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Research Paper

On a p-valent Multiplier Differential Operator

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ABSTRACT: *In this paper, we focus on inclusion properties for the multiplier transformation of the form*

$$D_{p,\alpha}^{m} f(z) = z^p + \sum_{n=p+1}^{\infty} \alpha \left(\frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{m} a_n z^n$$

usingthe principle of subordination.
MSC[2010]: *30C45*
KEYWORDS:*p-valent, differential operator, subordination, inclusion.*

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I. INTRODUCTION AND PRELIMINARIES

Let A denote the class of normalized univalent functions of the form

$$z + a_2 z^2 + a_2 z^2 + a_2 z^2 \ldots \qquad (1)$$

which are analytic in the unit disc $U = \{z: |z| < 1\}$.
For the functionfof the form (1), the following results are well known: fis said to be starlike respectively convex with respect to the origin, if, and only if,

$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, |z| < 1$

And

$\mathrm{Re}\left\{1 + \frac{zf'(z)}{f(z)}\right\} > 0, |z| < 1$

Remark 1.1. From the above, it obvious that fis convex if and only if zf′is starlike.Respectively, f is said to be starlike, convex, of order γif and only if

$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \ |z| < 1$

And

$\mathrm{Re}\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \gamma, \ |z| < 1$

Definition: Let f ϵ Aand g is starlike of orderγ i.e. $g \in S^*(\gamma)$then $f \in K(\beta, \gamma)$,if, and only if

$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta, z \in U.$

This functions are called close-to-convexfunction of order βtypeγ.
We denote by A_p, the class of function f ϵ Aof the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, p \geq 1 \qquad (2)$$

Definition: Let g(z)be analytic and univalent in Uand f(z)is analytic in U, then, f is said to be subordinate to g if there exists a Schwartz w(z)function which is analytic in U with $w(0) = 0$and $|w(z)| < 1$ for all z ϵ Usuch that $f(z) = g(w(z))$. This is expressed as$f \prec g$.
Moreover, suppose g is univalent in U, then the following equivalence holds[1,4,5,6,10]

$$f \prec g \Leftrightarrow f(0) = g(0) \text{and} f(U) \subset g(U)$$

Forf ∈ A, the following subclasses of starlike, convex and close-to-convexfunctions

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$S^*(\xi,\phi), C(\xi,\phi)$,and$K(\xi,\rho;\phi,\varphi)$of order ξ, are studied by severalauthors (6, 8, 10) and are respectively defined by:

$$S^*(\mu,\psi) = \left\{f \in A: \frac{1}{1-\mu}\left(\frac{zf'(z)}{f(z)} - \mu\right) \prec \psi(z), z \in U\right\}$$

$$C(\mu,\psi) = \left\{f \in A: \frac{1}{1-\mu}\left(1 + \frac{zf'(z)}{f(z)} - \mu\right) \prec \psi(z), z \in U\right\}$$

$$K(\mu,\zeta;\psi,\varphi) = \left\{f \in A: \frac{1}{1-\zeta}\left(\frac{f'(z)}{g(z)} - \zeta\right) \prec \varphi(z), z \in U, g(z) \in S^*(\mu,\psi)\right\}$$

For the function of the form

$$f^{\alpha}(z) = z^{\alpha} + \sum_{n=p+1}^{\infty} \alpha a_n z^{n+\alpha-1} \quad (3)$$

[5], obtained the multiplier transformation $D_{\alpha}^{m} f$ given by

$$D_{\alpha}^{m} f(z) = z + \sum_{n=2}^{\infty} \alpha \left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^{m} a_n z^n \quad (4)$$

Where$D_{\alpha}^{m+1} f(z) = (1-\lambda)D_{\alpha}^{m} f(z) + z\lambda(D_{\alpha}^{m} f(z))'$
Such that
$z\lambda(D_{\alpha}^{m} f(z))' = D_{\alpha}^{m+1} f(z)$-$(1-\lambda)D_{\alpha}^{m} f(z)$ (5)
We denote $D_{p,\alpha}^{m}$fby

$$D_{p,\alpha}^{m} f(z) = z^p + \sum_{n=p+1}^{\infty} \alpha \left(\frac{1+\lambda(n+\alpha-2}{1+\lambda(\alpha-1)}\right)^{m} a_n z^n \quad (6)$$

