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Review Paper

A Fixed Point Theorem in Multiplicative Metric Spaces

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Abstract *In this paper , we shall prove a common fixed point theorems for three self maps satisfying weakly C-Contractive condition in multiplicative metric space. Our results extend and unify some fixed point theorems in multiplicative metric space.*
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I. Introduction and Preliminaries

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of all real numbers, the set of all positive real numbers and the set of all natural numbers respectively.

In 2008 , Bashirov *et al.* [1] introduced the concept of multiplicative metric space as follows:

Definition 1.1 Let X be nonempty set. A multiplicative metric is a mapping $d : X \times X \longrightarrow \mathrm{R}^+$ satisfying the following conditions:

$d(x, y) \geq 1, \forall\ x, y \in X$ and $d(x, y) = 1$, if and only if $x = y$;

$d(x, y) = d(y, x)$ *for all* $x, y \in X$;

$d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Then the mapping d together with X, that is , (X, d) is a multiplicative metric space.

Example 1.2. Let R_+^n be the collection of all n-tuples of positive real numbers.Let $d^*(x, y)$: $\mathrm{R}_+^n \longrightarrow \mathrm{R}$ be defined as follows :

$$d^*(u, v) = \left|\frac{u_1}{v_1}\right|^* . \left|\frac{u_1}{v_1}\right|^* . \ldots . \left|\frac{u_n}{v_n}\right|^*$$

where $u = (u_1, u_2, u_3, \ldots\ldots u_n)$, $v = (v_1, v_2, v_3,\ldots,v_n) \in \mathrm{R}_+^n$ and $|.|^* : \mathrm{R}_+ \longrightarrow \mathrm{R}_+$ is defined by:

$$|k|^* = \begin{cases} k, if\ k \geq 1 \\ \frac{1}{k}, if\ k < 1 \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric space are satisfied and (R_+^n, d) is a multiplicative metric space.

Example 1.3. Let d : $\mathrm{R} \times \mathrm{R} \longrightarrow [1, \infty)$ be defined as $d(x, y) = a^{|x-y|}$, where $x, y \in \mathrm{R}$ and $a > 1$. Then d is a multiplicative metric and (R, d) is a multiplicative metric space.

Remark 1.4. We note that Example 1.2 is valid for positive real numbers and Example 1.3 is valid for all real numbers.

Example 1.5. Let (X, d) be a metric space. Define a mapping d_a on X^2 by
$d_a(x, y) =$

$$a^{d(x-y)} = 1, \quad \text{if } x = y;$$

a, if $x \neq y$,

where $x, y \in X$ and $a > 1$. Then d_a is called a discrete multiplicative metric and (X, d_a) is known as the discrete multiplicative metric space.

Example 1.6. Let $X = C^* [a, b]$ be the collection of all real-valued multiplicative functions on $[a, b] \subset \mathbf{R}_+$, then (X, d) is a multiplicative metric space with d defined by $d(f, g) = sup_{x\in [a,b]}|\frac{f(x)}{g(x)}|$ for arbitrary $f, g \in X$.

Definition 1.7. Let (X, d) be a multiplicative metric space.Then a sequence $\{x_n\}$ is said to be

1. multiplicative convergent to x of for every multiplicative open ball $B_\epsilon(x) = y \in X \backslash d(x, y) < \epsilon, \epsilon > 1$, there exists a natural number N such that $n \geq N$ then $x_n \in B_\epsilon(x)$ that is, $d(x_n, x) \to 1$ as $n \to \infty$.
2. a multiplicative cauchy sequence if for all $\epsilon > 1$ there exists $N \in \mathbb{N}$ such that $d(x_n.x_m) < \epsilon$, $\forall\ m, n > N$ that is $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

A multiplicative metric space is said to be complete if every multiplicative cauchy sequence in it is multiplicative convergent to $x \in X$.

Definition 1.8. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a multiplicative contraction if there exists a real constant $\lambda \in [0, 1)$ such that $d(fx, fy) \leq d^\lambda(x, y)$ for all $x, y \in X$.

II. Main Results

In this section, our aim is to prove a common fixed point theorem for three self-mpas in multiplicative metric spaces.

B.S. Chaudhary [3] gave the concept of weakly C-contractive mappings.

Definition 2.1. Let (X, d) be a multiplicative metric. A mapping $T : X \to X$ space is said to be C-Contractive , if there exists $\alpha \in (0, \frac{1}{2})$ such that $\forall\ x, y \in X$ the following inequality holds:
$d(Tx, Ty) \leq [d(x, Ty).d(y, Tx)]^\alpha$.

Definition 2.2. A mapping $T : X \to X$,where (X, d) is a multiplicative metric space is said to be weakly C- contractive if $\forall\ x, y \in X$,
$d(Tx, Ty) \leq [d(x, Ty)d(y, Tx)]^\alpha - \phi(d(x, Ty).d(y, Tx))$,

where $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\phi(x, y) = 0$ if and only if $x = y = 1$.

