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Research Paper

Research on Fractional Cauchy's Mean Value Theorem

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ABSTRACT:Based on *Jumarie's modified* Riemann-Liouville(*R-L*) *fractional derivative, this article studies the fractional Cauchy's mean value theorem. On the other hand, we also prove the fractional L'Hospital's rule by using fractional Cauchy's mean value theorem. The fractional analytic functionand the fractional Rolle's theoremplay important roles in this paper. In fact, these resultswe obtained are generalizations of those in traditional calculus.*
KEYWORDS: *JumarieModified R-L Fractional Derivative, Fractional Cauchy's Mean Value Theorem, Fractional L'Hospital's Rule,Fractional Analytic Function, Fractional Rolle's Theorem.*

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I. INTRODUCTION

In applied mathematics and mathematical analysis, a fractional derivative is a derivative of any arbitrary order, whether it is a real number or a complex number.The concept of fractional operators has beenintroduced almost simultaneously with the development of the classical calculus. The first known reference can befound in the letter of Leibniz and L'Hospital in 1695, which raised the question ofthe meaning of the semi-derivative. This problemhas thus attracted the interest of a lot of famous mathematicians, including Euler, Liouville, Laplace, Riemann, Riesz, Grünwald, Letnikov, Weyland many others.Since the 19th century, the theory of fractional calculus has developed rapidly, mainly as a foundation for a numberof applied disciplines, including fractional differential equations and fractionaldynamics. The applications of fractional calculus are very wide nowadays. Almost all modernengineering and science has been affected by the tools and techniques of fractional calculus. Forexample, it can find wide and fruitful applications in economics, viscoelasticity,statistical physics, quantum mechanics,robotics, control theory, electrical and mechanical engineering, bioengineering, and so on [1-12]. The applications of fractional calculus to fractional differential equations can refer to [13-15].

However, different from the traditional calculus, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definitions are Riemann-Liouville (R-L) fractional derivatives, Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [16-19].

In this paper, we prove the fractional Cauchy's mean value theorem based on Jumarie type of R-L fractional derivative.Moreover, we also use the fractional Cauchy's mean value theorem to prove the fractional L'Hospital's rule. On the other hand, the fractional Rolle's theorem and theconcept of fractional analytic function play important roles in this study. In fact, the fractional Cauchy'smean value theorem is the generalization of Cauchy'smean value theorem in classical calculus.

II. DEFINITIONS AND PROPERTIES

First,the fractional calculus used in this paper and some properties are introduced below.

Definition 2.1 ([20])**:** Suppose that $0<\alpha\leq 1$, and x_0 is a real number. TheJumarie'smodified R-L α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt. \quad (1)$$

where$\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([21])**:** *If*α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \quad (2)$$

and

$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0.$ (3)

Definition 2.3([22])**:**Let x, x_0, anda_k be real numbers for all k, $x_0 \in (a,b), 0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$can be expressed as anα-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha}$ on some open interval $(x_0 - r, x_0 + r)$, then $f_\alpha(x^\alpha)$ is calledα-fractional analytic atx_0, where r is the radius of convergence about x_0.Moreover, if $f_\alpha: [a,b] \to R$is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then $f_\alpha$is called an α-fractional analytic function on$[a,b]$.

Theorem 2.4(fractional Rolle's theorem)([23])**:** *Let*$0 < \alpha \leq 1$*and*$(-1)^\alpha = -1$. *If* $\varphi_\alpha(x^\alpha)$*is a*α*-fractional analytic function on closed interval* $[a,b]$ *with* $\varphi_\alpha(a^\alpha) = \varphi_\alpha(b^\alpha)$, *then there exists* $\xi \in (a,b)$ *such that* $\left({}_{\xi}D_x^{\alpha}\right)[\varphi_\alpha(x^\alpha)](\xi^\alpha) = 0$.

III. MAIN RESULT AND APPLICATION

In the following, we provethe major result of this paper.

Theorem 3.1(fractional Cauchy's mean value theorem)**:***Assume that*$0 < \alpha \leq 1$, $a < b$, *and*$f_\alpha(x^\alpha)$,$g_\alpha(x^\alpha)$*are* α*-fractional analytic functions on* $[a,b]$. *If*$\left({}_{c}D_x^{\alpha}\right)[g_\alpha(x^\alpha)](c) \neq 0$*for all* $c \in (a,b)$, *then there exists* $\xi \in (a,b)$*such that*

$$\frac{f_\alpha(b^\alpha) - f_\alpha(a^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)} = \frac{\left({}_{\xi}D_x^{\alpha}\right)[f_\alpha(x^\alpha)](\xi^\alpha)}{\left({}_{\xi}D_x^{\alpha}\right)[g_\alpha(x^\alpha)](\xi^\alpha)}. \quad (4)$$

Proof Let

$$\varphi_\alpha(x^\alpha) = f_\alpha(x^\alpha) - \frac{f_\alpha(b^\alpha) - f_\alpha(a^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)} \cdot g_\alpha(x^\alpha). \quad (5)$$

Then

$$\varphi_\alpha(a^\alpha) = f_\alpha(a^\alpha) - \frac{f_\alpha(b^\alpha) - f_\alpha(a^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)} \cdot g_\alpha(a^\alpha)$$

$$= \frac{g_\alpha(b^\alpha) \cdot f_\alpha(a^\alpha) - g_\alpha(a^\alpha) \cdot f_\alpha(b^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)}. \quad (6)$$