We denote byH_p, the class of all functions which are analytic and p-valentin U for which $\psi(U)$ is convex such that $\psi(0) = 1$ and $Re(\psi(z)) > 0, z\epsilon U$

We denote by $S^*(\mu,\psi), C(\mu,\psi)$, and $K(\mu,\zeta;\varphi,\psi)$ the subclasses of starlike, convex and close-to-convex functions of order μrespectively, for the function $\psi,\varphi\epsilon H_p$which are defined by:

$S_{p,\alpha}^{m}(\mu,\psi) = \{f \in A: D_{p,\alpha}^{m} f(z)\epsilon S^*(\mu,\psi)\}$,
$C_{p,\alpha}^{m}(\mu,\psi) = \{f\epsilon A: D_{p,\alpha}^{m} f(z)\epsilon C^*(\mu,\psi)\}$,
$K_{p,\alpha}^{m}(\mu,\zeta;\varphi,\psi) = \{f\epsilon A: D_{p,\alpha}^{m} f(z)\epsilon K(\mu,\zeta;\varphi,\psi\}$,

In this paper, we shall investigate inclusion properties for the multiplier transform $D_{p,\alpha}^{m} f$with respect to starlike, convex and close-to-convex functions usingprinciple of subordination.

Next, we give the preliminary results that we shall employ to prove ourmain results.

Lemma 1: [2, 3, 8, 10]: Let ϕbe convex, univalent in $U$with $\phi(0) = 1$ and
$Re\{k\phi(z) + \gamma\} \geqq 0,\ k,\gamma\epsilon C$. If p is analytic in $U$with$p(0) = 1$, then
$p(z) + \frac{zp'(z)}{kp(z)+\gamma} \prec \phi(z),\ z \in U$implies $p(z) \prec \phi(z),\ z \in U$

Lemma 2: [6, 10]: Let ϕ be convex, univalent in U and w be analytic in U with
Re $(w(z)) \geqq$. If p is analytic in $U$with $p(0) = \phi(0)$, then
$p(z) + w(z)zp'(z) \prec \phi(z),\ z \in U$Implies$p(z) \prec \phi(z),\ z \in U$

In what follows, we give some inclusion properties of the operator$D_{p,\alpha}^{m} f$usingthe principle of subordination.

Inclusion Properties

Theorem 1: Let f belongs to the analytic function of the form (1) and let
$\varphi \in H_p$with $Re\left\{\big(p(1-\mu)\big)\psi(z) + \mu + \frac{1-\lambda}{\lambda}\right\} > 0$. Then,

$$S_{p,\alpha}^{m+1}(\mu,\psi) \subset S_{p,\alpha}^{m}(\mu,\psi)$$

Proof: Let f belongs to the class $S_{p,\alpha}^{m+1}(\mu,\psi)$ and let

$$p(z) = \frac{1}{p(1-\mu)}\left(\frac{z\big(D_{p,\alpha}^{m} f(z)\big)'}{D_{p,\alpha}^{m} f(z)} - \mu\right) \quad (7)$$

Applying (5) in (7), we obtain:

$$\frac{D_{p,\alpha}^{m+1} f(z) - D_{p,\alpha}^{m+1} f(z) + \lambda D_{p,\alpha}^{m+1} f(z)}{\lambda D_{p,\alpha}^{m+1`} f(z)} = p(1-\mu)p(z) + \mu$$

From where we have

$$\frac{D_{p,\alpha}^{m+1}f(z)}{\lambda D_{p,\alpha}^{m}f(z)} = \big(p(1-\mu)\big)p(z) + \mu + \frac{1-\lambda}{\lambda} \tag{8}$$

