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Review Paper

On Quasi Generalized β- α -topological Group

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Abstract: In this paper, we introduced the concept of Quasi generalized β-α-topological group. Quasi generalized β-α-topological group have the translation mappings and the inversion mapping are $\mathcal{G}$-β-α continuous with respect to the generalized topology.

KEYWORDS: $\mathcal{G}$-α open, $\mathcal{G}$-α_β-open, $\mathcal{G}$-β-α-continuous, Quasi $\mathcal{G}$-β-α topological group

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I. Introduction

Topological groups are logically the combination of topological spaces and groups. A topological group is defined as a group binded with a topology such that the binary operations are continuous. A. csaszar [2] introduced the concept of generalized topology in 2002. In 2013 Muard Hussian et.al [16] introduced the concept of generalized topological groups. A.B. Khalaf and H.Z Ibrahim et.al [10] introduced some basic concepts of β-α-topological groups.Let A be a subset of a topological space (G,τ). A subset A of a topological space (G,τ) is called α-open if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is called α-closed. By αO(G,τ), we denoted the family of all α-open sets of G. An operation $\beta : \alpha O(G,\tau) \to P(G)$ [8] is a mapping satisfying the condition, $V \subseteq V^\beta$ for each $V \in \alpha O(G,\tau)$. We call the mapping β an operation on αO(G,τ). A subset A of G is called α_β-open set [8] if for each point $x \in A$, there exists an α-open set U of G containing x such that $U^\beta \subseteq A$.

In this paper, we introduced the concept of Quasi generalized β-α-topological group and investigate the related concepts. Throughout this paper Quasi generalized β-α-topological group is denoted by quasi $\mathcal{G}$-β-α-topological group.

II. Preliminaries

Definition: 2.1 [2] Let X be any set and let $\mathcal{G} \subseteq P(X)$ be a subfamily of power set of X.

Then $\mathcal{G}$ is called a generalized topology if $\emptyset \in \mathcal{G}$ and for any index set I , $U_{i\in I}\ O_i \in \mathcal{G}$, $O_i \in \mathcal{G}$, i ∈ I

Definition: 2.2 [3] The element of generalized are called $\mathcal{G}$-open sets. Similarly, generalized closed set (or) $\mathcal{G}$-closed, is defined as complement of a $\mathcal{G}$-open set.

Definition: 2.3 [2] Let X and Y be two $\mathcal{G}$-topological space . A mapping $f : X \to Y$ is called a $\mathcal{G}$-continuous on X if for any $\mathcal{G}$-open set in Y , f^{-1} (0) is $\mathcal{G}$-open in X.

Definition : 2.4 [16] A triple (G , ∗ , $\mathcal{G}$) is said to be an $\mathcal{G}$-topological group if (G ,∗) is a group, (G , $\mathcal{G}$) is a generalized topological space and,
(i) The multiplication mapping m: G × G → G defined by m(x, y) = $x * y$, x, y ∊ G is $\mathcal{G}$-continuous.
(ii) The inverse mapping i : G → G defined by i(x) = x^{-1} , x ∊ G, is $\mathcal{G}$ – continuous.

Definition: 2.5[10] Let (G, ∗) be a group and τ be a topology on G.
(i).The inversion map is β-α-continuous if given a ∊ G and O ∊ αO(G,τ) such that a^{-1}∊ O. then there is U ∊ αO(G,τ) with $a \in$ U and $(U^{\beta})^{-1} \subseteq O^{\beta}$, where $(U^{\beta})^{-1} = \{x^{-1}: x \in U^{\beta}\}$
(ii).The multiplication is jointly β-α-continuous in both variables if given a, b ∊ G and O ∊ αO (G,τ) such that $a * b$ ∊ O, then there exist U, V ∊ αO(G,τ) with a ∊ U, b ∊ V and $U^{\beta} * V^{\beta} \subseteq O^{\beta}$.
(iii). A triple (G, ∗, τ) is called a β-α-topological group if the inversion is β-α-continuous and the multiplication map is jointly β-α-continuous in both variables.

Definition 2.6 [17] Let (G, ∗, τ) be a $\mathcal{G}$-topological space. A triple (G ,∗ ,τ) is called a $\mathcal{G}$-β-α-topological group if the inversion map is $\mathcal{G}$-β-α-continuous and the multiplication map is jointly $\mathcal{G}$-β-α-continuous in both variable.

Definition 2.7 [16] Let (G, ∗) is a group and given $x \in G, L_x: G \to G$ defined by $L_x(y) = x * y$ and $R_x: G \to G$ defined by $R_x(y) = y * x$, denoted left and right translation by x, respectively.

Definition 2.8 [1] A quasi topological group G, is a group which is also a topological space if the following conditions are satisfied,
(i). Left translation $L_x: G \to G, x \in G$ and right translation $R_x: G \to G, x \in G$ are continous.
(ii). The inverse mapping $i: G \to G$ $defined\ by\ i(x) = x^{-1},\ x \in G$ is continous.

Quasi Generalized β-α-Topological Groups

Definition: 3.1 A quasi $\mathcal{G}$-β-α topological group G is a group which is also a $\mathcal{G}$-topological space if the following conditions are satisfied,
(i). Left translation $L_x : G \to G,\ x \in G$ and Right translation $R_x : G \to G,\ x \in G$ are $\mathcal{G}$-β-α continuous and
(ii).The inversion map is $\mathcal{G}$-β-α-continuous if given $a \in G$ and O ∊ $\mathcal{G}$-αO(G, $\mathcal{G}$) such that a^{-1}∊O, then there is U ∊ $\mathcal{G}$-αO(G, $\mathcal{G}$) with $a \in U$ and $(U^{\beta})^{-1} \subseteq O^{\beta}$ where $(U^{\beta})^{-1} = \{x^{-1}: x \in U^{\beta}\}$.

Example: 3.2 Let $(Z_4, +_4)$ is a group under addition and $\mathcal{G}$ = { Z_4 , ∅, {1,3}, {2,3}, {1,2}, {1,2,3} }.
Then $(Z_4, +_4, \mathcal{G})$ is the quasi $\mathcal{G}$-β-α topological group.

Theorem: 3.3 Let (G, ∗ , $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group and $\mathfrak{B}_e$ be the collection of all $\mathcal{G}$-α open neighbourhood of identity e of G. Then
(i). For every $O \in \mathfrak{B}_e$, there is an element U ∊ $\mathfrak{B}_e$ such that $(U^{\beta})^{-1} \subseteq O^{\beta}$.
(ii). For every $O \in \mathfrak{B}_e$, there is an element U ∊ $\mathfrak{B}_e$ such that $U^{\beta} * x \subseteq O^{\beta}$ and $x * U^{\beta} \subseteq O^{\beta}$, for each $x \in G$.

Proof: (i) Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group.
Therefore, for every $O \in \mathfrak{B}_e$ with $a^{-1} \in O$, there is U $\in \mathfrak{B}_e$ such that $a \in U$ and $(U^{\beta})^{-1} \subseteq O^{\beta}$,because the inverse mapping $i : G \rightarrow G$ is $\mathcal{G}$-β-α continous.
(ii). Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group.
Then for any $x \in G$ and $O \in \mathfrak{B}_e$ containing x, there exist U $\in \mathfrak{B}_e$ such that $R_x(U) = U^{\beta} * x \subseteq O^{\beta}$. Similarly, $L_x(U) = x * U^{\beta} \subseteq O^{\beta}$.

Theorem: 3.4 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group and g be any element of G. Then the right translation (R_g) and left translation (L_g) of G by g is a $\mathcal{G}$-α homeomorphism of the space G onto itself.
Proof: Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group.
To prove that Right translation (R_g) is a $\mathcal{G}$-α homeomorphism.
First we prove that R_g is a bijection.
We mapping $R_g: G \rightarrow G$ be defined by $R_g(x) = xg$.
Assume that $R_g(x) = R_g(y)$

$\Rightarrow xg = yg$

$\Rightarrow \quad x = y$

Hence R_g is one-one. Assume that $y \in G$, then the element yg^{-1} maps to y.
Hence R_g is surjective. Since G is a quasi $\mathcal{G}$-β-α-topological group,
R_g is a $\mathcal{G}$-α continuous, then ${R_g}^{-1}$ is also $\mathcal{G}$-α continuous.
Hence the Right translation (R_g) is a $\mathcal{G}$-α homeomorphism.
Similarly we can prove that the Left translation (L_g) is a $\mathcal{G}$-α homeomorphism.

