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Review Paper

Solution of Third Order Non-Homogeneous Ordinary Differential Equations by Lie Symmetry Method

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ABSTRACT: *In present paper, we solved a third order non-homogeneous ordinary differential equation with the use of Lie symmetry method (LSM). Third order non-homogeneous ordinary differential equation (ODE) have a vital role in applied mathematics, applied sciences etc. LSM reduces the order of third order ODE to second order ODE then second order ODE transform into first order linear ODE, which is solvable by any known method. LSM is the general method for solving ODE, it gives exact solution of the given ODE. Which is shown in this paper by an example.*
KEYWORDS: *Lie Symmetry method, Infinitesimal generators, Third order non-homogeneous ODE etc.*

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I. INTRODUCTION

Differential equations are mathematical equations that involve one or more derivatives of an unknown function. They describe the relationship between a function and its derivatives and are widely used to model various physical, biological and engineering etc. Differential equations play a crucial role in understanding dynamic systems. Differential equations are used in various scientific disciplines to model phenomena such as heat conduction, fluid dynamics, population growth, quantum mechanics and electrical circuits. Solving these equations can provide valuable insights into the behavior of systems and help to make predictions or optimize designs. Various methods, including analytical techniques and numerical simulations, are employed to solve differential equations depending on their complexity and nature. There are various methods for solving third order non-homogeneous ODE. In this work, we shall apply Lie symmetry method to find the solution of given differential equations.

The Lie symmetry method provides a systematic and elegant approach to solve differential equations, offering insights into their structure and yielding solutions that might be challenging to obtain through other methods. It has applications across various fields, including physics, engineering and mathematical modeling. The Lie symmetry method involves finding a Lie group of transformations that leaves the differential equation invariant. These transformations can be represented by a Lie algebra, and their corresponding infinitesimal generators are used to determine the symmetries.

II. LITERATURE REVIEW

F. M. Mahomed and P.G.L. Leach (1989) studied about the second order ODE and their Lie algebras. In this paper, they showed that a second order ODE can have exactly $0,1,2,3$ or 8 point symmetries. **F. M. Mahomed and P.G.L. Leach (1990)** studied about the symmetries of differential equations. In this work, they showed that any differential equations of order $(n \geq 3)$ can't have more than $(n+4)$ symmetries. Which is very useful for our research. **R.Z. Zhdanov (1998)** developed lie theory for first, second order ODE. He solved many examples of first and second order linear and non-linear differential equations. **S. Hasan and W. Salman Abd (2016)** used lie symmetry theory for the systems of ODE. They defined this method for homogeneous systems and linear differential equations of systems. **Mousa Illie and Jafar Biazar (2017)** used the lie symmetry theory for second-order fractional differential equation. They solved fractional differential equation with the help of Lie symmetry theory. **R. Mohanasubha and V.K. Chandrasekar(2017)** derived the linearized symmetry condition for second-order non-linear ODE. **G. Kumar (2019)** solved first order homogeneous ODE with use of Lie symmetry theory. **W. Khalid Jaber and K. Salman Hasan (2020)** In this paper, they used Lie symmetry for solving first and second order linear differential equations. They found the linearized symmetry condition for first order linear ODE. **N. Sharma and G. Kumar (2022)** described the lie symmetry method to solve linear ODE of first order. **A. Kumar and G. Kumar (2023)** solved third order linear ODE with the help of lie symmetry theory.

III. EXAMPLE

We consider a special type of third order ode which is given as

$$y''' = 1 \tag{1}$$

Let $$\mathrm{H}(y''', y'', y', y, x) = (y''' - 1) = 0 \tag{2}$$

we shall use third extension of $S^{[3]}$, thus we have

$$S^{[3]} = S^{[2]} + (\gamma''' - 3y'''\delta' - y'\delta''' - 3y''\delta'')\frac{\partial}{\partial y'''}$$

$$S^{[3]} = S^{[1]} + (\gamma'' - 2y''\delta' - y'\delta'')\frac{\partial}{\partial y''} + (\gamma''' - 3y'''\delta' - y'\delta''' - 3y''\delta'')\frac{\partial}{\partial y'''}$$

