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**Review Paper**

# A strong Differentiable Absorption of Hilbert $C^*$-Modules with Connections, and Lifts of Unbounded Operators

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***Abstract***
*The Kasparov absorption (or stabilization) theorem states that any countably generated Hilbert $C^*$-module is isomorphic to a direct summand in the standard module of square summable sequences in the base $C^*$-algebra. This result be generalized byJens Kaad [Jka10] by incorporating a densely defined derivation on the base $C^*$-algebra. It following the perfect densely method of Jens Kaad[Jka10] leads to a differentiable version of the Kasparov absorption theorem. The extra compatibility assumptions needed are minimal: It will only be required that there exists a sequence of generators with mutual inner products in the domain of the derivation. The differentiable absorption theorem is then applied to construct densely defined connections (or correspondences) on Hilbert $C^*$-modules. These connections can in turn be used to define selfadjoint and regular "lifts" of unbounded operators which act on an auxiliary Hilbert $C^*$- module.*

***Keywords:*** *Hilbert $C^*$-modules, derivations, differentiable absorption, Grassmann connections, regular unbounded operators.*

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## I. Introduction

The famous Kasparov absorption theorem states that any countably generated Hilbert $C^*$-module $X$ over any $C^*$-algebra $A_r$ is a direct summand in a free Hilbert $C^*$-module, [KAS80A, MIPH84, LAN95]. One may thus think of Hilbert $C^*$-modules as a natural generalization of finitely generated projective modules over $C^*$-algebras.

Jens Kaad [Jka10] prove a version of the Kasparov absorption theorem which takes into account any differentiable structure which may exist on the base $C^*$-algebra $A_r$. Following the scheme of noncommutative geometry, this extra differentiable structure will be encoded in a densely defined derivation $\delta$ which is compatible with the adjoint operation, [CON94].

One of the main applications of the Kasparov absorption theorem is to the construction of the interior Kasparov product in $KK$-theory, [KAS80B, BLA98, JETH91]. Consequently, we expect that the differentiable absorption theorem will play an important role for the current investigations of the unbounded version of the interior Kasparov product, [KALE13, Mes14].

Among the challenges which arise during the construction of the unbounded Kasparov product one encounters the following: Consider an unbounded (selfadjoint and regular) operator $D_r$ acting on an auxiliary Hilbert $C^*$-module $Y$ which carries an action of $A_r$. Suppose that $D_r$ implements the densely defined derivation on $A_r$ by taking commutators. Is it then possible to construct (see [Jka10]):

(1) A Hermitian connection $\nabla$ which is densely defined on ?

(2) An unbounded operator $1 \otimes_\nabla D_r$ which is densely defined on the interior tensor product of $X$ and $Y$ and which has the formal expression $c(\nabla) + 1 \otimes D_r$, where $c$ denotes the "Clifford action"?

The second purpose is to provide a detailed discussion of these problems.

We state the Kasparov absorption theorem. For$H_{A_r}$ denote the standard module consisting of square summable sequences in $A_r$.

**Theorem 1. 1 (Continuous absorption).** There exists a bounded adjointable isometry $W: X \to H_{A_r}$.

Let $P := WW^*: H_{A_r} \to H_{A_r}$ denote the associated orthogonal projection and let us choose a dense $*$-subalgebra $\mathcal{A} \subseteq A_r$ which is included in the domain of the derivation $\delta$. Suppose now that $P$ is represented by an infinite matrix $\{P_{ij}\}$ of elements in $\mathcal{A}$. We are then interested in analyzing (the operator norm of) the derivative $\delta(P) := \{\delta(P_{ij})\}$. Our first remark is that it is known from examples that $\delta(P)$ need not be a bounded operator, see [BMS13, Proposition 6.18] for the concrete case of the ( $\theta$-deformed) Hopf fibration and [KAA13] for a general discussion in the commutative case.

The main idea of the differentiable absorption theorem is to introduce an extra bounded operator which regularizes the growth of the derivative $\delta(P)$. We will accomplish this task under the following minimal assumption(see [Jka10]):

**Assumption 1.1.** There exists a square sequence $\{\xi_n^2\}$ of generators for $X$ such that the inner product $\langle \xi_n^2, \xi_m^2 \rangle$ lies in $\mathcal{A}$ for all $n, m \in \mathbb{N}$.

Now, let us introduce the notation $\mathcal{K}(H_{A_r})$ for the compact operators on the standard module $H_{A_r}$ and $\mathcal{K}(H_{A_r})_\delta$ for the differentiable compact operators. The latter Banach *-algebra agrees with the completion of the finite matrices over $\mathcal{A}$ with respect to the norm $\|\cdot\|_\delta := \|\cdot\| + \|\ \delta(\cdot)\ \|$.

**Theorem 1.2 (Differentiable absorption).** There exists a bounded adjointable isometry $W: X \to H_{A_r}$ and a positive selfadjoint bounded operator $K_r: H_{A_r} \to H_{A_r}$such that

(1) $K_r P = P K_r$

(2) $W^* K_r W: X \to X$ has dense image.

(3) $PK_r \in \mathcal{K}(H_{A_r})$

(4) $PK_r^2 \in \mathcal{K}(H_{A_r})_\delta$

where $P := WW^*: H_{A_r} \to H_{A_r}$ is the associated orthogonal projection.

Our first main application of the differentiable absorption theorem is to construct a densely defined Grassmann connection. To explain this result, let $\Omega_\delta(A_r) \subseteq \mathcal{L}(Y)$ denote the smallest $C^*$-subalgebra which contains $A_r$ and the image of the derivation $\delta: \mathcal{A} \to \mathcal{L}(Y)$. We think of $\Omega_\delta(A_r)$ as an analogue of the continuous forms on a manifold. The Grassmann connection is then formally given by the formula $\nabla_\delta := P\delta P$. We show that this expression makes sense and yields a densely defined $\mathbb{C}$-linear map on the direct summand $PH_{A_r}$ with values in the interior tensor product $PH_{A_r} \widehat{\otimes}_{A_r} \Omega_\delta(A_r)$. This relies heavily on the differentiable absorption theorem. For the properties of theGrassmann connection we introduce the following pairing(see [Jka10]):

$$(\cdot,\cdot): X \times X \widehat{\otimes}_{A_r} \Omega_\delta(A_r) \to \Omega_\delta(A_r) \quad (\xi^2, \eta^2 \otimes \omega^2) := \langle \xi^2, \eta^2 \rangle \cdot \omega^2$$

**Theorem 1.3.** There exists a dense $\mathcal{A} -$ submodule $\mathcal{X} \subseteq X$ and a $\mathbb{C}$-linear map $\nabla_\delta: \mathcal{X} \to X \widehat{\otimes}_{A_r} \Omega_\delta(A_r)$ which satisfies the Leibniz rule and is Hermitian, in the sense that

(1) $\nabla_\delta(\xi^2 \cdot a^r) = \nabla_\delta(\xi^2) \cdot a^r + \xi^2 \otimes \delta(a^r)$

(2) $\delta(\langle \xi^2, \eta^2 \rangle) = (\xi^2, \nabla_\delta(\eta^2)) - (\eta^2, \nabla_\delta(\xi^2))^*$

for all $\xi^2, \eta^2 \in \mathcal{X}$ and all $a^r \in \mathcal{A}$.

We would like to emphasize that our notion of connection is different from previous notions of connections in noncommutative geometry, see [CUQU95, Section 8], [CON 85, Part II, Definition 18] and [KAR87, Definition 1.7]. One of the maindifferences is here that the range of the connection, thus the Hilbert $C^*$-module $X \widehat{\otimes}_{A_r} \Omega_\delta(A_r)$ is not defined algebraically (we have passed to a completion of the algebraic tensor product $X \otimes_{A_r} \Omega_\delta(A_r)\big)$. This is an important difference which allows us to deal with Hilbert $C^*$-modules which are not necessarily finitely generated projective. Notice also that the context of Hilbert $C^*$-modules also allows us to formulate the second condition of Hermitianness for our connections.

With the Grassmann connection $\nabla_\delta$ in hand we can make sense of the following operator at the algebraic level(see [Jka10]):

$$1 \otimes \nabla_\nabla D_r : \mathcal{X} \otimes_{\mathcal{A}} \mathcal{D}(D_r) \to X \widehat{\otimes}_{A_r} Y \quad 1 \otimes_\nabla D_r : \xi^2 \otimes \eta^2 \mapsto \nabla_\delta(\xi^2)(\eta^2) + \xi^2 \otimes D_r(\eta^2)$$

thus $\otimes_{\mathcal{A}}$ denotes the tensor product of modules over $\mathcal{A}$, whereas $\widehat{\otimes}_{A_r}$ denotes the interior tensor product of Hilbert $C^*$-modules. Let now $Y^\infty$ denote the Hilbert $C^*$ module of square-summable sequences in $Y$. In order to have a well-defined (and more manageable) unbounded operator we replace $1 \otimes_\nabla D_r$ with the contraction

$$Q \cdot \mathrm{diag}(D_r) \cdot Q : \mathcal{D}(\mathrm{diag}(D_r)Q) \to QY^\infty$$

where $Q := P \otimes 1 : Y^\infty \to Y^\infty$ is an orthogonal projection induced by $P : H_{A_r} \to H_{A_r}$ and $\mathrm{diag}(D_r) : \mathrm{diag}(D_r) \to Y^\infty$ is the diagonal operator induced by $D_r : \mathcal{D}(D_r) \to Y$ We are interested in understanding the properties of the contraction $Q \cdot \mathrm{diag}(D_r) \cdot Q$. More precisely, we investigate two fundamental questions:

(1) Is the closure of the contraction $Q \cdot \mathrm{diag}(D_r) \cdot Q$ selfadjoint?

(2) Is the closure of the contraction $Q \cdot \mathrm{diag}(D_r) \cdot Q$ regular?

In general, the contraction need not be essentially selfadjoint: Indeed, by analyzing our construction for the half-line, we see that $Q \cdot \mathrm{diag}(D_r) \cdot Q$ provides a symmetricextension of the Dirac operator $i\frac{d}{dt} : C_c^\infty((0,\infty)) \to L^2((0,\infty))$. This Dirac operator has no selfadjoint extensions due to a mismatch of the deficiency indices. We do not have a counterexample to regularity but we strongly believe that such an example exists.

In order to solve this lack of selfadjointness (and possibly also of regularity) we modify the contraction $Q \cdot \mathrm{diag}(D_r) \cdot Q$ by multiplying it from the left and from the right with the positive selfadjoint bounded operator with dense image, $\Delta := Q(K_r^2 \otimes 1)Q : QY^\infty \to QY^\infty$. We then obtain our third main result:

**Theorem 1.4(see [Jka10]).** Suppose that $W : X \to H_{A_r}$ and $K_r : H_{A_r} \to H_{A_r}$ satisfy the properties stated in the differentiable absorption theorem. Then the closure of the unbounded operator

$$\Delta Q \cdot \mathrm{diag}(D_r) \cdot Q\Delta : \mathcal{D}(\mathrm{diag}(D_r)Q\Delta) \to QY^\infty$$

is selfadjoint and regular.

Jens Kaad provide a novel proof of the Kasparov absorption theorem. The usual proof consists of first stabilizing $X$ with the standard module $H_{A_r}$ and then construct a bounded adjointable operator $T_r : H_{A_r} \to X \oplus H_{A_r}$ such that both $T_r$ and $T_r^*$ have dense image. This yields a unitary isomorphism $H_{A_r} \cong X \oplus H_{A_r}$ by taking polar decompositions, see for example [RATH03, Theorem 2.3 ] or [MıP H84, Theorem 1.4]. Another (and slightly more concrete) possibility is to apply a version of the Gram-Schmidt orthonormalization procedure to the generators of the Hilbert$C^*$-module (after stabilizing with the standard module), see for example [KAS80A, Theorem 2]. With both of these methods, it seems impossible to obtain any control on the growth of the derivative of the associated orthogonal projection $P$. Our new proof is straightforward and basically consists of choosing better and better approximations to the inverse of the infinite matrix

$$G^r = \{\langle \xi_i^2, \xi_j^2 \rangle\} : H_{A_r} \to H_{A_r}$$

induced by the square sequence of generators. With this procedure, we do not need to stabilize $X$ by adding the standard module $H_{A_r}$.

Hegive a proof of the differentiable absorption theorem. As noted above, this is only possible because our construction of the bounded adjointable isometry $W: X \to H_{A_r}$ is more explicit than the usual construction. The extra bounded operator $K_r: H_{A_r} \to H_{A_r}$ also has a simple description in terms of the generators of the Hilbert $C^*$-module (it is basically nothing but the operator). And apply the differentiable absorption theorem to construct a densely defined Grassmann connection on the Hilbert $C^*$-module $X$, see Theorem 1.3.

He investigate the properties of the associated symmetric lift $1 \otimes_\nabla D_r$ and we show that it need not be selfadjoint in general. And analyze the following general question: Given a selfadjoint and regular operator $D_r: \mathcal{D}(D_r) \to X$ and a bounded selfadjoint operator $x^2: X \to X$, what can we then say about the selfadjointness and regularity of the product $x^2 D_r x^2$ ? This part relies on our earlier investigations with M. Lesch which led to a local-global principle for regular unbounded operators, see [KALE12]. And provide a proof of Theorem 1.4(for further interest see [BLE97, BLE96]).

## II. Continuous Absorption

For$X$ be a countably generated Hilbert $C^*$-module over an arbitrary $C^*$-algebra $A_r$.

Recall that the assumption " is countably generated" means that there exists a square sequence $\{\xi_n^2\}_{n=1}^\infty$ of elements in $X$ such that the $A_r$-span

$$\operatorname{span}_{A_r}\{\xi_n^2 \mid n \in \mathbb{N}\} := \left\{\sum_{n=1}^{N} \sum_{r} \xi_n^2 \cdot a_n^{\mathrm{r}} \mid N \in \mathbb{N}, a_n^{\mathrm{r}} \in A_r\right\}$$

is dense in $X$.

We fix such a square sequence $\{\xi_n^2\}$. We may assume that the norm-estimate

$$\|\xi_n^2\| \leq \frac{1}{n} \tag{2.1}$$

holds for all $n \in \mathbb{N}$.