Theorem: 3.5 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group and U be any $\mathcal{G}$-α open set in G. Then (i). $a * U$ and $U * a$ is $\mathcal{G}$-α open in G for all $a \in G$.
(ii). For any subset A of G, the sets $U * A$ and $A * U$ are $\mathcal{G}$-α open in G.
Proof: Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group.
(i). Let $x \in U * a$.
First we prove that x is a $\mathcal{G}$-α interior point of $U * a$.
Let $x = U * a$ for some $u \in U$.
Take U = $U * a * a^{-1}$

$= x * a^{-1}$

$\Rightarrow$ u = $x * a^{-1}$ for some $u \in U$.
Since $R_{a^{-1}} : G \rightarrow G$ is a $\mathcal{G}$-β-α continuous, using the Right translation ($R_{a^{-1}}$), we get $R_{a^{-1}}(x) = x * a^{-1} = u$.
Then for every $\mathcal{G}$-α open set containing u, there exists a $\mathcal{G}$-α open set M_x containing x such that $R_{a^{-1}}(M_x) \subseteq U$

$\Rightarrow M_x * a^{-1} \subseteq U$

$\Rightarrow M_x \subseteq U * a^{-1}$

$\Rightarrow x$ is a $\mathcal{G}$-α interior point of $U * a$.

Hence $U * a$ is $\mathcal{G}$-α open in G.
Similarly we can prove that $a * U$ is $\mathcal{G}$-α open in G.
(ii).Let A be the subset of G. Using by above result, we get $U * a$ is $\mathcal{G}$-α open in G.
Then $U * A = U * \cup_{a \in A} a = \cup_{a \in A}(U * a)$.
Hence $U * A$ is $\mathcal{G}$-α open in G.
Similarly we can prove that $A * U$ is $\mathcal{G}$-α open in G.

Theorem: 3.6 Suppose that a subgroup H of a quasi $\mathcal{G}$-β-α-topological group G contains a non-empty $\mathcal{G}$-open subset of G. Then H is $\mathcal{G}$-α open in G.
Proof: Let G be a quasi $\mathcal{G}$-β-α-topological group.
Let H be a subgroup of G and U be a $\mathcal{G}$-open non-empty subset of G with $U \subset H$.
Let $L_g(U) = U * g$ is $\mathcal{G}$-α open in G for each $g \in H$, *then* $H = \cup_{g \in H} U * g$.
Hence H is $\mathcal{G}$-α open in G.

Theorem: 3.7 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group. Then for every subset A of G and every $\mathcal{G}$-α_β-open neighbourhood O^β of the identity element of e, $\mathcal{G}$-α_β $Cl(A) \subseteq AO^\beta$.
Proof: Let G be a quasi $\mathcal{G}$-β-α-topological group and A be a subset of G.
Let $\mathcal{G}$-α_β-open neighbourhood U^β containing e such that $(U^\beta)^{-1} \subseteq O^\beta$.
Take $x \in \mathcal{G}$-α_β $Cl(A)$.
Then xU^β is an $\mathcal{G}$-α_β-open set containing x.
Now $a \in A \cap xU^\beta$, which implies $a = xb$ for some $b \in U^\beta$,
then $x = ab^{-1} \in A(U^\beta)^{-1} \subseteq AO^\beta$.
Hence $\mathcal{G}$-α_β $Cl(A) \subseteq AO^\beta$.

Theorem: 3.8 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group and H be a subgroup of G. If H is $\mathcal{G}$-α_β-open set then it is also $\mathcal{G}$-α_β-closed in G.
Proof: Let $\mathfrak{B}$ = {gh : $g \in G$} be the family of all left coset of H in G.
This family is a disjoint $\mathcal{G}$-α_β-open covering of G by left translation.
Therefore, every element of $\mathfrak{B}$ is $\mathcal{G}$-α_β closed in G.
In particular, H = eH is $\mathcal{G}$-$\alpha_\beta$closed in G.

Theorem: 3.9 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group and $\mathfrak{B}_e$ be the collection of all $\mathcal{G}$-α open of G. Then, for every $O \in \mathfrak{B}_e$, there is an element U $\in \mathfrak{B}_e$ such that

$(U^\beta)^2 \subseteq O^\beta$.
Proof: Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group.
Let $\mathfrak{B}_e$ be the collection of all $\mathcal{G}$-α open of G.
Then, for every $O \in \mathcal{G}$-α open with $a * b \in O$, there is U $\in \mathcal{G}$-α open such that $a \in U$ and $(U^\beta)^2 \subseteq O^\beta$.

Theorem: 3.10 Every quasi $\mathcal{G}$-β-α-topological group G has $\mathcal{G}$-$\alpha_\beta$open neighbourhood at identity element e consisting of symmetric neighbourhood.
Proof: For an arbitrary $\mathcal{G}$-$\alpha_\beta$open neighbourhood U^β of the identity e in G.
If $V^\beta = U^\beta \cap (U^\beta)^{-1}$, *then* $V^\beta = (V^\beta)^{-1}$.
The set V^β is an $\mathcal{G}$-$\alpha_\beta$open neighbourhood e.
Therefore V^β is a symmetric neighbourhood and $V^\beta \subset U^\beta$.

Theorem: 3.11 Let G be a quasi $\mathcal{G}$-β-α topological group. If N is a normal subgroup of G then $\overline{N}$ also a normal subgroup of G.
Proof : Let G be a quasi $\mathcal{G}$-β-α topological group.
To prove that $a\overline{N}a^{-1} \epsilon \overline{N}$ for all $a \in G$.
Since N is a normal subgroup of G, $aNa^{-1} \in N$ for all $a \epsilon G$.
$\Rightarrow \overline{aNa^{-1}} \subset \overline{N}$ for all $a \in G$.
$\Rightarrow a\overline{N}a^{-1} \subset \overline{N}$ for all $a \in G$.
$\Rightarrow a\overline{N}a^{-1} \in \overline{N}$ for all $a \in G$.
Which implies $\overline{N}$ is a normal subgroup of G.

Theorem 3.12 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α topological group. Then for any b ∈ G, A ⊆ G and B ⊆ G the following statements are true:

(i). $\mathcal{G}$-α$Cl_\beta(A) * $ b = $\mathcal{G}$-α$Cl_\beta(A * b)$ and $\mathcal{G}$-α$Cl_\beta(A) * B \subseteq$ $\mathcal{G}$-α$Cl_\beta(A * B)$.

(ii). $\mathcal{G}$-α$Int_\beta(A) * b =$ $\mathcal{G}$-α$Int_\beta(A * b)$ and $\mathcal{G}$-α$Int_\beta(A) * B \subseteq$ $\mathcal{G}$-α$Int_\beta(A * B)$.

Proof: (i) Consider y ∈$\mathcal{G}$-α$Cl_\beta(A) * b$.

Then y = $x * b$ where $x \in$ $\mathcal{G}$-α$Cl_\beta(A)$.

Let O ∈ $\mathcal{G}$-αO(G,τ) with $x * b$ ∈ O.

Since the right translation is $\mathcal{G}$-β-α-continuous, then there is a V ∈ $\mathcal{G}$-αO(G,τ) with $x \in V$ and $V^\beta * b \subseteq O^\beta$.

Since $x \in$ $\mathcal{G}$-α$Cl_\beta(A)$, there is a ∈ A∩V^β.

So $a * b \in V^\beta * b \subseteq O^\beta$.

Therefore $a * b \in A \cap O^\beta * b$, which implies y = $x * b$ ∈ $\mathcal{G}$-α$Cl_\beta(A * b)$.

Hence $\mathcal{G}$-α$Cl_\beta(A) * b \subseteq$ $\mathcal{G}$-α$Cl_\beta(A * b)$.

