Quest Journals
Journal of Research in Applied Mathematics
Volume 5 ~ Issue 1 (2018) pp: 11-15
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Almost Regular Matrix Summability ofDouble Conjugate Fourier series

Ahmadu Kiltho, Ado Balili And M. Abdullahi
Department of Mathematics and Statistics University of Maiduguri, Borno State, Nigeria
Corresponding Author: Ahmadu Kiltho

ABSTRACT: *The main object of this paper is to prove that a double conjugate Fourier series is four dimensional almost regular Hankel matrix summable.*
KEY WORDS:*Double conjugate, Fourier series, Matrix summability, Almost regular, Double limits, Four dimensional matrix*

Received 26 May, 2018; **A**ccepted 11 June 2018 © The Author (S) 2018. **P**ublished With Open Access At **Www.Questjournals.Org**

I INTRODUCTION

Double limit of a sequence(x_{nk}); $n, k = 1, 2, 3, ...$is a number $x$if for any$\varepsilon > 0$, there exists an integer$N(\varepsilon)$ such that for all$n, k > N(\varepsilon)$the inequality $|x_{nk} - x| < \varepsilon$ is valid. The notation is$x = \lim_{n,k\to\infty,} x_{nk}$. Iffor any$\varepsilon > 0$, there exists an$N(\varepsilon)$such that for all $n, k > N(\varepsilon)$the inequality$|x_{nk}| > \varepsilon$ is fulfilled, then the sequence(x_{nk}); $n, k = 1, 2, 3, ...$ has infinity as its limit, and we write$\lim_{n,k\to\infty,} x_{nk} = \infty$. The double limit of a sequence is a special case of the double limit of a function of two variables.Infinite matrix methods of summability are basically called transformations, because assigning limits to divergent sequences or series is by considering a transformation (or an infinite matrix) rather than the original sequences or series. In fact, the most important transformations are given by infinite matrices. It is necessary that these processes be described as follows:

For any single sequence(x_k) a new sequence (y_k) is defined as follows:

$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k$, for a matrix A(1.1)

where the matrix A is called the A-transform of the single sequence $x_k$provided (1.1) exists. Analogously, A-double transform of a double sequence $(x_{m,n})$ is given by

$y_{m,n} = \sum_{k=1,l=1}^{\infty} a_{m,n,k,l} x_{n,k}$, (1.2)

where, $a_{m,n,k,l} := A$ is the matrix of transformation, provided (1.2) exists Robison[9]

In Nigam and Sharma[8] it was proved that a double conjugate Fourier series is summable by a product of two infinite matrices. We wish to prove that a Double conjugate Fourier series is almost regular four dimensional matrix summable.

II BACKGROUND AND NOTATIONS

If $(x_n)$and $(y_n)$are two sequences such that a numbern_0 exists such that $\frac{|x_n|}{|y_n|} < K$whenever$n > n_0$, where K is independent of n, we say that (x_n) is "big order of" (y_n), and we write $(x_n) = O(y_n)$, thus, $\frac{15n+19}{1+n^2} = O\left(\frac{1}{1+n^2}\right)$. If, however the limit $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = 0$we write $(x_n) = o(y_n)$, read(x_n) is " small order of" (y_n).There are several such definitions in literatures.

A function $f(x, y)$ of two variables defined on a bounded rectangle $R := [a, b] \times [c, d]$is said to be of bounded variation over this rectangle if the following three quantities are finite:

$V_{11} = \sup_{\lambda_1\times\lambda_2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left|f(x_j, y_k) - f(x_{j-1}, y_k) - f(x_j, y_{k-1}) + f(x_{j-1}, y_{k-1})\right|$(2.1)

$V_{10} = \sup_{\lambda_1} \sum_{j=1}^{m} \left|f(x_j, c) - f(x_{j-1}, c)\right|$ (2.2)

$V_{01} = \sup_{\lambda_2} \sum_{k=1}^{n} |f(a, y_k) - f(a, y_{k-1})|$(2.3) where, $\lambda_1 : a = x_0 < x_1 < x_2 < ... < x_m = b$ and $\lambda_2 : c = y_0 < y_1 < y_2 < ... < y_m = d$ are arbitrary finite partitions. The sum $V(f) = V(f;R) := V_{11} + V_{10} + V_{01}$ is called the total variation over R, Moricz, [6]. We shall use the definition of functions of bounded variations in two variables because these types of functions guarantee the convergence double Fourier series.