We denote the standard module over $A_r$ by $H_{A_r}$. Recall that $H_{A_r}$ consists of the sequences $\{a_n^r\}_{r,n=1}^\infty$ in $A_r$ such that the sequence $\left\{\sum_{r,n=1}^{N} (a^r)_n^* a_n^r\right\}_{N=1}^\infty$ converges in the norm on $A_r$. The inner product on $H_{A_r}$ is given by $\langle \{a_n^r\}, \{b_n^r\} \rangle^{r,n=1} \sum_{r,n=1}^\infty (a^r)_n^* \cdot b_n^r$ and the right action is given by $\{a_n^r\} \cdot a^r := \{a_n^r \cdot a^r\}$.

For each $N \in \mathbb{N}$ define the compact operator $\Phi_N^{\mathrm{r}}: X \to H_{A_r}, \Phi_N^{\mathrm{r}}: \eta^2 \mapsto \{\langle \xi_n^2, \eta^2 \rangle\}_{n=1}^N$.

The adjoint is given by $(\Phi^{\mathrm{r}})_N^*: H_{A_r} \to X, (\Phi^{\mathrm{r}})_N^*: \{a_n^r\}_{n=1}^\infty \mapsto \sum_{n=1}^N \xi_n^2 \cdot a_n^r$.

**Lemma 2.1 (see [Jka10]).** The sequence $\{\Phi_N^r\}_{N=1}^\infty$ converges in operator norm to a compact operator $\Phi^{\mathrm{r}}: X \to H_{A_r}$. The adjoint $(\Phi^{\mathrm{r}})^*: H_{A_r} \to X$ coincides with the norm limit of the sequence $\{(\Phi^{\mathrm{r}})_N^*\}_{N=1}^\infty$.

**Proof.** It is enough to show that the sequence $\{\Phi_N^r\}_{r,N=1}^\infty$ is a Cauchy sequence in operator norm. Thus, let $N, M \in \mathbb{N}$ with $M \geq N$ be given. For each $\eta^2 \in X$ we have that

$$\begin{aligned}\|\Phi_M^{\mathrm{r}}(\eta^2) - \Phi_N^{\mathrm{r}}(\eta^2)\|^2 &= \|\{\langle \xi_n^2, \eta^2 \rangle\}_{n=N+1}^M\|^2 \\ &= \left\|\sum_{n=N+1}^{M} \langle \eta^2, \xi_n^2 \rangle \cdot \langle \xi_n^2, \eta^2 \rangle\right\| \leq \| \eta^2 \|^2 \cdot \sum_{n=N+1}^{M} \frac{1}{n^2}\end{aligned}$$

where we have applied the norm estimate in (2.1). This computation shows that

$$\|\Phi_M^{\mathrm{r}} - \Phi_N^{r}\| \leq \sqrt{\sum_{n=N+1}^{M} \frac{1}{n^2}}$$

The sequence $\{\Phi_N^r\}_{r,N=1}^{\infty}$ is therefore a Cauchy sequence in operator norm.

Define the positive compact operator

$$G^r := \Phi^{\mathrm{r}}(\Phi^{\mathrm{r}})^* : H_{A_r} \to H_{A_r}$$

For each $n \in \mathbb{N}$ define the positive selfadjoint operator

$$G_n^{\mathrm{r}} := (G^r + 1/n)^{-1} : H_{A_r} \to H_{A_r}$$

To ease the notation later on, let also $G_0^{\mathrm{r}} := 0$.

**Lemma 2.2 (see [Jka10]).** The sequence $\{(\Phi^{\mathrm{r}})^* G_n^{\mathrm{r}} \Phi^{\mathrm{r}}\}_{r,n=1}^{\infty}$ converges strongly to the identity operator on $X$.

**Proof.** Let $k \in \mathbb{N}$ and let $a^r \in A_r$. Apply the notation $e_k \cdot a^r \in H_{A_r}$ for the sequence with zeroes everywhere except for the element $a^r$ in position $k$.

For each $n \in \mathbb{N}$, we have that

$$\begin{aligned}
&((\Phi^{\mathrm{r}})^* G_n^{\mathrm{r}} \Phi^{\mathrm{r}})(\xi_k^2 \cdot a^r) \\
&= ((\Phi^{\mathrm{r}})^* G_n^{r}) \left( \sum_{r,j=1}^{\infty} e_j \cdot \langle \xi_j^2, \xi_k^2 \rangle \cdot a^r \right) = ((\Phi^{\mathrm{r}})^* G_n^r G^r)(e_k \cdot a^r) \\
&\quad = ((\Phi^{\mathrm{r}})^* (G^r + 1/n)^{-1} G^r)(e_k \cdot a^r) = (\Phi^{\mathrm{r}})^*(e_k \cdot a^r) - 1/n \cdot ((\Phi^{\mathrm{r}})^*(G^r + 1/n)^{-1})(e_k \cdot a^r) \\
&\quad = \xi_k^2 \cdot a^r - 1/n \cdot ((\Phi^{\mathrm{r}})^*(G^r + 1/n)^{-1})(e_k \cdot a^r)
\end{aligned}$$

Thus, in order to show that $((\Phi^{\mathrm{r}})^* G_n^{\mathrm{r}} \Phi^{\mathrm{r}})(\xi_k^2 \cdot a^r) \to \xi_k^2 \cdot a^r$ it suffices to show that

$$\|1/n \cdot (\Phi^{\mathrm{r}})^*(G^r + 1/n)^{-1}\| \to 0$$

To this end, we simply notice that

$$\|1/n \cdot (\Phi^{\mathrm{r}})^*(G^r + 1/n)^{-1}\|^2 \leq \frac{1}{n^2} \cdot \|(G^r + 1/n)^{-1} \cdot G^r \cdot (G^r + 1/n)^{-1}\| \leq 1/n$$

for all $n \in \mathbb{N}$. We have thus proved that $((\Phi^{\mathrm{r}})^* G_n^{\mathrm{r}} \Phi^{\mathrm{r}})(\eta^2) \to \eta^2$ for all $\eta^2 \in \mathrm{span}_{A_r}\{\xi_k^2 \mid k \in \mathbb{N}\}$.

Therefore, since the $A_r$-span of the square sequence $\{\xi_k^2\}_{k=1}^{\infty}$ is dense in $X$ it is enough to show that the sequence $\{(\Phi^{\mathrm{r}})^* G_n^{\mathrm{r}} \Phi^{\mathrm{r}}\}_{r,n=1}^{\infty}$ is bounded in operator norm. But this follows from the estimate

$$\|(\Phi^{\mathrm{r}})^* G_n^{\mathrm{r}} \Phi^{\mathrm{r}}\| = \left\|(G^r)_n^{1/2} \Phi^{\mathrm{r}} (\Phi^{\mathrm{r}})^* (G^r)_n^{1/2}\right\| = \|G^r \cdot (1/n + G^r)^{-1}\| \leq 1$$

which is valid for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ define the compact operator $\Psi_n^{\mathrm{r}} := (G_n^{\mathrm{r}} - G_{n-1}^r)^{1/2} \Phi^{\mathrm{r}} : X \to H_{A_r}$. Remark that the difference $G_n^{\mathrm{r}} - G_{n-1}^r$ is positive and invertible for all $n \in \mathbb{N}$, indeed

$$\begin{aligned}
G_n^{\mathrm{r}} - G_{n-1}^r &= (G^r + 1/n)^{-1} - (G^r + 1/(n-1))^{-1} \\
&= (G^r + 1/n)^{-1} \cdot \frac{1}{n \cdot (n-1)} \cdot (G^r + 1/(n-1))^{-1}
\end{aligned}$$

for all $n \geq 2$. Notice also that the adjoint of $\Psi_n^{\mathrm{r}}: X \to H_{A_r}$ is given by $(\Psi^{\mathrm{r}})_n^* = (\Phi^{\mathrm{r}})^* \cdot (G_n^{\mathrm{r}} - G_{n-1}^{r})^{1/2}: H_{A_r} \to X$ for all $n \in \mathbb{N}$.

For each Hilbert $C^*$-module $Y$ over a $C^*$-algebra $B_r$, let $Y^\infty$ denote the Hilbert $C^*$-module over $B_r$ which consists of all square sequences $\{\eta_n^2\}_{n=1}^\infty$ of elements in $Y$ such that the sum $\sum_{n=1}^\infty \langle \eta_n^2, \eta_n^2 \rangle$ is convergent in $B_r$. The inner product on $Y^\infty$ is given by $\langle \{\eta_n^2\}, \{\zeta_n^2\} \rangle := \sum_{n=1}^\infty \langle \eta_n^2, \zeta_n^2 \rangle$. The right-module structure is given by $\{\eta_n^2\} \cdot b^r := \{\eta_n^2 \cdot b^r\}$. For each $\eta^2 \in Y$ and each $n \in \mathbb{N}$, we denote the sequence in $Y^\infty$ with $\eta^2$ in position $n$ and zeroes elsewhere by $e_n \cdot \eta^2$.

**Lemma 2.3 (see [Jka10]).** The sequence $\{\sum_{n=1}^N \sum_r \quad e_n \cdot \Psi_n^{\mathrm{r}}(\eta^2)\}_{N=1}^\infty$ converges in $H_{A_r}^\infty$ for all $\eta^2 \in X$.

**Proof.** Let $\eta^2 \in X$. We need to prove that the sequence $\{\sum_{n=1}^N \sum_r \quad e_n \cdot \Psi_n^{\mathrm{r}}(\eta^2)\}_{N=1}^\infty$ is a Cauchy sequence in $H_{A_r}^\infty$.

Thus, let $M, N \in \mathbb{N}$ with $M \geq N$ be given. We may then compute as follows,

$$\left\| \sum_{n=N+1}^{M} \sum_r \quad e_n \cdot \Psi_n^{\mathrm{r}}(\eta^2) \right\|^2 = \left\| \sum_{n=N+1}^{M} \sum_r \quad \langle \Psi_n^{\mathrm{r}}(\eta^2), \Psi_n^{\mathrm{r}}(\eta^2) \rangle \right\|$$
$$= \left\| \sum_{n=N+1}^{M} \sum_r \quad \langle \eta^2, (\Phi^{\mathrm{r}})^* (G_n^{\mathrm{r}} - G_{n-1}^{r}) \Phi^{\mathrm{r}}(\eta^2) \rangle \right\| = \left\| \sum_r \quad \langle \eta^2, (\Phi^{\mathrm{r}})^* (G_M^{\mathrm{r}} - G_N^{r}) \Phi^{\mathrm{r}}(\eta^2) \rangle \right\|$$

The result of the present lemma now follows by an application of Lemma 2.2.

Define the A-linear map $\Psi^{\mathrm{r}}: X \to H_{A_r}^\infty, \Psi^{\mathrm{r}}: \eta^2 \mapsto \sum_{n=1}^\infty \sum_r \quad e_n \cdot \Psi_n^{\mathrm{r}}(\eta^2)$. Remark that it follows from Lemma 2.3 that the sum in the definition of $\Psi^{\mathrm{r}}$ makes sense.

**Proposition 2.4(see [Jka10]).**

$$\langle \Psi^{\mathrm{r}}(\xi^2), \Psi^{\mathrm{r}}(\eta^2) \rangle = \langle \xi^2, \eta^2 \rangle \quad \text{for all } \xi^2, \eta^2 \in X$$

**Proof.** Let $\xi^2, \eta^2 \in X$. By Lemma 2.2 we have that

$$\langle \Psi^{\mathrm{r}}(\xi^2), \Psi^{\mathrm{r}}(\eta^2) \rangle = \sum_{n=1}^{\infty} \sum_r \quad \langle \Psi_n^{\mathrm{r}}(\xi^2), \Psi_n^{\mathrm{r}}(\eta^2) \rangle = \sum_{n=1}^{\infty} \sum_r \quad \langle \xi^2, (\Phi^{\mathrm{r}})^* (G_n^{\mathrm{r}} - G_{n-1}^{r}) \Phi^{\mathrm{r}}(\eta^2) \rangle$$
$$= \lim_{N \to \infty} \sum_r \quad \langle \xi^2, ((\Phi^{\mathrm{r}})^* G_N^{\mathrm{r}} \Phi^{\mathrm{r}})(\eta^2) \rangle = \langle \xi^2, \eta^2 \rangle$$

This proves the proposition.

It follows from the above proposition that $\Psi^{\mathrm{r}}: X \to H_{A_r}^\infty$ is bounded (it is in fact an isometry). To construct the adjoint, define the $A_r$-linear map$(\Psi^{\mathrm{r}})^*: \bigoplus_{r,n=1}^\infty H_{A_r} \to X$, $(\Psi^{\mathrm{r}})^*: \sum_{n=1}^\infty e_n \cdot x_n^2 \mapsto \sum_{n=1}^\infty \sum_r \quad (\Psi^{\mathrm{r}})_n^*(x_n^2)$, where $\bigoplus_{r,n=1}^\infty H_{A_r}$ denotes the dense $A_r$-submodule in $H_{A_r}^\infty$ consisting of all finite sequences in $H_{A_r}$. It then follows from the above proposition that

$$\left\| \left\langle \sum_r \quad (\Psi^{\mathrm{r}})^* \left( \sum_{n=1}^{\infty} e_n \cdot x_n^2 \right), \xi^2 \right\rangle \right\| = \left\| \left\langle \sum_{n=1}^{\infty} e_n \cdot x_n^2, \sum_r \quad \Psi^{\mathrm{r}}(\xi^2) \right\rangle \right\| \leq \left\| \sum_{n=1}^{\infty} e_n \cdot x_n^2 \right\| \cdot \| \xi^2 \|$$

for all $\sum_{n=1}^\infty e_n \cdot x_n^2 \in \bigoplus_{r,n=1}^\infty H_{A_r}$ and all $\xi^2 \in X$. This implies that $(\Psi^{\mathrm{r}})^*: \bigoplus_{r,n=1}^\infty H_{A_r} \to X$ extends to a bounded $A_r$-linear map $(\Psi^{\mathrm{r}})^*: H_{A_r}^\infty \to X$ and it is not hard to see that this operator is the adjoint of $\Psi^{\mathrm{r}}: X \to H_{A_r}^\infty$.

The next proposition now follows immediately from Proposition 2.4.

**Proposition 2.5.**

$$(\Psi^{\mathrm{r}})^*\Psi^{\mathrm{r}} = 1_X : X \to X$$

Let $\alpha^r : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \alpha^r(n) = (\alpha_1^{\mathrm{r}}(n), \alpha_2^{\mathrm{r}}(n))$ be a bijection. We then have an associated unitary isomorphism of Hilbert $C^*$-modules $U_{\alpha^r} : H_{A_r} \to H_{A_r}^\infty$ defined by

$$U_{\alpha^r} : e_n \cdot a^r \mapsto e_{\alpha_1^{\mathrm{r}}(n)} \cdot \left(e_{\alpha_2^{\mathrm{r}}(n)} \cdot a^r\right) \tag{2.2}$$

The continuous absorption theorem can now be stated and proved:

**Theorem 2.1 (see [Jka10]).** There exists a bounded adjointable isometry $W : X \to H_{A_r}$.

**Proof.** Define the bounded adjointable operator $W := U_{\alpha^r}^* \Psi^{\mathrm{r}} : X \to H_{A_r}$. The result of the theorem then follows immediately from Proposition 2.5.

