Quest Journals
Journal of Research in Applied Mathematics
Volume 10 ~ Issue 12 (2024) pp: 01-19
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Direct Applications on Multilinear Theory of Strongly Singular Calderón–Zygmund Operators on Product of Respective Spaces

SuhaSoliman [(1)] and Shawgy Hussein[(2)]
(1) Sudan University of Science and Technology, Sudan.
(2) Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.

Abstract.
Yan Lin[47] study the multilinear strongly singular Calderón–Zygmund operator whose kernel is more singular adjacent the diagonal than that of the standard multilinearCalderón–Zygmund operator. Following [47] whoimproved the sharp maximal estimate of multilinear singular integrals, and determine, its boundedness on product of weighted and variable exponent Lebesgue spaces, so clearify, the endpoint estimates of $L^\infty \times \ldots \times L^\infty \to BMO$, $BMO \times \ldots \times BMO \to BMO$, and $LMO \times \ldots \times LMO \to LMO$ and are basically discussed for the multilinear strongly singular Calderón–Zygmund operator. Hence similarly improve equally the corresponding known for the standard multilinearCalderón–Zygmund operator. To catch the sharp maximal estimates we get rid of the proposed size condition for the kernel of the multilinear strongly singular Calderón–Zygmund operator that arised for the standard multilinearCalderón–Zygmund operator.With additional interest when dealing with the mean oscillation over balls with small radius to produce the stronger singularity in establishing the recognized mentioned ones.

Keywords: *Multilinear strongly singular, Calderón–Zygmund operator, Sharp maximal function, BMO function, LMO function*

***Received** 01 Dec., 2024; **Revised** 08 Dec., 2024; **Accepted** 10 Dec., 2024* *© The author(s) 2024.*
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I. Introduction

The strongly singular integral operator model is the multiplier operator defined by

$$\left(T_{1-\epsilon,1+\epsilon}\left(\sum_k f_k\right)\right)^{\wedge}(\xi)=\theta(\xi)\frac{e^{i|\xi|^{1-\epsilon}}}{|\xi|^{1+\epsilon}}\left(\sum_k \hat{f}_k(\xi)\right),$$

where $0<\epsilon<1$, and $\theta(\xi)$ is a standard cutoff function. This operator was studied by [23], [42], [45] and [15] and was named weakly- strongly singular Calderón- Zygmund operator by [15].
The convolution form of $T_{1-\epsilon,1+\epsilon}$ can be written as

$$T_{1-\epsilon,1+\epsilon}\left(\sum_k f_k\right)(x)=\mathrm{p.v.}\int\sum_k\frac{e^{i|\epsilon|^{-\frac{1-\epsilon}{\epsilon}}}}{|\epsilon|^{\frac{2(1+\epsilon)^2}{\epsilon(1-\epsilon)}}}\chi(|\epsilon|)f_k(x+\epsilon)d(x+\epsilon),$$

It was shown that$T_{1-\epsilon,1+\epsilon}$is bounded on$L^{1+\epsilon}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$by[23] and [42]when $\left|\frac{\epsilon-1}{\epsilon+1}\right|<\frac{1}{2}\left[\frac{\epsilon^2-\epsilon+2}{\epsilon^2+\epsilon+2}\right]$ and $T_{1-\epsilon,1+\epsilon}$is not bounded on$L^{1+\epsilon}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ by [45] if $\left|\frac{\epsilon-1}{\epsilon+1}\right|>\frac{1}{2}\left[\frac{\epsilon^2-\epsilon+2}{\epsilon^2+\epsilon+2}\right]$. At the critical exponent $p_0=\frac{1}{4}\left[\frac{\epsilon^2+3\epsilon+2}{\epsilon^2+\epsilon+2}\right]$, [15] proved that $T_{1-\epsilon,1+\epsilon}$ is bounded from $L^{p_0}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ to Lorentz space $L^{p_0,p_0'}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$, where p_0' is the dual exponent i.e$\frac{1}{p_0}+\frac{1}{p_0'}=1$.
When$\epsilon\geq 0$, the sharp endpoint estimate for strongly singular operators were established by [16] using the duality of Hardy space H^1 and BMO space. The weighted norm estimates were established by [6].

The authors in [2] introduced the strongly singular non- convolution operator which is called strongly singular Calderón- Zygmund operator, whose properties are similar to those of classical Calderón-Zygmund operators, but the kernel is more singular near the diagonal than that of the standard case.

Definition 1.1. Let $T : S \to S'$ be a bounded linear operator. T is called a *stronglysingularCalderónZygmundoperator* if the following conditions are satisfied.

(1) T can be extended into a continuous operator from $L^2\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ into itself.

(2) There exists a function $K(x, x+\epsilon)$ continuous away from the diagonal $\{(x, x+\epsilon) : \epsilon = 0\}$ such that

$$|K(x,x+\epsilon) - K(x,x+2\epsilon)| + |K(x+\epsilon,x) - K(x+2\epsilon,x)| \leq (1+\epsilon)\frac{|\epsilon|^{\delta}}{|2\epsilon|^{\frac{2(1+\epsilon)+\delta}{(1-\epsilon)}}},$$

if $2|\epsilon|^{1-\epsilon} \leq |2\epsilon|$ for some $0 < \delta \leq 1$ and $0 < \epsilon < 1$. And

$$\langle Tf_k, g_k\rangle = \int \int \sum_k K\,(x, x+\epsilon) f_k(x+\epsilon) g_k(x) d(x+\epsilon) dx,$$

for $f_k, g_k \in S$ with disjoint supports.

(3) For some $0 \leq \epsilon < 1$, both T and its conjugate operator T^* can be extended into continuous operators from $L^{1+\epsilon}$ to L^2, where $\epsilon = 0$ or 1.

Following a suggestion of Stein, Alvarez and Milman showed that the pseudo- differential operator with symbol in the Hörmander's class $S^{-(1+\epsilon)}_{1-\epsilon,\delta}$, where $0 < \delta \leq \alpha < 1$ and $0 \leq \epsilon < 1$, is included in the strongly singular Calderón- Zygmund operator. This fact shows that the research of strongly singular Calderón- Zygmund operators has important value both on the theory of singular integrals in harmonic analysis and related subjects in PDE.

[2], [3] discussed the boundedness of the strongly singular Calderón- Zygmund operator on Lebesgue spaces. [29] and [33] established the sharp maximal estimates and endpoint estimates of the strongly singular Calderón- Zygmund operator. For the boundedness of strongly singular Calderón-Zygmund operators and related topics, see [4], [29]–[34], [40], [46] and so on.

[47] focus on the multilinear form of the strongly singular Calderón-Zygmund operator. The multilinearCalderón- Zygmund theory was first studied by [7–9]. The study of multilinear singular integrals was motivated not only as generalizations of the theory of linear ones but also its natural appearance in harmonic analysis. In recent years, this topic has received increasing attentions and well development, such as the systemic treatment of multilinearCalderón- Zygmund operators by [19], [20] and multilinear fractional integrals by [25] and from various points of view. See [5], [21], [22], [24], [27], [28], [36]–[39].

We now review briefly the definition of the multilinearCalderón-Zygmund operator. For $m \in \mathbb{N}^+$ and $K(y_{s-1}, y_s, \dots, y_{s+m-1})$ be a function defined away from the diagonal $y_{s-1} = y_s = \cdots = y_{s+m-1}$ in $\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)^{m+1}$. T is an m-linear operator defined on product of test functions such that for K, the integral representation below is valid

$$T((f_k)_s, \dots, (f_k)_{s+m-1})(x) = \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \cdots \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \sum_k K\,(x, y_s, \dots, y_{s+m-1}) \prod_{j_0=1}^{m} (f_k)_{j_0}\left((x+\epsilon)_{j_0}\right) dy_s \cdots dy_{s+m-1}, \tag{1.1}$$

where $(f_k)_{j_0} (j_0 = s, \dots, s+m-1)$ are smooth functions with compact support and $x \notin \bigcap_{j=s}^{s+m-1} \mathrm{s\,upp}(f_k)_{j_0}$.

Especially, we call K a *standard* m *-linearCalderón-Zygmundkernel* if it satisfies the following size and smoothness estimates.

$$|K(y_{s-1}, y_s, \dots, y_{s+m-1})| \leq \frac{1+\epsilon}{\left(\sum_{k_0,l=0}^{s+m-1} |\, y_{K_{s-1}} - y_{s+l-1}|\right)^{m(\frac{2(1+\epsilon)}{1-\epsilon})}}, \tag{1.2}$$

for some $\epsilon \geq 0$ and all $(y_{s-1}, y_s, \dots, y_{s+m-1}) \in \left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)^{s+m}$ away from the diagonal.

$$\begin{aligned}&\left|K\left(y_{s-1}, \dots, y_{s+j-1}, \dots, y_{s+m-1}\right) - K\left(y_{s-1}, \dots, y'_{s+j-1}, \dots, y_{s+m-1}\right)\right| \\ &\leq \frac{(1+\epsilon)|y_{s+j-1} - y'_{s+j-1}|^{\varepsilon}}{\left(\sum_{k,l=0}^{s+m-1} |\, y_{s+k-1} - y_{s+l-1}|\right)^{m(\frac{2(1+\epsilon)}{1-\epsilon})+\varepsilon}},\end{aligned} \tag{1.3}$$

for some $\varepsilon > 0$,whenever $0 \leq s+j-1 \leq s+m-1$ and

$$|y_{s+j-1} - y'_{s+j-1}| \leq \frac{1}{2} \max_{s-1 \leq k_0 \leq s+m-1} |y_{s+j-1} - y_{s-1}|.$$