From (8), we obtain

$$\frac{(D_{p,\alpha}^{m+1}f(z))'}{D_{p,\alpha}^{m+1}f(z)} = \frac{(D_{p,\alpha}^{m`}f(z))'}{D_{p,\alpha}^{m`}f(z)} + \frac{\big(p(1-\mu)\big)p'(z)}{\big(p(1-\mu)\big)p(z) + \mu + \frac{1-\lambda}{\lambda}} \tag{9}$$

But

$$\frac{(D_{p,\alpha}^{m}f(z))'}{D_{p,\alpha}^{m`}f(z)} = \frac{\big(p(1-\mu)\big)p(z) + \mu}{z} \tag{10}$$

Using (9) and (10), we obtain

$$\frac{1}{p(1-\mu)}\left(\frac{z(D_{p,\alpha}^{m+1}f(z))'}{D_{p,\alpha}^{m+1}f(z)} - \mu\right) = p(z) + \frac{zp^{'}(z)}{\big(p(1-\mu)\big)p(z) + \mu + \frac{1-\lambda}{\lambda}} \tag{11}$$

Applying Lemma 1 to (11) shows that

$$p(z) \prec \phi(z), i.e. f\epsilon D_{p,\alpha}^{m+1`}f(z)$$

Thus,

$$S_{\alpha}^{m+1}(\mu,\psi) \subset S_{\alpha}^{m}(\mu,\psi)$$

Theorem 2: Let f belongs to the analytic function of the form (1) and let$\psi \in H_p$ with Re$\big\{\big(p(1-\mu)\big)\psi(z) +$ μ+1−$\lambda\lambda$> 0. Then,

$$C_{p,\alpha}^{m+1}(\mu,\psi) \subset C_{p,\alpha}^{m}(\mu,\psi)$$

Proof: From Remark 1, we have

$$f\epsilon C_{p,\alpha}^{m+1}(\mu,\psi) \Leftrightarrow zf'\epsilon S_{p,\alpha}^{m+1}(\mu,\psi)$$

and from Theorem 1, we have

$$\begin{aligned} f\epsilon C_{p,\alpha}^{m+1}(\mu,\psi) &\Leftrightarrow zf'\epsilon S_{p,\alpha}^{m+1}(\mu,\psi) \subset S_{p,\alpha}^{m}(\mu,\psi \\ &\Rightarrow zf'\epsilon S_{p,\alpha}^{m}(\mu,\psi) \\ &\Rightarrow f\epsilon C_{p,\alpha}^{m}(\mu,\psi) \end{aligned}$$

Thus,

$$C_{p,\alpha}^{m+1}(\mu,\psi) \subset C_{p,\alpha}^{m}(\mu,\psi)$$

The function $\psi(z) = \frac{1-Az}{1+Bz}$is analytic and satisfies $\psi(0) = 1$. Thus, we have the following corollaries.

Corollary 3: Let $f\epsilon A$ and $\psi(z) = \frac{1-Az}{1+Bz}, -1 \le B \le A \le 1$ in Theorem 1.

Then

$$S_{p,\alpha}^{m+1}(\mu,A,B) \subset S_{p,\alpha}^{m}(\mu,A,B)$$

Corollary 4: Let$f\epsilon A$ and $\psi(z) = \frac{1-Az}{1+Bz}, -1 \le B \le A \le 1$ in Theorem 2.