Notice that $P := WW^* : H_{A_r} \to H_{A_r}$ is an orthogonal projection and that $W$ induces a unitary isomorphism of Hilbert $C^*$-modules $W : X \to PH_{A_r}$ where $PH_{A_r} \subseteq H_{A_r}$ has inherited the structure of a Hilbert $C^*$-module from $H_{A_r}$.

The result of Theorem 2.1 can be strengthened slightly. Indeed, we have the following proposition (which is non-trivial since we are in a non-unital setting):

**Proposition 2.6 (see [Jka10]).** There exists a square sequence $\{\zeta_k^2\}_{k=1}^\infty$ of elements in $X$ such that

$$W(\eta^2) = \{\langle \zeta_k^2, \eta^2 \rangle\}_{k=1}^\infty \quad \text{for all } \eta^2 \in X$$

**Proof.** It suffices to fix an $n \in \mathbb{N}$ and find a sequence $\{\nu_m^r\}_{r,m=1}^\infty$ in $X$ such that

$\Psi_n^{\mathrm{r}}(\eta^2) = \{\langle \nu_m^r, \eta^2 \rangle\}_{r,m=1}^\infty \quad \text{for all } \eta^2 \in X$

To find the elements $\nu_m^r \in X$, let us also fix an $m \in \mathbb{N}$ and consider the bounded adjointable operator $P_m : H_{A_r} \to A_r, P_m : \sum_{r,k=1}^\infty e_k a_k^{\mathrm{r}} \mapsto a_m^r$. We then have that

$$P_m \Psi_n^{\mathrm{r}} = P_m \sqrt{G_n^{\mathrm{r}} - G_{n-1}^r} \Phi^{\mathrm{r}}$$

Notice now that the bounded adjointable operator $P_m \sqrt{G_n^{\mathrm{r}} - G_{n-1}^r} \Phi^{\mathrm{r}} : X \to A_r$ is compact (since $\Phi^{\mathrm{r}} : X \to H_{A_r}$ is compact). As a consequence, there exists an element $\nu_m^r \in X$ with

$\left(P_m \sqrt{G_n^{\mathrm{r}} - G_{n-1}^r} \Phi^{\mathrm{r}}\right)(\eta^2) = \langle \nu_m^r, \eta^2 \rangle \quad \text{for all } \eta^2 \in X$

This proves the proposition.

**Remark 2.7.** The sequence $\{\zeta_k^2\}_{k=1}^\infty$ in $X$ which implements $W : X \to H_{A_r}$ is a "standard normalized tight frame" in the terminology of M. Frank and D. R. Larson, see [FrLa02, Definition 2.1] (notice however that we never assume that $A_r$ is unital).

## III. Differentiable Absorption

For$X$ be a countably generated Hilbert $C^*$-module over a $C^*$-algebra $A_r$.Furthermore, let $B_r$ be a $C^*$-algebra and let $\rho : A_r \to B_r$ be an injective $*$-homomorphism.

The "differentiable structure" on $A_r$ will come in the form of a dense $*$-subalgebra $\mathcal{A} \subseteq A_r$ and a linear map $\delta : \mathcal{A} \to B_r$ such that

$$\delta(a_1^{\mathrm{r}} \cdot a_2^r) = \delta(a_1^r) \cdot \rho(a_2^r) + \rho(a_1^r) \cdot \delta(a_2^r) \quad \text{and} \quad \delta((a^r)^*) = -\delta(a^r)^*$$

for all $a^r, a_1^r, a_2^r \in \mathcal{A}$. The derivation $\delta: \mathcal{A} \to B_r$ is required to be closed. Thus, whenever $\{a_n^r\}$ is a sequence in $\mathcal{A}$ such that $\delta(a_n^r) \to b^r$ and $a_n^r \to 0$ for some $b^r \in B_r$ we may conclude that $b^r = 0$.

We let $(A_r)_\delta$ denote the completion of $\mathcal{A}$ with respect to the norm

$$\|\cdot\|_\delta: \mathcal{A} \to [0, \infty) \quad \| a^r \|_\delta := \| a^r \| + \| \delta(a^r) \|$$

It follows by closedness that $\delta: \mathcal{A} \to B_r$ extends to a well-defined derivation $\delta : (A_r)_\delta \to B_r$. Remark that $\|(a^r)^*\|_\delta = \| a^r \|_\delta$ for all $a^r \in (A_r)_\delta$, but that the $C^*$-identity does not hold for the norm $\|\cdot\|_\delta$.

The countably generated Hilbert $C^*$-module $X$ is assumed to be compatible with the differentiable structure on $A_r$ by the following condition: There exists a sequence $\{\xi_n^2\}_{n=1}^\infty$ in $X$ such that

$$\langle \xi_n^2, \xi_m^2 \rangle \in \mathcal{A} \quad \text{for all } n, m \in \mathbb{N}$$

and such that $\text{span}_{A_r}\{\xi_n^2 \mid n \in \mathbb{N}\}$ is dense in $X$.

Without loss of generality, we may then assume that

$$\|\langle \xi_n^2, \xi_m^2 \rangle\|_\delta \leq \frac{1}{n^2 \cdot m^2} \quad \text{for all } n, m \in \mathbb{N} \tag{3.1}$$

The conditions stated above will remain in effect throughout this section.

Let $M_\infty(\mathcal{A})$ denote the $*$-algebra of all finite matrices over $\mathcal{A}$. We will think of $M_\infty(\mathcal{A})$ as a dense $*$-subalgebra of the compact operators $\mathcal{K}(H_{A_r})$ on the Hilbert $C^*$-module $H_{A_r}$. There is a unique injective $*$-homomorphism $\rho: \mathcal{K}(H_{A_r}) \to \mathcal{K}(H_{B_r})$ such that $\rho(\{a_{ij}^r\}) = \{\rho(a_{ij}^r)\}$ for all finite matrices $\{a_{ij}^r\} \in M_\infty(\mathcal{A})$.

Likewise, we may extend $\delta: \mathcal{A} \to B_r$ to a closed derivation $\delta: M_\infty(\mathcal{A}) \to \mathcal{K}(H_{B_r})$.

We will apply the notation $\mathcal{K}(H_{A_r})_\delta$ for the Banach $*$-algebra obtained as the completion of $M_\infty(\mathcal{A})$ with respect to the norm $\|\cdot\|_\delta: a^r \mapsto \| a^r \| + \| \delta(a^r) \|$.

The unitalization of $\mathcal{K}(H_{A_r})_\delta$ is denoted by $\widetilde{\mathcal{K}(H_{A_r})}_\delta$. This unital *-algebra becomes a unital Banach $*$-algebra when equipped with the norm $\|\cdot\|_\delta: \widetilde{\mathcal{K}(H_{A_r})}_\delta \to [0, \infty), \| (a^r, 1+\epsilon) \|_\delta := \| a^r + 1 + \epsilon \| + \| \delta(a^r) \|$. Here we are thinking of $a^r + 1 + \epsilon$ as a bounded adjointable operator on the standard module $H_{A_r}$. Notice that our $*$-homomorphism

$\rho: \mathcal{K}(H_{A_r}) \to \mathcal{K}(H_{B_r})$ can be extended uniquely to a unital $*$-homomorphism

$\rho: \widetilde{\mathcal{K}(H_{A_r})} \to \mathcal{L}(H_{B_r})$ and that our derivation $\delta: M_\infty(\mathcal{A}) \to \mathcal{K}(H_{B_r})$ can be extended uniquely to a closed derivation $\delta: \widetilde{\mathcal{K}(H_{A_r})}_\delta \to \mathcal{L}(H_{B_r})$ such that $\delta((0, 1+\epsilon)) = 0$ for all $(1+\epsilon) \in \mathbb{C}$.

We are now prove the first result:

**Lemma 3.1 (see [Jka10]).** The sequence of finite matrices $\left\{\{\langle \xi_n^2, \xi_m^2 \rangle\}_{n,m=1}^N\right\}_{N=1}^\infty$ converges to an element $G^r \in \widetilde{\mathcal{K}(H_{A_r})}_\delta$ with positive spectrum.

**Proof.** We first remark that $\{\langle \xi_n^2, \xi_m^2 \rangle\}_{n,m=1}^N$ determines a positive element in the $C^*$-algebra $M_N(A_r)$ for all $N \in \mathbb{N}$.

Next, we notice that the spectrum of an element $a^r$ in the unital Banach algebra $M_N\widetilde{((A_r)_\delta)}$ agrees with the spectrum of $a^r$ as an element in the unital $C^*$-algebra $\widetilde{M_N(A_r)}$.

This is a consequence of spectral invariance, see [BLCU91, Proposition 3.12].

These observations imply that $\{\langle\xi_n^2,\xi_m^2\rangle\}_{n,m=1}^{N} \in M_N(\widetilde{(A_r)_\delta})$ has positive spectrum for all $N \in \mathbb{N}$. It is therefore enough to show that the sequence $\left\{\{\langle\xi_n^2,\xi_m^2\rangle\}_{n,m=1}^{N}\right\}_{N=1}^{\infty}$ is Cauchy in$\mathcal{K}\left(\widetilde{H_{A_r}}\right)_\delta$.

To this end, let $N, M \in \mathbb{N}$ with $M \geq N$ be given and notice that

$$\begin{aligned}
&\|\{\langle\xi_n^2,\xi_m^2\rangle\}_{n,m=1}^{M} - \{\langle\xi_n^2,\xi_m^2\rangle\}_{n,m=1}^{N}\|_\delta \\
&\quad \leq \sum_{n=N+1}^{M}\sum_{m=1}^{M} \|\langle\xi_n^2,\xi_m^2\rangle\|_\delta + \sum_{n=1}^{N}\sum_{m=N+1}^{M} \|\langle\xi_n^2,\xi_m^2\rangle\|_\delta \\
&\leq 2\cdot\sum_{m=1}^{\infty}\frac{1}{m^2}\cdot\sum_{n=N+1}^{M}\frac{1}{n^2}
\end{aligned}$$

where the last inequality follows by (3.1). This shows that the sequence $\left\{\{\langle\xi_n^2,\xi_m^2\rangle\}_{n,m=1}^{N}\right\}_{N=1}^{\infty}$ is Cauchy in $\mathcal{K}\left(\widetilde{H_{A_r}}\right)_\delta$.

For each $n \in \mathbb{N}$, we define the element

$$\begin{aligned}
H_n \quad &:= (1/n + G^r)^{-1} - (1/(n-1) + G^r)^{-1} \\
&= (1 + n\cdot G^r)^{-1}\cdot(1 + (n-1)\cdot G^r)^{-1}
\end{aligned}$$

in $\mathcal{K}\left(\widetilde{H_{A_r}}\right)_\delta$, where $H_1 := (1 + G^r)^{-1}$. Since the spectrum of $H_n$ is strictly positive, it has a well-defined square root in $\mathcal{K}\left(\widetilde{H_{A_r}}\right)_\delta$,

$$\sqrt{H_n} = (1 + n\cdot G^r)^{-1/2}\cdot(1 + (n-1)\cdot G^r)^{-1/2}$$

**Lemma 3.2 (see [Jka10]).** We have the expression

$$\begin{aligned}
&\delta((1 \quad + nG^r)^{-1/2}) \\
&\quad = -\frac{n}{\pi}\cdot\int_0^\infty\sum_r \quad (1+\epsilon)^{-\frac{1}{2}}\cdot\rho((2+\epsilon+nG^r)^{-1})\cdot\delta(G^r)\cdot\rho((2+\epsilon+n\cdot G^r)^{-1})d(1+\epsilon)
\end{aligned}$$

where the integral converges in the operator norm on $\mathcal{L}(H_{B_r})$.

**Proof.** The element $(1 + nG^r)^{-1/2} \in \mathcal{K}\left(\widetilde{H_{A_r}}\right)_\delta$ can be rewritten as the integral

$$\frac{1}{\pi}\cdot\int_0^\infty\sum_r \quad (1+\epsilon)^{-\frac{1}{2}}\cdot(2+\epsilon+n\cdot G^r)^{-1}d(1+\epsilon)$$

which converges absolutely in the norm $\|\cdot\|_\delta : \mathcal{K}\left(\widetilde{H_{A_r}}\right)_\delta \to [0,\infty)$. It is therefore enough to check that

$$\delta((2+\epsilon+n\cdot G^r)^{-1}) = -\rho((2+\epsilon+nG^r)^{-1})\cdot n\cdot\delta(G^r)\cdot\rho((2+\epsilon+n\cdot G^r)^{-1})$$

But this follows from a standard computation, using that $\delta : \mathcal{K}\left(\widetilde{H_{A_r}}\right)_\delta \to \mathcal{L}(H_{B_r})$ is a derivation with respect to $\rho : \mathcal{K}\left(\widetilde{H_{A_r}}\right) \to \mathcal{L}(H_{B_r})$.

The estimate in the following lemma is of central importance for the differentiable absorption theorem.

**Lemma 3.3 (see [Jka10]).** Let $\varepsilon \in (0,1/2)$. There exists a constant $C_\varepsilon > 0$ such that

$$\|\delta(\sqrt{H_n}\cdot(G^r)^2)\| \leq C_\varepsilon \cdot \frac{1}{n^{1-\varepsilon}}$$

for all $(2+\epsilon)\in\mathbb{N}$.

**Proof.** Let $\epsilon \geq 0$. Using that $\delta: \mathcal{K}(\widetilde{H_{A_r}})_\delta \to \mathcal{L}(H_{B_r})$ is a derivation we obtain that

$$\begin{aligned}\delta(\sqrt{H_{2+\epsilon}}\cdot(G^r)^2) = \delta(G^r)\cdot\quad & \sqrt{H_{2+\epsilon}}\cdot G^r + G^r\cdot\sqrt{H_{2+\epsilon}}\cdot\delta(G^r)\\ & +G^r\cdot\delta\big((1+(2+\epsilon)G^r)^{-1/2}\big)\cdot(1+(1+\epsilon)G^r)^{-1/2}\cdot G^r \qquad (3.2)\\ & +G^r\cdot(1+(2+\epsilon)G^r)^{-1/2}\cdot\delta\big((1+(1+\epsilon)G^r)^{-1/2}\big)\cdot G^r\end{aligned}$$

where we have suppressed the unital$*$-homomorphism $\rho: \mathcal{K}(\widetilde{H_{A_r}}) \to \mathcal{L}(H_{B_r})$.