According to [20], if an m-linear operator T defined by (1.1) associated with a standard m- linear Calderón-Zygmund kernel K , and satisfies either of the following two conditions for given numbers $1 \leq t_s, t_{s+1}, \ldots, t_{s+m-1}, t < \infty$ with $\frac{1}{t} = \frac{1}{t_s} + \frac{1}{t_{s+1}} + \cdots + 1/t_{s+m-1}$,
(C1) T maps $L^{t_s,1} \times \ldots \times L^{t_{s+m-1},1}$ into $L^{t,\infty}$ if $\epsilon > 0$,
(C2) T maps $L^{t_s,1} \times \ldots \times L^{t_{s+m-1},1}$ into L^1 if $\epsilon = 0$,
where $L^{t_s,1}, \ldots, L^{t_{s+m-1},1}$ and $L^{t,\infty}$ are Lorentz spaces, then we say that T is a *standard* m *-linearCalderónZygmundoperator*.
We are interested in the *multilinearstronglysingular*Calderón-Zygmund operator defined as follows.
Definition 1.2. Let T be an m -linear operator defined by (1.1). T is called an m *-linearstronglysingularCalderón-Zygmundoperator* if the following conditions are satisfied.
(i) For some $\varepsilon > 0$ and $0 \leq \epsilon < 1$,

$$|K(x, y_s, \ldots, y_{s+m-1}) - K(x', y_s, \ldots, y_{s+m-1})| \leq \frac{(1+\epsilon)|x - x'|^{\varepsilon}}{(|x - y_s| + \cdots + |x - y_{s+m-1}|)^{m(\frac{2(1+\epsilon)}{1-\epsilon}) + \varepsilon/(1-\epsilon)}}, \tag{1.4}$$

whenever $|x - x'|^{1-\epsilon} \leq \frac{1}{2}\max_{1 \leq j \leq m}|x - y_{s+j-1}|$.
(ii) For some given numbers $1 \leq r_s, \ldots, r_{s+m-1} < \infty$ with $\frac{1}{r} = \frac{1}{r_s} + \cdots + 1/r_{s+m-1}$, T maps $L^{r_s} \times \ldots \times L^{r_{s+m-1}}$ into $L^{r,\infty}$.
(iii) For some given numbers $1 \leq l_s, \ldots, l_{s+m-1} < \infty$ with $1/l = 1/l_s + \cdots + 1/l_{s+m-1}$, T maps $L^{l_s} \times \ldots \times L^{l_{s+m-1}}$ into $L^{1+\epsilon,\infty}$, where $0 < l/1 + \epsilon \leq 1 - \epsilon$.
The following remarks are in order to illustrate common points and the differences between the standard multilinearCalderón-Zygmund operator and the multilinear strongly singular Calderón-Zygmund operator (see [47]).
Remark 1.1. It is easy to see that the condition (1.3) implies the condition (1.4) when $\epsilon = 0$. In this special case, we can take $l_{s+j-1} = r_{s+j-1}, j = s, \ldots, s+m-1$, and $(1+\epsilon) = l = r$ in (3) of Definition 1.2. Then the condition (3) of Definition 1.2 is completely in agreement with the condition (2), thus we can remove it in this situation. We can say that the multilinear strongly singular Calderón-Zygmund operator generalizes the standard multilinearCalderón-Zygmund operator.
Remark 1.2. However, for the case $0 \leq \epsilon < 1$, the kernel of the multilinear strongly singular CalderónZygmund operator defined by Definition 1.2 is more singular than that of the standard one near the diagonal. This is the reason why we call it "strongly singular and the stronger singularities will bring up new difficulties and new techniques are needed to handle such operators.
Remark 1.3. It should be pointed out that we do not need any size condition like (1.2) for the kernel of the multilinear strongly singular Calderón-Zygmund operator to obtain our main results in this paper.
Remark 1.4. On the other hand, by comparing Definition 1.1 with Definition 1.2, one can find out that the linear strongly singular Calderón-Zygmund operator satisfies the conditions of Definition 1.2 for the situations $m = 1$ and $0 \leq \epsilon < 1$. Thus, the multilinear strongly singular Calderón- Zygmund operators also generalize the linear ones.
In what follows, for $0 \leq \epsilon \leq \infty, \frac{1+\epsilon}{\epsilon}$ is the conjugate index of $1+\epsilon$. $E^c = \mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}} \backslash E$ is the complementary set of E. $(1+\epsilon)$'s will be constants which are independent of the main parameter and may vary from line to line. We will always denote by $B(x, 1+\epsilon)$ the ball centered at x with radius $1+\epsilon, \epsilon \geq 0, (1+\epsilon)B(x, 1+\epsilon) = B(x, (1+\epsilon)^2)$ for $\epsilon \geq 0, |B(x, 1+\epsilon)|$ the Lebesgue measure of $B(x, 1+\epsilon)$ and $(f_k)_{B(x,1+\epsilon)} = \frac{1}{|B(x,1+\epsilon)|}\int_{B(x,1+\epsilon)} \sum_k \quad f_k\,(x+\epsilon)d(x+\epsilon)$.
The sharp maximal pointwise estimate for the multilinear strongly singular Calderón-Zygmund operator will be established. And as applications, we can obtain the boundedness of the multilinear strongly singular Calderón-Zygmund operator on product of weighted Lebesgue spaces and product of variable exponent Lebesgue spaces, respectively. Three kinds of endpoint estimates for the multilinear strongly singular Calderón-Zygmund operator will be discussed. We will establish the boundedness of $L^\infty \times \ldots \times L^\infty \to BMO$, $BMO \times \ldots \times BMO \to BMO$, and $LMO \times \ldots \times LMO \to LMO$ types, respectively.

II. Sharp Maximal Pointwise Estimates and Applications

2.1. Sharp Maximal Pointwise Estimates

The definition and properties of BMO functions inspire us naturally to study the sharp maximal function $f_k^{\#}$, associated to any locally integrable function f_k. It is defined by

$$M^{\#}\left(\sum_k f_k\right)(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B\sum_k |f_k(x+\epsilon)-(f_k)_B|d(x+\epsilon)$$
$$\sim\sup_{B\ni x}\inf_{\propto\in x}\frac{1}{|B|}\int_B|f_k(x+\epsilon)-a|dx,$$

where the supremum is taken over all balls B containing x.

By M we will denote the Hardy-Littlewood maximal operator defined by

$$M\left(\sum_k f_k(x)\right)=\sup_{B\ni x}\frac{1}{|B|}\int_B\sum_k |f_k(x+\epsilon)|d(x+\epsilon).$$

For $0\le\epsilon<\infty$, we use $M_{1+\epsilon}$ and $M_{1+\epsilon}^{\#}$ to denote the operators $M_{1+\epsilon}(\sum_k f_k)=\sum_k [M(|f_k|^{1+\epsilon})]^{1/1+\epsilon}$ and $M_{1+\epsilon}^{\#}(\sum_k f_k)=\sum_k [M^{\#}(|f_k|^{1+\epsilon})]^{1/1+\epsilon}$, respectively.

The pointwise estimate for the sharp maximal function of the multilinear strongly singular Calderón- Zygmund operator is established by means of other classes of maximal functions. We have the following.

Theorem 2.1 (see [47]). *Let T bean m -linearstronglysingularCalderón-Zygmundoperatorand* $s=\max\{r_s,\ldots,r_{s+m-1},l_s,\ldots,l_{s+m-1}\}$, *where$r_jandl_j$aregivenasinDefinition*1.2,$j=s,\ldots,s+m-1$. *If*$0<\delta<1/m$, *then*

$$M_\delta^{\#}(T\left(\sum_k \overrightarrow{f_k}\right)(x)\le(1+\epsilon)\prod_{j_0=s}^{s+m-1}\sum_k M_s\left((f_k)_{j_0}\right)(x),$$

forallm-tuples$\vec{f_k}=((f_k)_s,\ldots,(f_k)_{s+m-1})$*ofboundedmeasurablefunctionswithcompactsupport*.

For the special case $\epsilon=0$, as we have discussed in Remarks 1.1-1.4, the condition (3) of Definition 1.2 can be removed. Then the standard multilinearCalderón-Zygmund operator satisfies the conditions of Definition 1.2 by taking $r_s=\cdots=r_{s+m-1}=1$ since the $L^1\times\ldots.\times L^1\to L^{1/m,\infty}$ boundedness obtained in [20]. Thus in this special case, $s=\max\{r_s,\ldots,r_{s+m-1},l_s,\ldots,l_{s+m-1}\}=1$. Then we can obtain the corresponding sharp maximal estimate for the standard multilinear Calderón-Zygmund operator as a corollary of Theorem 2.1.

Corollary 2.1. *LetTbeastandardm-linearCalderón-Zygmundoperator*. *If*$0<\delta<1/m$, *then*

$$M_\delta^{\#}\left(T\left(\sum_k \vec{f_k}\right)\right)(x)\le(1+\epsilon)\prod_{j_0=s}^{s+m-1}\sum_k M\left((f_k)_{j_0}\right)(x),$$

forallm-tuples$\vec{f_k}=((f_k)_s,\ldots,(f_k)_{s+m-1})$*ofboundedmeasurablefunctionswithcompactsupport*.

Remark 2.1. It should be pointed out that the sharp maximal estimate for the standard multilinearCalderón-Zygmund operator obtained in Corollary 2.1 was established earlier in [38]. However, differently from the method in [38], here we do not need any size condition assumption for the kernel.

We need the following two lemmas(see [47]).

Lemma 2.1 (***See*[17,27]).** *Let* $0\le\epsilon<\infty$, *thenthereisapositiveconstant* $(1+\epsilon)=C_{1+\epsilon,1+2\epsilon}$*suchthatforanymeasurablefunction*f_k*therehas*

$$|Q|^{-\frac{1}{1+\epsilon}}\|\sum_k f_k\|_{L(Q)}(1+\epsilon)\le(1+\epsilon)|Q|^{-\frac{1}{1+2\epsilon}}\sum_k \|f_k\|_{L^{1+2\epsilon,\infty}(Q)}.$$

Lemma 2.2. *Let* $\delta>0,x\in\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}$ *and* f_k *bealocallyintegrablefunction*. *Thenforanyball* $B=B(x_0,1+2\epsilon)$*containingxwith*$\epsilon\ge0$, *thereis*

$$\int_{B^c}\sum_k\frac{|f_k(x+\epsilon)|}{|x_0-x-\epsilon|^{\left(\frac{2(1+\epsilon)}{1-\epsilon}\right)+\delta}}d(x+\epsilon)\le(1+\epsilon)^{1-\delta}\sum_k M(f_k)(x),$$

where$(1+\epsilon)$*isapositiveconstantindependentof*f_k,x,x_0*and*$(1+\epsilon)$.