Summability of double conjugate Fourier series using regularity conditions of product of twoinfinite matrices was carried out by Nigam and Sharma[8]. The necessary and sufficient conditions imposed on almost-regular matrices in the ordinary sense are due to King[2], while the analogue of the almost-regular matrices for four-dimensional matrices were due to Moricz and Rhaodes[4]. The notion of almost convergence for ordinary or single sequences was given by Lorentz[1]and for double sequences by Moricz and Rhoades [4] after 40 years!

In the work of Moricz [5], partial sums to double conjugate Fourier series was defined including its double conjugate function. In Mursaleen and Savas [7] four dimensional almost regular matrix was defined and characterized, with impositions of five conditions. Also, Nigam and Sharma [8] defined three different double conjugate Fourier series. The third definition in terms of double variables (x, y) will be used in this study, which was put in different form in Moricz [5].

Let $f(x,y)$ be a function of a period 2π with respect to each variablesx, y. If $f(x,y)$ is Lebesgue integrable over the square $\Delta: -\pi \leq x \leq \pi, -\pi \leq y\pi$. Then we can associate f with its double Fourier series:

$\sum_{j,k=-\infty}^{+\infty} \hat{f}(j,k) e^{i(jx+ky)}$; (2.4)

where,

$$\hat{f}(j,k) := \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} f(u,v) e^{-i(ju+kv)} dudv \quad (2.5)$$

(Marcinkiewicz and Zygmund [3])

The double conjugate Fourier series to the double Fourier series (2.4) is given by

$$\sum_{j,k=-\infty}^{+\infty} (-isgnj)(-isgnk) \hat{f}(j,k) e^{i(jx+ky)}, \quad \text{(Moricz[5])}$$

The rectangular partial sum of the series (2.5) is defined as: $\bar{S}_{mn}(f;x,y) = \sum_{|j|=-m,|k|=-n}^{m,n} (-isgnj)(-isgnk) \hat{f}(j,k) e^{i(jx+ky)}$ $= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \psi_{xy}(f;u,v) \widetilde{D}_m(u) \widetilde{D}_n(v)$; (2.6)

where,$\varphi(x,y,u,v) = \psi_{xy}(f;u,v) = f(x-u,y-v) - f(x+u,y-v) - f(x-u,y+v) + f(x+u,y+v) - \delta_{xy}(f)$

is a function of two variables of bounded variations; and

$\widetilde{D}_m(u)$is the Dritchlet kernel. In Moricz[6] the following limit exists as a number in Δ: $\lim_{u,v\to 0+} \psi_{xy}(f;u,v) := \delta_{xy}(f)$.

The corresponding conjugate function to (2.4) is defined in Nigam and Sharma [8] as follows:

$$\tilde{f}(x,y) = -\frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \psi_{xy} \frac{du}{2\tan u/2} \frac{dv}{2\tan v/2} \, (2.7)$$

Since the study intends to prove a result on four dimensional matrix summability of double conjugate Fourier series, the following characterizations are quite useful:

Lemma 3.1(Mursaleen and Savas [7]): A four dimensional matrix $A = (a_{jk}^{mn})$ is almost regular if, and only if

(i) $\lim_{p,q\to\infty} \alpha(j,k,p,q,s,t) = 0$ $(j,k = 0,1,2,...)$, uniformly in s, t = 0, 1, 2, …;

(ii) $\lim_{p,q\to\infty} \sum_{j=0,k=0}^{\infty,\infty} \alpha(j,k,p,q,s,t) = 1$, uniformly in $s,t = 0,1,...$;