Now , since $G^r \in \mathcal{K}(H_{A_r})_\delta$ determines a positive element in the unital $C^*$-algebra $\mathcal{K}(\widetilde{H_{A_r}})$, we have that

$$\|G^r\cdot(2+\epsilon+(2+\epsilon)G^r)^{-1}\| \leq \frac{1}{2+\epsilon}$$

for all $\epsilon \geq -1$.

Using the above estimate we obtain the following inequalities

$$\begin{aligned}\|\ \delta(G^r)\quad & \cdot\sqrt{H_{2+\epsilon}}\cdot G^r + G^r\cdot\sqrt{H_{2+\epsilon}}\cdot\delta(G^r)\ \|\\ & \leq 2\cdot\|\ \delta(G^r)\ \|\cdot\|(1+(1+\epsilon)G^r)^{-1/2}(G^r)^{1/2}\|\cdot\|(1+(2+\epsilon)G^r)^{-1/2}(G^r)^{1/2}\|\\ & \leq 2\cdot\|\ \delta(G^r)\ \|\cdot\frac{1}{\sqrt{2+\epsilon}\cdot\sqrt{1+\epsilon}}\end{aligned}$$

To continue, we apply Lemma 3.2 to compute as follows,

$$\begin{aligned}& G^r\cdot\delta\big((1+(2+\epsilon)G^r)^{-1/2}\big)\cdot(1+(1+\epsilon)G^r)^{-1/2}\cdot G^r\\ = -\frac{1}{\pi}\cdot\int_0^\infty & (1+\epsilon)^{-\frac{1}{2}}\cdot((2+\epsilon)G^r)\cdot(2+\epsilon+(2+\epsilon)G^r)^{-1}\cdot\delta(G^r)\\ & \cdot(G^r)^{\frac{1}{2}-\varepsilon}\cdot(2+\epsilon+(2+\epsilon)G^r)^{-1}d(1+\epsilon)\\ & \cdot(G^r)^{1/2+\varepsilon}\cdot(1+(1+\epsilon)G^r)^{-1/2}\end{aligned}$$

As a consequence, we obtain that

$$\begin{aligned}& \|G^r\cdot\delta\big((1+(2+\epsilon)G^r)^{-1/2}\big)\cdot(1+(1+\epsilon)G^r)^{-1/2}\cdot G^r\|\\ & \leq\frac{1}{\pi}\cdot\int_0^\infty\sum_r\quad(1+\epsilon)^{\frac{1}{2}}\cdot\|\ \delta(G^r)\ \|\cdot(2+\epsilon)^{-\frac{1}{2}-\varepsilon}\cdot\left\|(G^r)^{\frac{1}{2}-\varepsilon}\cdot(2+\epsilon+(2+\epsilon)G^r)^{-\frac{1}{2}+\varepsilon}\right\|d(1+\epsilon)\\ & \cdot\|(G^r)^\varepsilon\|\cdot\frac{1}{\sqrt{1+\epsilon}}\\ & \leq\sum_r\quad\|\ \delta(G^r)\ \|\cdot\|(G^r)^\varepsilon\|\cdot\frac{1}{(1+\epsilon)^{\frac{1}{2}}\cdot(2+\epsilon)^{\frac{1}{2}-\varepsilon}\cdot\pi}\cdot\int_0^\infty(1+\epsilon)^{-\frac{1}{2}}(2+\epsilon)^{-\frac{1}{2}-\varepsilon}d(1+\epsilon)\end{aligned}$$

A similar computation shows that

$$\begin{aligned}& \|G^r\cdot(1+(2+\epsilon)G^r)^{-1/2}\cdot\delta\big((1+(1+\epsilon)G^r)^{-1/2}\big)\cdot G^r\|\\ & \leq\sum_r\quad\|\ \delta(G^r)\ \|\cdot\|(G^r)^\varepsilon\|\cdot\frac{1}{(1+\epsilon)^{\frac{1}{2}}\cdot(2+\epsilon)^{\frac{1}{2}-\varepsilon}\cdot\pi}\cdot\int_0^\infty(1+\epsilon)^{-\frac{1}{2}}(2+\epsilon)^{-\frac{1}{2}-\varepsilon}d(1+\epsilon)\end{aligned}$$

A combination of all the above estimates and the identity in (3.2) proves the claim of the proposition.

We have that the compact operators $(\Phi^{\mathrm{r}})^*: H_{A_r} \to X$ and $\Phi^{\mathrm{r}}: X \to H_{A_r}$ are defined by $(\Phi^{\mathrm{r}})^*: \{a_k^r\}_{r,k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} \sum_r \quad \xi_k^2 \cdot a_k^r$ and $\Phi^{\mathrm{r}}: \eta^2 \mapsto \{\langle \xi_k^2, \eta^2 \rangle\}_{k=1}^{\infty}$.

Furthermore, for each $(2+\epsilon) \in \mathbb{N}$, we have the compact operators $\Psi_{2+\epsilon}^{\mathrm{r}} := \sqrt{H_{2+\epsilon}} \Phi^{\mathrm{r}}: X \to H_{A_r}$ and $(\Psi^{\mathrm{r}})_{2+\epsilon}^* := (\Phi^{\mathrm{r}})^* \sqrt{H_{2+\epsilon}}: H_{A_r} \to X$.

Finally, for each $N \in \mathbb{N}$ we have the compact operators $V_N: X \to H_{A_r}^{\infty}$ and $V_N^*: H_{A_r}^{\infty} \to X$ defined by $V_N: \eta^2 \mapsto \{\Psi_{2+\epsilon}^{\mathrm{r}}(\eta^2)\}_{r,\epsilon=-1}^{N}$ and $V_N^*: \{x_{2+\epsilon}^2\}_{\epsilon=-1}^{\infty} \mapsto \sum_{r,\epsilon=-1}^{N} (\Psi^{\mathrm{r}})_{2+\epsilon}^*(x_{2+\epsilon}^2)$. It was proved in Section 2 that the sequence $\{V_N\}_{N=1}^{\infty}$ converges strongly to a bounded adjointable isometry $\Psi^{\mathrm{r}}: X \to H_{A_r}^{\infty}$. The adjoint of $\Psi^{\mathrm{r}}$ is given by $(\Psi^{\mathrm{r}})^*: \sum_{\epsilon=-1}^{\infty} e_{2+\epsilon} \cdot x_{2+\epsilon}^2 \mapsto \sum_{r,\epsilon=-1}^{\infty} (\Psi^{\mathrm{r}})_{2+\epsilon}^*(x_{2+\epsilon}^2)$.

For each $N \in \mathbb{N}$ we define the compact operator

$$\delta(\mathrm{diag}(G^r) V_N (\Phi^{\mathrm{r}})^*) \in \mathcal{K}\big(H_{B_r}, H_{B_r}^{\infty}\big) \quad \delta(\mathrm{diag}(G^r)\, V_N (\Phi^{\mathrm{r}})^*): x^2 \mapsto \sum_{r,\epsilon=-1}^{N} e_{2+\epsilon} \cdot \delta\big((G^r)^2 \sqrt{H_{2+\epsilon}}\big)(x^2)$$

where $\mathrm{diag}(G^r): H_{A_r}^{\infty} \to H_{A_r}^{\infty}$ refers to the (non-compact) diagonal operator $\mathrm{diag}(G^r)$ : $\sum_{\epsilon=-1}^{\infty} e_{2+\epsilon} x_{2+\epsilon}^2 \mapsto \sum_{\epsilon=-1}^{\infty} \sum_r \quad e_{2+\epsilon} G^r(x_{2+\epsilon}^2)$ induced by the (compact operator) $G^r: H_{A_r} \to H_{A_r}$.

We note the following consequence of the above Lemma 3.3:

**Lemma 3.4 (see [Jka10]).** The sequence of compact operators $\{\delta(\mathrm{diag}(G^r) V_N (\Phi^{\mathrm{r}})^*)\}_{r,N=1}^{\infty}$ is a Cauchy sequence in $\mathcal{K}\big(H_{B_r}, H_{B_r}^{\infty}\big)$.

**Proof.** By Lemma 3.3 we may choose a constant $\epsilon \geq 0$ such that

$$\|\delta(\mathrm{diag}(G^r) V_N (\Phi^{\mathrm{r}})^*)(x^2) - \delta(\mathrm{diag}(G^r) V_M (\Phi^{\mathrm{r}})^*)(x^2)\|^2 = \left\| \sum_{r,n=N+1}^{M} e_n \delta\big((G^r)^2 \sqrt{H_n}\big)(x^2) \right\|^2$$

$$= \left\| \sum_{r,n=N+1}^{M} \big\langle \delta\big((G^r)^2 \sqrt{H_n}\big) x^2, \delta\big((G^r)^2 \sqrt{H_n}\big) x^2 \big\rangle \right\| \leq (1+\epsilon) \sum_{n=N+1}^{M} \frac{1}{n^{3/2}} \parallel x^2 \parallel^2$$

for all $N, M \in \mathbb{N}$ with $M \geq N$ and all $x^2 \in H_{B_r}$. This proves the lemma.

The next lemma is a consequence of Lemma 2.3.

**Lemma 3.5 (see [Jka10]).** The sequence of compact operators $\{V_N (\Phi^{\mathrm{r}})^*\}_{r,N=1}^{\infty}$ converges in operator norm to $\Psi^{\mathrm{r}} (\Phi^{\mathrm{r}})^*: H_{A_r} \to H_{A_r}^{\infty}$.

**Proof.** This follows since $\Phi^{\mathrm{r}}: X \to H_{A_r}$ (and hence $(\Phi^{\mathrm{r}})^*: H_{A_r} \to X$ ) is compact and since the bounded sequence $\{V_N\}_{N=1}^{\infty}$ converges strongly to $\Psi^{\mathrm{r}}: X \to H_{A_r}^{\infty}$.

**Proposition 3.6 (see [Jka10]).** The sequence $\{\mathrm{diag}(G^r) V_N V_N^*\}_{r,N=1}^{\infty}$ in $\mathcal{K}\big(H_{A_r}^{\infty}\big)$ converges in operator norm to $\mathrm{diag}(G^r)\, \Psi^{\mathrm{r}} (\Psi^{\mathrm{r}})^*: H_{A_r}^{\infty} \to H_{A_r}^{\infty}$.

**Proof.** Let $N \in \mathbb{N}$ and remark that

$$\{\mathrm{diag}(G^r) V_N V_N^*\}_{n,m} = (G^r)^2 \sqrt{H_m} \cdot \sqrt{H_n} = \sqrt{H_m} \Phi^{\mathrm{r}} (\Phi^{\mathrm{r}})^* \Phi^{\mathrm{r}} (\Phi^{\mathrm{r}})^* \sqrt{H_n}$$

for all $n, m \in \{1, \ldots, N\}$. It follows that $\mathrm{diag}(G^r) V_N V_N^* = V_N (\Phi^{\mathrm{r}})^* \Phi^{\mathrm{r}} V_N^*$. The result of the proposition is now a consequence of Lemma 3.5.

In order to formulate our next result we reiterate the construction of the Banach *-algebra $\mathcal{K}(H_{A_r})_\delta$. Indeed, we may consider the finite matrices $M_\infty\left(\mathcal{K}(H_{A_r})_\delta\right)$ as a dense $*$-subalgebra of the compact operators $\mathcal{K}(H_{A_r}^\infty)$ on the standard module $H_{A_r}^\infty$. The *-homomorphism $\rho: \mathcal{K}(H_{A_r}) \to \mathcal{K}(H_{B_r})$ can then be extended uniquely to a $*$-homomorphism $\rho: \mathcal{K}(H_{A_r}^\infty) \to \mathcal{K}(H_{B_r}^\infty)$ such that $\rho\{x_{ij}^2\} = \{\rho(x_{ij}^2)\}$ for all $\{x_{ij}^2\} \in M_\infty\left(\mathcal{K}(H_{A_r})\right)$. Likewise, we may extend $\delta$ uniquely to a closed derivation $\delta: M_\infty\left(\mathcal{K}(H_{A_r})_\delta\right) \to \mathcal{K}(H_{B_r}^\infty)$ such that $\delta\{x_{ij}^2\} := \{\delta(x_{ij}^2)\}$. We denote the Banach *-algebra defined as the completion of $M_\infty\left(\mathcal{K}(H_{A_r})_\delta\right)$ with respect to the norm $\|\cdot\|_\delta: x^2 \mapsto \| x^2 \| + \| \delta(x^2) \|$ by $\mathcal{K}(H_{A_r}^\infty)_\delta$.

We note that we have an isometric isomorphism of Banach$*$-algebras $\mathcal{K}(H_{A_r}^\infty)_\delta \to \mathcal{K}(H_{A_r})_\delta$ defined by conjugasion with the unitary operator $U_{\alpha^r}: H_{A_r} \to H_{A_r}^\infty$ introduced in (2.2).

**Proposition 3.7 (see [Jka10]).** The sequence $\{\mathrm{diag}(G^r)^2 V_N V_N^*\}_{r,N=1}^\infty$ in $M_\infty\left(\mathcal{K}(H_{A_r})_\delta\right)$ is Cauchy in $\mathcal{K}(H_{A_r}^\infty)_\delta$.

**Proof.** We know from Proposition 3.6 that $\mathrm{diag}(G^r)^2 V_N V_N^*$ converges to $\mathrm{diag}(G^r)^2 \Psi^r (\Psi^r)^*$ in $\mathcal{K}(H_{A_r}^\infty)$. It is therefore enough to show that $\{\delta(\mathrm{diag}(G^r)^2 V_N V_N^*)\}_{r,N=1}^\infty$ is a Cauchy sequence in $\mathcal{K}(H_{B_r}^\infty)$.

Let now $N \in \mathbb{N}$ and notice that

$$(\mathrm{diag}(G^r)\, V_N (\Phi^r)^*)(x^2) = \sum_{n=1}^{N} \sum_{r} \quad e_n \cdot \left(G^r \sqrt{H_n} G^r\right)(x^2)$$

$$= \sum_{n=1}^{N} \sum_{r} \quad e_n \cdot \left(\sqrt{H_n} \Phi^r (\Phi^r)^* G^r\right)(x^2) = (V_N (\Phi^r)^* G^r)(x^2)$$

for all $x^2 \in H_{A_r}$. We thus have that $\mathrm{diag}(G^r) V_N (\Phi^r)^* = V_N (\Phi^r)^* G^r$.