Jungck and Rhoades [5] introduced the notion of weakly compatible maps as follows:

Definition 2.3. Let T and S be two self mappings of a multiplicative metric space (X, d) , T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx \Rightarrow TSx = STx$.

Theorem 2.4. Let (X, d) be a complete multiplicative metric space and let E be a non empty closed subset of X. Let $T, S : E \to E$ be such that

$$d(Tx, Sy) \le \{(d(Rx,Sy) + d(Ry,Tx)\}^{\frac{1}{2}} - \phi(d(Rx, Sy), d(Ry, Tx)), \qquad (1)$$

for every pair $(x, y) \in X \times X$,where $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\phi(x, y) = 0$ if and only if $x = y = 1$ and $R : E \to X$ satisfying the following hypothesis

$TE \subseteq RE$ and $SE \subseteq RE$,

The pairs (T, R) and (S, R) are weakly compatible. In addition, assume that $R(E)$ is a closed subset of X. Then T and R and S have a unique common fixed point.

Proof:- Let $x_0 \in E$ be arbitrary ,using (1) ,∃ two sequences $\{x_n\}$ and $\{y_n\}$ such that

$y_0 = Tx_o = Rx_1$, $y_1 = Sx_1 = Rx_2$, $y_2 = Tx_2 = Rx_3 \ldots, y_{2n} = Tx_{2n} = Rx_{2n+1}$,

$y_{2n+1} = Sx_{2n+1} = Rx_{2n+2}, \ldots$.

We complete the proof in three steps

Step 1

We will prove that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 1$.

By making use of equation (1), for n = 2k, we have

$$d(y_{2k}, y_{2k+1}) = d(Tx_{2k}, Sx_{2k+1})$$
$$\le [d(Rx_{2k}, Sx_{2k+1}).d(Rx_{2k+1}, T_{2k})]^{1/2} - \phi(d(Rx_{2k}, Sx_{2k+1})d(Rx_{2k+1}, Tx_{2k}))$$
$$= [d(y_{2k-1}, y_{2k+1})d(y_{2k}, y_{2k})]^{1/2} - \phi(d(y_{2k-1}, y_{2k+1})d(y_{2k}, y_{2k})) \qquad (2)$$
$$\le d(y_{2k-1}, y_{2k+1})^{1/2}$$
$$\le [d(y_{2k-1},y_{2k}).d(y_{2k}, y_{2k+1})]^{1/2}$$

Hence

$$d(y_{2k+1}, y_{2k}) \le d(y_{2k}, y_{2k+1})^{1/2}$$

If $n = 2K+1$, then similarly we can prove

$$d(y_{2k+2}, y_{2k+1}) \le d(y_{2k+1}, y_{2k})$$

Thus $d(y_{n+1}, y_n)$ is decreasing sequence of non negative real numbers and hence it is convergent.

Also we assume that

$$\lim_{n\to\infty}(d(y_{n+1}, y_n)) = r$$

Therefore

$$d(y_{n+1}, y_n) \le d(y_{n-1}, y_{n+1})^{1/2} \le d(y_{n-1}, y_n).d(y_n, y_{n-1})^{1/2} \qquad (3)$$

If $n \to \infty$,then we have

$r \leq \lim_{n\to\infty}[d(y_{n-1}, y_{n+1})]^{1/2} \leq r$

Therefore $\lim_{n\to\infty} d(y_{n-1}, y_{n+1}) = r^2$.

$d(y_{2k+1}, y_{2k}) = d(Tx_{2k}, Sx_{2k+1})$

$$\leq d(y_{2k-1}, y_{2k+1}).d(y_{2k}, y_{2k})^2 - \phi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})) \quad (4)$$

Now if $k \to \infty$ and using the continuity of ϕ we obtain

$r \leq r - \phi(r^2, 1)$

And consequently $\phi(r^2, 1) = 0$ gives us that

$$r = \lim_{n\to\infty} d(y_n, y_{n+1}) = 1. \quad (5)$$

Step 2:

Here we shall prove that $\{y_n\}$ is a Cauchy sequence.

Since $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$,

It is sufficient to show that the sub-sequence $\{y_{2n}\}$ is a Cauchy sequence.

Suppose that $\{y_{2n}\}$ is not Cauchy sequence.

then $\exists\ \epsilon > 0$ for we can find sub-sequence $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of y_{2n} such that $n(k)$ is the least index for which $n(k) > m(k) > k$ and $d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon$

This means that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \epsilon \quad (6)$$

From triangular inequality, we have

$$\epsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)-2}).d(y_{2m(k)-2}, y_{2n(k)-1}).d(y_{2n(k)-1}, y_{2n(k)}) \quad (7)$$

Letting $k \to \infty$ and using (5) we can conclude that

$\epsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq \epsilon.1.1 = \epsilon$

Therefore, we get

$$d(y_{2m(k)}, y_{2n(k)}) = \epsilon \quad (8)$$

Moreover we have

$$| d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)}) | \leq d(y_{2n(k)}, y_{2n(k)+1}) \quad (9)$$