$$S^{[3]} = \delta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} + (\gamma' - y'\delta')\frac{\partial}{\partial y'} + (\gamma'' - y'\delta'' - 2y''\delta')\frac{\partial}{\partial y''} + (\gamma''' - 3y'''\delta' - y'\delta''' - 3y''\delta'')\frac{\partial}{\partial y'''} \tag{3}$$

now we operate $S^{[3]}$ on equation (2) then we get

$$S^{[3]}[H] = 0$$

$$\left[\delta(y'''') + \gamma.0 + (\gamma' - y'\delta').0 + (\gamma'' - y'\delta'' - 2y''\delta').0 + (\gamma''' - 3y'''\delta' - 3y''\delta'' - y'\delta''').(1)\right] = 0$$

$$[\delta(y'''') + (\gamma''' - 3y'''\delta' - y'\delta''' - 3y''\delta'')] = 0 \tag{4}$$

Differentiate equation (2) with respect to $x,$ we get

$$\frac{\partial H}{\partial x} = y'''' = 0 \tag{5}$$

Putting the values from equation (5) in equation (4) we get

$$[\delta.0 + (\gamma''' - 3y'''\delta' - y'\delta''' - 3y''\delta'')] = 0$$

$$[\gamma''' - 3y'''\delta' - y'\delta''' - 3y''\delta''] = 0 \tag{6}$$

The derivatives of γ and δ are

$$\gamma' = \frac{\partial\gamma}{\partial x} + y'\frac{\partial\gamma}{\partial y} \tag{7}$$

since $\left\{\frac{d\gamma}{dx} = \frac{\partial\gamma}{\partial x} + \frac{\partial\gamma}{\partial y}.\left(\frac{\partial y}{\partial x}\right)\right\}$

$$\gamma'' = \frac{d}{dx}\left(\frac{\partial\gamma}{\partial x} + y'\frac{\partial\gamma}{\partial y}\right)$$

$$\gamma'' = \frac{\partial^2\gamma}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial\gamma}{\partial x}\right).\frac{\partial y}{\partial x} + y'\left\{\frac{\partial}{\partial x}\left(\frac{\partial\gamma}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\gamma}{\partial y}\right).\frac{\partial y}{\partial x}\right\} + \frac{\partial\gamma}{\partial y}.y''$$

$$\gamma'' = \frac{\partial^2\gamma}{\partial x^2} + \frac{\partial^2\gamma}{\partial x\partial y}.y' + y'.\frac{\partial^2\gamma}{\partial x\partial y} + y'^2.\frac{\partial^2\gamma}{\partial y^2} + \frac{\partial\gamma}{\partial y}.y''$$

$$\gamma'' = \frac{\partial^2\gamma}{\partial x^2} + 2y'\frac{\partial^2\gamma}{\partial x\partial y} + y'^2\frac{\partial^2\gamma}{\partial y^2} + \frac{\partial\gamma}{\partial y}y'' \tag{8}$$

and

$$\gamma''' = \frac{\partial^3 \gamma}{\partial x^3} + 3y'\frac{\partial^3 \gamma}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \gamma}{\partial x \partial y} + y'''\frac{\partial \gamma}{\partial y} + 3y'^2\frac{\partial^3 \gamma}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \gamma}{\partial y^2} + y'^3\frac{\partial^3 \gamma}{\partial y^3} \quad (9)$$

Similarly, we have

$$\delta' = \frac{\partial \delta}{\partial x} + y'\frac{\partial \delta}{\partial y} \quad (10)$$

$$\delta'' = \frac{d}{dx}\left(\frac{\partial \delta}{\partial x} + y'\frac{\partial \delta}{\partial y}\right)$$

$$\delta'' = \frac{\partial^2 \delta}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial \delta}{\partial x}\right).\frac{\partial y}{\partial x} + y'\left\{\frac{\partial}{\partial x}\left(\frac{\partial \delta}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \delta}{\partial y}\right).\frac{\partial y}{\partial x}\right\} + \frac{\partial \delta}{\partial y}.y''$$

$$\delta'' = \frac{\partial^2 \delta}{\partial x^2} + \frac{\partial^2 \delta}{\partial x \partial y}.y' + y'.\frac{\partial^2 \delta}{\partial x \partial y} + y'^2.\frac{\partial^2 \delta}{\partial y^2} + \frac{\partial \delta}{\partial y}.y''$$

$$\delta'' = \frac{\partial^2 \delta}{\partial x^2} + 2y'\frac{\partial^2 \delta}{\partial x \partial y} + y'^2\frac{\partial^2 \delta}{\partial y^2} + \frac{\partial \delta}{\partial y}y'' \quad (11)$$