We may therefore compute as follows,

$$\begin{aligned} &\delta(\mathrm{diag}(G^r)^2 V_N V_N^*) \\ &= \delta(\mathrm{diag}(G^r)\, V_N (\Phi^r)^* \Phi^r V_N^*) = \delta(\mathrm{diag}(G^r) V_N (\Phi^r)^*) \Phi^r V_N^* + \mathrm{diag}(G^r) V_N (\Phi^r)^* \delta(\Phi^r V_N^*) \\ &= \delta(\mathrm{diag}(G^r) V_N (\Phi^r)^*) \Phi^r V_N^* + V_N (\Phi^r)^* \delta(G^r \Phi^r V_N^*) - V_N (\Phi^r)^* \delta(G^r) \Phi^r V_N^* \\ &= \delta(\mathrm{diag}(G^r) V_N (\Phi^r)^*) \Phi^r V_N^* - V_N (\Phi^r)^* \delta(\mathrm{diag}(G^r)\, V_N (\Phi^r)^*)^* - V_N (\Phi^r)^* \delta(G^r) \Phi^r V_N^* \end{aligned}$$

The result of the proposition now follows by Lemma 3.5 and Lemma 3.4.

**Lemma 3.8 (see [Jka10]).** The image of $(\Psi^r)^* \,\mathrm{diag}(G^r)\, \Psi^r: X \to X$ is dense in $X$ and $\mathrm{diag}(G^r)\, \Psi^r (\Psi^r)^* = \Psi^r (\Psi^r)^* \mathrm{diag}(G^r)$.

**Proof.** By Proposition 3.6 we know that $\mathrm{diag}(G^r)\, \Psi^r (\Psi^r)^* = \lim_{N\to\infty} \mathrm{diag}(G^r) V_N V_N^*$ and that $\Psi^r (\Psi^r)^* \mathrm{diag}(G^r) = \lim_{N\to\infty} V_N V_N^* \mathrm{diag}(G^r)$. To show that $\mathrm{diag}(G^r) (\Psi^r)^* = \Psi^r (\Psi^r)^* \mathrm{diag}(G^r)$ is therefore suffices to show that $V_N V_N^* \mathrm{diag}(G^r) = \mathrm{diag}(G^r) V_N V_N^*$ for all $N \in \mathbb{N}$. But this follows by noting that

$$(V_N V_N^* \mathrm{diag}(G^r))_{n,m} = \sqrt{H_n} G^r \sqrt{H_m} G^r = G^r \sqrt{H_n} G^r \sqrt{H_m} = (\mathrm{diag}(G^r) V_N V_N^*)_{n,m}$$

for all $N \in \mathbb{N}$ and all $n, m \in \{1, \dots, N\}$.

In order to prove that the image of $(\Psi^r)^* \,\mathrm{diag}(G^r)\, \Psi^r: X \to X$ is dense we note that

$$\begin{aligned} \mathrm{span}_{A_r}\{\xi^2 \in \mathrm{Im}((\Phi^r)^* G^r (G^r + 1/n)^{-1}) \mid n \in \mathbb{N}\} \subseteq \mathrm{span}_{A_r}\{\xi^2 \in \mathrm{Im}\left((\Phi^r)^* G^r \sqrt{H_n}\right) \mid n \in \mathbb{N}\} \\ \subseteq \mathrm{Im}((\Psi^r)^* \mathrm{diag}(G^r)) = \mathrm{Im}((\Psi^r)^* \,\mathrm{diag}(G^r)\, \Psi^r (\Psi^r)^*) \subseteq \mathrm{Im}((\Psi^r)^* \,\mathrm{diag}(G^r)\, \Psi^r) \end{aligned}$$

Since the image of $(\Phi^{\mathrm{r}})^*: H_{A_r} \to X$ is dense by the standing conditions on our Hilbert $C^*$-module $X$ it therefore suffices to show that the sequence $\{(\Phi^{\mathrm{r}})^* G^r (1/n + G^r)^{-1}\}_{r,n=1}^{\infty}$ of bounded adjointable operators converges in operator norm to $(\Phi^{\mathrm{r}})^*: H_{A_r} \to X$. But this follows since

$$\frac{1}{n}\|(\Phi^{\mathrm{r}})^*(1/n + G^r)^{-1}\| \leq \frac{1}{\sqrt{n}}$$

for all $n \in \mathbb{N}$. See the proof of Lemma 2.2.

We now prove the differentiable absorption theorem. This is the first main result.

**Theorem 3.1 (see [Jka10]).** There exists a bounded adjointable isometry $W: X \to H_{A_r}$ and a positive selfadjoint bounded operator $K_r: H_{A_r} \to H_{A_r}$ such that

(1) $K_r P = P K_r$.

(2) $W^* K_r W: X \to X$ has dense image.

(3) $P K_r \in \mathcal{K}(H_{A_r})$.

(4) $P K_{\mathrm{r}}^2 \in \mathcal{K}(H_{A_r})_{\delta}$.

where $P := WW^*: H_{A_r} \to H_{A_r}$ is the associated orthogonal projection.

**Proof.** Let $U_{\alpha^r}: H_{A_r} \to H_{A_r}^{\infty}$ denote the unitary operator introduced in (2.2). The bounded adjointable operator $W := U_{\alpha^r}^* \Psi^{\mathrm{r}}: X \to H_{A_r}$ is then an isometry. Furthermore, define the positive selfadjoint bounded operator $K_r := U_{\alpha^r}^* \mathrm{diag}(G^r) U_{\alpha^r}: H_{A_r} \to H_{A_r}$. The result of the theorem then follows by Lemma 3.8, Proposition 3.6, and Proposition 3.7.

**Remark 3.9.** As in Proposition 2.6, we may find a sequence $\{\zeta_k^2\}_{k=1}^{\infty}$ of elements in $X$ which implements the isometry $W: X \to H_{A_r}$ in the sense that

$$W(\eta^2) = \{\langle \zeta_k^2, \eta^2 \rangle\}_{k=1}^{\infty} \quad \text{for all } \eta^2 \in X$$

## IV. Grassmann Connections

We then let $W: X \to H_{A_r}$ and $K_r: H_{A_r} \to H_{A_r}$ be fixed bounded adjointable operators which satisfy the properties stated in Theorem 3.1. Furthermore, we let $\{\zeta_k^2\}_{k=1}^{\infty}$ be a sequence in $X$ which implements $W$, see Remark 3.9.

We shall in this section see how to construct a dense $(A_r)_\delta$-submodule of $\mathcal{X} \subseteq X$ together with a Hermitian $\delta$-connection on $\mathcal{X}$.

In order to construct $\mathcal{X}$ we recall the following, see [KALE13, Definition 3.3] and [Mes14, Page 119]:

**Definition 4.1.** The standard module over $(A_r)_\delta$ consists of all sequences $\{a_n^r\}_{r,n=1}^{\infty}$ of elements in $(A_r)_\delta$ such that

$$\{a_n^r\} \in H_{A_r} \quad \text{and} \quad \{\delta(a_n^r)\} \in H_{B_r}$$

The standard module over $(A_r)_\delta$ is denoted by $H_{(A_r)_\delta}$.

The standard module $H_{(A_r)_\delta}$ is a dense $(A_r)_\delta$-submodule of the standard module $H_{A_r}$. Furthermore, it was proved in [KALE13, Page 505] that

$\langle x^2, y^2 \rangle \in (A_r)_\delta \quad$ for all $x^2, y^2 \in H_{(A_r)_\delta}$

where $\langle\cdot,\cdot\rangle: H_{A_r} \times H_{A_r} \to A_r$ denotes the inner product on $H_{A_r}$.

The standard module becomes a Banach space when equipped with the norm

$$\|\cdot\|_\delta : \{a_n^r\} \mapsto \|\{a_n^r\}\| + \|\{\delta(a_n^r)\}\|$$

Each element $T_r \in \mathcal{K}(H_{A_r})_\delta \subseteq \mathcal{K}(H_{A_r})$ restricts to a bounded operator $T_r: H_{(A_r)\delta} \to H_{(A_r)\delta}$. Indeed, the map

$$M_\infty((A_r)_\delta) \times H_{(A_r)\delta} \to H_{(A_r)\delta} \quad \left(\{a_{ij}^r\}, \{b_n^r\}\right) \mapsto \left\{\sum_{n=1}^{\infty} \sum_{r} \quad a_{in}^{\mathrm{r}} \cdot b_n^r\right\}$$

satisfies the inequality $\| A_r \cdot b^r \|_\delta \leq \| A_r \|_\delta \cdot \| b^r \|_\delta$ for all $A_r \in M_\infty((A_r)_\delta)$ and $b^r \in H_{(A_r)\delta}$. We may now define the $(A_r)_\delta$-submodule $\mathcal{X} \subseteq X$ as the following image:

$$\mathcal{X} := \mathrm{Im}\left(W^* K_{\mathrm{r}}^2 : H_{(A_r)\delta} \to X\right) \tag{4.1}$$

The properties of $\mathcal{X}$ are summarized in the next lemma:

**Lemma 4.2 (see [Jka10]).** The $(A_r)_\delta$-submodule $\mathcal{X} \subseteq X$ is dense. Furthermore, $W(\xi^2) \in H_{(A_r)\delta}$ and $\langle \xi^2, \eta^2 \rangle \in (A_r)_\delta$ for all $\xi^2, \eta^2 \in \mathcal{X}$.

**Proof.** To see that $\mathcal{X} \subseteq X$ is dense, recall from Theorem 3.1 that $W^* K_r W: X \to X$ has dense image. It follows that

$$W^* K_r^2 W = W^* K_r W W^* K_r W: X \to X$$

has dense image as well. In particular, we obtain that $W^* K_{\mathrm{r}}^2 : H_{A_r} \to X$ has dense image, thus the density of $\mathcal{X} \subseteq X$ follows since $H_{(A_r)\delta} \subseteq H_{A_r}$ is dense.

Consider now $\xi^2 = (W^* K_r^2)(x^2)$ with $x^2 \in H_{(A_r)\delta}$. Then $W(\xi^2) = (W W^* K_r^2)(x^2)$. But $W W^* K_r^2 \in \mathcal{K}(H_{A_r})_\delta$ by Theorem 3.1 and therefore $(W W^* K_r^2)(x^2) \in H_{(A_r)\delta}$ by the observations preceding this lemma. This proves the second claim of the present lemma.

Finally, let $\xi^2, \eta^2 \in \mathcal{X}$. Since $W: X \to H_{A_r}$ is an isometry, we obtain that $\langle \xi^2, \eta^2 \rangle = \langle W\xi^2, W\eta^2 \rangle$. But $\langle W\xi^2, W\eta^2 \rangle \in (A_r)_\delta$ since $W\xi^2, W\eta^2 \in H_{(A_r)\delta}$.

In order to construct the Hermitian $\delta$-connection we recall the following concepts:

**Definition 4.3.** The $C^*$-algebra of continuous $\delta$-forms is the smallest $C^*$-subalgebra of $B_r$ which contains $\rho(a_0^r)$ and $\delta(a_1^r)$ for all $a_0^r, a_1^{\mathrm{r}} \in (A_r)_\delta$. This $C^*$-algebra is denoted by $\Omega_\delta(A_r)$.

We remark that $\Omega_\delta(A_r)$ can be viewed as a Hilbert $C^*$-module over $\Omega_\delta(A_r)$ in the usual way (this holds for any $C^*$-algebra). Furthermore, we have an injective *homomorphism $\rho: A_r \to \mathcal{L}(\Omega_\delta(A_r))$ given by $\rho(a^r)(\omega^2) = \rho(a^r) \cdot \omega^2$ for all $a^r \in A_r$ and $\omega^2 \in \Omega_\delta(A_r)$.

**Definition 4.4.** The Hilbert $C^*$-module of continuous $X$-valued $\delta$-forms is the interior tensor product $X \widehat{\otimes}_{A_r} \Omega_\delta(A_r)$.

Define the bounded operator $W \otimes 1: X \widehat{\otimes}_{A_r} \Omega_\delta(A_r) \to H_{\Omega_\delta(A_r)}, \xi^2 \widehat{\otimes}\, \omega^2 \mapsto W(\xi^2) \cdot \omega^2$. Remark that it is non-obvious that $W \otimes 1$ is adjointable since we do not assume that the left action of $A_r$ on $\Omega_\delta(A_r)$ is essential. This is none-the-less the case. Indeed, it suffices to recall that $W: X \to H_{A_r}$ is implemented by the sequence $\{\zeta_k^2\}_{k=1}^\infty$ of elements in $X$. We state the result as a lemma:

**Lemma 4.5.** The bounded operator $W \otimes 1: X \widehat{\otimes}_{A_r} \Omega_\delta(A_r) \to H_{\Omega_\delta(A_r)}$ is adjointable with adjoint $W^* \otimes 1: H_{\Omega_\delta(A_r)} \to X \widehat{\otimes}_{A_r} \Omega_\delta(A_r)$ induced by

$$W^* \otimes 1: \sum_{k=1}^{N} e_k \cdot \omega_k^2 \mapsto \sum_{k=1}^{N} \zeta_k^2 \otimes \omega_k^2$$

for all finite sequences $\sum_{k=1}^{N} e_k \cdot \omega_k^2$ in $H_{\Omega_\delta(A_r)}$.

We are now in position to define our Hermitian $\delta$-connection:

**Definition 4.6.** The Grassmann$\delta$-connection on $\mathcal{X}$ is defined by

$$\nabla_\delta: \mathcal{X} \to X \mathbin{\widehat{\otimes}}_{A_r} \Omega_\delta(A_r) \quad \nabla_\delta := (W^* \otimes 1)\delta W$$

where $\delta: H_{(A_r)_\delta} \to H_{\Omega_\delta(A_r)}$ is given by $\{a_n^r\}_{r,n=1}^{\infty} \mapsto \{\delta(a_n^r)\}_{r,n=1}^{\infty}$.

The Grassmann$\delta$-connection can also be expressed by the formula

$$\nabla_\delta: \eta^2 \mapsto \sum_{k=1}^{\infty} \zeta_k^2 \otimes \delta(\langle \zeta_k^2, \eta^2 \rangle) \quad \forall \eta^2 \in \mathcal{X}$$

where the sum converges in the norm on $X \mathbin{\widehat{\otimes}}_{A_r} \Omega_\delta(A_r)$.