Proof of Theorem 2.1. In order to simplify the proof, we only consider the situation when $m=2$. Actually, the similar procedure works for all other situations.

Let $(f_k)_1,(f_k)_2$ be bounded measurable functions with compact support, then for any ball $B=B(x_0,r_B)$ containing x with $r_B>0$, we will consider two cases, respectively.

Case 1:$r_B\ge1$.

Write

$$\begin{aligned}(f_k)_l=(f_k)_1\chi_{2B}+(f_k)_1\chi_{(2B)^c}\coloneqq(f_k)_1^1+(f_k)_1^2,(f_k)_2\\=(f_k)_2\chi_{2B}+(f_k)_2\chi_{(2B)^c}:=(f_k)_2^1+(f_k)_2^2.\end{aligned}\quad(2.1)$$

Take a $c_0=T((f_k)_1^2,(f_k)_2^1)(x_0)+T((f_k)_1^1,(f_k)_2^2)(x_0)+T((f_k)_1^2,(f_k)_2^2)(x_0)$, then

$$\left(\frac{1}{|B|}\int_B \sum_k \; ||T((f_k)_1,(f_k)_2)(x+2\epsilon)|^\delta - |c_0|^\delta| d(x+2\epsilon)\right)^{1/\delta}$$
$$\leq \left(\frac{1}{|B|}\int_B \sum_k \; |T((f_k)_1,(f_k)_2)(x+2\epsilon) - c_0|^\delta d(x+2\epsilon)\right)^{\frac{1}{\delta}}$$
$$\leq (1+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k \; |T((f_k)_1^1,(f_k)_2^1)(x+2\epsilon)|^\delta d(x+2\epsilon)\right)^{\frac{1}{\delta}} + (1$$
$$+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k \; |T((f_k)_1^2,(f_k)_2^1)(x+2\epsilon) - T((f_k)_1^2,(f_k)_2^1)(x_0)|^\delta d(x+2\epsilon)\right)^{\frac{1}{\delta}} + (1$$
$$+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k \; |T((f_k)_1^1,(f_k)_2^2)(x+2\epsilon) - T((f_k)_1^1,(f_k)_2^2)(x_0)|^\delta d(x+2\epsilon)\right)^{\frac{1}{\delta}} + (1$$
$$+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k \; |T((f_k)_1^2,(f_k)_2^2)(x+2\epsilon) - T((f_k)_1^2,(f_k)_2^2)(x_0)|^\delta d(x+2\epsilon)\right)^{1/\delta} :$$
$$= \sum_{j=1}^4 I_j.$$

Notice that $0<\delta<r<\infty$, where r is given as in Definition 1.2. It follows from Lemma 2.1 and (2) of Definition 1.2 that

$$I_1 \leq (1+\epsilon)|B|^{-\frac{1}{\delta}}\sum_k \; \|T((f_k)_1^1,(f_k)_2^1)\|_{L^\delta(B)} \leq (1+\epsilon)|B|^{-\frac{1}{r}}\sum_k \; \|T((f_k)_1^1,(f_k)_2^1)\|_{L^{r,\infty}(B)}$$
$$\leq (1+\epsilon)\left(\frac{1}{|2B|}\int_{2B}\sum_k \; |(f_k)_1(y_s)|^{r_1}dy_s\right)^{\frac{1}{r_1}}\left(\frac{1}{|2B|}\int_{2B}\sum_k \; |(f_k)_2(y_{s+1})|^{r_2}dy_{s+1}\right)^{\frac{1}{r_2}}$$
$$\leq (1+\epsilon)M_r\sum_k \; ((f_k)_1)(x)M_{r_2}((f_k)_2)(x) \leq (1+\epsilon)M_s\sum_k \; ((f_k)_1)(x)M_s((f_k)_2)(x).$$

For $(x+2\epsilon)\in B$ and $y_s\in(2B)^c$, there is $|x+2\epsilon-x_0|^{1-\epsilon}\leq r_B^{1-\epsilon}\leq r_B\leq\frac{1}{2}|y_s-x_0|$. By Hölder's inequality, the condition of the kernel in (1) of Definition 1.2 and Lemma 2.2, we have

$$I_2 \leq (1+\epsilon)\frac{1}{|B|}\int_B \sum_k \; |T((f_k)_1^2,(f_k)_2^1)(x+2\epsilon) - T((f_k)_1^2,(f_k)_2^1)(x_0)| d(x+2\epsilon)$$
$$\leq (1+\epsilon)\frac{1}{|B|}\int_B \int_{(2B)^c}\int_{2B}\sum_k \; |K(x+2\epsilon,y_s,y_{s+1}) - K(x_0,y_s,y_{s+1})||(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s d(x$$
$$+2\epsilon)$$
$$\leq (1+\epsilon)\frac{1}{|B|}\int_B \int_{(2B)^c}\int_{2B}\sum_k \; \frac{|x+2\epsilon-x_0|^\varepsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}|(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s d(x+2\epsilon)$$
$$\leq (1+\epsilon)r_B^\varepsilon\sum_k \; \left(\int_{2B}|(f_k)_2(y_{s+1})|dy_{s+1}\right)\left(\int_{(2B)^{1+\epsilon}}\frac{|(f_k)_1(y_s)|}{|x_0-y_s|^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_s\right)$$
$$\leq (1+\epsilon)r_B^\varepsilon\sum_k \; |B|M((f_k)_2)(x)r_B^{-\left(\frac{2(1+\epsilon)+\xi j}{1-\epsilon}\right)}M((f_k)_1)(x) \leq \sum_k \; (1+\epsilon)M((f_k)_1)(x)M((f_k)_2)(x)r_B^{\varepsilon:-\frac{\epsilon}{1-\epsilon}}$$
$$\leq (1+\epsilon)\sum_k \; M_s((f_k)_1)(x)M_s((f_k)_2)(x).$$

Similarly we can get that

$$I_3 \leq (1+\epsilon)\sum_k \; M_s((f_k)_1)(x)M_s((f_k)_2)(x).$$

For $(x+2\epsilon)\in B$ and $y_s\in(2B)^c$, there are $|x+2\epsilon-x_0|^{1-\epsilon}\leq\frac{1}{2}|y_s-x_0|$ and $|x+2\epsilon-x_0|^{1-\epsilon}\leq\frac{1}{2}|y_{s+1}-x_0|$. It follows from Hölder's inequality, (1) of Definition 1.2 and Lemma 2.2 that

$$I_4 \le (1+\epsilon)\frac{1}{|B|}\int_B \sum_k |T((f_k)_1^2,(f_k)_2^2)(x+2\epsilon) - T((f_k)_1^2,(f_k)_2^2)(x_0)|d(x+2\epsilon)$$
$$\le (1+\epsilon)\frac{1}{|B|}\int_B \int_{(2B)^c}\int_{(2B)^c}\sum_k |K(x+2\epsilon,y_s,y_{s+1})$$
$$- K(x_0,y_s,y_{s+1})||(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s d(x+2\epsilon)$$
$$\le (1+\epsilon)\frac{1}{|B|}\int_B \int_{(2B)^c}\int_{(2B)^c}\sum_k \frac{|x+2\epsilon-x_0|^{\varepsilon}}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}|(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s d(x+2\epsilon)$$
$$\le (1+\epsilon)r_B^{\xi j}\sum_k \left(\int_{(2B)^c}\frac{|(f_k)_1(y_s)|}{|x_0-y_s|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_s\right)\left(\int_{(2B)^c}\frac{|(f_k)_2(y_{s+1})|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_{s+1}\right)$$
$$\le (1+\epsilon)r_B^{\varepsilon}r_B^{-\frac{\epsilon}{2(1-\epsilon)}}\sum_k M((f_k)_1)(x)r_B^{-\frac{\epsilon}{2(1-\epsilon)}}M((f_k)_2)(x) \le (1+\epsilon)\sum_k M_s((f_k)_1)(x)M_s((f_k)_2)(x).$$

Case 2: $0 < r_B < 1$.

Denote by $\tilde{B} = B(x_0, r_B^{1-\epsilon})$. Write

$$(f_k)_1 = (f_k)_1\chi_{2B^-} + (f_k)_1\chi_{(2\overline{B})^c} := (\widetilde{f_k})_1^1 + (\widetilde{f_k})_1^2, (f_k)_2$$
$$= (f_k)_2\chi_{2B^-} + (f_k)_2\chi_{(2\overline{B})^c} := (\widetilde{f_k})_2^1 + (\widetilde{f_k})_2^2. \quad (2.2)$$

Take a $\tilde{c}_0 = T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^1\big)(x_0) + T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^2\big)(x_0) + T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)(x_0)$, then

$$\left(\frac{1}{|B|}\int_B \sum_k ||T((f_k)_1,(f_k)_2)(x+2\epsilon)|^{\delta} - |\tilde{c}_0|^{\delta}|d(x+2\epsilon)\right)^{1/\delta}$$
$$\le (1+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k |T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^1\big)(x+2\epsilon)|^{\delta}d(x+2\epsilon)\right)^{\frac{1}{\delta}} + (1$$
$$+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k |T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^1\big)(x+2\epsilon) - T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^1\big)(x_0)|^{\delta}d(x+2\epsilon)\right)^{\frac{1}{\delta}} + (1$$
$$+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k |T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^2\big)(x+2\epsilon) - T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^2\big)(x_0)|^{\delta}d(x+2\epsilon)\right)^{\frac{1}{\delta}} + (1$$
$$+\epsilon)\left(\frac{1}{|B|}\int_B \sum_k |T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)(x+2\epsilon) - T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)(x_0)|^{\delta}d(x+2\epsilon)\right)^{1/\delta} :$$
$$= \sum_{j=1}^{4}\widetilde{I_j}.$$