Conversely, let $x * b$ ∈ $\mathcal{G}$-α$Cl_\beta(A * b)$.

Let O ∈ $\mathcal{G}$-αO(G,τ) such that x ∈ O .

Since $x = x * b^{-1} *$ b ∈ O, there is a V ∈ $\mathcal{G}$-αO(G,τ) with $x * b$ ∈V and $V^\beta * b^{-1} \subseteq O^\beta$.

Since $x * b \in$ $\mathcal{G}$-α$Cl_\beta(A)$ ∩ V, $a * b \in A \cap V^\beta * b$.

So $(a * b) * b^{-1} \in V^\beta * b^{-1} \subseteq O^\beta$.

Hence a ∈A ∩ O^β and $x \in$ $\mathcal{G}$-α$Cl_\beta(A)$.

Therefore $x * b \in$ $\mathcal{G}$-α$Cl_\beta(A) * b$ and $\mathcal{G}$-α$Cl_\beta(A * b) \subseteq$ $\mathcal{G}$-α$Cl_\beta(A) * b$.

consequently $\mathcal{G}$-α$Cl_\beta(A) * b =$ $\mathcal{G}$-α$Cl_\beta(A * b)$.

Now, we take $b \in$ B ,

then $\mathcal{G}$-α$Cl_\beta(A) * b =$ $\mathcal{G}$-α$Cl_\beta(A * b) \subseteq$ $\mathcal{G}$-α$Cl_\beta(A * B)$ as $A * b \subseteq A * B$.

Hence $\mathcal{G}$-α$Cl_\beta(A) * B = \cup_{b\in B}$ $\mathcal{G}$-α$Cl_\beta(A) * b = \cup_{b\in B}$ $\mathcal{G}$-α$Cl_\beta(A * b) \subseteq$ $\mathcal{G}$-α$Cl_\beta(A * B)$

(ii). Consider $x * b \in$ $\mathcal{G}$-α$Int_\beta(A) * b$, then there exists an $\mathcal{G}$-α-open set O that contains x such that $O^\beta \subseteq$ B.

Since $x = x * b^{-1} * b$ ∈ O, there exists an $\mathcal{G}$-α-open set V with $x * b$ ∈ V and $V^\beta * b^{-1} \subseteq O^\beta$, thus $x * b$ ∈V ⊆ $V^\beta \subseteq O^\beta * b \subseteq A *$ b.

Therefore $x * b \in$ $\mathcal{G}$-α$Int_\beta(A)$ and $\mathcal{G}$-α$Int_\beta(A) * b \subseteq$ $\mathcal{G}$-α$Int_\beta(A * b)$.

Conversely, let $x * b \in$ $\mathcal{G}$-α$Int_\beta(A * b)$,

then there is an $\mathcal{G}$-α-open set O such that $x * b$ ∈ O and $O^\beta \subseteq a *$ B.

There is a $\mathcal{G}$-α-open set V containing x with $V^\beta * b \subseteq O^\beta$.

Hence x ∈ V ⊆ $V^\beta \subseteq O^\beta * b^{-1} \subseteq$ B.

Let $x \in$ $\mathcal{G}$-α$Int_\beta(A)$.

Therefore, $x * b \in$$\mathcal{G}$-α$Int_\beta(A) * b$ and $\mathcal{G}$-α$Int_\beta(A * b) \subseteq$ $\mathcal{G}$-α$Int_\beta(A) * b$.

Consequently, $\mathcal{G}$-α$Int_\beta(A) * b =$ $\mathcal{G}$-α$Int_\beta(A * b)$.

Now, we take $b \in B$, then $\mathcal{G}$-α$Int_\beta(A) * b =$ $\mathcal{G}$-α$Int_\beta(A * b) \subseteq$ $\mathcal{G}$-α$Int_\beta(A * B)$.

Hence $\mathcal{G}$ -α$Int_\beta(A) * B = \cup_{b\in B}$ $\mathcal{G}$-α$Int_\beta(A) * b = \cup_{b\in B}$ $\mathcal{G}$-α$Int_\beta(A * b) \subseteq$ $\mathcal{G}$-α$Int_\beta(A * B)$.

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Theorem: 3.13 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group and A $\subseteq$ G , B $\subseteq$ G. If A is arbitrary and B is $\mathcal{G}$-α_β-open, then $A * B$ and $B * A$ are $\mathcal{G}$-α-open

Proof : Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group.

Let B be $\mathcal{G}$-$\alpha_\beta$open. Then $\mathcal{G}$-$\alpha_\beta Int(B)$ = B.

Consider $a \in A$, then $a * B = a * \mathcal{G}\text{-}\alpha_\beta Int(B)$

$= \mathcal{G}\text{-}\alpha_\beta Int(a * B)$.

Hence, $A * B = A * \mathcal{G}\text{-}\alpha_\beta Int(B)$

$= \cup_{a\in A}\, a * \mathcal{G}\text{-}\alpha Int_\beta(B)$

$= \cup_{a\in A}\, \mathcal{G}\text{-}\alpha Int_\beta(a * B)$.

Then A $*$ B is the union of $\mathcal{G}$-α-open sets.Hence A $*$ B is $\mathcal{G}$-α-open.

Let $a \in A$ and then by theorem 3.12, we have $B * a = \mathcal{G}\text{-}\ \alpha Int_\beta(B) * a = \mathcal{G}\text{-}\alpha Int_\beta(B * a)$.

Hence $B * A = \mathcal{G}\text{-}\alpha Int_\beta(B)$*A

$= \mathcal{G}\text{-}\alpha Int_\beta(B) * \cup_{a\in A}\, a$

$= \cup_{a\in A}\, \mathcal{G}\text{-}\alpha Int_\beta(B * a)$.

Then $B * A$ is the union of $\mathcal{G}$-αopen sets . Hence $B * A$ is $\mathcal{G}$-α-open.

Theorem: 3.14 Let G be a quasi $\mathcal{G}$-β-α topological group and $\mathcal{B}_e$ a base of the space G at the identity element e_G. Then for every subset A of $\mathcal{G}$-α open,

$$\mathcal{G}\text{-}\alpha_\beta\, Cl(A) = \cap\{\, AO^\beta : O \in \mathcal{B}_e\}.$$

Proof: Let G be a quasi $\mathcal{G}$-β-α topological group and for every subset A of $\mathcal{G}$-α open.

By theorem 3.13, assume $x \notin \mathcal{G}\text{-}\alpha_\beta\, Cl(A)$.

Then there exists $O \in \mathcal{B}_e$, such that $x \notin AO^\beta$.

Since $x \notin \mathcal{G}\text{-}\alpha_\beta\, Cl(A)$, then there exists a $\mathcal{G}$-α open U of e_G such that $(xU) \cap A = \emptyset$.

Take $O \in \mathcal{B}_e$, satisfying the condition $(O^\beta)^{-1} \subseteq U$.

Then $(x(O^\beta)^{-1}) \cap A = \emptyset$. Clearly that $x \notin AO^\beta$.

Theorem 3.15 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α topological group then

(i) If β is $\mathcal{G}$-α-open , then $\mathcal{G}\text{-}\alpha Cl_\beta(A) * \mathcal{G}\text{-}\alpha Cl_\beta(B) \subseteq \mathcal{G}\text{-}\alpha Cl_\beta(A * B)$ for all A,B $\subseteq$ G.

(ii) $\mathcal{G}\text{-}\alpha Int_\beta(A) * \mathcal{G}\text{-}\alpha Int_\beta(B) \subseteq$ A * $\mathcal{G}\text{-}\alpha Int_\beta(B) \cap \mathcal{G}\text{-}\alpha Int_\beta(A) * B \subseteq$ A $*$ $\mathcal{G}\text{-}\alpha Int_\beta(B)$ $\cup\, \mathcal{G}\text{-}\alpha Int_\beta(A) * B \subseteq \mathcal{G}\text{-}\alpha Int_\beta(A * B)$.

(iii) If A is arbitrary and B is $\mathcal{G}$- α_β-open, then A * B and B * A are $\mathcal{G}$-α-open.