(iii) $\lim_{p,q\to\infty} \sum_{j=0}^{\infty} |\alpha(j,k,p,q,s,t)| = 0$, $(k = 0,1,2,...)$, uniformly in $s,t = 0,1,...$;

(iv) $\lim_{p,q\to\infty} \sum_{k=0}^{\infty} |\alpha(j,k,p,q,s,t)| = 0$, $(j = 0,1,2,...)$, uniformly in $s,t = 0,1,...$;

(v) $\sum_{j=0,k=0}^{\infty,\infty} |a_{jk}^{mn}| \leq M < \infty$, $(m,n = 0,1,2,...)$; where,

$\alpha(j,k,p,q,s,t) = \frac{1}{pq} \sum_{m=s,n=t}^{s+p-1,\ t+q-1} a_{jk}^{mn}$.

Before we prove this result we need to fix some notations. Let A_{mn} be an almost regular infinite matrix satisfying the conditions (i) – (v) as above.Suppose $m(k)$ and $n(k)$ are non-decreasing sequences of counting numbers satisfying the limits $\lim_{k\to\infty} m(k) = \infty$ and $\lim_{k\to\infty} n(k) = \infty$.

Define,

$\varphi(x,y,s,t) := \int_0^s \int_0^t |\psi(x,y,u,v| \, dudv$.

Then

$$\int_{1/p(m(k))}^{\pi} \frac{\varphi(x,y,s,\pi)}{t^2} dt = o(\ln p(m(k)) \ln q(n(k)) dudv, \quad k \to \infty \quad (2.8)$$

*Corresponding Author: *Ahmadu Kiltho*
Department of Mathematics and Statistics University of Maiduguri, Borno State, Nigeria

Let$f \in [-\pi,\pi;-\pi,\pi]$. Then at any point $(x,y) \in [-\pi,\pi;-\pi,\pi]$,

$\lim_{s,t\to 0} \frac{\varphi(x,y,s,t)}{ts} = 0$ (2.9)

And for every$\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\varphi(x,y,\delta,\delta)}{\delta^2} < \varepsilon$$

The conjugate Dritchlet kernel is defined as: $\breve{D}_m(u) := \sum_{j=1}^{m} \sin ju = \frac{\cos(u/2) - \cos(m+1/2)u}{2\sin(u/2)}$.

III MAIN RESULT

The study intends to prove the following theorem:

Theorem 3:Let $f(x,y)$, a 2π period, be a function of two variables integrable in the sense of Lebesgue over the squareΔ. Let $\sum_{j,k=-\infty}^{+\infty}(-isgnj)(-isgnk)\hat{f}(j,k)e^{i(jx+ky)}$ be its double conjugate Fourier series. Further, let $A = (a_{jk}^{mn})$ be almost regular matrix. Then the sequence of partial sums of double conjugate function $\{\bar{S}_{mn}(f;x,y)\}$ is A-summable to its conjugate function, $\tilde{f}(x,y)$.

Proof: Now,

$$\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\left\{\tilde{S}_{mn}(f;xy) - \hat{f}(x,y)\right\}$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\varphi(x,y,u,v)\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\,\widetilde{D}_i(u)\widetilde{D}_j(v)dudv +$$

$$\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\varphi(x,y,u,v)\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\,\widetilde{D}_i(u)\widetilde{D}_j(v)dudv$$

or,

$$A_{mn}\tilde{S}_{mn}(f;xy) - \hat{f}(x,y \le \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\varphi(x,y,u,v)\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\,\widetilde{D}_i(u)\widetilde{D}_j(v)dudv +$$

$$\frac{\delta_{xy}(f)}{\pi^2}\int_0^{\pi}\int_0^{\pi}\varphi(x,y,u,v)\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\,\widetilde{D}_i(u)\widetilde{D}_j(v)dudv$$

$= J_1 + J_2$,say.

where,

$\frac{\delta_{xy}(f)}{\pi^2} = \lim_{m,n\to\infty}\frac{\tilde{S}_{m,n}(f;x,y)}{(\log m)(\log n)}$.