Notice that $0 < \delta < 1+\epsilon < \infty$ and $0 < l/1+\epsilon \le 1-\epsilon$, where l and $(1+\epsilon)$ are given as in Definition 1.2. It follows from Lemma 2.1 and (3) of Definition 1.2 that

$$\tilde{I} \le (1+\epsilon)|B|^{-\frac{1}{\delta}}\sum_k \|T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^1\big)\|_{L^{\delta}(B)} \le (1+\epsilon)|B|^{-\frac{1}{1+\epsilon}}\sum_k \|T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^1\big)\|_{L^{1+\epsilon,\infty}(B)}$$
$$\le (1+\epsilon)|B|^{-\frac{1}{1+\epsilon}}|\tilde{B}|^{\frac{1}{l}}\sum_k \left(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|(f_k)_1(y_s)|^{l_1}dy_s\right)^{\frac{1}{l_1}}\left(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|(f_k)_2(y_{s+1})|^{l_2}dy_{s+1}\right)^{\frac{1}{l_2}}$$
$$\le (1+\epsilon)r_B^{\left(\frac{2(1-\epsilon^2-l)}{l(1-\epsilon)}\right)}\sum_k M_{l_1}((f_k)_1)(x)M_{l_2}((f_k)_2)(x)$$
$$\le (1+\epsilon)\sum_k M_s((f_k)_1)(x)M_s((f_k)_2)(x).$$

For $(x+2\epsilon) \in B$ and $y_s \in \big(2\tilde{B}\big)^c$, there is $|x+2\epsilon-x_0|^{1-\epsilon} \le r_B^{1-\epsilon} \le \frac{1}{2}|y_s - x_0|$. By Hölder's inequality, (1) of Definition 1.2 and Lemma 2.2, we have

$$\tilde{I}_2 \leq (1+\epsilon)\frac{1}{|B|}\int_B \sum_k |T((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^1)(x+2\epsilon) - T((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^1)(x_0)|d(x+2\epsilon)$$

$$\leq (1+\epsilon)\frac{1}{|B|}\int_B \int_{(2\tilde{B})^{1+\epsilon}}\int_{2B^-}\sum_k \frac{|x+2\epsilon-x_0|^\varepsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}|(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s d(x$$

$$+2\epsilon) \leq (1+\epsilon)r_B^\varepsilon \sum_k \left(\int_{2B^-}|(f_k)_2(y_{s+1})|dy_{s+1}\right)\left(\int_{(2B^-)^{1+\epsilon}}\frac{|(f_k)_1(y_s)|}{|x_0-y_s|^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_s\right)$$

$$\leq (1+\epsilon)r_B^{\xi i}|\tilde{B}|\sum_k M((f_k)_2)(x)(r_B^{1-\epsilon})^{-\left(\frac{2(1+\epsilon)+\varepsilon i}{1-\epsilon}\right)}M((f_k)_1)(x)$$

$$\leq (1+\epsilon)\sum_k M_s((f_k)_1)(x)M_s((f_k)_2)(x)\,.$$

Similarly we can get that

$$\tilde{I}_3 \leq (1+\epsilon)M_s((f_k)_1)(x)M_s((f_k)_2)(x)\,.$$

For $(x+2\epsilon)\in B$ and $y_s, y_{s+1}\in (2\tilde{B})^c$, there are $|x+2\epsilon-x_0|^{1-\epsilon}\leq \frac{1}{2}|y_s-x_0|$ and $|x+2\epsilon-x_0|^{1-\epsilon}\leq \frac{1}{2}|y_{s+1}-x_0|$. It follows from Hölder's inequality, (1) of Definition 1.2 and Lemma 2.2 that

$$\widetilde{I_4} \leq (1+\epsilon)\frac{1}{|B|}\int_B \sum_k |T((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2)(x+2\epsilon) - T((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2)(x_0)|d(x+2\epsilon)$$

$$\leq (1+\epsilon)\frac{1}{|B|}\int_B \int_{(2B^-)^c}\int_{(2B^-)^{1+\epsilon}}\sum_k \frac{|x+2\epsilon-x_0|^\epsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}|(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s d(x$$

$$+2\epsilon) \leq (1+\epsilon)r_B^\varepsilon \sum_k \left(\int_{(2\tilde{B})^c}\frac{|(f_k)_1(y_s)|}{|x_0-y_s|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_s\right)\left(\int_{(2\tilde{B})^{1+\epsilon}}\frac{|(f_k)_2(y_{s+1})|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_{s+1}\right)$$

$$\leq (1+\epsilon)r_B^\varepsilon (r_B^{1-\epsilon})^{-\frac{\epsilon}{2(1-\epsilon)}}\sum_k M((f_k)_1)(x)(r_B^{1-\epsilon})^{-\frac{\epsilon}{2(1-\epsilon)}}M((f_k)_2)(x) \leq (1+\epsilon)M_s((f_k)_1)(x)M_s((f_k)_2)(x)\,.$$

Thus, combining the estimates in both cases, there is

$$\sum_k M_\delta^\#(T((f_k)_1,(f_k)_2))(x) \sim \sup_{B\ni x}\inf_{a\in 1+\epsilon}\left(\frac{1}{|B|}\int_B \sum_k \left||T((f_k)_1,(f_k)_2)(x+2\epsilon)|^\delta - a\right|d(x+2\epsilon)\right)^{\frac{1}{\delta}}$$

$$\leq (1+\epsilon)\sum_k M_s((f_k)_1)(x)M_s((f_k)_2)(x)\,,$$

which completes the proof of the theorem.

2.2. Boundednesson Product of Weighted LebesgueSpaces

A non- negative measurable function w is said to be in the Muckenhoupt class $A_{1+\epsilon}$ with $0\leq\epsilon<\infty$ if for every cube Q in $\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}$, there exists a positive constant C independent of Q such that

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{\epsilon}}dx\right)^\epsilon \leq (1+\epsilon),$$

where Q denotes a cube in $\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}$ with the side parallel to the coordinate axes. When $\epsilon=0$, a non- negative measurable function w is said to belong to A_1, if there exists a constant $\epsilon\geq 0$ such that for any cube Q,

$$\frac{1}{|Q|}\int_Q w(x+\epsilon)d(x+\epsilon) \leq (1+\epsilon)w(x)\,, \qquad a.e.\, x\in Q.$$

Denote by $A_\infty = \bigcup_{\epsilon\geq 0}A_{1+\epsilon}$. The well known property of the Muckenhoupt class is that if $w\in A_{1+\epsilon}$ with $0\leq\epsilon<\infty$, then $w\in A_r$ for all $\epsilon>0$, and $w\in A_{1+\epsilon}$ for some $\epsilon>0$.

By means of the pointwise estimate for the sharp maximal function, we can establish the boundedness of the multilinear strongly singular Calderón- Zygmund operator on product of weighted Lebesgue spaces as follows.

Theorem 2.2 (see [47]). *Let T bean m - linearstronglysingularCalderón- Zygmundoperatorand $s=\max\{r_s,\ldots,r_{s+m-1},l_s,\ldots l_{s+m-1}\}$, where$r_jandl_j$aregivenasinDefinition1.2, $j=s,\ldots,s+m-1$. Thenforany$s<p_s,p_{s+m-1}<\infty$ with $1/p=1/p_s+\cdots+1/p_{s+m-1}$, T canbeextendedintoaboundedoperatorfrom $L^{p_s}(w_s)\times\ldots\times L^{p_{s+m-1}}(w_{s+m-1})$intoL$^p(w)$, where*

$(w_s,\ldots,w_{s+m-1})\in(A_{p_s/s},\ldots,A_{p_{s+m-1}/s})andw=\prod_{j=s}^{s+m-1}w^{\frac{p}{jp_j}}$

For the special case $\epsilon = 0$, as we have discussed in Section 2.1, the standard multilinearCalderónZygmund operator satisfies the conditions of Definition 1.2 by taking $r_s = \cdots = r_{s+m-1} = 1$. Thus we can obtain the corresponding weighted estimate for the standard multilinear Calderón-Zygmund operator as a corollary of Theorem 2.2.

Corollary 2.2. Let T be a standard m-linear Calderón-Zygmund operator.

Then for any $1 < p_s, \dots, p_{s+m-1} < \infty$ with $1/p = 1/p_s + \cdots + 1/p_{s+m-1}$, T is bounded from $L^{p_s}(w_s) \times \dots \times L^{p_{s+m-1}}(w_{s+m-1})$ into $L^p(w)$, where $(w_s, \dots, w_{s+m-1}) \in \left(A_{p_s}, \dots, A_{p_{s+m-1}}\right)$ and $w = \prod_{j=s}^{s+m-1} w^{\frac{p}{jp_j}}$

Remark 2.2. It should be pointed out that in the special case $\epsilon = 0$, the result we obtain in Corollary 2.2 is in agreement with the one obtained in [18]. From this point of view, we can conclude that the range of indexes in Theorem 2.2 is reasonable.

We need the following two necessary lemmas(see [47]).

Lemma 2.3 (See [27]). Let $0 \le \epsilon < \infty$ and $w \in A_\infty$. Then there exists a constant $\epsilon \ge 0$ depending only on the A_∞ constant of w such that

$$\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} [M_\delta(f_k)(x)]^{1+\epsilon}\, w(x)dx \le (1+\epsilon) \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \left[M_\delta^\#(f_k)(x)\right]^{1+\epsilon} w(x)dx,$$

foreveryfunction$f_k$suchthattheleft- handsideisfinite.

Lemma 2.4 (*See* [*18*]).*For* $(w_s, \dots, w_{s+m-1}) \in \left(A_{p_s}, A_{p_{s+m-1}}\right)$ *with* $1 \le p_s, \dots, p_{s+m-1} < \infty$, *andfor* $0 < \theta_s, \theta_{s+m-1} < 1$*suchthat*$\theta_s + \cdots + \theta_{s+m-1} = 1$, *wehave*$w_s^{\theta_s} \cdots w_{s+m-1}^{\theta_{s+m-1}} \in A_{\max\{p_s,\dots,p_{s+m-1}\}}$.