And

$$\delta''' = \frac{\partial^3 \delta}{\partial x^3} + 3y'\frac{\partial^3 \delta}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \delta}{\partial x \partial y} + y'''\frac{\partial \delta}{\partial y} + 3y'^2\frac{\partial^3 \delta}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \delta}{\partial y^2} + y'^3\frac{\partial^3 \delta}{\partial y^3} \quad (12)$$

Substitute all values of derivatives in equation (6) we get

$$\left[\frac{\partial^3 \gamma}{\partial x^3} + 3y'\frac{\partial^3 \gamma}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \gamma}{\partial x \partial y} + y'''\frac{\partial \gamma}{\partial y} + 3y'^2\frac{\partial^3 \gamma}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \gamma}{\partial y^2} + y'^3\frac{\partial^3 \gamma}{\partial y^3}\right.$$

$$-3y''\left(\frac{\partial^2 \delta}{\partial x^2} + 2y'\frac{\partial^2 \delta}{\partial x \partial y} + y'^2\frac{\partial^2 \delta}{\partial y^2} + y''\frac{\partial \delta}{\partial y}\right)$$

$$-y'\left(\frac{\partial^3 \delta}{\partial x^3} + 3y'\frac{\partial^3 \delta}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \delta}{\partial x \partial y} + y'''\frac{\partial \delta}{\partial y} + 3y'^2\frac{\partial^3 \delta}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \delta}{\partial y^2} + y'^3\frac{\partial^3 \delta}{\partial y^3}\right)$$

$$\left.-3y'''\left(\frac{\partial \delta}{\partial x} + y'\frac{\partial \delta}{\partial y}\right)\right] = 0 \quad (13)$$

$$\left[\frac{\partial^3 \gamma}{\partial x^3} + 3y'\frac{\partial^3 \gamma}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \gamma}{\partial x \partial y} + y'''\frac{\partial \gamma}{\partial y} + 3y'^2\frac{\partial^3 \gamma}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \gamma}{\partial y^2} + y'^3\frac{\partial^3 \gamma}{\partial y^3}\right.$$

$$-3y''\frac{\partial^2 \delta}{\partial x^2} - 6y'y''\frac{\partial^2 \delta}{\partial x \partial y} - 3y'^2 y''\frac{\partial^2 \delta}{\partial y^2} - 3(y'')^2\frac{\partial \delta}{\partial y} - 3y'''\frac{\partial \delta}{\partial x} - 3y'y'''\frac{\partial \delta}{\partial y}$$

$$\left.-y'\frac{\partial^3 \delta}{\partial x^3} - 3y'^2\frac{\partial^3 \delta}{\partial x^2 \partial y} - 3y'y''\frac{\partial^2 \delta}{\partial x \partial y} - y'y'''\frac{\partial \delta}{\partial y} - 3y'^3\frac{\partial^3 \delta}{\partial x \partial y^2} - 3y'^2 y''\frac{\partial^2 \delta}{\partial y^2} - y'^4\frac{\partial^3 \delta}{\partial y^3}\right] = 0 \quad (14)$$

Now we equate the coefficient of various powers of y, y', y'', y''' etc.

Coefficient of $y'y'''$:

$$-4\frac{\partial \delta}{\partial y} = 0$$

$$\delta = a(x) \quad (15)$$

Coefficient of y''' :

$$\left(\frac{\partial \gamma}{\partial y} - 3\frac{\partial \delta}{\partial x}\right) = 0$$

$$\frac{\partial\gamma}{\partial y}=3a'$$

$$\gamma=3a'y+b(x) \tag{16}$$

Coefficient of y'' :

$$\left(3\frac{\partial^2\gamma}{\partial x\partial y}-3\frac{\partial^2\delta}{\partial x^2}\right)=0$$

$$\frac{\partial^2\gamma}{\partial x\partial y}=\frac{\partial^2\delta}{\partial x^2}$$

$$3a''=a''$$

$$a(x)=h_1x+h_2 \tag{17}$$

Coefficient of constants:

$$\frac{\partial^3\gamma}{\partial x^3}=0$$

$$\left(3a''''y+b'''\right)=0 \tag{18}$$

Equate constant

$$b'''=0$$

After solving, we get

$$b(x)=h_3\frac{x^2}{2}+h_4x+h_5 \tag{19}$$

Where h_1,h_2,h_3,h_4,h_5 are arbitrary constants.

