mathrm{M_{kl}})$ be a double Musielak-Orlicz function and $\mathrm{p} = (\mathrm{p_{kl}})$ be a double sequence of strictly positive real numbers. Then the space $\Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$ is linear space over the field $\mathbb{C}$ of complex numbers.

Proof. Let $x = (x_{kl}), y = (y_{kl}) \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1, ρ_2 such that

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho_1}\right)\right]^{p_{kl}} \to 0, \text{as } m, n \to \infty \qquad (3.1)$$

And

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v y_{kl}|^{1/k+l}}{\rho_2}\right)\right]^{p_{kl}} \to 0, \text{as } m, n \to \infty \qquad (3.2)$$

In order to prove the result, we need to find ρ_3 such that

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v (\alpha x_{kl}+\beta y_{kl})|^{1/k+l}}{\rho_3}\right)\right]^{p_{kl}} \to 0, \text{as } m, n \to \infty \qquad (3.3)$$

Let $\rho_3 = \max(2|\alpha|^{1/k+l}\rho_1, 2|\beta|^{1/k+l}\rho_2)$. Since $\mathcal{M} = (\mathrm{M_{kl}})$ is non decreasing, convex and q is a seminorm, so by using inequality (1.3), we have

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v (\alpha x_{kl}+\beta y_{kl})|^{1/k+l}}{\rho_3}\right)\right]^{p_{kl}}$$
$$\leq \frac{1}{mn}\sum_{k,l=1,1}^{m,n}(kl)^{-s}\left[M_{kl}\left(q\left\{|\alpha|^{1/k+l}\frac{|\Delta^m_v x_{kl}|^{1/k+l}}{\rho_3} + |\beta|^{1/k+l}\frac{|\Delta^m_v y_{kl}|^{1/k+l}}{\rho_3}\right\}\right)\right]^{p_{kl}}$$
$$\leq \frac{1}{mn}\sum_{k,l=1,1}^{m,n}(kl)^{-s}\left[M_{kl}\left(q\left\{\frac{|\Delta^m_v x_{kl}|^{1/k+l}}{\rho_1} + \frac{|\Delta^m_v y_{kl}|^{1/k+l}}{\rho_2}\right\}\right)\right]^{p_{kl}}$$
$$\leq C\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho_1}\right)\right]^{p_{kl}} + C\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho_2}\right)\right]^{p_{kl}}$$
$$\to 0, \quad m, n \to \infty$$

Thus $\alpha x_{kl} + \beta y_{kl} \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$, showing that it is linear space.

Theorem 3.2 Let $\mathcal{M} = (M_{kl})$ be a double Musielak-Orlicz function and $p = (p_{kl})$ be a double sequence of strictly positive real numbers. Then the space $\Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$ is a paranormed space with paranorm defined by

$$g(x) = inf\left\{\rho_H^{p_{kl}} : \sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}/H} \leq 1\right\} uniformly\ in\ m, n > 0, \rho > 0.$$

Where $H = max\{1, sup_{kl}\, p_{kl}\}$

Proof. It is clear $g(x) \geq 0, g(x) = g(-x) and\ g(\theta) = 0, \theta\ is\ the\ zero\ sequence\ of\ X$. Let $x_{kl}, y_{kl} \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$. Let $\rho_1, \rho_2 > 0$ be such that

$$\sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_1}\right)\right]^{p_{kl}/H} \leq 1$$

And $sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_2}\right)\right]^{p_{kl}/H} \leq 1$

Let $\rho = \rho_1 + \rho_2$, then by using minkowski inequality, we have

$$\sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m (x_{kl} + y_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}/H}$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_1}\right)\right]^{p_{kl}/H} + \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_2}\right)\right]^{p_{kl}/H}$$

≤ 1

Hence

$$g(x+y) \leq inf\left\{(\rho_1+\rho_2)^{p_{mn}/H}\colon\right\}\sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m (x_{kl} + y_{kl})|^{1/k+l}}{\rho_1+\rho_2}\right)\right]^{p_{kl}/H} \leq 1\, \rho_1,\rho_2 > 0, m,n \in \mathbb{N}$$

$$\leq inf\left\{(\rho_1)^{p_{mn}/H}\colon \sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m (x_{kl} + y_{kl})|^{1/k+l}}{\rho_1}\right)\right]^{p_{kl}/H} \leq 1\,, \rho_1 > 0, m,n \in \mathbb{N}\right\}$$

$$+inf\left\{(\rho_2)^{p_{mn}/H}\colon sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m (x_{kl} + y_{kl})|^{1/k+l}}{\rho_2}\right)\right]^{p_{kl}/H} \leq 1\,, \rho_2 > 0, m,n \in \mathbb{N}\right\}$$

Thus we have $g(x+y) \leq g(x) + g(y)$. Hence g satisfies the triangle inequality.

Now

$$g(\lambda x) = inf\left\{(\rho)^{p_{mn}/H}\colon \sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\lambda\Delta_v^m x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}/H} \leq 1\,, \rho > 0, m,n \in \mathbb{N}\right\}$$

$$= inf\left\{(r|\lambda|)^{p_{mn}/H}\colon \sup_{kl\geq 1}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{r}\right)\right]^{p_{kl}/H} \leq 1\,, r > 0, m,n \in \mathbb{N}\right\}$$

Where $r = \frac{\rho}{|\lambda|},\ \ since\ |\lambda|^{p_{mn}} \leq max(1, |\lambda|^{sup\ kl})$. Hence $\Gamma^2_{\mathcal{M}}(\Delta_v^m, p, q, s)$ is a paranormed space.