Proof of Theorem 2.2. It follows from Lemma 2.4 that $w \in A_{\max\{p_s/1+\epsilon,\dots,p_{s+m-1}/1+\epsilon\}} \subset A_\infty$. For every $j = s, \dots, s+m-1, w_j \in A_{p_j/1+\epsilon}$ and $p_j > 1+\epsilon$ imply that the Hardy- Littlewood maximal operator M is bounded on $L^{p_j/1+\epsilon}(w_j)$.

Take a $(1+2\epsilon)$ such that $0 \le \epsilon < 1/m$, then by Lemma 2.3 and Theorem 2.1, we have

$$\begin{aligned}
\left\|\sum_k T(\vec{f_k})\right\|_{L^{1+\epsilon}(w)} &\le \sum_k \left\|M_{1+2\epsilon}\left(T(\vec{f_k})\right)\right\|_{L^{1+\epsilon}(w)} \le (1+\epsilon)\sum_k \left\|M_{1+2\epsilon}^\#\left(T(\vec{f_k})\right)\right\|_{L^{1+\epsilon}(w)} \\
&\le (1+\epsilon)\left\|\prod_{j=s}^{s+m-1}\sum_k M_{1+\epsilon}\left((f_k)_j\right)\right\|_{L^{1+\epsilon}(w)} \le (1+\epsilon)\prod_{j=s}^{s+m-1}\sum_k \left\|M_{1+\epsilon}\left((f_k)_j\right)\right\|_{L^{p_j}(w_j)} \\
&= (1+\epsilon)\prod_{j=s}^{s+m-1}\sum_k \left\|M\left(|(f_k)_j|^{(1+\epsilon)}\right)\right\|_{L^{\frac{p_j}{1+\epsilon}}(w_j)}^{\frac{1}{1+\epsilon}} \\
&\le (1+\epsilon)\prod_{j=s}^{s+m-1}\sum_k \left\||(f_k)_j|^{(1+\epsilon)}\right\|_{L^{\frac{p_j}{1+\epsilon}}(w_j)}^{\frac{1}{1+\epsilon}} = (1+\epsilon)\prod_{j=s}^{s+m-1}\sum_k \left\|(f_k)_j\right\|_{L^{p_j}(w_j)},
\end{aligned}$$

which gives the desired result.

2.3. Boundednesson Product of Variable Exponent LebesgueSpaces

The theory of variable exponent function spaces has been intensely investigated recently since some elementary results were established by [26]. For the properties of Calderón-Zygmund operators and many classical operators on the variable exponent Lebesgue space, see [10,11,13,14,26].

Definition 2.1. Let $p(\cdot) : \mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}} \to [1,\infty)$ be a measurable function. The variable exponent Lebesgue space, $L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$, is defined by

$$L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right) = \{f_k \text{ is measurable} : \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \sum_k \left(\frac{|f_k(x)|}{\frac{2}{\epsilon}}\right)^{p(x)} dx < \infty$$

$$(f_j)\text{or some constant } \frac{2}{\epsilon} > 0\}.$$

It is well known that the set $L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ becomes a Banach space with respect to the norm

$$\left\|\sum_k f_k\right\|_{L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)} = \inf\{\frac{2}{\epsilon} > 0 : \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \sum_k \left(\frac{|f_k(x)|}{\frac{2}{\epsilon}}\right)^{p(x)} dx \le 1\}.$$

Denote by $\mathcal{P}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}} \to [1,\infty)$ such that

$$1 < p_-:=\operatorname{essinf}_{x\in\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} p(x) \text{ and } p_+:=\operatorname{esssup}_{x\in\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} p(x) < \infty,$$

and by $\mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ the set of all $p(\cdot)\in\mathcal{P}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ such that the Hardy- Littlewood maximal operator M is bounded on $L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$.

By means of the pointwise estimate for the sharp maximal function, we can establish the boundedness of the multilinear strongly singular Calderón- Zygmund operator on product of variable exponent Lebesgue spaces as follows.

Theorem 2.3 (see [47]). Let $p(\cdot), p_s(\cdot), \dots, p_{s+m-1}(\cdot)\in\mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ with $1/p(\cdot) = 1/p_s(\cdot) + +1/p_{s+m-1}(\cdot)$. Let $(1+\epsilon)_0^j$ be given as in Lemma 2.5 for $p_j(\cdot), j = s, \dots, s+m-1$. Let T be an m-linear strongly singular Calderón-Zygmund operator and $s = \max\{r_s, \dots, r_{s+m-1}, l_s, \dots, l_{s+m-1}\}$, where r_j and l_j are given as in Definition 1.2, $j = s, \dots, s+m-1$. If $s \le \min_{s\le j\le s+m-1}(1+\epsilon)_0^j$, then T can be extended into a bounded operator from $L^{p_s(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)\times\dots\times L^{p_{s+m-1}(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ into $L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$.

For the special case $\epsilon = 0$, as we have discussed in Section 2.1, the standard multilinearCalderónZygmund operator satisfies the conditions of Definition 1.2 by taking $r_s = \dots = r_{s+m-1} = 1$. Thus we can obtain the corresponding boundedness for the standard multilinear Calderón- Zygmund operator on product of variable exponent Lebesgue spaces as a corollary of Theorem 2.3.

Corollary 2.3. Let $p(\cdot), p_s(\cdot), \dots, p_{s+m-1}(\cdot)\in B\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ with $1/p(\cdot) = 1/p_s(\cdot) + +1/p_{s+m-1}(\cdot)$ and T be a standard m -linear Calderón-Zygmund operator. Then T is bounded from $L^{p_s(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)\times\dots\times L^{p_{s+m-1}(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ into $L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$.

To proof Theorem 2.3, we need the following necessary lemmas(see [47]).

Lemma 2.5 (*See*[12]).Let $p(\cdot)\in\mathcal{P}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$. Then M is bounded on $L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ if and only if $M_{1+\epsilon}$ is bounded on $L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ for some $0 < \epsilon < \infty$.

Lemma 2.6(See[35]).*Let* $p(\cdot), p_s(\cdot), \dots, p_{s+m-1}(\cdot)\in\mathcal{P}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ *sothat* $\frac{1}{p(x)} = \frac{1}{p_s(x)} + \dots + 1/p_{s+m-1}(x)$. *Thenforany*$(f_k)_j\in L^{p_j(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right), j = s, \dots, s+m-1$, *therehas*

$$\left\|\prod_{j=s}^{s+m-1}\sum_k (f_k)_j\right\|_{L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)} \le 2^{m-1}\prod_{j=s}^{s+m-1}\sum_k \|(f_k)_j\|_{L^{p_j(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)}.$$

Lemma 2.7 (*See*[13]).*Givenafamily*$\mathcal{F}$ *oforderedpairsofmeasurablefunctions, supposeforsomefixed*$0\le\epsilon<\infty$, *every*$(f_k, g_k)\in\mathcal{F}$*andevery*$w\in A_1$,

$$\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}}\sum_k |f_k(x)|^{1+\epsilon}w(x)dx \le C_0\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}}\sum_k |g_k(x)|^{1+\epsilon}w(x)dx.$$

Let $p(\cdot)\in\mathcal{P}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ *with* $(1+\epsilon)\le p_-\cdot$ *If* $\left(\frac{p(\cdot)}{1+\epsilon}\right)'\in\mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$, *thenthereexistsaconstant* $\epsilon\ge 0$*suchthatforall*$(f_k, g_j)\in\mathcal{F}$, $\|\sum_k f_k\|_{L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)} \le (1+\epsilon)\sum_k \|g_k\|_{L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)}$.

Lemma 2.8(*See* [*13*]).*If*$p(\cdot)\in\mathcal{P}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$, *then*$C_0^\infty\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$*isdensein*$L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$.

Lemma 2.9(*See* [*12*]).*Let*$p(\cdot)\in\mathcal{P}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$. *Thenthefollowingconditionsareequivalent.*

(i) $p(\cdot)\in\mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$.

(ii) $p'(\cdot)\in\mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$.

(iii)$\frac{p(.)}{1+\epsilon}.\in\mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$for some $1 < 1+\epsilon < p_-$

(iv)$\left(\frac{p(.)}{p_0}\right)'\in\mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$for some $1 < 1+\epsilon < p_-\cdot$

Then, we are able to prove Theorem 2.3.

Proof of Theorem 2.3. Since $p(\cdot) \in \mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$, then by Lemma 2.9, there exists a $(1+\epsilon)$ such that $0 \le \epsilon < p_-$ and $\left(\frac{p(\cdot)}{1+\epsilon}\right)' \in \mathcal{B}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$. Take a $(1+2\epsilon)$ such that $0 < 1+2\epsilon < 1/m$. For any $w \in A_1$, it follows from Lemma 2.3 and Theorem 2.1 that

$$\begin{aligned}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \sum_k |T(\overrightarrow{f_k})(x)|^{1+\epsilon} w(x)dx &\le \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \sum_k [M_\delta\left(T(\overrightarrow{f_k})\right)(x)]^{1+\epsilon} w(x)dx \\ &\le (1+\epsilon)\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \sum_k [M^{\#}_{1+2\epsilon}\left(T(\overrightarrow{f_k})\right)(x)]^{1+\epsilon} w(x)dx \\ &\le (1+\epsilon)\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \left[\prod_{j=s}^{s+m-1} \sum_k M_{1+\epsilon}\left((f_k)_j\right)(x)\right]^{1+\epsilon} w(x)dx \\ &\le (1+\epsilon)\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}} \left[\prod_{j=s}^{s+m-1} \sum_k M_{(1+\epsilon)_0^j}\left((f_k)_j\right)(x)\right]^{1+\epsilon} w(x)dx\end{aligned}$$

holds for all m- tuples $\overrightarrow{f_k} = ((f_k)_s, \dots, (f_k)_{s+m-1})$ of bounded measurable functions with compact support. Applying Lemma 2.7 to the pair $(T\left(\overrightarrow{f_k}, \prod_{j=s}^{s+m-1} M_{(1+\epsilon)_0^j}\left((f_k)_j\right)\right)$, we have

$$\|\sum_k T(\overrightarrow{f_k})\|_{L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)} \le (1+\epsilon)\|\prod_{j=s}^{s+m-1}\sum_k M_{(1+\epsilon)_0^j}\left((f_k)_j\right)\|_{L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)}$$

Noticing the choice of $(1+\epsilon)_0^j$, we can get that $M_{(1+\epsilon)_0^j}$ is bounded on $L^{p_j(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)$ by Lemma 2.5, $j = s, \dots, s+m-1$. Then it follows from Lemma 2.6 that

$$\begin{aligned}\|\sum_k T(\overrightarrow{f_k})\|_{L^{p(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)} &\le (1+\epsilon)\prod_{j=s}^{s+m-1}\sum_k \|M_{(1+\epsilon)_0^j}\left((f_k)_j\right)\|_{L^{p_j(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)} \\ &\le (1+\epsilon)\prod_{j=s}^{s+m-1}\sum_k \|(f_k)_j\|_{L^{p_j(\cdot)}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right)}.\end{aligned}$$

This completes the proof of the theorem.