Suppose J_2 takes the form:

$$J_2 = \frac{1}{\pi^2}\left(\int_0^{\frac{1}{p}} + \int_{\frac{1}{p}}^{\delta} + \int_{\delta}^{\pi}\right)\left(\int_0^{\frac{1}{q}} + \int_{\frac{1}{q}}^{\delta} + \int_{\delta}^{\pi}\right)\varphi(x,y,u,v)$$

$$\times \sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\;\widetilde{D}_i(u)\widetilde{D}_j(v)dudv$$

$$= \frac{1}{\pi^2}\left(J_A + J_B + J_C + J_D + J_E + J_F + J_G + J_H + J_J\right)\varphi(x,y,u,v) \times$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\,\widetilde{D}_i(u)\widetilde{D}_j(v)dudv$$

where, $J_A = \frac{1}{\pi^2}\int_0^{1/p}\int_0^{1/q}\varphi(x,y,u,v)\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\,\widetilde{D}_i(u)\widetilde{D}_j(v)dudv$, etc.

Since $p := p(m(k))$and $q := q(n(k)$, k can be chosen such that $1/p$, $1/q<\delta$, and it is well known that $\left|\widetilde{D}_k(t)\right| \le k$ and $\left|\widetilde{D}_k(t)\right| \le \frac{2}{\pi t}, 0 < t \le \pi$. So, we have

$|J_A| \le \frac{p(m(k))q(n(k)A_{m(k))n(k))}}{\pi^2}\int_0^{1/p}\int_0^{1/q}\varphi(x,y,u,v)dudv$ (3.1)

$$< \frac{\varepsilon A_{m(k))n(k))}}{\pi^2}$$

$= \frac{\varepsilon}{\pi^2}$, because of almost regularity condition (ii).

Next, we consider J_B noting that

$J_B = \frac{1}{\pi^2}\int_0^{1/p}\int_{1/q}^{\delta}\varphi(x,y,u,v)\sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\,\breve{D}_i(u)\,\breve{D}_j(v)dudv$.

Let, $g(u,v) = \sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\;\breve{D}_i(u)\breve{D}_j(v)dudv$. Then,

$$\int_{1/q}^{\delta} g(u,v)dv = \sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\;\breve{D}_j(u)\int_{1/q}^{\delta}\frac{\cos(v/2) - \cos(j+1/2)v}{2\sin(v/2)}dv$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\;\breve{D}_j(u)\left(\left[\ln(\sin\frac{v}{2} + \frac{1}{2j}(cosjv + sinjvcot\,(\frac{v}{2})\right]_{1/q}^{\delta} - \int_{1/q}^{\delta}\frac{\sin jv}{sin^2(\frac{v}{2})}dv\right)$$

*Corresponding Author: *Ahmadu Kiltho*

Department of Mathematics and Statistics University of Maiduguri, Borno State, Nigeria

$$= \sum_{i=1}^{p}\sum_{j=1}^{q} a_{mnij}\ \breve{D}_j(u)\left(\left[ln\left(\frac{sin\delta/2}{\sin 1/q} + \frac{1}{2j}(cosj\delta - \cos\frac{j}{q}) + \frac{1}{2j}(\sin j\delta cot\delta/2 - sinj/qcot\ 1/2q\right)\right] - \int_{1/q}^{\delta} \frac{\sin jv}{sin^2(\frac{v}{2})}dv\right)$$

Integrating J_B by parts gives,

$$J_B = \frac{1}{\pi^2}\int_0^{1/p}\left[\varphi(x,y,u,v)h(u,v)|_{1/q}^{\delta} - \int_{1/q}^{\delta} g(u,v)\frac{\partial}{\partial u}\varphi(x,y,u,v)du\right]dv$$