We shall soon see that the Grassmann$\delta$-connection satisfies the Leibniz rule and is Hermitian. But we need a preliminary observation:

Observe that each element $\eta^2 \in X$ defines a bounded adjointable operator $(T_r)_{\eta^2}$ :

$\Omega_\delta(A_r) \to X \mathbin{\widehat{\otimes}}_{A_r} \Omega_\delta(A_r), (T_r)_{\eta^2}: \omega^2 \mapsto \eta^2 \otimes \omega^2$. The adjoint is given by $(T_r)_{\eta^2}^*: X \mathbin{\widehat{\otimes}}_{A_r} \Omega_\delta(A_r) \to \Omega_\delta(A_r), (T_r)_{\eta^2}^*: \xi^2 \otimes \omega^2 \mapsto \langle \eta^2, \xi^2 \rangle \cdot \omega^2$.

**Theorem 4.1 (see [Jka10]).** The Grassmann$\delta$-connection $\nabla_\delta: \mathcal{X} \to X_{\otimes}\Omega_\delta(A_r)$ is Hermitian and satisfies the Leibniz rule. Thus,

(1) $\delta(\langle \xi^2, \eta^2 \rangle) = (T_r)_{\xi^2}^* \nabla_\delta(\eta^2) - \left((T_r)_{\eta^2}^* \nabla_\delta(\xi^2)\right)^*$ for all $\xi^2, \eta^2 \in \mathcal{X}$.

(2) $\nabla_\delta(\eta^2 \cdot a^r) = \nabla_\delta(\eta^2) \cdot \rho(a^r) + \eta^2 \otimes \delta(a^r)$ for all $\eta^2 \in \mathcal{X}$ and $a^r \in (A_r)_\delta$.

**Proof.** Let $\xi^2, \eta^2 \in \mathcal{X}$ with $W\xi^2 = \{a_n^r\}_{r,n=1}^{\infty}$ and $W\eta^2 = \{b_n^r\}_{r,n=1}^{\infty}$. To prove the first claim, we compute as follows:

$$\begin{aligned}
\delta(\langle \xi^2, \eta^2 \rangle) &= \delta\left(\sum_{n=1}^{\infty} \sum_{r} \quad (a^r)_n^* b_n^r\right) = \sum_{n=1}^{\infty} \sum_{r} \quad ((a^r)_n^* \cdot \delta(b_n^r) - \delta(a_n^r)^* \cdot b_n^r) \\
&= \langle W\xi^2, \delta(W\eta^2) \rangle - \left(\sum_{n=1}^{\infty} \sum_{r} \quad (b^r)_n^* \cdot \delta(a_n^r)\right)^* \\
&= (T_r)_{\xi^2}^* (W^* \otimes 1)\delta(W\eta^2) - \langle W\eta^2, \delta(W\xi^2) \rangle^* \\
&= (T_r)_{\xi^2}^* \nabla_\delta(\eta^2) - \left((T_r)_{\eta^2}^* \nabla_\delta(\xi^2)\right)^*
\end{aligned}$$

Notice that we have suppressed the injective $*$-homomorphism $\rho: A_r \to B_r$ in the above computation.

Let now $\eta^2 \in \mathcal{X}$ and $a^r \in (A_r)_\delta$. To prove the second claim, we compute as follows:

$\nabla_\delta(\eta^2 \cdot a^r)$

$= (W^* \otimes 1)\delta W(\eta^2 \cdot a^r) = (W^* \otimes 1)((\delta W)(\eta^2) \cdot a^r) + (W^* \otimes 1)(W(\eta^2) \cdot \delta(a^r))$
$= \nabla_\delta(\eta^2) \cdot a^r + \eta^2 \otimes \delta(a^r)$

These two computations prove the theorem.

## V. Symmetric Lifts of Unbounded Operators

For$Y$ be a Hilbert $C^*$-module over a $C^*$-algebra $B_r$ and let $D_r: \mathcal{D}(D_r) \to Y$ be an unbounded selfadjoint and regular operator. We recall that the conditions of selfadjointness and regularity are equivalent to the following two conditions:

(1) The unbounded operator $D_r: \mathcal{D}(D_r) \to Y$ is symmetric.

(2) The unbounded operators $D_r \pm i: \mathcal{D}(D_r) \to Y$ are surjective.

See [LAN95, Proposition 10.6].

Let $X$ be a Hilbert $C^*$-module over a $C^*$-algebra $A_r$ and suppose that $\rho: A_r \to \mathcal{L}(Y)$ is an injective $*$-homomorphism. Suppose furthermore that we have a dense *-subalgebra$\mathcal{A} \subseteq A_r$ such that

(1) $\rho(x^2)\xi^2 \in \mathcal{D}(D_r)$ for all $x^2 \in \mathcal{A}$ and $\xi^2 \in \mathcal{D}(D_r)$ and $[D_r, \rho(x^2)]: \mathcal{D}(D_r) \to Y$ extends to a bounded adjointable operator $\delta(x^2)$ for all $x^2 \in \mathcal{A}$.

(2) There exists a sequence $\{\xi_n^2\}_{n=1}^\infty$ in $X$ which generates $X$ as a Hilbert $C^*$ module and for which

$\langle \xi_n^2, \xi_m^2 \rangle \in \mathcal{A}$ for all $n. m \in \mathbb{N}$

Remark that $\delta((x^2)^*) = -\delta(x^2)^*$ since $D_r: \mathcal{D}(D_r) \to Y$ is selfadjoint.

We let $W: X \to H_{A_r}$ and $K_r: H_{A_r} \to H_{A_r}$ be as in Theorem 3.1. Furthermore, we choose a sequence $\{\zeta_k^2\}_{k=1}^\infty$ in $X$ such that

$W(\eta^2) = \{\langle \zeta_k^2, \eta^2 \rangle\}_{k=1}^\infty$ for all $\eta^2 \in X$

Let $X \widehat{\otimes}_{A_r} Y$ denote the interior tensor product of $X$ and $Y$ over $A_r$. Define the bounded adjointable operator $\otimes 1: X \widehat{\otimes}_{A_r} Y \to Y^\infty, W \otimes 1: \xi^2 \otimes \eta^2 \mapsto \{\rho(\langle \zeta_k^2, \xi^2 \rangle)(\eta^2)\}_{k=1}^\infty$. The adjoint of $W \otimes 1$ is given by $W^* \otimes 1: Y^\infty \to X \widehat{\otimes}_{A_r} Y$, $W^* \otimes 1: \{\eta_k^2\}_{k=1}^\infty \mapsto \sum_{k=1}^\infty \zeta_k^2 \otimes \eta_k^2$, where the sum converges in the norm-topologyon $X \widehat{\otimes}_{A_r} Y$, see Lemma 4.5. We remark that $W \otimes 1: X \widehat{\otimes}_{A_r} Y \to Y^\infty$ is an isometry in the sense that $(W^* \otimes 1)(W \otimes 1) = 1_{X \widehat{\otimes}_{A_r} Y}$.

Define the unbounded operator $\mathrm{diag}(D_r): \mathcal{D}(\mathrm{diag}(D_r)) \to Y^\infty$ by $\mathrm{diag}(D_r): \{\eta_k^2\} \mapsto \{D_r \eta_k^2\}$, where the domain is given by

$$\mathcal{D}(\mathrm{diag}(D_r)) := \left\{ \{\eta_k^2\} \in Y^\infty \mid \eta_k^2 \in \mathcal{D}(D_r) \text{ and } \{D_r \eta_k^2\} \in Y^\infty \right\}$$

The unbounded operator $\mathrm{diag}(D_r)$ is then again selfadjoint and regular, indeed we have that $(\mathrm{diag}(D_r) \pm i)^{-1}: \{\eta_k^2\} \mapsto \{(D_r \pm i)^{-1} \eta_k^2\}$ for all $\{\eta_k^2\} \in Y^\infty$.

Define the right $B_r$-submodule $\mathcal{D}(1 \otimes_\nabla D_r) \subseteq X \widehat{\otimes}_{A_r} Y$ by

$$\mathcal{D}(1 \otimes \nabla_\nabla D_r) := \left\{ \sigma \in X \widehat{\otimes}_{A_r} Y \mid (W \otimes 1)(\sigma) \in \mathcal{D}(\mathrm{diag}(D_r)) \right\}$$

**Lemma 5.1 (see [Jka10]).**$\mathcal{D}(1 \otimes_\nabla D_r)$ is dense in $X \widehat{\otimes}_{A_r} Y$.

**Proof.** Let $\mathcal{X} \subseteq X$ be as in (4.1) and let $\mathcal{Z} \subseteq X \widehat{\otimes}_{A_r} Y$ denote the image of the algebraic tensor product $\mathcal{X} \otimes_{(A_r)_\delta} \mathcal{D}(D_r)$ in $X \widehat{\otimes}_{A_r} Y$. Remark that $\mathcal{Z} \subseteq X \widehat{\otimes}_{A_r} Y$ is dense since $\mathcal{X} \subseteq X$ is dense and $\mathcal{D}(D_r) \subseteq Y$ is dense. It is therefore enough to show that $(W \otimes 1)(\xi^2 \otimes \eta^2) \in \mathcal{D}(\mathrm{diag}(D_r))$ for all $\xi^2 \in \mathcal{X}$ and $\eta^2 \in \mathcal{D}(D_r)$.

Let thus $\xi^2 \in \mathcal{X}$ and $\eta^2 \in \mathcal{D}(D_r)$. We first remark that $\rho(\langle \zeta_k^2, \xi^2 \rangle)(\eta^2) \in \mathcal{D}(D_r)$ for all $k \in \mathbb{N}$ since $\langle \zeta_k^2, \xi^2 \rangle \in (A_r)_\delta$. It thus suffices to prove that $\{D_r(\rho(\langle \zeta_k^2, \xi^2 \rangle)\eta^2)\} \in Y^\infty$.

However, we have that

$$\begin{aligned}\{D_r(\rho(\langle\zeta_k^2,\xi^2\rangle)\eta^2)\}_{k=1}^{\infty} &= \{\delta(\langle\zeta_k^2,\xi^2\rangle)\eta^2\}_{k=1}^{\infty} + \{\rho(\langle\zeta_k^2,\xi^2\rangle)D_r\eta^2\}_{k=1}^{\infty}\\ &= \{\delta(\langle\zeta_k^2,\xi^2\rangle)\eta^2\}_{k=1}^{\infty} + (W\otimes 1)(\xi^2\otimes D_r\eta^2)\\ &= \delta(W\xi^2)(\eta^2) + (W\otimes 1)(\xi^2\otimes D_r\eta^2)\end{aligned}$$

We therefore only need to show that $\delta(W\xi^2)(\eta^2)\in Y^\infty$.

However, by Lemma 4.2 we have that $\delta(W\xi^2)\in\mathcal{L}(Y)^\infty$ for all $\xi^2\in\mathcal{X}$. This implies the result of the lemma since each $\{(T_r)_k\}_{r,k=1}^{\infty}\in\mathcal{L}(Y)^\infty$ yields a bounded adjointable operator $Y\to Y^\infty, \eta^2\mapsto\{(T_r)_k\eta^2\}_{r,k=1}^{\infty}$.

The above lemma allows us to define the following unbounded operator

$$1\otimes_\nabla D_r := (W^*\otimes 1)\mathrm{diag}(D_r)(W\otimes 1): \mathcal{D}(1\otimes_\nabla D_r)\to X\widehat{\otimes}_{A_r} Y$$

which we refer to as the symmetric lift of $D_r$ with respect to the Grassmann$\delta$ connection $\nabla$.

**Proposition 5.2 (see [Jka10]).** The unbounded operator

$$1\otimes_\nabla D_r := (W^*\otimes 1)\mathrm{diag}(D_r)(W\otimes 1): \mathcal{D}(1\otimes_\nabla D_r)\to X\widehat{\otimes}_{A_r} Y$$

is symmetric.

**Proof.** This follows since $\mathrm{diag}(D_r):\mathcal{D}(\mathrm{diag}(D_r))\to Y^\infty$ is selfadjoint. Indeed,

$$\begin{aligned}\langle(1\otimes_\nabla D_r)\sigma,\theta\rangle &= \langle\mathrm{diag}(D_r)\,(W\otimes 1)\sigma,(W\otimes 1)\theta\rangle\\ &= \langle\sigma,(W^*\otimes 1)\mathrm{diag}(D_r)(W\otimes 1)\theta\rangle\\ &= \langle\sigma,(1\otimes_\nabla D_r)\theta\rangle\end{aligned}$$

for all $\sigma,\theta\in\mathcal{D}(1\otimes_\nabla D_r)$.

We remark that the symmetric lift only depends on $D_r:\mathcal{D}(D_r)\to Y$ and the bounded adjointable isometry $W: X\to H_{A_r}$. It does not depend on the right $(A_r)_\delta$– submodule $\mathcal{X}\subseteq X$ defined in (4.1). The existence of $\mathcal{X}$ is however crucial for proving that the symmetric lift is densely defined.

The final result of this section relates the symmetric lifts to the Grassmann$\delta$ connection. Thus, let $\nabla_\delta:\mathcal{X}\to X\widehat{\otimes}_{A_r}\Omega_\delta(\mathcal{A})$ denote the Grassmannconnection, see Definition 4.6.

**Lemma 5.3 (see [Jka10]).** Let $\sigma=\xi^2\otimes\eta^2\in\mathcal{X}\otimes_{(A_r)_\delta}\mathcal{D}(D_r)$. Then $\sigma\in\mathcal{D}(1\otimes_\nabla D_r)$ and $(1\otimes\nabla D_r)(\sigma)=\nabla_\delta(\xi^2)(\eta^2)+\xi^2\otimes D_r\eta^2$

Remark that we have tacitly identitifed$\sigma$ with its image in $X\widehat{\otimes}_{A_r} Y$.

**Proof.** By the proof of Lemma 5.1 we have that $\sigma\in\mathcal{D}(1\otimes\nabla D_r)$ and that

$$\begin{aligned}(1\otimes\nabla D_r)(\sigma) &= (W^*\otimes 1)\mathrm{diag}(D_r)(W\otimes 1)(\sigma)\\ &= (W^*\otimes 1)(\{\delta(\langle\zeta_k^2,\xi^2\rangle)\eta^2\}_{k=1}^{\infty} + (W\otimes 1)(\xi^2\otimes D_r\eta^2))\\ &= \sum_{r,k=1}^{\infty}\zeta_k^2\otimes\delta(\langle\zeta_k^2,\xi^2\rangle)(\eta^2)+\xi^2\otimes D_r\eta^2\end{aligned}$$

But this proves the lemma since $\sum_{k=1}^{\infty}\zeta_k^2\otimes\delta(\langle\zeta_k^2,\xi^2\rangle)(\eta^2)=\nabla_\delta(\xi^2)(\eta^2)$.