III. Endpoint Estimate

3.1. Boundedness of $L^\infty \times \dots \times L^\infty \to BMO$ Type

We will focus on the behaviors when $p_s = \cdots = p_{s+m-1} = \infty$ and establish the endpoint estimate for the multilinear strongly singular Calderón- Zygmund operator from $L^\infty \times \dots \times L^\infty$ into BMO.

Theorem 3.1 (see [47]). *LetTbeanm- linearstronglysingularCalderón- Zygmundoperator*, $(1+\epsilon)$*begivenasin* (3) *ofDefinition*1.2*and*$\epsilon > 0$. *ThenTcanbeextendedintoaboundedoperatorfrom*$L^\infty \times \dots \times L^\infty$*intoBMO*.

Proof. We only give the proof when $m = 2$ and omit other situations since their similarities.

Let $(f_k)_1, (f_k)_2 \in L^\infty$, then for any ball $B = B(x_0, r_B)$ with $r_B > 0$, we will consider two cases, respectively.

Case 1: $r_B \ge 1$.

Use the same decomposition as (2.1) and choose the same c_0, then

$$\begin{aligned}&\frac{1}{|B|}\int_B \sum_k |T((f_k)_1, (f_k)_2)(x) - c_0|dx \\ &\le \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^1, (f_k)_2^1)(x)|dx + \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^2, (f_k)_2^1)(x) \\ &- T((f_k)_1^2, (f_k)_2^1)(x_0)|dx + \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^1, (f_k)_2^2)(x) - T((f_k)_1^1, (f_k)_2^2)(x_0)|dx \\ &+ \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^2, (f_k)_2^2)(x) - T((f_k)_1^2, (f_k)_2^2)(x_0)|dx := \sum_{j=1}^4 J_j.\end{aligned}$$

Denote by $s = \max\{r_s, \dots, r_{s+m-1}, l_s, \dots, l_{s+m-1}\}$, where r_j and l_j are given as in Definition 1.2, $j = s, \dots, s+m-1$. Take $p_s, \dots, p_{s+m-1}$ such that $\max\{s, s+m-1\} < p_s, \dots, p_{s+m-1} < \infty$. Let $\frac{1}{p} = \frac{1}{p_s} + \cdots + 1/p_{s+m-1}$,

then $0 < \epsilon < \infty$. It follows from Theorem 2.2 that T is bounded from $L^{p_s} \times ... \times L^{p_{s+m-1}}$ into $L^{1+\epsilon}$.
By Hölder's inequality and the $L^{p_s} \times L^{p_{s+1}} \to L^{1+\epsilon}$ boundedness of T, we have

$$J_1 \leq \left(\frac{1}{|B|}\int_B \sum_k |T((f_k)_1^1,(f_k)_2^1)(x)|^{1+\epsilon}dx\right)^{\frac{1}{1+\epsilon}} \leq (1+\epsilon)|B|^{-\frac{1}{1+\epsilon}}\sum_k \|(f_k)_1^1\|_{L^{p_1}}\|(f_k)_2^1\|_{L^{p_2}}$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

For $x \in B$ and $y_s \in (2B)^c$, there is $|x-x_0|^{1-\epsilon} \leq \frac{1}{2}|y_s - x_0|$. By the condition of the kernel in (1) of Definition 1.2, we have

$$J_2 \leq \frac{1}{|B|}\int_B \int_{(2B)^{1+\epsilon}}\int_{2B}\sum_k |K(x,y_s,y_{s+1}) - K(x_0,y_s,y_{s+1})||(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s dx$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty \frac{1}{|B|}\int_B\int_{(2B)^{1+\epsilon}}\int_{2B}\frac{|x-x_0|^\varepsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_{s+1}dy_s dx$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty r_B^\varepsilon|B|\int_{(2B)^c}\frac{1}{|x_0-y_s|^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_s$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

Similarly we can get that

$$J_3 \leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

For $x \in B$ and $y_s, y_{s+1} \in (2B)^c$, there are $|x-x_0|^{1-\epsilon} \leq \frac{1}{2}|y_s - x_0|$ and $|x-x_0|^{1-\epsilon} \leq \frac{1}{2}|y_{s+1}-x_0|$. It follows from (1) of Definition 1.2 that

$$J_4 \leq \frac{1}{|B|}\int_B\int_{(2B)^{1+\epsilon}}\int_{(2B)^{1+\epsilon}}\sum_k |K(x,y_s,y_{s+1}) - K(x_0,y_s,y_{s+1})||(f_k)_1(y_s)||(f_k)_2(y_{s+1})|dy_{s+1}dy_s dx$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty\frac{1}{|B|}\int_B\int_{(2B)^{1+\epsilon}}\int_{(2B)^{1+\epsilon}}\frac{|x-x_0|^\varepsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_{s+1}dy_s dx$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty r_B^\varepsilon\left(\int_{(2B)^c}\frac{1}{|x_0-y_s|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_s\right)\left(\int_{(2B)^c}\frac{1}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_{s+1}\right)$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty r_B^\varepsilon r_B^{-\frac{\varepsilon}{2(1-\epsilon)}}r_B^{-\frac{\varepsilon}{2(1-\epsilon)}} \leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

Case 2: $0 < r_B < 1$.
Denote by $\tilde{B} = B(x_0, r_B^{1-\epsilon})$. Use the same decomposition as (2.2) and choose the same $\tilde{c}_0$, then

$$\frac{1}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x) - \tilde{c}_0|dx$$
$$\leq \frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^1\big)(x)|dx + \frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^1\big)(x)$$
$$- T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^1\big)(x_0)|dx + \frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^2\big)(x) - T\big((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^2\big)(x_0)|dx$$
$$+ \frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)(x) - T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)(x_0)|dx := \sum_{j=1}^4 \tilde{J}_j.$$

Notice that $0 < \epsilon < \infty$ and $0 < l/1+\epsilon \leq 1-\epsilon$, where l is given as in Definition 1.2. It follows from Lemma 2.1 and (3) of Definition 1.2 that

$$\tilde{J}_1 \leq (1+\epsilon)|B|^{-1}\sum_k \|T((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^1)\|_{L^1(B)} \leq (1+\epsilon)|B|^{-\frac{1}{1+\epsilon}}\sum_k \|T((\widetilde{f_k})_1^1,(\widetilde{f_k})_2^1)\|_{L^{1+\epsilon,\infty}(B)}$$
$$\leq (1+\epsilon)|B|^{-\frac{1}{1+\epsilon}}\sum_k \|(\widetilde{f_k})_1^1\|_{L^{l_1}}\|(\widetilde{f_k})_2^1\|_{L^{l_2}} \leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty r_B^{\frac{2(1-\epsilon^2-l)}{l(1-\epsilon)}}$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

For $x \in B$ and $y_s \in (2\tilde{B})^c$, there is $|x-x_0|^{1-\epsilon} \leq \frac{1}{2}|y_s - x_0|$. By the condition of the kernel in (1) of Definition 1.2, we have

$$\tilde{J}_2 \leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty \frac{1}{|B|}\int_B \int_{(2\tilde{B})^{1+\epsilon}}\int_{2\tilde{B}} \frac{|x-x_0|^\varepsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_{s+1}dy_s dx$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty r_B^\varepsilon|\tilde{B}|\int_{(2\tilde{B})^c}\frac{1}{|x_0-y_s|^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_s$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

Similarly we can get that

$$\tilde{J}_3 \leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

For $x \in B$ and $y_s, y_{s+1} \in (2\tilde{B})^c$, there are $|x-x_0|^{1-\epsilon} \leq \frac{1}{2}|y_s-x_0|$ and $|x-x_0|^{1-\epsilon} \leq \frac{1}{2}|y_{s+1}-x_0|$. It follows from (1) of Definition 1.2that

$$\tilde{J}_4 \leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty \frac{1}{|B|}\int_B \int_{(2B^-)^{1+\epsilon}}\int_{(2B^-)^{1+\epsilon}} \frac{|x-x_0|^{\varepsilon:}}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_{s+1}dy_s dx$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty r_B^\varepsilon\left(\int_{(2B^-)^{1+\epsilon}}\frac{1}{|x_0-y_s|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_s\right)\left(\int_{(2B^-)^{1+\epsilon}}\frac{1}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_{s+1}\right)$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty.$$

Thus, combining the estimates in both cases, there is

$$\sum_k \|T((f_k)_1,(f_k)_2)\|_{BMO} \sim \sup_B \inf_{a\in 1+\epsilon} \frac{1}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x)-a|dx$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_\infty\|(f_k)_2\|_\infty,$$

which completes the proof of the theorem.

3.2. Boundedness of BMO × ... × BMO → BMOType

The definition of the *BMO* space is well known. The most useful property of the BMO function is the classical John- Nirenberg inequality, which shows that functions in the BMO space are locally exponentially integrable.