Proof:

(i) By Theorem 3.7 , A $*$ $\mathcal{G}\text{-}\alpha Cl_\beta(B) \subseteq \mathcal{G}\text{-}\alpha Cl_\beta(A * B)$ which implies $\mathcal{G}\text{-}\alpha Cl_\beta(A * \mathcal{G}\text{-}\alpha Cl_\beta(B) \subseteq \mathcal{G}\text{-}\alpha Cl_\beta(A * B)$.By Theorem 3.23(1), we have $\mathcal{G}\text{-}\alpha Cl_\beta(A) * \mathcal{G}\text{-}\alpha Cl_\beta(B) \subseteq \mathcal{G}\text{-}\alpha Cl_\beta(A * \mathcal{G}\text{-}\alpha Cl_\beta(B)) \subseteq \mathcal{G}\text{-}\alpha Cl_\beta(A * B)$.

(ii) Since $\mathcal{G}$-α$Int_\beta(A) \subseteq$ A, thus $\mathcal{G}$-α$Int_\beta(A) *$ $\mathcal{G}$-α$Int_\beta(B) \subseteq$ A$*$ $\mathcal{G}$-α$Int_\beta(B)$. Similarly $\mathcal{G}$ -α$Int_\beta(A) *$ $\mathcal{G}$ -α$Int_\beta(B) \subseteq \mathcal{G}$ -α$Int_\beta(A) * B$. Thus $\mathcal{G}$ -α$Int_\beta(A) *$ $\mathcal{G}$ -α$Int_\beta(B) \subseteq$ A * $\mathcal{G}$ -α$Int_\beta(B) \cap \mathcal{G}$-α$Int_\beta(A) * B \subseteq$ A * $\mathcal{G}$-α$Int_\beta(B) \cup \mathcal{G}$- α$Int_\beta(A) * B$. By Theorem 3.23(ii), A $*$ $\mathcal{G}$-α$Int_\beta(B) \subseteq \mathcal{G}$-α$Int_\beta(A * B)$. By the definition, we get $\mathcal{G}$-α$Int_\beta(A) * B \subseteq$ $\mathcal{G}$-α$Int_\beta(A * B)$. Therefore, A$*$ $\mathcal{G}$-α$Int_\beta(B) \cup \mathcal{G}$- α$Int_\beta(A) * B \subseteq \mathcal{G}$-α$Int_\beta(A * B)$.

(iii) By Theorem 3.7, $A * B$ is $\mathcal{G}$-α-open. Let $a \in A$. By the definition then by Theorem 3.23(ii), we have $B * a = \mathcal{G}$-α$Int_\beta(B) * a = \mathcal{G}$ -α$Int_\beta(B * a)$. Hence $B * A = \mathcal{G}$ -α$Int_\beta(B)$*A = $\mathcal{G}$ -α$Int_\beta(B) * \cup_{a\in A} a$=$\cup_{a\in A} a$ $\mathcal{G}$-α$Int_\beta(B * a)$. Thus $B * A$ is the union of $\mathcal{G}$ -α-open sets and therefore, $B * A$ is $\mathcal{G}$ -α-open.

Theorem 3.16 Suppose that G, M and L are quasi $\mathcal{G}$-β-α topological group and that $\pi : G \to M$ and $\mu : G \to L$ are homomorphism such that $\mu(G) = L$ and $ker\mu \subset ker\, \pi$.

Then there exists homomorphism $g : L \to M$ such that $\pi = g \circ \mu$. In addition, for each $\mathcal{G}$-neighbourhood U of the identity element e_M in M, there exists a $\mathcal{G}$-neighbourhood V of the identity element e_L in L such that $\mu^{-1}(V) \subset \pi^{-1}(U)$, then g is $\mathcal{G}$-β-α-continuous.

Proof :

Let U be $\mathcal{G}$-neighbourhood of e_M in M.

Consider that $\mathcal{G}$-neighbourhood V of the identity element e_L in L such that,

$$W = \mu^{-1}(V) \subset \pi^{-1}(U).$$
$$\pi(W) = \pi(\mu^{-1}(V)) \subset \pi(\pi^{-1}(U))$$
$$\pi(W) = g(V) \subset U$$
$$g(V) \subset U.$$

We have g is $\mathcal{G}$-β-α-continuous at the identity element of L.

Hence g is $\mathcal{G}$-β-α-continuous.

Corollary : 3.17 Let $\phi : G \to H$ and $\psi : G \to K$ be a quasi $\mathcal{G}$-β-α-continuous homomorphism of a quasi $\mathcal{G}$-β-α topological group G, H and K such that $\psi(G) = K$ and $ker\,\psi \subset ker\,\phi$. If the homomorphism ψ is $\mathcal{G}$-α-open, then there exists a $\mathcal{G}$-β-α-continuous homomorphism, $f : K \to H$ such that $\phi = f \circ \psi$.

Proof:

consider the homomorphism $f : K \to H$ such that $\phi = f \circ \psi$.

Take an arbitrary $\mathcal{G}$-α-open set V in H. Then $f^{-1}(V) = \psi(\phi^{-1}(V))$.

Since ϕ is $\mathcal{G}$-β-α-continuous and ψ is an $\mathcal{G}$-α-open map, $f^{-1}(V)$ is $\mathcal{G}$-α-open in K.

Hence f *is* $\mathcal{G}$-β-α-continuous

Proposition 3.18 :

Let G be a quasi $\mathcal{G}$-β-α-topological group. Every neighbourhood U of e contain an $\mathcal{G}$-α-open symmetric neighbourhood V of e such that $VV \subset U$.

Proof :

Consider that U' be the $\mathcal{G}$-α-interior of U.
Take the multiplication mapping $\pi : U' \times U' \to G$.
$since\ \pi$ is $\mathcal{G}$-β-α-continuous,$\pi^{-1}(U')$ is $\mathcal{G}$-α-open and contain (e,e).
Hence there are $\mathcal{G}$-α-open sets $V_1, V_2 \subset U$ such that (e,e) ∈ $V_1 \times V_2$,and $V_1V_2 \subset U$.
Let $V_3 = V_1 \cap V_2, then\ V_3V_3 \subset U$ and V_3 is an $\mathcal{G}$-α-open neighbourhood of e.
Let V = $V_3 \cap {V_3}^{-1}$, which is $\mathcal{G}$-α-open, contain e and V is symmetric and satisfies $VV \subset U$.

Theorem 3.19 :

Let p : $G \to H$ be a $\mathcal{G}$-β-α-continuous homomorphism of a quasi $\mathcal{G}$-β-α-topological groups. Suppose that the image p(U) contains a non-empty $\mathcal{G}$-α-open set in H, for each $\mathcal{G}$-α-open neighbourhood U of the neutral element e_G in G.Then the homomorphism p is $\mathcal{G}$-α-open.

Proof.

We prove that the neutral element e_H of H is in the $\mathcal{G}$-α-interior of p(U), for each $\mathcal{G}$-α-open neighbourhood U of e_G in G. Choose an $\mathcal{G}$-α-open neighbourhood V of e_G such that $V^{-1}V \subset U$. Consider that p(V) contains a non-empty $\mathcal{G}$-α-open set W ∈ H.
Then $W^{-1}W$ is an $\mathcal{G}$-α-open neighbourhood of e_H and we have that $W^{-1}W \subset p(V)^{-1}p(V) = p(V^{-1}V) \subset p(U)$. Consider an arbitrary element $y \in p(U)$, where U is an arbitrary non-empty $\mathcal{G}$-α-open set in G.We can determine $x \in U$ with $p(x) = y$ and $\mathcal{G}$-α-open neighbourhood V of e_G in G such that $xV \subset U$. Take W be an $\mathcal{G}$-α-open neighbourhood of e_H with W ⊂ p(V). Then the set yW contains y, it is $\mathcal{G}$-α-open in H and, $yW \subset p(xV) \subset p(U)$. Hence p(U) is $\mathcal{G}$-α-open in H.