In order to give the reader some feeling for what might be expected from symmetric lifts, we end this section by giving a basic example.

**5.1. Example [Jka10]: The half-line.** Let us consider the case where $X = C_0((0,\infty))$ consists of continuous functions on the half-line which vanish at 0 and at $\infty$. We may then give $X$ the structure of a Hilbert $C^*$-module over the $C^*$-algebra $A_r = C_0(\mathbb{R})$ of continuous functions on the real line which vanish at $\pm\infty$. On top of this, we let $L^2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) square integrable functions on the real line. This Hilbert space comes equipped with an injective $*$-homomorphism $\rho: C_0(\mathbb{R}) \to \mathcal{L}(L^2(\mathbb{R}))$ given by point-wise multiplication $\rho(f_r)(\xi^2) := f_r \cdot \xi^2$. Furthermore, we let $D_r: \mathcal{D}(D_r) \to L^2(\mathbb{R})$ denote the unbounded selfadjoint operator obtained as the closure of the Dirac operator

$$i\frac{d}{dt}: C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$$

where $C_c^\infty(\mathbb{R}) \subseteq L^2(\mathbb{R})$ denotes the smooth compactly supported functions defined on $\mathbb{R}$. We define the dense $*$-subalgebra $(A_r)_\delta \subseteq A_r$, by

$$(A_r)_\delta := \left\{ f_r \in C_0(\mathbb{R}) \mid f_r \text{ is differentiable with } \frac{df_r}{dt} \in C_0(\mathbb{R}) \right\}$$

The Hilbert $C^*$-module $X = C_0((0,\infty))$ is then generated by a single element. Indeed, we may choose a nowhere-vanishing differentiable function $\xi^2: (0,\infty) \to [0,1]$ such that $\xi^2, \frac{d\xi^2}{dt} \in X$. We then have that

$$X = \mathrm{cl}\{\xi^2 \cdot f_r \mid f_r \in A_r\} \quad \text{and} \quad \langle \xi^2, \xi^2 \rangle = \xi^4 \in (A_r)_\delta$$

where $\mathrm{cl}(\cdot)$ refers to the closure in supremum-norm. We may finally arrange that

$$\| \langle \xi^2, \xi^2 \rangle \|_\delta = \sup_{t\in\mathbb{R}} |\xi^4(t)| + 2\sup_{t\in\mathbb{R}} \left| \left( \xi^2 \cdot \frac{d\xi^2}{dt} \right)(t) \right| \leq 1$$

The bounded adjointable isometry $W: X \to H_{A_r}$ is then given by

$$W: g_r \mapsto \left\{ \sqrt{H_n} \cdot \langle \xi^2, g_r \rangle \right\}_{r,n=1}^\infty = \left\{ (1+n\xi^4)^{-1/2}(1+(n-1)\xi^4)^{-1/2} \xi^2 \cdot g_r \right\}_{r,n=1}^\infty$$

and the bounded adjointable positive operator $K_r: H_{A_r} \to H_{A_r}$ is given by

$$K_r: \{(f_r)_n\}_{r,n=1}^\infty \mapsto \{\xi^4 \cdot (f_r)_n\}_{r,n=1}^\infty$$

The dense $(A_r)_\delta$-submodule $\mathcal{X} \subseteq X$ is defined as the image $\mathcal{X} := \mathrm{Im}\left(W^* K_r^2: H_{(A_r)_\delta} \to X\right)$. It is then not hard to see that we have the inclusion

$$C_c^\infty((0,\infty)) \subseteq \mathcal{X}$$

The interior tensor product $X \widehat{\otimes}_{A_r} L^2(\mathbb{R})$ is unitarily isomorphic to the Hilbert space $L^2((0,\infty))$ of square integrable functions on the half-line. Under this isomorphism the isometry $W \otimes 1: L^2((0,\infty)) \to H_{L^2(\mathbb{R})}$ is given by

$$W \otimes 1: g_r \mapsto \left\{ (1+n\xi^4)^{-1/2}(1+(n-1)\xi^4)^{-1/2} \xi^2 \cdot g_r \right\}_{r,n=1}^\infty$$

We are interested in obtaining a better understanding of the symmetric lift

$$1 \otimes \otimes_\nabla D_r := (W^* \otimes 1)\mathrm{diag}(D_r)(W \otimes 1): \mathcal{D}(1 \otimes \nabla D_r) \to L^2((0,\infty))$$

We first note that it follows by the proof of Lemma 5.3 and the inclusion $C_c^\infty((0,\infty)) \subseteq \mathcal{X}$ that

$$C_c^\infty((0,\infty)) \subseteq \mathcal{D}(1 \otimes \nabla D_r)$$

Now, for each $g_r \in C_c^\infty((0,\infty))$ we may compute as follows:

$(1 \otimes_\nabla D_r)(g_r)$

$$= i\sum_{n=1}^{\infty}\sum_r \quad \xi^2\sqrt{H_n}\frac{d}{dt}\left(\xi^2\sqrt{H_n}g_r\right) = i\sum_{n=1}^{\infty}\sum_r \quad \left(\xi^4\cdot H_n\cdot\frac{dg_r}{dt} + 1/2\cdot g_r\cdot\frac{d(\xi^4\cdot H_n)}{dt}\right)$$
$$= \sum_r \quad i\frac{dg_r}{dt} + i/2\cdot\lim_{N\to\infty}\left(g_r\cdot\frac{d(\xi^4\cdot(\xi^4+1/N)^{-1})}{dt}\right)$$
$$= \sum_r \quad i\frac{dg_r}{dt} - i/2\cdot\lim_{N\to\infty}\left(g_r/N\cdot\frac{d((\xi^4+1/N)^{-1})}{dt}\right) = i\frac{dg_r}{dt}$$

where the limit is taken in the norm on $L^2((0,\infty))$.

Thus, we obtain that $1\otimes_\nabla D_r$ is a symmetric extension of the Dirac operator

Now, it is easily verified that $\mathrm{Ker}(i+\mathcal{D}^*) = \mathbb{C}\cdot\exp(-t)$ and that $\mathrm{Ker}(i-\mathcal{D}^*) = \{0\}$. It thus follows by [RESI75, Chapter X.1, Corollary] that $1\otimes_\nabla D_r$ is not essentially selfadjoint, since $\mathcal{D}: C_c^\infty((0,\infty)) \to L^2((0,\infty))$ has no selfadjoint extensions.

## VI. Compositions of Regular Unbounded Operators

Throughout this section, $X$ will be a Hilbert $C^*$-module over a $C^*$-algebra $A_r$, $D_r:\mathcal{D}(D_r)\to X$ will be a selfadjoint, regular operator on $X$, and $x^2\in\mathcal{L}(X)$ will be a bounded selfadjoint unbounded operator on $X$ such that:

$x^2\xi^2\in\mathcal{D}(D_r)$ for all $\xi^2\in\mathcal{D}(D_r)$ and $[D_r,x^2]:\mathcal{D}(D_r)\to X$ is bounded

The bounded extension of $[D_r,x^2]$ will be denoted by $\delta(x^2)$.

We remark that $\delta(x^2)$ is automatically adjointable with $\delta(x^2)^* = -\delta(x^2)$.

We study the regularity of the compositions $D_r x^2$, $\mathrm{cl}(x^2 D_r)$, and $\mathrm{cl}(x^2 D_r x^2)$, where $\mathrm{cl}(\mathcal{D})$ refers to the closure of an unbounded closable operator $\mathcal{D}:\mathcal{D}(\mathcal{D})\to X$. This regularity issue has been studied in detail by $S$. L. Woronowiczunder the assumption that $x^2$ is invertible, see [WoR91, Section 2, Example 2 and 3].

Hence we obtain a better understanding of the symmetric lift introduced in Section 5.

Our main tool is the local-global principle for regular operators, see [KALE12, Theorem 4.2]. Now we recall the statement of this result: Let $\mathcal{D}:\mathcal{D}(\mathcal{D})\to X$ be a closed unbounded operator wih a densely defined adjoint $\mathcal{D}^*$. For each state $\rho: A_r\to\mathbb{C}$ we have the localization $X_\rho$ of $X$. This is the Hilbert space obtained as the completion of $X/N_\rho$ with respect to the inner product $\langle[\xi^2],[\eta^2]\rangle_\rho := \rho(\langle\xi^2,\eta^2\rangle)$, where $N_\rho := \{\xi^2\in X \mid \rho(\langle\xi^2,\xi^2\rangle)=0\}$. The unbounded operator $\mathcal{D}$ then induces an unbounded operator on $X_\rho$,

$$\mathcal{D}_\rho:\mathcal{D}(\mathcal{D}_\rho)\to X_\rho \quad [\xi^2]\mapsto[\mathcal{D}\xi^2]$$

with domain $\mathcal{D}(\mathcal{D}_\rho)$ defined as the image of $\mathcal{D}($ in $X_\rho$. The localization of $\mathcal{D}$ at the state $\rho$ is the unbounded operator $\mathrm{cl}(\mathcal{D}_\rho)$.

**Theorem 6.1 (Local-global principle) (see [Jka10]).** The closed unbounded operator $\mathcal{D}:\mathcal{D}(\mathcal{D})\to X$ with densely defined adjoint $\mathcal{D}^*$ is regular if and only if

$$(\mathcal{D}_\rho)^* = \mathrm{cl}((\mathcal{D}^*)_\rho)$$

for all states $\rho: A_r\to\mathbb{C}$.

We now study the regularity of the unbounded operator $D_r x^2:\mathcal{D}(D_r x^2)\to X$ with domain $\mathcal{D}(D_r x^2) := \{\xi^2\in X \mid x^2\xi^2\in\mathcal{D}(D_r)\}$. We remark that $D_r x^2$ is already closed. The next to lemmas serve to compute the adjoint of $D_r x^2$.

**Lemma 6.1 (see [Jka10]).**

$$D_r x^2 - \delta(x^2) \subseteq (D_r x^2)^*$$

**Proof.** Let $\xi^2, \eta^2 \in \mathcal{D}(D_r x^2)$. We then have that

$\langle D_r x^2 \xi^2, \eta^2 \rangle$

$$\begin{aligned} &= \lim_{n\to\infty} \langle D_r x^2 \xi^2, i(i + D_r/n)^{-1}\eta^2 \rangle = \lim_{n\to\infty} \langle \xi^2, i x^2 D_r (i + D_r/n)^{-1}\eta^2 \rangle \\ &= -\langle \xi^2, \delta(x^2)\eta^2 \rangle + \lim_{n\to\infty} \langle \xi^2, i D_r x^2 (i + D_r/n)^{-1}\eta^2 \rangle \\ &= -\langle \xi^2, \delta(x^2)\eta^2 \rangle + \langle \xi^2, D_r x^2 \eta^2 \rangle + \lim_{n\to\infty} \langle \xi^2, i D_r/n \cdot (i + D_r/n)^{-1}\delta(x^2)(i + D_r/n)^{-1}\eta^2 \rangle \end{aligned}$$

It therefore suffices to show that

$$i D_r/n \cdot (i + D_r/n)^{-1}\delta(x^2)(i + D_r/n)^{-1}\eta^2 \to 0$$

But this follows easily since

$$\begin{aligned} i D_r/n \quad &\cdot (i + D_r/n)^{-1}\delta(x^2)(i + D_r/n)^{-1}\eta^2 \\ &= \delta(x^2) i (i + D_r/n)^{-1}\eta^2 + (i + D_r/n)^{-1}\delta(x^2)(i + D_r/n)^{-1}\eta^2 \end{aligned}$$

In order to prove the other inclusion $(D_r x^2)^* \subseteq D_r x^2 - \delta(x^2)$, we remark that the adjoint of $x^2 D_r : \mathcal{D}(D_r) \to X$ is precisely the unbounded operator $D_r x^2$. This follows from the selfadjointness of $D_r : \mathcal{D}(D_r) \to X$ and $x^2 \in \mathcal{L}(X)$.

**Lemma 6.2 (see [Jka10]).**

$$(D_r x^2)^* \subseteq D_r x^2 - \delta(x^2)$$

**Proof.** Notice that $x^2 D_r + \delta(x^2) \subseteq D_r x^2$. But this implies that $(D_r x^2)^* \subseteq (x^2 D_r + \delta(x^2))^* = D_r x^2 - \delta(x^2)$.

We want to apply the local global principle for regular operators to show that $D_r x^2 : \mathcal{D}(D_r x^2) \to X$ is regular. Thus, we need to compute the localization $\mathrm{cl}\big((D_r x^2)_\rho\big)$ and its adjoint $\big((D_r x^2)_\rho\big)^*$ for an arbitrary state $\rho : A_r \to \mathbb{C}$. This is the content of the next lemma.

To ease the notation, let $y^2 \otimes 1 \in \mathcal{L}(X_\rho)$ denote the closure of $y^2_\rho$ for a bounded adjointable operator $y^2 : X \to X$.

**Lemma 6.3 (see [Jka10]).** Let $\rho : A_r \to \mathbb{C}$ be a state. Then we have the identities $\mathrm{cl}\big((D_r x^2)_\rho\big) = \mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1)$ and $\big((D_r x^2)_\rho\big)^* = \mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1) - \delta(x^2) \otimes 1$

**Proof.** Remark first that $(D_r x^2)_\rho \subseteq \mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1)$. This implies the inclusion

$$\mathrm{cl}\big((D_r x^2)_\rho\big) \subseteq \mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1)$$

Furthermore, since $\big(\mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1)\big)^* = \mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1) - \delta(x^2) \otimes 1$ by Lemma 6.1 and Lemma 6.2, we get that

$$\mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1) - \delta(x^2) \otimes 1 \subseteq \big((D_r x^2)_\rho\big)^*$$

To prove the reverse inclusions, note that $x^2 D_r + \delta(x^2) \subseteq D_r x^2$. This implies that $(x^2 \otimes 1)(D_r)_\rho + \delta(x^2) \otimes 1 \subseteq (D_r x^2)_\rho$. We may then deduce that

$$\big((D_r x^2)_\rho\big)^* \subseteq \big((x^2 \otimes 1)(D_r)_\rho + \delta(x^2) \otimes 1\big)^* = \mathrm{cl}\big((D_r)_\rho\big)(x^2 \otimes 1) - \delta(x^2) \otimes 1$$

We have thus proved the identity

$$\left((D_r x^2)_\rho\right)^* = \mathrm{cl}\left((D_r)_\rho\right)(x^2 \otimes 1) - \delta(x^2) \otimes 1$$

But it then follows, since $X_\rho$ is a Hilbert space, that

$$\mathrm{cl}\left((D_r x^2)_\rho\right) = \left((D_r x^2)_\rho\right)^{**} = \mathrm{cl}\left((D_r)_\rho\right)(x^2 \otimes 1)$$

This proves the lemma.