A well-known result in [43] showed that the classical Calderón- Zygmund operator is bounded on the BMO space. Lin and Lu also gave the *BMO*boundedness of the linear strongly singular Calderón- Zygmund operator in [33]. These conclusions essentially depend on the cancellation condition of the kernel, which can be expressed as $T_1 = 0$. A natural question is: if we give some kinds of cancellation conditions to the multilinear strongly singular Calderón- Zygmund operator, whether it can also be bounded on the product of BMO spaces? Actually, the answer is affirmative.

Theorem 3.2 (see [47]).*LetTbeanm- linearstronglysingularCalderón- Zygmundoperator,* $(1+\epsilon)$*begivenasin* (3) *ofDefinition* 1.2 *and* $\epsilon > 0$. *If* $((f_k)_s, \ldots, (f_k)_{s+j-1}, 1, (f_k)_{s+j-1}, \ldots, (f_k)_{s+m-1}) = 0, j = s, \ldots, s+m-1$, *thenTcanbeextendedintoaboundedoperatorfrom*$BMO \times \ldots \times BMO$*intoBMO.*

We need the following two lemmas to develop the proof of Theorem 3.2.

Lemma 3.1(***See***[**33**]).*Let*f_k*beafunctioninBMO. Suppose*$0 \leq \epsilon < \infty, x \in \mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}$, *and*$\epsilon \geq 0$. *Then*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r)}\sum_k ||f_k(x+\epsilon)-(f_k)_{B(x,r_2)}|^{1+\epsilon}d(x+\epsilon)\right)^{\frac{1}{1+\epsilon}}$$

$$\leq (1+\epsilon)\left(1+|\ln\frac{r_1}{r_2}|\right)\sum_k \|f_k\|_{BMO},$$

where $\epsilon \geq 0$ *isindependentof* $f_k, x, 1+\epsilon$ *and* $(1+2\epsilon)$.

Lemma 3.2. *Let* $\delta > 0$ *and* $f_k \in BMO$. *Thenforanyball* $B = B(x, 1+\epsilon)$ *with* $\epsilon \geq 0$ *and* $x \in \mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}$, *thereis*

$$\int_{B^{1+\epsilon}}\sum_k \frac{|f_k(x+\epsilon)-(f_k)_B|}{|\epsilon|^{\frac{2(1+\epsilon)}{1-\epsilon}+\delta}} d(x+\epsilon) \leq (1+\epsilon) r^{-\delta}\sum_k \|f_k\|_{BMO},$$

where $(1+\epsilon)$ *isapositiveconstantindependentof* f_k, x *and* $(1+\epsilon)$.

Now, we prove the main result.

Proof of Theorem 3.2. We only give the proof when $m = 2$ and omit other situations since their similarities. Let $(f_k)_1, (f_k)_2 \in BMO$, then for any ball $B = B(x_0, r_B)$ with $r_B > 0$, we will consider two cases, respectively.

Case 1: $r_B \geq 1$.

Write

$$(f_k)_1 = ((f_k)_i)_{2B} + ((f_k)_i - ((f_k)_i)_{2B})\chi_{2B} + ((f_k)_i - ((f_k)_i)_{2B})\chi_{(2B)^c} := (f_k)_1^1 + (f_k)_1^2 + (f_k)_1^3,$$

$$(f_k)_2 = ((f_k)_2)_{2B} + ((f_k)_2 - ((f_k)_2)_{2B})\chi_{2B} + ((f_k)_2 - ((f_k)_2)_{2B})\chi_{(2B)^c} := (f_k)_2^1 + (f_k)_2^2 + (f_k)_2^3.$$

It follows from the hypothesis of the theorem that

$$T((f_k)_1, (f_k)_2) = T((f_k)_1^2, (f_k)_2^2) + T((f_k)_1^2, (f_k)_2^3) + T((f_k)_1^3, (f_k)_2^2) + T((f_k)_1^3, (f_k)_2^3).$$

Take a $d_0 = T((f_k)_1^2, (f_k)_2^3)(x_0) + T((f_k)_1^3, (f_k)_2^2)(x_0) + T((f_k)_1^3, (f_k)_2^3)(x_0)$, then

$$\begin{aligned}\frac{1}{|B|}\int_B \sum_k & |T((f_k)_1, (f_k)_2)(x) - d_0| dx \\ &\leq \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^2, (f_k)_2^2)(x)| dx + \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^2, (f_k)_2^3)(x) \\ &- T((f_k)_1^2, (f_k)_2^3)(x_0)| dx + \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^3, (f_k)_2^2)(x) - T((f_k)_1^3, (f_k)_2^2)(x_0)| dx \\ &+ \frac{1}{|B|}\int_B \sum_k |T((f_k)_1^3, (f_k)_2^3)(x) - T((f_k)_1^3, (f_k)_2^3)(x_0)| dx := \sum_{j=1}^4 L_j.\end{aligned}$$

Choose $p_s, \ldots, p_{s+m-1}, 1+\epsilon$ the same as in the proof of Theorem 3.1. Then T is bounded from $L^{p_s} \times \ldots \times L^{p_{s+m-1}}$ into $L^{1+\epsilon}$ and $0 < \epsilon < \infty$. It follows from Hölder's inequality that

$$\begin{aligned} L_1 &\leq \left(\frac{1}{|B|}\int_B \sum_k |T((f_k)_1^2, (f_k)_2^2)(x)|^{1+\epsilon} dx\right)^{\frac{1}{1+\epsilon}} \\ &\leq (1+\epsilon)\sum_k \left(\frac{1}{|2B|}\int_{2B} |(f_k)_1(y_s) - ((f_k)_1)_{2B}|^{p_1} dy_s\right)^{\frac{1}{p_1}} \left(\frac{1}{|2B|}\int_{2B} |(f_k)_2(y_{s+1}) \right. \\ &\left. - ((f_k)_2)_{2B}|^{p_2} dy_{s+1}\right)^{\frac{1}{p_2}} \leq (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO}.\end{aligned}$$

For $x \in B$ and $y_{s+1} \in (2B)^c$, there is $|x - x_0|^{1-\epsilon} \leq \frac{1}{2}|y_{s+1} - x_0|$. By the condition of the kernel in Definition 1.2 and Lemma 3.2, we have

$$L_2 \le \frac{1}{|B|}\int_B \int_{(2B)^c}\int_{2B}\sum_k |K(x,y_s,y_{s+1})-K(x_0,y_s,y_{s+1})| \times |(f_k)_1(y_s)-((f_k)_1)_{2B}||(f_k)_2(y_{s+1}) - ((f_k)_2)_{2B}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)\frac{1}{|B|}\int_B \int_{(2B)^c}\int_{2B}\sum_k \frac{|x-x_0|^{\varepsilon}}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}} \times |(f_k)_1(y_s) - ((f_k)_1)_{2B}||(f_k)_2(y_{s+1})-((f_k)_2)_{2B}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)r_B^{\varepsilon}\sum_k \left(\int_{2B}|(f_k)_1(y_s)-((f_k)_1)_{2B}|dy_s\right)\left(\int_{(2B)^c}\frac{|(f_k)_2(y_{s+1})-((f_k)_2)_{2B}|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_{s+1}\right)$$

$$\le (1+\epsilon)r_B^{\varepsilon}\sum_k \|(f_k)_1\|_{BMO}|B|r_B^{-\left(\frac{3\varepsilon+2}{1-\epsilon}\right)}\|(f_k)_2\|_{BMO}$$

$$\le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO}.$$

Similarly we can get that

$$L_3 \le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO}.$$

For $x\in B$ and $y_s, y_{s+1}\in (2B)^c$, there are $|x-x_0|^{1-\epsilon}\le \frac{1}{2}|y_s-x_0|$ and $|x-x_0|^{1-\epsilon}\le\frac{1}{2}|y_{s+1}-x_0|$. It follows from (1) of Definition 1.2 and Lemma 3.2 that

$$L_4 \le \frac{1}{|B|}\int_B\int_{(2B)^{1+\epsilon}}\int_{(2B)^{1+\epsilon}}\sum_k |K(x,y_s,y_{s+1})-K(x_0,y_s,y_{s+1})|\times|(f_k)_1(y_s)-((f_k)_1)_{2B}||(f_k)_2(y_{s+1}) - ((f_k)_2)_{2B}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)\frac{1}{|B|}\int_B\int_{(2B)^c}\int_{(2B)^c}\sum_k \frac{|x-x_0|^{\varepsilon:}}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}\times|(f_k)_1(y_s)-((f_k)_1)_{2B}||(f_k)_2(y_{s+1}) - ((f_k)_2)_{2B}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)r_B^{\varepsilon}\sum_k\left(\int_{(2B)^{1+\epsilon}}\frac{|(f_k)_1(y_s)-((f_k)_1)_{2B}|}{|x_0-y_s|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_s\right)\left(\int_{(2B)^{1+\epsilon}}\frac{|(f_k)_2(y_{s+1})-((f_k)_2)_{2B}|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_{s+1}\right)$$

$$\le (1+\epsilon)r_B^{\varepsilon}r_B^{-\frac{\epsilon}{2(1-\epsilon)}}\sum_k \|(f_k)_1\|_{BMO}r_B^{-\frac{\varepsilon}{2(1-\epsilon)}}\|(f_k)_2\|_{BMO}\le(1+\epsilon)\sum_k \|(f_k)_i\|_{BMO}\|(f_k)_2\|_{BMO}.$$

Case 2: $0<r_B<1$.