And

$$\varphi_\alpha(b^\alpha) = f_\alpha(b^\alpha) - \frac{f_\alpha(b^\alpha) - f_\alpha(a^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)} \cdot g_\alpha(b^\alpha)$$

$$= \frac{g_\alpha(b^\alpha) \cdot f_\alpha(a^\alpha) - g_\alpha(a^\alpha) \cdot f_\alpha(b^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)}. \quad (7)$$

Therefore, $\varphi_\alpha(a^\alpha) = \varphi_\alpha(b^\alpha)$. By fractional Rolle's theorem, there exists $\xi \in (a,b)$ such that$\left({}_{\xi}D_x^{\alpha}\right)[\varphi_\alpha(x^\alpha)](\xi^\alpha) = 0$. Thus,

$$\left({}_{\xi}D_x^{\alpha}\right)[f_\alpha(x^\alpha)](\xi^\alpha) = \frac{f_\alpha(b^\alpha) - f_\alpha(a^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)} \cdot \left({}_{\xi}D_x^{\alpha}\right)[g_\alpha(x^\alpha)](\xi^\alpha). \quad (8)$$

And hence,

$$\frac{f_\alpha(b^\alpha) - f_\alpha(a^\alpha)}{g_\alpha(b^\alpha) - g_\alpha(a^\alpha)} = \frac{\left({}_{\xi}D_x^{\alpha}\right)[f_\alpha(x^\alpha)](\xi^\alpha)}{\left({}_{\xi}D_x^{\alpha}\right)[g_\alpha(x^\alpha)](\xi^\alpha)}.$$ Q.e.d.

Next, we provide an application of fractional Cauchy's mean value theorem.

Theorem 3.2(fractional L'Hospital's rule)**:***Assume that*$0 < \alpha \leq 1$, p*is a real number, and*$f_\alpha(x^\alpha)$,$g_\alpha(x^\alpha)$*are* α*-fractional analytic functions at* p. *If* $\lim_{x \to p} f_\alpha(x^\alpha) = 0$, $\lim_{x \to p} g_\alpha(x^\alpha) = 0$, *and* $\lim_{x \to p} \frac{\left({}_{p}D_x^{\alpha}\right)[f_\alpha(x^\alpha)]}{\left({}_{p}D_x^{\alpha}\right)[g_\alpha(x^\alpha)]}$ *exists. Then*$\lim_{x \to p} \frac{f_\alpha(x^\alpha)}{g_\alpha(x^\alpha)}$*exists, and*

$$\lim_{x \to p} \frac{f_\alpha(x^\alpha)}{g_\alpha(x^\alpha)} = \lim_{x \to p} \frac{\left({}_{p}D_x^{\alpha}\right)[f_\alpha(x^\alpha)]}{\left({}_{p}D_x^{\alpha}\right)[g_\alpha(x^\alpha)]}. \quad (9)$$

Proof We may assume that $f_\alpha(p^\alpha) = g_\alpha(p^\alpha) = 0$. Thus, by fractional Cauchy's mean value theorem, there exists

ξ between p and x such that

$$\frac{f_\alpha(x^\alpha)}{g_\alpha(x^\alpha)}=\frac{f_\alpha(x^\alpha)-f_\alpha(p^\alpha)}{g_\alpha(x^\alpha)-g_\alpha(p^\alpha)}=\frac{\left({}_{\xi}D_x^\alpha\right)[f_\alpha(x^\alpha)](\xi^\alpha)}{\left({}_{\xi}D_x^\alpha\right)[g_\alpha(x^\alpha)](\xi^\alpha)}. \qquad (10)$$

Therefore,

$$\lim_{x\to p}\frac{f_\alpha(x^\alpha)}{g_\alpha(x^\alpha)}=\lim_{x\to p}\frac{f_\alpha(x^\alpha)-f_\alpha(p^\alpha)}{g_\alpha(x^\alpha)-g_\alpha(p^\alpha)}=\lim_{x\to p}\frac{\left({}_{\xi}D_x^\alpha\right)[f_\alpha(x^\alpha)](\xi^\alpha)}{\left({}_{\xi}D_x^\alpha\right)[g_\alpha(x^\alpha)](\xi^\alpha)}=\lim_{x\to p}\frac{\left({}_{p}D_x^\alpha\right)[f_\alpha(x^\alpha)]}{\left({}_{p}D_x^\alpha\right)[g_\alpha(x^\alpha)]}. \quad \text{Q.e.d.}$$

Remark 3.3: In Theorem 3.2, the real number c can be replaced by c^+, c^-, $+\infty$, or $-\infty$. And $\lim_{x\to c} f_\alpha(x^\alpha)$ and $\lim_{x\to c} g_\alpha(x^\alpha)$ may be $+\infty$, or $-\infty$.

IV. CONCLUSION

This papermainly studiesthe fractional Cauchy's mean value theorem based on Jumarie type ofR-L fractional derivative.In addition, we use the fractional Cauchy's mean value theorem to prove the fractional L'Hospital's rule. The fractional Rolle's theorem and the fractional analytic function play important roles in this article.In the future, we will continue to use fractional Cauchy's mean value theoremand fractional L'Hospital's rule to solve the problems in fractional calculus and applied mathematics.

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