By equation (15) we have

$$\delta=h_1x+h_2 \tag{20}$$

By equation (16) we have

$$\gamma=3a'y+b(x)$$

$$\gamma=3(h_1x+h_2)'y+\left(h_3\frac{x^2}{2}+h_4x+h_5\right)$$

$$\gamma=3(h_1)y+\left(h_3\frac{x^2}{2}+h_4x+h_5\right) \tag{21}$$

The Generator S of the infinitesimal transformation is of the form,

$$S=\delta\frac{\partial}{\partial x}+\gamma\frac{\partial}{\partial y}$$

$$S=\left(h_1x+h_2\right)\frac{\partial}{\partial x}+\left(3h_1y+h_3\frac{x^2}{2}+h_4x+h_5\right)\frac{\partial}{\partial y} \tag{22}$$

Which is five parameter symmetry are given as

$$S_1=\left(x\frac{\partial}{\partial x}+3y\frac{\partial}{\partial y}\right)$$

$$S_2=\frac{\partial}{\partial x}$$

$$S_3=\frac{x^2}{2}\frac{\partial}{\partial y}$$

$$S_4=x\frac{\partial}{\partial y}$$

$$S_5=\frac{\partial}{\partial y} \tag{23}$$

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These all are infinitesimal generators for equation (1)

Now, we take

$$S_4 = x\frac{\partial}{\partial y}$$

The given equation has five Lie one parameter symmetry (23)

Third order prolongation is obtained as follows

$$S_4^{[0]} = x.\frac{\partial}{\partial y} + 0.\frac{\partial}{\partial x} = x\frac{\partial}{\partial y} \tag{24}$$

$$S_4^{[1]} = S_4^{[0]} + (1 - y'.0)\frac{\partial}{\partial y'} = x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \tag{25}$$

$$S_4^{[2]} = S_4^{[1]} + (0 - 2y''.0 - y'.0)\frac{\partial}{\partial y''}$$

$$S_4^{[2]} = x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} - 0.\frac{\partial}{\partial y''} \tag{26}$$

$$S_4^{[3]} = S_4^{[2]} + (0 - 3y'''.0 - 3y''.0 - y'.0)\frac{\partial}{\partial y'''}$$

$$S_4^{[3]} = x\frac{\partial}{\partial y} + 0.\frac{\partial}{\partial x} + \frac{\partial}{\partial y'} - 0.\frac{\partial}{\partial y''} + 0.\frac{\partial}{\partial y'''} \tag{27}$$

Now, we must solve

$$\frac{dx}{0} = \frac{dy}{x} = \frac{dy'}{1} = \frac{dy''}{0} = \frac{dy'''}{0} \tag{28}$$

Case 1st :

$$\frac{dx}{0} = \frac{dy}{x}$$

$$x = u \tag{29}$$

Where u is a constant.

Case 2nd :

$$\frac{dy}{x} = \frac{dy'}{1}$$

Integrating on both side, we get

$$\int \frac{dy}{u} = \int \frac{dy'}{1}$$

$$\frac{y}{u} = y' + v_1$$

$$v_1 = \frac{y}{x} - y' \tag{30}$$

Where v_1 is a constant.

Case 3rd :

$$\frac{dy'}{1} = \frac{dy''}{0}$$

On solving, we get

$$y'' = v \tag{31}$$

Where v is a constant.

Equation (1) can be reduced as follows

$$\frac{dv}{du} = \frac{D_x(v)}{D_x(u)}$$

$$\frac{dv}{du} = \frac{y'''}{1} \tag{32}$$

Using equation (1) in equation (32) then

$$\frac{dv}{du} = 1$$
$$dv = du$$

Integrating on bothsides

$$v = u + t_1$$
$$y'' = x + t_1$$
$$y' = \frac{x^2}{2} + t_1 x + t_2$$
$$y = \frac{x^3}{6} + t_1 \frac{x^2}{2} + t_2 x + t_3 \qquad (33)$$

Where t_1, t_2, t_3 are arbitrary constants.