Definition 3.20

Let (G,∗) be a group and τ be a $\mathcal{G}$ -topology on G. A mapping β: $\mathcal{G}$ -αO(G, $\mathcal{G}$)→P(G) is called $\mathcal{G}$ -α-left operation if for any $a \in G$and whenever V, $(a * V)^{-1} \in \mathcal{G}$ -αO(G, $\mathcal{G}$),then $((a * V)^{-1})^{\beta}=(a * V^{\beta})^{-1}$.

In the result we denote the collection of all $\mathcal{G}$-α open set containing the identity e by Ω

Theorem 3.21

Let (G,*) be a group ,τ be a $\mathcal{G}$ -topology on G,(G, $\mathcal{G}$) be $\mathcal{G}$ -α_β-regular and β be an $\mathcal{G}$ -α-left operation, then for any subset A of G, $\mathcal{G}$ -αCl_β(A) = $\cap_{V \in \Omega}$A ∗ V^β = $\cap_{V \in \Omega}$ V^β ∗ A.

Proof.

Let $x \in \mathcal{G}$ -αCl_β(A) and V∈Ω, then e∈V implies that e $\in V^{-1}$ which is $\mathcal{G}$ -α-open. Since e = $x^{-1} * x$, there is an $\mathcal{G}$ -α-open set U containing x with $x^{-1} * U^\beta \subseteq V^{-1^\beta}$. There is $a \in A \cap U^\beta$,thus $x^{-1} * a \in V^{-1^\beta}$ and $a^{-1} * x \in V^\beta$. Therefore $x = a * (a^{-1} * x) \in A * V^\beta$. Hence $\mathcal{G}$ -αCl_β(A) $\subseteq \cap_{V\in\Omega}$A $* V^\beta$.

Conversely, let $x \in \cap_{V\in\Omega}$A $* V^\beta$ and O∈ $\mathcal{G}$ -α-O(G, $\mathcal{G}$) such that $x = x * e \in O$. then there is V∈Ω with $x * V^\beta \subseteq O^\beta$.Since e∈V and V is $\mathcal{G}$ -α-open, implies e$\in V^{-1}$ and V^{-1} is also $\mathcal{G}$ -α-open, we have $V^{-1} \in \Omega$ and $x \in A * V^{-1^\beta}$. For some $a \in A, a^{-1} * x \in V^{-1^\beta}$, then $x^{-1} * a \in V^\beta$.Since $a = x * (x^{-1} * a) \subseteq x * V^\beta \subseteq O^\beta$,$a \in A \cap O^\beta$,implies $x \in \mathcal{G}$-αCl_β(A).
Therefore, $\cap_{V\in\Omega}$A $* V^\beta \subseteq \mathcal{G}$-α$Cl_\beta$(A). and , $\mathcal{G}$ -αCl_β(A)=$\cap_{V\in\Omega}$A $* V^\beta$. Similarly, $\mathcal{G}$ -αCl_β(A) = $\cap_{V\in\Omega} V^\beta *$ A. Hence $\mathcal{G}$ -αCl_β(A)=$\cap_{V\in\Omega}$A $* V^\beta = \cap_{V\in\Omega} V^\beta *$ A.

Theorem 3.22

Let (G,*) be a group and (G, $\mathcal{G}$) be a $\mathcal{G}$-α_β-regular space, then the following statements are true:.

(i). For any $\in G$, the left translation l_a : G → G defined by $l_a(x) = a * x$, is an $\mathcal{G}$ -α-(β,β)-homeomorphism of G onto G.

(ii) For two elements x *and* y in G, there exists an $\mathcal{G}$-α-(β,β)-homeomorphism f of G onto itself such that f(x) = y.

Proof.

(i) Let $a, x, y \in G$ and $l_a(x) = l_a(y)$, then $a * x = a * y$ and x = y.Therefore, l_a is one-to-one. Since G is a group, for every $x \in G, a^{-1} * x \in G$, thus $l_a(a^{-1} * x) = a * (a^{-1} * x) = x$. Hence l_a is onto. Let O be an $\mathcal{G}$ -α_β-open set, then l_a(O) = $a * O$.
By Theorem 3.13 ,so $a * O \in \mathcal{G}$ -$\alpha O(G,\tau)_\beta$. Hence l_a is $\alpha_{(\beta,\beta)}$-open. Let O be an $\mathcal{G}$-α_β-open set, then ${l_a}^{-1}$(O) = $a^{-1} * O$.By Theorem 3.13 and since G is $\mathcal{G}$-α_β-regular, So $a^{-1} * O \in \mathcal{G}$-$\alpha O(G,\tau)_\beta$. Hence, l_a is $\mathcal{G}$ -α-(β,β)-continuous .Thus, l_a is an $\mathcal{G}$-α-(β,β)-homeomorphism.

(ii). Let $x, y \in G$ and $y * x^{-1} \in G$. Define f = $l_{y*x^{-1}}$: G→G as above, then $l_{y*x^{-1}}$ is $\mathcal{G}$-α-(β,β)-homeomorphism, and $l_{y*x^{-1}}(x) = y * x^{-1} * x$ = y..

Theorem 3.23 :

Suppose that G is a quasi $\mathcal{G}$-β-α topological group. Then, for each $\mathcal{G}$-α-compact subset F of G such that e ∉ F, there exists an $\mathcal{G}$-α_β-open neighbourhood O(F) of F and an $\mathcal{G}$-α_β-open neighbourhood O(e) of e such that O(F) ∩ $O(e)^{-1}$ = ∅.

Proof :
Let γ = { $V_x x : x \in F$ }. Choose the $\mathcal{G}$-α_β-open neighbourhood V_x of e such that $x^{-1} \notin V_x^2$.
Then $V_x\, x \cap V_x^{-1}$ = ∅. Therefore γ = { $V_x x : x \in F$ } is a family of $\mathcal{G}$-α_β-open sets in G of $\mathcal{G}$-α-covering the $\mathcal{G}$-α-compact set F, there exists a finite subset K of F such that
F ⊂ $\cup_{x\in K} V_x\, x$. Put O(e) = $\cup_{x\in K} V_x$ and O(F) = $\cup_{x\in K} V_x\, x$. Then O(e) is an $\mathcal{G}$-α_β-open neighbourhood of e, O(F) is an $\mathcal{G}$-α_β-open neighbourhood of F, and O(F) ∩ $O(e)^{-1}$ = ∅.

Theorem 3.24.
Let A and B be non-empty subsets of a quasi $\mathcal{G}$-β-α-topological group (G, *, $\mathcal{G}$) and β be identity, then
(i). If A and B are $\mathcal{G}$-α_β-connected, then A * B is $\mathcal{G}$-α_β-connected
(ii). If B is $\mathcal{G}$-α_β-connected, and A * b is $\mathcal{G}$-α_β-connected for some b ∈ B, then A * B is $\mathcal{G}$ -α_β-connected.
(iii). If B is $\mathcal{G}$-α_β-connected and A ⊆ B^{-1}, then e ∈ A * B and A* B is $\mathcal{G}$ -α_β-connected.
(iv). If A is $\mathcal{G}$ -α_β-connected, then A^{-1} * A and A * A^{-1} are $\mathcal{G}$-α_β-connected.

Proof.

1. Let f : (G, $\mathcal{G}$) × (G, $\mathcal{G}$) → (G, $\mathcal{G}$) defined by f(a,b) = a * b for all a, b ∈ G , so f is $\mathcal{G}$ -α-β-continuous implies f is $\mathcal{G}$ -$\alpha_{(\beta,\beta)}$-continuous. Since A and B are $\mathcal{G}$-α_β-connected in G,we have A × B is $\mathcal{G}$-α_β-connected in G × G. Since the $\mathcal{G}$-$\alpha_{(\beta,\beta)}$- continuous image of an $\mathcal{G}$-α_β-connected set is $\mathcal{G}$-α_β-connected. Hence f (A × B) = A * B and A* B is $\mathcal{G}$-α_β-connected.