Where, $h(u,v)|_{1/q}^{\delta} = \int_{1/q}^{\delta} g(u,v)du$. So,

$$J_B = \frac{1}{\pi^2}\int_0^{1/p}\varphi(x,y,u,v)h(u,v)|_{1/q}^{\delta}dv - \frac{1}{\pi^2}\int_0^{1/p}\left(\int_{1/q}^{\delta} g(u,v)\frac{\partial}{\partial u}\varphi(x,y,u,v)du\right)dv$$

$$= \frac{1}{\pi^2}h(u,v)|_{\frac{1}{q}}^{\delta}\int_{\frac{1}{q}}^{\delta}\varphi(x,y,u,v)dv - \frac{1}{\pi^2}\left(\int_0^{\frac{1}{p}}\left[g(u,v)\lambda(u,v)|_{\frac{1}{q}}^{\delta}\right]dv - \frac{1}{\pi^2}\int_0^{1/p}\left(\int_{1/q}^{\delta}\varphi(x,y,u,v)\frac{\partial}{\partial u}g(u,v)du\right)\right)du$$

Where, $\lambda(u,v)|_{1/q}^{\delta} = \int_{1/q}^{\delta}\frac{\partial}{\partial u}\varphi(x,y,u,v)du$.

$\Rightarrow |J_B| \leq \frac{2p(m(k))}{\pi^2}\int_0^{1/p}\int_{1/q}^{\delta}\frac{\varphi(x,y,u,v)}{v}dudv \quad \leq \frac{2\varepsilon}{\pi^3}\left(1 + \int_{\frac{1}{q}}^{\delta}\frac{dv}{v}\right)$

$= O(\ln q(n(k))$ (3.2)

Similarly,

$|J_C| \leq \frac{2p(m(k))}{\pi^3}\int_0^{1/p}\int_{\delta}^{\pi}\frac{\varphi(x,y,u,v)}{v}dudv$ (3.3)

$$\leq \frac{2p(m(k))}{\pi^4}\int_0^{\pi}\int_0^{1/p}|\varphi(x,y,u,v)|dudt + \frac{2p(m(k))}{\pi^3}\int_{\delta}^{\pi}\frac{1}{v^2}\int_{\delta}^{\pi}\int_0^{1/p}|\varphi(x,y,u,v)|dudt$$

$$= \frac{2p\big(m(k)\big)}{\pi^3}\left(\frac{1}{\pi}\varphi\left(x,y,\frac{1}{p\big(m(k)\big)},\pi\right)\right) + \varphi(x,y,1/(p(m(k)),\pi)\int_{\delta}^{\pi}\frac{dv}{v}$$

$= o(\ln p(m(k))\ln q(n(k))$; as

$\varphi\left(x,y,\frac{1}{p(m(k))},\pi\right)\cdot p\big(m(k)\big) = o(\ln p(m(k))\ln q(n(k))$.

Now,

$J_D = \frac{1}{\pi^2}\int_{1/p}^{\delta}\int_0^{1/q}\varphi(x,y,u,v)\sum_{i=1}^{p}\sum_{j=1}^{q}a_{mnij}\ \breve{D}_i(u)\ \breve{D}_j(v)dudv \qquad \Rightarrow |J_D| = 0,$ from (2.8).

Repeating same processes of integration by parts we can have the following:

$$|J_E| = \frac{4}{\pi^4}\left(\int_{1/q}^{\delta}|\varphi(x,y,u,v)|\frac{dv}{v}\right)\frac{du}{u}$$

$\leq M\varepsilon\ln p\big(m(k)\big).\ln q(n(k))$ (3.4)

Where, *M* in (3.4) is any constant but fixed as in condition (v) of almost regularity of A_{mn}.

$|J_F| = o(\ln p\big(m(k)\big).lnq(n(k))$ (3.5)

$|J_F| \leq \frac{4}{\pi^4\delta^2}\int_0^{\pi}|\varphi(x,y,u,v)|\ dudv = O(1)$ (3.6)

While, $|J_G| = |J_H| = |J_J| = 0$ as for $|J_D|$.