Equation (33) gives us the solution of equation (1) by the help of Lie symmetry method.

IV. CONCLUSION

In this work, we solved third order non-homogeneous ordinary differential equations with the use of Lie symmetry theory. Firstly, we found determining equations, it has five-dimensional symmetry algebra. In which, we took S_4 which is solvable by Lagrange's method. Now, the order of differential equation has been converted into two. After solving this, given differential equation became of first order ODE which is solvable easily. Thus, we found the solution of given third order non-homogeneous ODE with Lie symmetry theory.

REFRENCES

[1]. **D.H. Sattinger and O.L. Weaver, (1986).** "Lie group and algebras with application of physics, geometry and mechanics", *Springer-Verlag*, Berlin, New York.

[2]. **P.J. Olver, P. Rosenau, (1987).** "Group-invariant solutions of differential equations". *SIAM J. Appl. Math.* Vol. 47, Page (263-278).

[3]. **F.M. Mahomed and P.G.L. Leach, (1987).** "Symmetries of nonlinear differential equations and linearisation". *J. Phys. A.,* Vol. 20, Page (277-292).

[4]. **G. Bluman, (1989). "**Symmetries and Differential equations". *Springer-Verlag*, New York.

[5]. **H. Stephani, (1989).** "Differential equations: their solutions using symmetries". *Cambridge University Press*, New Yor and London.

[6]. **F. M. Mahomed and P.G.L. Leach, (1989).** "Lie algebras associated with second order Ordinary differential equations". *J. Math. Phys.*, Vol. 30, Page (2770-2777).

[7]. **G. Bluman, (1990).** "A reduction algorithm for an ordinary differential equation admitting a solvable Lie group". *SIAM Journal of applied Mathematics.* Vol. 50, Page (1689-1705)

[8]. **P.G.L. Leach and F.M. Mahomed, (1990).** "Symmetry Lie algebras of nth order Ordinary differential equations". *J. Math. Anal. Appl.,* Vol. 151, Page (80-107)

[9]. **J.M. Hill and M.P. Edwards, (1992). "**Some Diffusion with absorption" *Journal of applied Mathematics*. Vol. 48, Page (283-304).

[10]. **P.J. Olver, (1993).** "Applications of lie groups to differential equations". *Springer,* New York.

[11]. **F. Oliveri and A. Donato, (1994).** "Linearization procedure of nonlinear first order Systems of PDE's by means of canonical variables related to Lie groups of point transformations". *J. Math. Anal. Appl.,* Vol. 188, Page (552-568).

[12]. **S.C. Anco., (1996).** "Derivation of conservation laws from non-local Symmetries of differential equations". *Journal of Mathematical physics.* Vol. 37, Page (236 1-2375).

[13]. **Berkovich L.M., (1996).** "The method of an exact linearization of n-order ordinary Differential equations". *J. Nonlinear Math. Phys.*, Vol. 3(3), Page (341-350).

[14]. **G. Baumann, (1997).** "Symmetry analysis of differential equations with mathematica". *Mathl. Comput. Modelling,* Vol. 25(8), Page (25-37).

[15]. **L. Dresner, (1999).** "Applications of Lie's theory of ordinary and partial differential Equations". *Institute of physics publishing*, Bristol.

[16]. **N. H. Ibragimov, (1999).** "Elementary Lie group analysis and Ordinary Differential Equations". *John Willey and Sons*, New York.

[17]. **P.J. Olver, (2000).** "Application of Lie groups to differential equations". *Springer*, New York.

[18]. **M.C. Nucci and V. Torrisi, (2001).** "Application of lie group analysis to Mathematical model which describes HIV transmission, in the geometrical Study of differential equation*". American Mathematical society,* Vol. 285, Page (11-20).

[19]. **P. G. L. Leach and G.P. Flessas, (2001).** "Complete symmetry groups of Ordinary differential equations and their integrals: some basic Considerations". *Journal of mathematical analysis and applications.* Vol. 262(1), Page (256-273).

[20]. **Muriel, C., Romero, J.L., (2001).** "New methods of reduction for ordinary. differential equations". *IMA J. Appl. Math.* Vol. 66, Page (111-125).