Then

$$C_{p,\alpha}^{m+1}(\mu,A,B) \subset C_{p,\alpha}^{m}(\mu,A,B)$$

Theorem 5: Let f belongs to the analytic function of the form (1) and let $\psi,\varphi \in H_p$ with Re$\left\{\big(p(1-\mu)\big)\psi(z) + \mu + \frac{1-\lambda}{\lambda}\right\} > 0$. Then,

$$K_{p,\alpha}^{m+1}(\mu,\zeta;\varphi,\psi) \subset K_{p,\alpha}^{m}(\mu,\zeta;\varphi,\psi)$$

Proof. Let$f \in K_{p,\alpha}^{m+1}(\mu,\zeta;\varphi,\psi)$, then there must exist a function $g \in S_{p,\alpha}^{m+1}(\mu,\zeta;\varphi,\psi$)such that Re$\left\{\frac{z(D_{p,\alpha}^{m+1}f(z))'}{D_{p,\alpha}^{m+1}g(z)}\right\} > \zeta, z \in U$

That is, we should have

$$\frac{1}{p(1-\zeta)}\left(\frac{z(D_{p,\alpha}^{m+1}f(z))'}{D_{p,\alpha}^{m+1}g(z)} - \zeta\right) \prec \varphi, z \in U$$

Let

$$p(z) = \frac{1}{p(1-\zeta)}\left(\frac{z(D_{p,\alpha}^{m}f(z))'}{D_{p,\alpha}^{m}g(z)} - \zeta\right) \tag{12}$$

From (5), we have

$$z\left(D_{p,\alpha}^{m}f(z)\right)^{'} = \frac{D_{p,\alpha}^{m+1}f(z) - (1-\lambda)D_{p,\alpha}^{m}f(z)}{\lambda}$$

Now, from (5) we have:

$$\frac{D_{p,\alpha}^{m+1}f(z)}{\lambda} = \frac{1-\lambda}{\lambda}\left(D_{p,\alpha}^{m}f(z)\right) + ((p-\zeta)p(z)+\zeta)D_{p,\alpha}^{m}g(z)$$

This implies that

$$\frac{z(D_{p,\alpha}^{m+1}f(z))'}{\lambda}=\frac{1-\lambda}{\lambda}z\left(D_{p,\alpha}^{m}f(z)\right)'+\left(p(1-\zeta)zp'^{(z)}\right)D_{p,\alpha}^{m}g(z)$$
$$+\left(p(1-\zeta)p(z)+\zeta\right)z\left[D_{p,\alpha}^{m}g(z)\right]' \qquad (13)$$

Also, by Theorem 1$g\in S_{p,\alpha}^{m+1}(\mu,\psi)\Rightarrow g\in S_{p,\alpha}^{m}(\zeta,\psi)$

Now, let

$$q(z)=\frac{1}{p(1-\zeta)}\left(\frac{z(D_{p,\alpha}^{m}g(z))'}{D_{p,\alpha}^{m}g(z)}-\zeta\right) \qquad (14)$$

Using (5) in (14), we obtain

$$\frac{D_{p,\alpha}^{m+1}g(z)}{\lambda D_{p,\alpha}^{m}g(z)}=\left(p(1-\zeta)\right)q(z)+\zeta+\frac{1-\lambda}{\lambda} \qquad (15)$$

and further, from (13) and (15), we obtain

$$\frac{z(D_{p,\alpha}^{m+1}f(z))'}{D_{p,\alpha}^{m+1}g(z)}=\left(p(1-\zeta)\right)p(z)+\zeta+\frac{\left(p(1-\mu)\right)p'(z)}{\left(p(1-\mu)\right)q(z)+\mu+\frac{1-\lambda}{\lambda}} \qquad (16)$$

But

Algebraic manipulation in (16) gives

$$\frac{1}{p(1-\zeta)}\left(\frac{z(D_{p,\alpha}^{m+1}f(z))'}{D_{p,\alpha}^{m+1}f(z)}-\zeta\right)=p(z)+\frac{zp'(z)}{\left(p(1-\mu)\right)q(z)+\mu+\frac{1-\lambda}{\lambda}} (11)$$

Thus, making

$$\frac{1}{\left(p(1-\mu)\right)q(z)+\mu+\frac{1-\lambda}{\lambda}}=w(z)$$

and apply Lemma 2, we have

$$p(z)\prec\varphi(z), \text{i.e.} f\in K_{p,\alpha}^{m+1}(\mu,\zeta;\varphi,\psi)$$

This proves the theorem.

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