2. Suppose A is not $\mathcal{G}$-α_β -connected, then there exist $\mathcal{G}$-α_β-separated sets U and V such that A = U ∪ V, implies that A * b = U * b ∪ V * b. If x ∈ U * b ∩ V * b, then x = u * b and x = v * b for some u ∈ U and v ∈ V, x = u * b = v * b implies u = v. This is contradiction ,since U and V are disjoint, therefore U * b ∩ V * b = ∅. Since $\mathcal{G}$-αCl_β (U * b) = $\mathcal{G}$ - αCl_β(U) * b and $\mathcal{G}$ - αCl_β(V * b) = $\mathcal{G}$-α$Cl_\beta(V) * b$, using by (i) we get $\mathcal{G}$ - αCl_β(U * b) ∩ V * b = ∅ and U * b∩ $\mathcal{G}$ - αCl_β (V*b) = ∅. Then U * b and V * b are $\mathcal{G}$ - αCl_β-separated sets whose union is A * b, but this contradicts the fact that A * b is $\mathcal{G}$- αCl_β-connected. Therefore A is $\mathcal{G}$- αCl_β -connected. Since A and B are $\mathcal{G}$- αCl_β-connected, by one (i) A * B is $\mathcal{G}$- αCl_β-connected.

3. Let x ∈ A, then x^{-1} ∈ B implies e = $x * x^{-1}$ ∈ A * B.
Let x, y ∈ A, then {x}, {y} and B are $\mathcal{G}$- αCl_β-connected, thus {x} * B = x * B and {y} * B = y * B are $\mathcal{G}$- αCl_β-connected by (i). Since x, y ∈ A, x^{-1}, y^{-1} ∈ B e = $x * x^{-1}$ = y * y^{-1} ∈ x * B∩y * B. Thus $x *$ B and y *B are not $\mathcal{G}$- αCl_β-separated for every x, y ∈ A. Hence {x *B: x ∈ A} is a collection of $\mathcal{G}$-αCl_β-connected subsets of G such that no two members are $\mathcal{G}$ -αCl_β-separated. Therefore $U_{x\in A} x * B = A * B$ is $\mathcal{G}$-αCl_β -connected.

4. Since inversion is $\mathcal{G}$-$\alpha_{(\beta,\beta)}$-continuous and A is $\mathcal{G}$ - αCl_β-connected, so A^{-1} is $\mathcal{G}$- αCl_β-connected. By (i) A and A^{-1} $\mathcal{G}$- αCl_β -connected implies A^{-1} * A and A * A^{-1} are $\mathcal{G}$- αCl_β-connected.

4 Quotients on quasi generalized β-α-topological group

Definition 4.1. Let (G, * , $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-toplogical group and S be a normal subgroup of a group G and the mapping $\pi : G \to$ G/S be defined by $\pi(g) = g * S$,for each $g \in G$. In the set G/S, we define a family $\mathcal{G}'$ and $\mathcal{G}$-αO(G/S, $\mathcal{G}'$) of subset as follows:
$\mathcal{G}'$ = {O ⊆ G/S : π^{-1}(O) ∊ $\mathcal{G}$ } and
$\mathcal{G}$-α O(G/S , $\mathcal{G}'$) = {O ⊆ G/S : π^{-1}(O) ∊ $\mathcal{G}$-α O(G , $\mathcal{G}$) }.
From the operation β which is defined on $\mathcal{G}$-α O(G/S , $\mathcal{G}$), we defined the operation $\beta_{G/S}$: $\mathcal{G}$-αO(G/S,$\mathcal{G}'$)→P(G/S) as follows:
$(\pi(U))^{\beta_{G/S}} = \pi(U^\beta)$ for every U∊ $\mathcal{G}$-α O(G, $\mathcal{G}$) and $\pi(U)$∊ $\mathcal{G}$-α O(G/S, $\mathcal{G}'$).

Example:4.2 Let $(Z_3, +_3)$ is a group under addition and $\mathcal{G}$ = { Z_3, ∅, {0,2},{1,2},{0},{0,1}} is a quasi $\mathcal{G}$-β-α topological group and
we define β on $\mathcal{G}$-αO(Z_3, $\mathcal{G}$) by A^β= cl(A) for each A ∊ $\mathcal{G}$-αO(Z_3, $\mathcal{G}$)

Let S = {0,3}, so Z_3/S = {S,1+S,2+S} Then $\mathcal{G}'$= {∅, Z_3/S,{S},{1+S,2+S},{S,1+S},{S,2+S}}.
Then , $\{S\}^{\beta_{Z_3/S}} = \pi(\{0,3\})^{\beta_{Z_3/S}} = \pi(\{0,3\}^\beta) = \pi(Z_3) = Z_3$/S

$\{1+S, 2+S\}^{\beta_{Z_3/S}} = \pi(\{1,2,0,3\})^{\beta_{Z_3/S}} = \pi(\{1,2,0,3\}^{\beta}) = \pi(Z_3) = Z_3/S$,

$\{S, 1+S\}^{\beta_{Z_3/S}} = \pi(\{1,0,3\})^{\beta_{Z_3/S}} = \pi(\{1,2,0,3\}^{\beta}) = \pi(Z_3) = Z_3/S$,

$\{S, 2+S\}^{\beta_{Z_3/S}} = \pi(\{1,2,0,3\})^{\beta_{Z_3/S}} = \pi(\{1,2,0,3\}^{\beta}) = \pi(Z_3) = Z_3/S$,

Therefore, $(Z_3/S, +_3, \mathcal{G}')$ is a quasi $\mathcal{G}$- $\beta_{Z_3/S}$-α-topological group.

Theorem 4.3. Let (G, ∗, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group and let S be normal subgroup of G. If (G, $\mathcal{G}$) is $\mathcal{G}$-α_{β}-regular, then (G/S, * , $\mathcal{G}'$) is a quasi $\mathcal{G}$-$\beta_{G/S}$-α-topological group.

Proof Let G be an irresolute $\mathcal{G}$-β-α-topological group and $\mathcal{G}$-α_{β}-regular. First we prove that $\pi(U) \in \mathcal{G}$-S_{α}O(G/S, $\mathcal{G}'$) for every U ∈ $\mathcal{G}$-S_{α}O(G, $\mathcal{G}$).By the definition of $\mathcal{G}$-S_{α}O(G/S, $\mathcal{G}'$), $\pi(U) \in \mathcal{G}$-S_{α}O(G/S,$\mathcal{G}'$) at $\pi^{-1}(\pi(U)) \in \mathcal{G}$-$S_{\alpha}$O(G, $\mathcal{G}$). Let the mapping $\pi : G \to G/S$ be defined by $\pi(g) = g * S$,for each g∈G, then the mapping is $\mathcal{G}$-homomorphism, we have $\pi^{-1}(\pi(g)) = g * S$, for each g∈G.
Therefore $\pi^{-1}(\pi(U)) = \cup_{g\in U}\, g * S = U * S$.

By theorem 3.13, we have U * S∈ $\mathcal{G}$-S_{α}O(G, $\mathcal{G}$) with U∈ $\mathcal{G}$-S_{α}O(G, $\mathcal{G}$) .Since (G, $\mathcal{G}$) is $\mathcal{G}$-$S_{\alpha\beta}$-regular, U∈ $\mathcal{G}$-S_{α}O(G, $\mathcal{G}$) = $\mathcal{G}$-$S_{\alpha\beta}$O(G, $\mathcal{G}$). Since $\pi(U) \in \mathcal{G}$-S_{α}O(G/S, $\mathcal{G}'$) for every U∈ $\mathcal{G}$-S_{α}O(G, $\mathcal{G}$), therefore $\pi^{-1}(\pi(U)) \in \mathcal{G}$-$S_{\alpha}$O(G, $\mathcal{G}$).

Next we show that the multiplication mapping (a,b) → $a * b$ is jointly $\mathcal{G}$-$\beta_{G/S}$-α-continuous in both variables (G/S, $\mathcal{G}'$)X (G/S, $\mathcal{G}'$) → (G/S, $\mathcal{G}'$).