[21]. **B.J. Cantwell, (2002).** "Introduction to symmetry analysis*". Cambridge University Press,* New York.

[22]. **G. W. Bluman and S. C. Anco., (2002).** "Symmetries and Integration methods for Differential equations". *Springer-Verlag*, New York.

[23]. **P.E. Hydon, (2005).** "Symmetry analysis of initial value problems*". Journal of mathematical analysis and applications,* Vol. 309, Page (103-116).

[24]. **N. Ibragimov and S. Meleshko, (2005).** "Linearization of third-order ordinary Differential equations". *J. Math. Anal. Appl.,* Vol. 308(1), Page (266-289).

[25]. **K. Eradmann and M.J. Wildon, (2006).** "Introduction to Lie algebras*". Springer.*

[26]. **J. Starrett, (2007).** "Solving differential equations by symmetry groups". *American Mathematical Monthly,* Vol. 114 (9) Page (778-792).

[27]. **F M Mahomed, (2007).** "Symmetry group classification of ordinary Differential equations: Survey of some results". *Math. Meth. Appl. Sci.* Vol. 30, Page (1995-2012).

[28]. **F. Oliveri and R. Vitolo, (2007).** "On differential equations characterized by their Lie point Symmetries". *J. Math. Anal. Appl.,* Vol. 332, Page (767-786).

[29]. **F. Oliveri and R. Vitolo, (2007).** "Differential equations uniquely determined by algebras of point symmetries". *Theor. Math. Phys.,* Vol. 151, Page (843-850).

[30]. **R. Gilmore, (2008).** "Lie groups, physics and geometry. An introduction for physicists, Engineers and chemists". Cambridge University Press, New York.

[31]. **G.W. Blumann and A.F. Cheviakov, (2009).** "Applications of symmetry methods to partial Differential equations". *Springer,* New York

[32]. **F. M. Mahomed and A. Qadir, (2009).** "Invariant linearization criteria for systems of Cubically nonlinear second order differential equations". *J. Nonlinear Math. Phys.,* Vol. 16, Page (283-298).

[33]. **F. Oliveri, (2010).** "Lie symmetries of differential equations: Classical results and Contributions". *Symmetry,* Vol. 2(2), Page (658-706).

[34]. **M. Ac and N. Bildik, (2013).** "On the Lie symmetries of first order ordinary differential equations". *AIP Conf. Proc.,* Vol. 1558, Page (2575-2578).

[35]. **G. Bluman and S. Kumei, (2013).** "Symmetries and differential equations". *Springer* Science *and Business Media*, Vol. 81.

[36]. **S. V. Meleshko and S. Moyo, (2014).** "On the group classifications of systems of two linear Second order ordinary differential equations with constant coefficients". *J. Math. Anal. Appl.,* Vol. 410, Page (341-347).

[37]. **D. J. Arrigo, (2015).** "Symmetry analysis of differential equations: an introduction". *John Willey and Sons,* New York.

[38]. **Carinena, J.F., Falceto,F. and Grabowski, J. (2016).** "Solvability of a Lie algebra of vector fields implies their integrability by quadratures". *Journal of physics A: Mathematical and theoretical.* Vol. 49 (42).

[39]. **Sameer Qasim Hasan and Wissam Salman Abd, (2016).** "Solutions of Ordinary differential system by Using symmetry group". *American Journal of applied mathematics,* Vol. 4(4), Page (186-191).

[40]. **Alexander Bihlo and Roman O. Popovych, (2017).** "Group classification of linear Evolution equations". *J. Math. Anal. Appl.,* Vol. 448(2), Page (982-1005).

[41]. **Ahmad Y. Al-Dweik and R.S. Alassar, (2018**). "Linearization of third-order ordinary Differential equations via point transformations". *Mathematical methods in Appl. Sci.,* Vol. 41(1).

[42]. **Jervin Zen Lobo and Y.S. Valaulikar, (2019).** "Lie symmetries of first order neutral Differential equations". *Journal of applied Mathematics and computational Mechanics*, Vol. 18(1), Page (29-40).

[43]. **Mobeenmunir and Muhammad athar, (2021).** "Lie symmetry of generalized equal width Wave equations". *AIMS Mathematics*, Vol. 6(11), Page (12148-12165).