Theorem 3.3 Let $\mathcal{M}' = (M'_{kl})$ and $\mathcal{M}'' = (M''_{kl})$ be two double Musielak-Orlicz functions. Then $\Gamma^2_{\mathcal{M}'}(\Delta_v^m, p, q, s) \cap \Gamma^2_{\mathcal{M}''}(\Delta_v^m, p, q, s) \subseteq \Gamma^2_{\mathcal{M}'+\mathcal{M}''}(\Delta_v^m, p, q, s)$

Proof. Let $x \in \Gamma^2_{\mathcal{M}'}(\Delta_v^m, p, q, s) \cap \Gamma^2_{\mathcal{M}''}(\Delta_v^m, p, q, s)$. Then there exists ρ_1, ρ_2 such that

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_1}\right)\right]^{p_{kl}} \to 0, as\ m,n \to \infty \tag{3. 4}$$

And

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_2}\right)\right]^{p_{kl}} \to 0, as\ m,n \to \infty \tag{3.5}$$

Let $\rho = min\left(\frac{1}{\rho_1}, \frac{1}{\rho_2}\right)$. Then we have

$$\frac{1}{mn}\sum_{k,l\geq 1,1}^{m,n}(kl)^{-s}\left[(M'_{kl} + M''_{kl})\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}}$$

$$\leq K\frac{1}{mn}\sum_{k,l\geq 1,1}^{m,n}(kl)^{-s}\left[M'_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_1}\right)\right]^{p_{kl}} + K\frac{1}{mn}\sum_{k,l\geq 1,1}^{m,n}(kl)^{-s}\left[M''_{kl}\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho_2}\right)\right]^{p_{kl}}$$

$\to 0\ as\ m,n \to \infty$

By (3.4) and (3.5). Then

$$\frac{1}{mn}\sum_{k,l\geq 1,1}^{m,n}(kl)^{-s}\left[(M'_{kl} + M''_{kl})\left(\frac{q|\Delta_v^m x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0\ as\ m,n \to \infty.$$

Therefore

$x \in \Gamma^2_{\mathcal{M}'+\mathcal{M}''}(\Delta^m_v, p, q, s)$

Theorem 3.4 Suppose $\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \leq |x_{kl}|^{1/k+l}$ Then

$$\Gamma^2 \subset \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s).$$

Proof. Let $x \in \Gamma^2$. Then we have

$$|x_{kl}|^{1/k+l}, as\ k, l \to \infty \quad (3.6)$$

But

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \leq |x_{kl}|^{1/k+l}$$

By the assumption above, it implies that

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0, as\ m, n \to \infty$$

By (3.6)

Then $x \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$, and hence

$$\Gamma^2 \subset \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s).$$

Theorem 3.5 $\Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$ is solid

Proof. Let $|x_{kl}| \leq |y_{kl}|$ and $(y_{kl}) \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$

since $\mathcal{M} = (M_{kl})$ is non decreasing, it implies that

$$\frac{1}{mn}\sum_{k,l\geq 1,1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \leq \frac{1}{mn}\sum_{k,l\geq 1,1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v y_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}}$$

Since $y \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$. Then

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v y_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0, as\ m, n \to \infty$$

And

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0, as\ m, n \to \infty$$

Therefore

$x \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$. Hence the result.

Theorem 3.6 (i) Let $0 < inf p_{kl} \leq 1$. Then $\Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s) \subset \Gamma^2_{\mathcal{M}}(\Delta^m_v,\ q, s)$.

(ii) Let $1 \leq p_{kl} \leq sup_{kl}\, p_{kl} < \infty$. Then $\Gamma^2_{\mathcal{M}}(\Delta^m_v, q, s) \subset \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$.

Proof. (i) Let $x \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$. Then

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0, as\ m, n \to \infty \quad (3.7)$$

Since $0 < inf p_{kl} \leq 1$.

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right] \leq \frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \quad (3.8)$$

From (3.7) and (3.8), it follows that $x \in \Gamma^2_{\mathcal{M}}(\Delta^m_v,\ q, s)$.

Thus $\Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s) \subset \Gamma^2_{\mathcal{M}}(\Delta^m_v,\ q, s)$.

(ii) Let $p_{kl} \geq 1\ for\ each\ k, l\ and\ sup_{kl}\, p_{kl}$ and let $x \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, q, s)$. Then

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0, as\ m, n \to \infty \quad (3.9)$$

Since $1 \leq p_{kl} \leq sup_{kl}\, p_{kl} < \infty$, we have

$$\frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \leq \frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]$$

$$\Longrightarrow \frac{1}{mn}\sum_{k,l=1}^{m,n}(kl)^{-s}\left[M_{kl}\left(\frac{q|\Delta^m_v x_{kl}|^{1/k+l}}{\rho}\right)\right]^{p_{kl}} \to 0, as\ m, n \to \infty$$

This implies that $x \in \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s)$. Therefore

$$\Gamma^2_{\mathcal{M}}(\Delta^m_v, q, s) \subset \Gamma^2_{\mathcal{M}}(\Delta^m_v, p, q, s).$$

IV. CONCLUSION

There are several extensions of some concept of single sequence spaces to double sequence spaces by some authors. We also extended the generalized entire difference sequence space $\Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$ to double sequence space $\Gamma_{\mathcal{M}}^2(\Delta_v^m, p, q, s)$, which is solid and the intersection of the spaces defined by two double Musielak-Orlicz functions is identical with the space defined by the addition of the two given functions.

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