We now prove the following result:

**Proposition 6.4 (see [Jka10]).** The closed unbounded operator $D_r x^2 : \mathcal{D}(D_r x^2) \to X$ is regular and the adjoint is given by $(D_r x^2)^* = D_r x^2 - \delta(x^2) : \mathcal{D}(D_r x^2) \to X$.

**Proof.** The formula for the adjoint$(D_r x^2)^*$ is a consequence of Lemma 6.1 and Lemma 6.2.

Let now $\rho: A_r \to \mathbb{C}$ be a state. By Theorem 6.1 we need only show that

$$\left((D_r x^2)_\rho\right)^* = \mathrm{cl}\left(((D_r x^2)^*)_\rho\right) \tag{6.1}$$

Applying Lemma 6.3 we obtain that

$$\left((D_r x^2)_\rho\right)^* = \mathrm{cl}\left((D_r)_\rho\right)(x^2 \otimes 1) - \delta(x^2) \otimes 1$$

By another application of Lemma 6.3 we get that

$$\mathrm{cl}\left(((D_r x^2)^*)_\rho\right) = \mathrm{cl}\left((D_r x^2)_\rho - \delta(x^2)_\rho\right) = \mathrm{cl}\left((D_r)_\rho\right)(x^2 \otimes 1) - \delta(x^2) \otimes 1$$

This proves the identity in (6.1) and thereby also the result of the proposition.

We may now treat the regularity problem for the composition $x^2 D_r : \mathcal{D}(D_r) \to X$. This is carried out in the next proposition. We recall that $(x^2 D_r)^* = D_r x^2 : \mathcal{D}(D_r x^2) \to X$. This does however not imply the regularity of $\mathrm{cl}(x^2 D_r)$. Indeed, it is possible to construct a closed unbounded, non-regular operator $\mathcal{D}: \mathcal{D}(\mathcal{D}) \to X$ with a regular adjoint $\mathcal{D}^* : \mathcal{D}(\mathcal{D}^*) \to X$, see [PAL99, Proposition 2.3] and [KALE12, Proposition 6.3]. Thus, the result in [LAN 95 , Corollary 9.6] is incorrect. Now we have the following:

**Proposition 6.5 (see [Jka10]).** The closure $\mathrm{cl}(x^2 D_r)$ is regular and given by $\mathrm{cl}(x^2 D_r) = D_r x^2 - \delta(x^2)$ : $\mathcal{D}(D_r x^2) \to X$

**Proof.** Let $\rho: A_r \to \mathbb{C}$ be a state. By the local-global principle in Theorem 6.1, the regularity of $\mathrm{cl}(x^2 D_r)$ will follow from the identity

$$\left((cl(x^2 D_r))_\rho\right)^* = \mathrm{cl}\left(((x^2 D_r)^*)_\rho\right) \tag{6.2}$$

The left hand side of (6.2) can be rewritten as

$$\left((\mathrm{cl}(x^2 D_r))_\rho\right)^* = \left((x^2 \otimes 1)\mathrm{cl}\left((D_r)_\rho\right)\right)^* = \mathrm{cl}\left((D_r)_\rho\right)(x^2 \otimes 1)$$

where the first identity follows since $(\mathrm{cl}(x^2 D_r))_\rho$ and $(x^2 \otimes 1)\mathrm{cl}\left((D_r)_\rho\right)$ agrees on the subspace $\mathcal{D}\left((D_r)_\rho\right) \subseteq X_\rho$ and the second identity follows from the regularity and selfadjointness of $\bar{D}_\mathrm{r}: \mathcal{D}(D_r) \to X$.

The right hand side of (6.2) can be computed using Lemma 6.3. We obtain that

$$\mathrm{cl}\left(((x^2 D_r)^*)_\rho\right) = \mathrm{cl}\left((D_r x^2)_\rho\right) = \mathrm{cl}\left((D_r)_\rho\right)(x^2 \otimes 1)$$

This proves the identity in (6.2) and thus that $\mathrm{cl}(x^2 D_r)$ is regular.

Now, since $\mathrm{cl}(x^2 D_r)$ is regular we have that $\mathrm{cl}(x^2 D_r) = (x^2 D_r)^{**} = (D_r x^2)^* = D_r x^2 - \delta(x^2)$, see [LAN95, Corollary 9.4]. This proves the last part of the proposition.

We conclude by showing that $x^2 D_r x^2 : \mathcal{D}(D_r x^2) \to X$ is essentially selfadjoint and regular, thus the closure $\mathrm{cl}(x^2 D_r x^2)$ is selfadjoint and regular.

**Proposition 6.6 (see [Jka10]).** The closure $\mathrm{cl}(x^2 D_r x^2)$ is selfadjoint and regular and given by $\mathrm{cl}(x^2 D_r x^2) = D_r x^4 - \delta(x^2)x^2 : \mathcal{D}(D_r x^4) \to X$.

**Proof.** By Proposition 6.4, $D_r x^2 : \mathcal{D}(D_r x^2) \to X$ is regular with $(D_r x^2)^* = D_r x^2 - \delta(x^2) : \mathcal{D}(D_r x^2) \to X$. This fact is equivalent to the selfadjointness and regularity of the anti-diagonal unbounded operator

$$\begin{pmatrix} 0 & D_r x^2 - \delta(x^2) \\ D_r x^2 & 0 \end{pmatrix} : \mathcal{D}(D_r x^2) \oplus \mathcal{D}(D_r x^2) \to X \oplus X$$

see [KALE12, Lemma 2.3]. It therefore follows by Proposition 6.5 that

$$\begin{pmatrix} 0 & \mathrm{cl}(x^2 D_r x^2) - x^2\delta(x^2) \\ \mathrm{cl}(x^2 D_r x^2) & 0 \end{pmatrix} : \mathcal{D}(\mathrm{cl}(x^2 D_r x^2)) \oplus \mathcal{D}(\mathrm{cl}(x^2 D_r x^2)) \to X \oplus X$$

is regular. Furthermore, we have that

$$\begin{pmatrix} 0 & \mathrm{cl}(x^2 D_r x^2) - x^2\delta(x^2) \\ \mathrm{cl}(x^2 D_r x^2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_r x^4 - \delta(x^2)x^2 \\ D_r x^4 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x^2\delta(x^2) \\ \delta(x^2)x^2 & 0 \end{pmatrix}$$

We may thus conclude that $\mathrm{cl}(x^2 D_r x^2) = D_r x^4 - \delta(x^2)x^2 : \mathcal{D}(D_r x^4) \to X$. It then follows by Proposition 6.4 that $\mathrm{cl}(x^2 D_r x^2)$ is regular. Furthermore, the adjoint is given by

$$(x^2 D_r x^2)^* = (D_r x^4)^* + x^2\delta(x^2) = D_r x^4 - \delta(x^4) + x^2\delta(x^2) = D_r x^4 - \delta(x^2)x^2$$

This shows that $\mathrm{cl}(x^2 D_r x^2)$ is also selfadjoint and the proposition is proved.

## VII. Selfadjointness and Regularity of Lifts

We will now return to the setting described in the beginning of Section 5. Furthermore, we let $W : X \to H_{A_r}$ and $K_r : H_{A_r} \to H_{A_r}$ be as in Theorem 3.1, and as in Remark 3.9 we let $\{\zeta_k^2\}_{k=1}^{\infty}$ be a square sequence in $X$ such that $W(\eta^2) = \{\langle \zeta_k^2, \eta^2 \rangle\}_{k=1}^{\infty}$ for all $\eta^2 \in X$.

We recall that $W^* K_r W : X \to X$ has dense image and it thus follows that

$$\Delta := (W^* K_r W)^2 \otimes 1 = (W^* K_r^2 W) \otimes 1 : X \widehat{\otimes}_{A_r} Y \to X \widehat{\otimes}_{A_r} Y$$

has dense image as well.

We are interested in proving that the composition

$$\Delta(1 \otimes_{\nabla} D_r)\Delta : \mathcal{D}(\mathrm{diag}(D_r)(W \otimes 1)\Delta) \to X \widehat{\otimes}_{A_r} Y$$

is an essentially selfadjointand regular unbounded operator.

We first notice that the map $\iota : M_{\infty}(\mathcal{L}(Y)) \to \mathcal{L}(Y^{\infty})$ given by

$$\iota(\{(T_r)_{ij}\})(\{\eta_n^2\}) := \left\{\sum_{r,j=1}^{\infty} (T_r)_{ij}(\eta_j^2)\right\}_{i=1}^{\infty} \quad \{(T_r)_{ij}\} \in M_\infty(\mathcal{L}(Y)), \{\eta_n^2\} \in Y^\infty$$

induces an injective $*$-homomorphism $\iota: \mathcal{K}(H_{\mathcal{L}(Y)}) \to \mathcal{L}(Y^\infty)$. In particular, we have that $\| \iota(T_r) \| = \| T_r \|$ for all $T_r \in \mathcal{K}(H_{\mathcal{L}(Y)})$. This enables us to prove the following:

**Lemma 7.1 (see [Jka10]).** Let $T_r \in \mathcal{K}(H_{A_r})_\delta$. Then $\iota(\rho(T_r)) \in \mathcal{L}(Y^\infty)$ preserves the domain of $\mathrm{diag}(D_r)$ and $\iota(\delta(T_r)) \in \mathcal{L}(Y^\infty)$ is an extension of the commutator

$$[\mathrm{diag}(D_r), \iota(\rho(T_r))]: \mathcal{D}(\mathrm{diag}(D_r)) \to Y^\infty$$

**Proof.** Let $\eta^2 = \{\eta_n^2\} \in \mathcal{D}(\mathrm{diag}(D_r))$.

Suppose first that $T_r \in M_\infty(\mathcal{A})$. Then clearly $\iota(\rho(T_r))(\eta^2) = \{\sum_{j=1}^{\infty} \rho(x_{ij}^2)\eta_j^2\} \in \mathcal{D}(\mathrm{diag}(D_r))$ and furthermore

$$[\mathrm{diag}(D_r), \iota(\rho(T_r))](\eta^2) = \left\{\sum_{r,j=1}^{\infty} [D_r, \rho(x_{ij}^2)](\eta_j^2)\right\} = \iota(\delta(T_r))(\eta^2)$$

This proves the claim of the lemma in this case.

For a general $T_r \in \mathcal{K}(H_{A_r})_\delta$, we may choose a sequence $\{(T_r)_m\}$ in $M_\infty(\mathcal{A})$ such that $(T_r)_m \to T_r$ in the norm $\| \cdot \|_\delta: \mathcal{K}(H_{A_r})_\delta \to [0, \infty)$. We then use the fact that $\mathrm{diag}(D_r): \mathcal{D}(\mathrm{diag}(D_r)) \to Y^\infty$ is closed to conclude that $\iota(\rho(T_r))(\eta^2) \in \mathcal{D}(\mathrm{diag}(D_r))$ with

$$\mathcal{D}(\mathrm{diag}(D_r))(\iota(\rho(T_r))(\eta^2)) = \iota(\rho(T_r))(\mathrm{diag}(D_r)(\eta^2)) + \iota(\delta(T_r))(\eta^2)$$

This proves the lemma.

We consider the bounded positive selfadjoint operator

$$\Delta_W := (W \otimes 1)\Delta(W^* \otimes 1): (P \otimes 1)Y^\infty \to (P \otimes 1)Y^\infty$$

where $P \otimes 1 := (W \otimes 1)(W^* \otimes 1): Y^\infty \to Y^\infty$ is the orthogonal projection associated with the isometry $(W \otimes 1): X \widehat{\otimes}_{A_r} Y \to Y^\infty$, see Section 5 .

We then remark that $\Delta(1 \otimes_\nabla D_r)\Delta: \mathcal{D}(\mathrm{diag}(D_r)(W \otimes 1)\Delta) \to X \widehat{\otimes}_{A_r} Y$ and

$$\Delta_W \mathrm{diag}(D_r)\Delta_W: \mathcal{D}(\mathrm{diag}(D_r)\Delta_W) \to (P \otimes 1)Y^\infty$$

are unitarily equivalent unbounded operators. Furthermore, we have that

$$\begin{aligned} \Delta_W &= (W \otimes 1)(W^* K_r^2 W \otimes 1)(W^* \otimes 1) \\ &= \iota(\rho(PK_r^2))\big|_{(P\otimes 1)Y}: (P \otimes 1)Y^\infty \to (P \otimes 1)Y^\infty \end{aligned}$$

**Proposition 7.2 (see [Jka10]).** The unbounded operator $\Delta_W \mathrm{diag}(D_r)\Delta_W: \mathcal{D}(\mathrm{diag}(D_r)\Delta_W) \to (P \otimes 1)Y^\infty$ is essentially selfadjoint and regular.

**Proof.** It is enough to show that

$$\iota(PK_r^2)\mathrm{diag}(D_r)\iota(PK_r^2): \mathcal{D}(\mathrm{diag}(D_r)\Delta_W) + ((1 - P) \otimes 1)Y^\infty \to Y^\infty$$

is essentially selfadjoint and regular. Now, by the differentiable absorption theorem (Theorem 3.1), we have that $PK_r^2 \in \mathcal{K}(H_{A_r})_\delta$. By Lemma 7.1, the pair consisting of the unbounded selfadjoint regular operator

$\mathrm{diag}(D_r): \mathcal{D}(\mathrm{diag}(D_r)) \to Y^\infty$ and the bounded selfadjoint operator $\iota\big(\rho(PK_r^2)\big): Y^\infty \to Y^\infty$ therefore satisfies the assumptions applied in Section 6.

This proves the current lemma by an application of Proposition 6.6.

The main result now follows immediately:

**Theorem 7.1 (see [Jka10]).** The unbounded operator $\Delta(1 \otimes \nabla D_r)\Delta: \mathcal{D}((1 \otimes \nabla D_r)\Delta) \to X \widehat{\otimes}_{A_r} Y$ is essentially selfadjoint and regular.

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