And

$$| d(y_{2nk}, y_{2m(k)-1}) - d(y_{2nk}, y_{2m(k)}) | \leq d(y_{2mk}, y_{2m(k)-1}) \quad (10)$$

And

$$| d(y_{2nk}, y_{2m(k)-2}) - d(y_{2nk}, y_{2m(k)-1}) | \leq d(y_{2m(k)-2}, y_{2m(k)-1}) \quad (11)$$

Using (5), (8), (9), (10) and (11), we get

$$\lim_{k\to\infty} d(y_{m(k)-1}, y_{2n(k)}) = \lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1})$$
$$= \lim_{k\to\infty} d(y_{2m(k)-2}, y_{2n(k)}) = \epsilon \quad (12)$$

Now, from (1) we have

$$d(y_{2m(k)-1}, y_{2n(k)}) = d(Tx_{2n(k)}, Sx_{2m(k)-1})$$
$$\leq [d(Rx_{2n(k)}, Sx_{2m(k-1)}).d(Rx_{2m(k-1)}, Tx_{2n(k)})]^{1/2}$$
$$- \phi(d(Rx_{2n(k)}, Sx_{2m(k)-1})d(Rx_{2m(k+1)}, Tx_{2n(k)}))$$
$$\leq [d(y_{2n(k-1)}, y_{2m(k-1)}).d(y_{2m(k-2)}, y_{2n(k)})]^{1/2}$$
$$- \phi(d(y_{2n(k-1)}, y_{2m(k-1)}), d(y_{2m(k-2)}, y_{2nk})) \quad (13)$$

Letting $k \to \infty$ in the above inequality, using (12) and the continuity of ϕ , we have

$\epsilon \leq [(\epsilon.\epsilon)]^{1/2} - \phi(\epsilon,\epsilon)$

$\epsilon \leq \epsilon - \phi(\epsilon, \epsilon)$

$\phi(\epsilon, \epsilon) \leq 0$

$\phi(\epsilon, \epsilon) = 0$

And from the last inequality $\phi(\epsilon, \epsilon) = 0$.

By our assumption about ϕ , we have

$\epsilon = 1$, which is contradiction (because $\epsilon > 1$).

STEP (3):

Here, we shall show that S , T and R have a common fixed point.

Since (X, d) is complete and $\{y_n\}$ is Cauchy , then$\exists\ z \in X$ such that $\lim_{n\to\infty} y_n = z$

Since E is closed and $y_n \subseteq E$, we have $z \in E$.

By assumption $R(E)$ is closed , so$\exists\ u \in E$ such that $z = Ru$.

For all $n \in \mathbb{N}$, we have

$$d(Tu, y_{2n+1}) = d(Tu, Sx_{2n+1})$$
$$\leq [d(Ru, Sx_{2n+1}).d(Rx_{2n+1}, Tu)]^{1/2}$$
$$- \phi(d(Ru, Sx_{2n+1})d(Rx_{2n+1}, Tu)) \quad (14)$$
$$\leq [d(z, y_{2n+1}).d(y_{2n}, Tu)]^{1/2} - \phi(d(Ru, Sx_{2n+1})d(Rx_{2n+1}, Tu))$$

Making $n \to \infty$, we get

$d(Tu, z) \leq [d(z, z).d(z, Tu)]^{1/2} - \phi(d(Ru, z).d(z, Tu))$ and hence

$\phi(1, d(z, Tu)) \leq [d(z, z).d(z, Tu)]^{1/2} - d(Tu, z) \leq 0$, that is,

$\phi(1, d(z, Tu)) = 0$

Therefore $d(z, Tu) = 1$.

Therefore $Tu = z$.

Similarly, we get

$Su = z, so\ Tu = Su = Ru = z$

Since the pairs (R, T) and (R, S) are weakly compatible, we have $Tz = Sz = Rz$.

Now we can have

$$d(Tz, y_{2n+1}) = d(Tz, Sx_{2n+1})$$
$$\leq [d(Rz, Sx_{2n+1}).d(Rx_{2n+1}, Tz)]^{1/2}$$
$$- \phi(d(Rz, Sx2n + 1)d(Rx_{2n+1}, Tz))$$
$$\leq [d(Rz, y_{2n+1}).d(y_{2n}, Tz)]^{1/2} - \phi(d(Rz, y_{2n+1}), d(y_{2n}, Tu)) \quad (15)$$

Making $n \to \infty$ and since $Tz = Sz = Rz$, we obtain

$d(Tz, z) = [d(Tz, z).d(z, Tz)]^{1/2} - \phi(d(Tz, z), d(z, Tz))$ (16)

Hence $\phi(d(Tz, z), d(z, Tz)) = 1$ and so $d(Tz, z) = 1$.

Therefore $Tz = z$ and $Tz = Sz = Rz$.

We conclude that $Tz = Sz = Rz = z$

Uniqueness of the common fixed point follows from (1).

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