Let O∈ $\mathcal{G}$-αO(G/S, $\mathcal{G}'$) and let $a, b \in$ G/S such that $a * b \in$ O. Let $x, y \in$ G satisfy $a = \pi(x)$ and $b = \pi(y)$. Since π is a homomorphism, so $\pi(x*y) = \pi(x) * \pi(y) = a * b \in O$ and thus $x * y \in \pi^{-1}(0)$. Since O∈ $\mathcal{G}$-αO(G/S, $\mathcal{G}'$), we have $\pi^{-1}(O) \in \mathcal{G}$-αO(G, $\mathcal{G}$). Since (G, ∗, $\mathcal{G}$) is a $\mathcal{G}$-β-α-topological group and $x * y \in \pi^{-1}(0) \in \mathcal{G}$-αO(G, τ), there exist U,V ∈ $\mathcal{G}$ -αO(G, $\mathcal{G}$) such that $x \in U$ and $y \in V$ and $U^{\beta} * V^{\beta} \subseteq \pi^{-1}(O)^{\beta}$. Again since π is homomorphism,wehave $\pi(U^{\beta} * V^{\beta}) = \pi(U^{\beta}) * \pi(V^{\beta})$

since $U^{\beta} * V^{\beta} \subseteq \pi^{-1}(O)^{\beta}$, *we have* $\pi(U^{\beta} * V^{\beta}) \subseteq \pi((\pi^{-1}(O)^{\beta})$andtherefore
$\pi(U^{\beta}) * \pi(V^{\beta}) \subseteq \pi((\pi^{-1}(O)^{\beta})$ implies $(\pi(U))^{\beta_{G/S}} * \pi(V)^{\beta_{G/S}} \subseteq \pi(\pi^{-1}(O)))^{\beta_{G/S}} = O^{\beta_{G/S}}$.
Hence, we have that $\pi(U) \in \mathcal{G}$-αO(G/S, $\mathcal{G}'$) an $\pi(V) \in \mathcal{G}$-αO(G/S, $\mathcal{G}'$). Since $a = \pi(x) \in \pi(U)$ and $b = \pi(y) \in \pi(V)$, we have shown that the multiplication mapping is jointly $\mathcal{G}$-$\beta_{G/S}$-α- -continuous in both variables.

Now, we have to show that the inversion mapping $a \to a^{-1}$ is $\mathcal{G}$-$\beta_{G/S}$-α-continuous (G/S, $\mathcal{G}'$) → (G/S, $\mathcal{G}'$).Let a∈ G/S and O ∈ αO(G/S, $\mathcal{G}'$) such that $a^{-1} \in$ O. let x∈ G such that $a^{-1} = \pi(x^{-1})$ and $a = \pi(x)$. Then $\pi(x^{-1}) = a^{-1} \in O$ and thus $x^{-1} \in \pi^{-1}(O)$. Since $\pi^{-1}(O) \in \mathcal{G}$-αO(G, $\mathcal{G}$), there is an $\mathcal{G}$-α-open set U such that x∈ U and $(U^{\beta})^{-1} \subseteq (\pi^{-1}(O))^{\beta}$. Now $\pi(x) = a \in \pi(U)$, and $\pi(U) \in \mathcal{G}$-αO(G/S, $\mathcal{G}'$). Since π is homomorphism, so $\pi(U^{\beta^{-1}}) \subseteq (\pi^{-1}(O))^{\beta}$ implies $\pi(U^{\beta})^{-1} \subseteq \pi(\pi^{-1}(O))^{\beta}$ and hence $(\pi(U)^{\beta_{G/S}})^{-1} \subseteq \pi(\pi^{-1}(O)))^{\beta_{G/S}} = O^{\beta_{G/S}}$. Therefore the inversion is $\mathcal{G}$-$\beta_{G/S}$-α-continuous and hence (G/S, * , $\mathcal{G}'$) is quasi $\mathcal{G}$-$\beta_{G/S}$-α-topological group.

Theorem 4.4 Let (G, $*$, $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group. If A is an $\mathcal{G}$ -α_β-closed subset of G, then the normalizer of A is $\mathcal{G}$-α_β-closed subgroups of G.

Proof. Let N = { : $x * A = A * x$} denote the normalizer of A and let y ∈ N ,

then $y * A = A * y$ implies $y^{-1} * A = A * y^{-1}$,thus y^{-1} ∈ N. If $x, y \in N$,

$$then\ (x * y^{-1}) * A = x * (y^{-1} * A)$$
$$= x * (A * y^{-1})$$
$$= (x * A) * y^{-1}$$
$$= A * (x * y^{-1}).$$

Hence $x * y^{-1}$∈ N and N is a subgroup. Let r ∈ $\mathcal{G}$ -αCl_β(N) and $r * a \epsilon r * A$ for $a \in A$. Let O ∈ $\mathcal{G}$-αO(G, $\mathcal{G}$) such that $r * a * r^{-1}$∈ O, then there are $\mathcal{G}$-α-open sets U and V such that $r \in U, a \in V\ and\ U^\beta * V^\beta * U^{\beta^{-1}} \subseteq O^\beta$. There is n ∈ U^β∩ N, thus $n * a * n^{-1} \in O^\beta$. Since n ∗ A = A ∗ n, so $n * a * n^{-1} \in A \cap O^\beta$. Thus $r * a * r^{-1} \in$ $\mathcal{G}$-αCl_β(A) = A, hence $r * a * r^{-1}$ ∈ A. Then $(r * a * r^{-1})$ ∗ r = r ∗ a ∈ A ∗ r and r ∗ A ⊆ A ∗ r. Similarly A ∗ r ⊆ r ∗ A and so r ∗ A = A ∗ r. Hence r ∈ N and N is $\mathcal{G}$ -α_β-closed.

Theorem 4.5 Let (G, ∗ , $\mathcal{G}$) be a quasi $\mathcal{G}$-β-α-topological group, (G, $\mathcal{G}$) be $\mathcal{G}$-α_β-regular and S be a normal subgroup of G, π be the natural mapping of G onto G/S and let U and V be an $\mathcal{G}$-α-open subset of G such that e∈U,e∈V and $V^{-1} * V$⊆ U. Then $\mathcal{G}$ -$\alpha_{\beta_{G/S}} Cl(\pi(V)) \subseteq \pi\ (U)$.

Proof.

Let any $x \in G$ and $\pi(x) \in$ $\mathcal{G}$-$\alpha_{\beta_{G/S}} Cl(\pi(V))$. Since V ∗ x is an $\mathcal{G}$-α-open set containing x and the mapping $\pi : G \to G/S$ be defined by $\pi(V) = V * x$ is $\mathcal{G}$-$\alpha_{\beta_{G/S}}$-open, then $\pi\ (V * x)$ is an $\mathcal{G}$-$\alpha_{\beta_{G/S}}$-open set containing $\pi(x)$. Here $\pi(V * x) \cap \pi(V) \neq \emptyset$, we have $\pi(a * x) = \pi\ (b)$ for some $a \in V$ and $b \in V$, ie) $a * x = b * h$, for some h∈ S,
which implies $x = (a^{-1} * b) * h \in U * S$, since $a^{-1} * b \in V^{-1} * V \subseteq U$.
Therefore, $\pi(x) \in \pi\ (U * S) = \pi(U)$. Hence $\mathcal{G}$ -$\alpha_{\beta_{G/S}} Cl(\pi(V)) \subseteq \pi\ (U)$.

Theorem 4.6 Let (G, ∗ , $\mathcal{G}$) be an iresolute $\mathcal{G}$-β-α-topological group, (G, $\mathcal{G}$) be $\mathcal{G}$-α_β-regular and S be a normal subgroup of G, the quotient space G/S is $\mathcal{G}$-$\alpha_{\beta_{G/S}}$-regular.

Proof.

Let the mapping $\pi : G \to G/S$ be defined by $\pi(V) = V * x$.
Let W be a $\mathcal{G}$-α-open set of $\pi(e)$ in G/H, where e is the neutral element of G.