Before we calculate J_2, let us define $z_i := \int_0^{\pi}\widetilde{D}_i(u)du$, which yields $\ln i$ as i approaches infinity. So, for each $\varepsilon > 0$ there exists a number *N* with *i*>*N* such that

$1 - \varepsilon < \frac{z_i}{\ln i} < 1 + \varepsilon$.

So, it implies that

$$\int_0^{\pi}\int_0^{\pi}\sum_{i=1}^{p}\sum_{j=1}^{q}a_{mnij}\ \widetilde{D}_i(u)\widetilde{D}_j(v)dudv = \sum_{i=1}^{p}\sum_{j=1}^{q}a_{mnij}\ z_i z_j$$

*Corresponding Author: *Ahmadu Kiltho*
Department of Mathematics and Statistics University of Maiduguri, Borno State, Nigeria

$$= \sum_{i=1}^{N} \sum_{j=1}^{q} a_{mnij}\, z_i z_j + \sum_{i=1}^{N} \sum_{j=N+1}^{q} a_{mnij}\, z_i z_j + \sum_{i=N+1}^{p} \sum_{j=1}^{N} a_{mnij}\, z_i z_j + \sum_{i=N+1}^{p} \sum_{j=N+1}^{q} a_{mnij}\, z_i z_j$$

$$\leq M_1^2(N)A_{mn} + (1+\varepsilon)M_1(N)A_{mn}\, lnq\big(n(k)\big) + (1+\varepsilon)M_1(N)A_{mn}\, lnp\big(m(k)\big) + (1+\varepsilon)^2 A_{mn}\, lnp\big(m(k)\big)lnq\big(n(k)\big)$$

with $M_1(N) \coloneqq \max_{0\leq i\leq N}|z_i|$. So that

$$\lim_{k\to\infty} \frac{\pi^2 J_2(k)}{\delta_{xy}(f)\ln p(m9k)\ln q(n(k)} \leq 1$$

Combining (2.7) – (3.6) the desired result is obtained. ■

IV CONCLUSION

This paper has shown that another way to sum double conjugate Fourier series is an application of double matrix summability.

ACKNOWLEDGEMENT

The authors thank the anonymous referees for their valuable suggestions which led to the improvement of the paper

REFERENCES

[1]. G. G. Lorentz, "A contributions to the theory of divergent sequences", Acta Math., ,(1948), 167 – 190.
[2]. J. P. King,Almost summable sequences, Proc. Amer. Math. Soc., 17(1966),1219 – 1225
[3]. J. Marcinkiewicz and A. Zygmund, On the summability of double Fourier series, Fund. Math.,32(1939), 122 – 132
[4]. F. Moricz and B. E Rhoades., Almost convergence of double sequences and strong regularity of summability matrices, Math.Proc. Camb. Phil. Soc.,104,(1988), 283 – 294.
[5]. F. Moricz, Approximation by rectangular partial sum of double conjugate Fourier series, J. Approx. Theory, 103 (2000), 130 – 150.
[6]. F. Moricz,Extension of a theorem of Ferenc Lukas from single to double conjugate series, J. Anal. Appl.,259(2001), 582 – 595.
[7]. M. Mursaleen and E. Savas, Almost regular matrices for double sequences, StudiaSci. Math. Hung., 40 (2003), 205 – 212.
[8]. H. K. Nigamand K. Sharma, On double summability of double conjugate Fourier series, Int. J. .Math. Sci. (2012), 1 – 15
[9]. G. M. Robison,Divergent double sequences and series, Trans. Amer. Math. Soc., ,30 (1926), 139 – 150

Ahmadu Kiltho ."Almost Regular Matrix Summability of Double Conjugate Fourier series." Quest Journals Journal of Research in Applied Mathematics , vol. 05, no. 01, 2018, pp. 11–15.

*Corresponding Author: *Ahmadu Kiltho*
Department of Mathematics and Statistics University of Maiduguri, Borno State, Nigeria