By the continuity of π, we can find an $\mathcal{G}$-α-open set U of e in G such that $\pi(U)$⊂ W.
Since G is a quasi $\mathcal{G}$-β-α-topological group, we can choose an $\mathcal{G}$-α-open set V of e such that $V^{-1} * V$⊆ U. Then, by theorem 4.5, $\overline{\pi(V)}$ ⊂ π(U) ⊂ W.
Since $\pi(V)$ is a $\mathcal{G}$-$\alpha_{\beta_{G/S}}$-open neighbourhood of $\pi(e)$, the $\mathcal{G}$-$\alpha_{\beta_{G/S}}$-regular of G/S at the point $\pi(e)$ is satisfy. Hence the space G/S is $\mathcal{G}$-$\alpha_{\beta_{G/S}}$-regular

Theorem 4.7 Let (G, * , $\mathcal{G}$) be a $\mathcal{G}$-β-α-topological group and β be identity.

If G_e is $\mathcal{G}$-α_β-component subset of G such that e∈G_e, then G_e is $\mathcal{G}$-α_β-closed normal subgroup.

Proof.

Let $a \in G_e$ and $a * G_e$ is $\mathcal{G}$-α_β-connected. Since G_e is $\mathcal{G}$-α_β-closed as it is an $\mathcal{G}$-α_β-component. Thus there is an $\mathcal{G}$-α_β-component C of G such that $a \in G_e \subseteq$ C. If C ≠ G_e, then C and G_e are separated, but a∈$C \cap G_e$. Therefore C = G_e and a∈$G_e \subseteq G_e$. Let b∈ G_e, since $a^{-1} * G_e$ is $\mathcal{G}$-α_β-connected and e∈$a^{-1} * G_e$, so $a^{-1} * G_e \subseteq G_e$. Thus $a^{-1} * b \in G_e$ and $b \in a * G_e$. Hence $G_e \subseteq a * G_e$ and $G_e = a * G_e$. Therefore G_e is a subgroup.

If $x \in G$, then $x * G_e * x^{-1}$ is $\mathcal{G}$-α_β-connected and $e \in x * G_e * x^{-1}$ implies $x * G_e * x^{-1} \subseteq G_e$. Similarly $x^{-1} * G_e * x \subseteq G_e$, thus $G_e \subseteq x * G_e * x^{-1}$. Therefore $G_e = x * G_e * x^{-1}$ and G_e is normal.

Theorem 4.8 Let (G, ∗, $\mathcal{G}$) be a $\mathcal{G}$-β-α-topological group and A be a subset of G. If G is $\mathcal{G}$-α-β-T_2, then the centralizer of A is $\mathcal{G}$-α_β-closed subgroups of G.

Proof. Let C = $\{ x : x * a = a * x \text{ for all } a \in A \}$ denote the centralizer of A.
Let y ∈ C, then $y * a = a * y$ for every $a \in A$.
Therefore $a * y^{-1} = y^{-1} * a$ for every $a \in A$, thus $y^{-1} \in$ C.
Let $x, y \in C$ and $a \in A$, then

$$\begin{aligned}(x * y^{-1}) * a &= x * (y^{-1} * a)\\ &= x * (a * y^{-1})\\ &= (x * a) * y^{-1}\\ &= a * (x * y^{-1}).\end{aligned}$$

Hence $x * y^{-1} \in$ C and C is a subgroup.
Let p ∈ $\mathcal{G}$ -αCl_β(C). Let $a \in A$ and O ∈ $\mathcal{G}$-αO(G, $\mathcal{G}$) such that $p * a * p^{-1} \in$ O, then there are $\mathcal{G}$ -α-open sets U and V such that $p \in U, a \in V$ and $U^\beta * V^\beta * U^{\beta^{-1}} \subseteq O^\beta$. Since there is $x \in U^\beta \cap$ C , $x * a * x^{-1} \in O^\beta$, but $x * a = a * x$, thus $a = x * a * x^{-1} \in O^\beta$. Therefore for every O ∈ $\mathcal{G}$ -αO(G, $\mathcal{G}$) such that $p * a * p^{-1} \in$ O, then $a \in O^\beta$.
Suppose $p * a * p^{-1} \neq a$. Since G is $\mathcal{G}$-α-β-T_2, then there are $\mathcal{G}$-α-open sets K and L such that $a \in k$, $p * a * p^{-1} \in$ L and $K^\beta \cap L^\beta = \emptyset$, but $p * a * p^{-1} \in$ L implies $a \in L^\beta$. This is a contradiction and thus, $a = p * a * p^{-1}$ and p∈ C. Hence C is $\mathcal{G}$ -α_β-closed.

Proposition 4.9:

The map π is a quasi $\mathcal{G}$-β-α-continuous and $\mathcal{G}$-α-open, the space (G/S,$\mathcal{G}'$) is $\mathcal{G}$-α-homogeneous. Moreover, if the Subgroup S is normal then the multiplication $xSyS = xyS$ in G/S is $\mathcal{G}$-β-α-continuous and (G/S, $\mathcal{G}'$) is a quasi $\mathcal{G}$-β-α-topological group.

Proof :

Let the map π be the $\mathcal{G}$-β-α-continuous. If U ⊂ G is an $\mathcal{G}$-α_β-open set then $\pi^{-1}\pi(U) = US$ and we have π(U) is $\mathcal{G}$-α_β-open.The space G/S is $\mathcal{G}$-α-homogeneous, because the translation $l_a : xS \rightarrow axS$ are $\mathcal{G}$-α-homeomorphism. Assume S be a normal subgroup of the group G. Let U' is a neighbourhood of the point $c' = a'b'$ then $c = ab$ for some representatives a, b, c from the classes a', b', c' respectively. For a neighbourhood U = $\pi^{-1}(U') \ni$ c there exist neighbourhoods $V_1(a)$ and $V_2(b)$ such that $V_1(a)V_2(b) \subset$ U. Hence $\pi(V_1(a))\pi\,(V_2(b)) \subset U'$ and G/S is a para $\mathcal{G}$-β-α-topological group.

Proposition 4.10 :

If S is $\mathcal{G}$-β-α-compact then the map π is $\mathcal{G}$-α-closed. If the space (G, $\mathcal{G}$) is Hausdorff then the space (G/S, $\mathcal{G}'$) is Hausdorff. If the space (G, $\mathcal{G}$) is $\mathcal{G}$-α-regular then the space (G/S, $\mathcal{G}'$) is $\mathcal{G}$-α-regular.

Proof

Let F be a $\mathcal{G}$-α-closed subset of the group G and $x' \in$ G/S\ $\pi(K)$.
Consider an arbitrary point $x \in \pi^{-1}(x')$. Then $xS \cap K = \emptyset$.
By Proposition 3.18 there exists an $\mathcal{G}$-α-open neighbourhood U of the unit such that $UxS \cap K = \emptyset$. Then $x' \in \pi(Ux)$ and $\pi(Ux) \cap \pi(F) = \emptyset$ thus the map π is $\mathcal{G}$-α-closed.
Consider that G be Hausdorff and $x'_1, x'_2 \in G/S$. To prove that the space G/S is Hausdorff.
Choose an arbitrary points $x_i \in \pi^{-1}(x'_i)$. Then $x_1S \cap x_2S = \emptyset$.
By Proposition 3.18 there exists an $\mathcal{G}$-α-open neighbourhood U of the unit such that $Ux_1S \cap Ux_2S = \emptyset$. Then$x'_i \in \pi(Ux_i)$ and $\pi(Ux_1) \cap \pi(Ux_2) = \emptyset$ thus the space G/S is Hausdorff.Let G be $\mathcal{G}$-α-regular. K' be a $\mathcal{G}$-α-closed subset of G/S and $x' \in$ G/S\ K'.
To prove that the space G/S is $\mathcal{G}$-α-regular. Choose an arbitrary points $x \in \pi^{-1}(x')$. Then $x \notin \pi^{-1}(K')$. By Proposition 3.18 and uniformity of G imply that there exists an $\mathcal{G}$-α-open neighbourhood U of the unit such that $\overline{U}x \cap \pi^{-1}(K')=\emptyset$.
Then$x' \in \pi(Ux) and \overline{\pi(Ux)} \cap K' = \emptyset$, so the space G/S is $\mathcal{G}$-α-regular.

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