Denote by $\tilde{B}=B(x_0,r_B^{1-\epsilon})$. Write

$$(f_k)_1 = ((f_k)_1)_{2B^-} + ((f_k)_1-((f_k)_1)_{2B^-})\chi_{2B^-} + ((f_k)_1-((f_k)_1)_{2B^-})\chi_{(2\overline{B})^c}$$
$$:= (\widetilde{f_k})_1^1 + (\widetilde{f_k})_1^2 + (\widetilde{f_k})_1^3, \tag{3.1}$$
$$(f_k)_2 = ((f_k)_2)_{2B^-} + ((f_k)_2-((f_k)_2)_{2B^-})\chi_{2B^-} + ((f_k)_2-((f_k)_2)_{2B^-})\chi_{(2\overline{B})^c}$$
$$:= (\widetilde{f_k})_2^1 + (\widetilde{f_k})_2^2 + (\widetilde{f_k})_2^3. \tag{3.2}$$

It follows from the hypothesis of the theorem that

$$T((f_k)_1,(f_k)_2) = T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big) + T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^3\big) + T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^2\big) + T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^3\big).$$

Take a $\tilde{d}_0 = T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^3\big)(x_0) + T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^2\big)(x_0) + T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^3\big)(x_0)$, then

$$\frac{1}{|B|}\int_B\sum_k |T((f_k)_1,(f_k)_2)(x)-\tilde{d}_0|dx$$

$$\le \frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)(x)|dx + \frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^3\big)(x) - T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^3\big)(x_0)|dx + \frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^2\big)(x) - T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^2\big)(x_0)|dx$$

$$+\frac{1}{|B|}\int_B\sum_k |T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^3\big)(x) - T\big((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^3\big)(x_0)|dx := \sum_{j=1}^{4}\tilde{L}_j.$$

Notice that $0<\epsilon<\infty$ and $0<l/1+\epsilon\le 1-\epsilon$, where l is given as in Definition 1.2. It follows from Lemma

2.1 and (3) of Definition 1.2 that

$$\tilde{L}_1 \le (1+\epsilon)\left|B\right|^{-1}\sum_k \left\| T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)\right\|_{L^1(B)} \le (1+\epsilon)\left|B\right|^{-\frac{1}{1+\epsilon}}\sum_k \left\| T\big((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\big)\right\|_{L^{1+\epsilon,\infty}(B)}$$

$$\le (1+\epsilon)\left|B\right|^{-\frac{1}{1+\epsilon}}\left|\tilde{B}\right|^{\frac{1}{l}}\left(\frac{1}{|2\tilde{B}|}\int_{2B^-}\sum_k |(f_k)_1(y_s)-((f_k)_1)_{2B^-}|^{l_1}dy_s\right)^{\frac{1}{l_1}}$$

$$\times\left(\frac{1}{|2\tilde{B}|}\int_{2B^-}\sum_k |(f_k)_2(y_{s+1})-((f_k)_2)_{2B^-}|^{l_2}dy_{s+1}\right)^{\frac{1}{l_2}} \le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO} r_B^{2\left(\frac{1-\epsilon^2-l}{l(1-\epsilon)}\right)}$$

$$\le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO}.$$

For $x\in B$ and $y_{s+1}\in (2\tilde{B})^c$, there is $|x-x_0|^{1-\epsilon}\le \frac{1}{2}|y_{s+1}-x_0|$. By the condition of the kernel in Definition 1.2 and Lemma 3.2, we have

$$\tilde{L}_2 \le (1+\epsilon)\frac{1}{|B|}\int_B\int_{(2B^-)^c}\int_{2B^-}\sum_k \frac{|x-x_0|^{\varepsilon}}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}\times|(f_k)_1(y_s)-((f_k)_1)_{2\tilde{B}}||(f_k)_2(y_{s+1}) - ((f_k)_2)_{2\tilde{B}}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)r_B^{\varepsilon}\left(\int_{2\tilde{B}}\sum_k |(f_k)_1(y_s) - ((f_k)_1)_{2B^-}|dy_s\right)\left(\int_{(2B^-)^c}\sum_k \frac{|(f_k)_2(y_{s+1})-((f_k)_2)_{2B^-}|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_{s+1}\right)$$

$$\le (1+\epsilon)\sum_k r_B^{\varepsilon}\|(f_k)_1\|_{BMO}|\tilde{B}|(r_B^{1-\epsilon})^{-\left(\frac{3\varepsilon+2}{1-\epsilon}\right)}\|(f_k)_2\|_{BMO}$$

$$\le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO}.$$

Similarly we can get that

$$\tilde{L}_3 \le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO}.$$

For $x\in B$ and $y_s, y_{s+1}\in (2\tilde{B})^c$, there are $|x-x_0|^{1-\epsilon}\le\frac{1}{2}|y_s-x_0|$ and $|x-x_0|^{1-\epsilon}\le\frac{1}{2}|y_{s+1}-x_0|$. It follows from (1) of Definition 1.2 and Lemma 3.2 that

$$\tilde{L}_4 \le (1+\epsilon)\frac{1}{|B|}\int_B\int_{(2\tilde{B})^c}\int_{(2\tilde{B})^c}\sum_k \frac{|x-x_0|^{\varepsilon}}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}\times|(f_k)_1(y_s)-((f_k)_1)_{2B^-}||(f_k)_2(y_{s+1}) - ((f_k)_2)_{2B^-}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)r_B^{\varepsilon}\left(\int_{(2\tilde{B})^C}\sum_k \frac{|(f_k)_1(y_s)-((f_k)_1)_{2\tilde{B}}|}{|x_0-y_s|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_s\right)\left(\int_{(2B^-)^{1+\epsilon}}\sum_k \frac{|(f_k)_2(y_{s+1})-((f_k)_2)_{2\tilde{B}}|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_{s+1}\right)$$

$$\le (1+\epsilon)r_B^{\varepsilon}(r_B^{1-\epsilon})^{-\frac{\epsilon}{2(1-\epsilon)}}\sum_k \|(f_k)_1\|_{BMO}(r_B^{1-\epsilon})^{-\frac{\epsilon}{2(1-\epsilon)}}\|(f_k)_2\|_{BMO} \le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO}.$$

Thus, combining the estimates in both cases, there is

$$\sum_k \|T((f_k)_1,(f_k)_2)\|_{BMO} \sim \sup_B \inf_{a\in 1+\epsilon}\frac{1}{|B|}\int_B\sum_k |T((f_k)_1,(f_k)_2)(x)-a|dx$$

$$\le (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO},$$

which completes the proof of the theorem.

3.3. *BoundednessofLMO* $\boldsymbol{\times \ldots \times LMO \to LMO}$***type***

The *LMO* space is essentially a special case of the function space introduced by [41]. Some properties of the *LMO* function are similar to those of the BMO function. But there are also some new interesting phenomena for the *LMO* function itself. See [1].

Definition 3.1.*LMO* is a subspace of BMO, equipped with the semi- norm

$$\sum_k [(f_k)]_{LMO} = \sup_{0<r<1} \frac{1+|\ln r|}{|B_r|}\int_{B_r} \sum_k |(f_k)(x)-(f_k)_{B_r}|dx + \sup_{r\geq 1} \frac{1}{|B_r|}\int_{B_r} \sum_k |(f_k)(x)-(f_k)_{B_r}|dx,$$

where B_r denotes by the ball in $\mathbb{R}^{\frac{2(1+\epsilon)}{1-\epsilon}}$ with radius r.
For $0 \leq \epsilon < \infty$, define

$$\sum_k [(f_k)]_{LMO1+\epsilon} = \sup_{0<r<\frac{1}{2}} (1+|\ln r|)\left(\frac{1}{|B_r|}\int_{B_r} \sum_k |(f_k)(x)-(f_k)_{B_r}|^{1+\epsilon}dx\right)^{\frac{1}{1+\epsilon}}$$

The authors in [33,41,44] obtained the LMO- boundedness of classical Calderón- Zygmund operators and linear strongly singular Calderón- Zygmund operators, respectively. We will establish the boundedness of the multilinear strongly singular Calderón- Zygmund operator on product of LMO spaces.

Theorem 3.3 (see [47]). *LetTbeanm- linearstronglysingularCalderón- Zygmundoperator,* $(1+\epsilon)$*begivenasin* (3) *ofDefinition* 1.2 *and* $\epsilon > 0$. *If* $\left((f_k)_1, \ldots, (f_k)_{j-1}, 1, (f_k)_{j+1}, \ldots, (f_k)_m\right) = 0, j = s, \ldots, s+m-1$, *thenTcanbeextendedintoaboundedoperatorfrom* $LMO \times \ldots \times LMO$ *intoLMO*.

To proveTheorem 3.3, we need the following lemmas.

Lemma 3.3(*See* [1]). *If* $f_k \in LMO$, *thenforany* $0 \leq \epsilon < \infty$, *thereexistsaconstant* $\epsilon \geq 0$*dependingonlyon*$\frac{2(1+\epsilon)}{1-\epsilon}$*and*$(1+\epsilon)$*suchthat*

$$[f_k]_{LMO_{1+\epsilon}} \leq (1+\epsilon)[f_k]_{LMO}.$$

Lemma 3.4(*See*[33]). *Let*$\delta > 0$*and*$f_k \in LMO$. *ThenforanyballB* $= B(x,r)$ *with*$0 < r < \frac{1}{2}$,

$$\int_{B^c} \sum_k \frac{|f_k(x+\epsilon)-(f_k)_B|}{|-\epsilon|^{\frac{2(1+\epsilon)}{1-\epsilon}+\delta}} d(x+\epsilon) \leq (1+\epsilon)r^{-\delta}(1+|\ln r|)^{-1}\sum_k [f_k]_{\mathrm{LMO}},$$

where$\epsilon \geq 0$*isindependentof*f_k, *xandr*.

Lemma 3.5. *Foranya,* $(a+\epsilon) \in \mathbb{R}$, *thereis*

$$1+|2a+\epsilon| \geq (1+|a|)^{-1}(1+|a+\epsilon|).$$

Proof of Theorem 3.3. We only consider the situation when $m = 2$. Actually, the similar procedure works for all other situations.

For $(f_k)_1, (f_k)_2 \in LMO$ and any ball $B = B(x_0, r_B)$ with $r_B \geq 1$, it follows from Theorem 3.2 and Definition 3.1 that

$$\frac{1}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x) - \left(T((f_k)_1,(f_k)_2)\right)_B|dx \leq \sum_k \|T((f_k)_1,(f_k)_2)\|_{BMO}$$
$$\leq (1+\epsilon)\sum_k \|(f_k)_1\|_{BMO}\|(f_k)_2\|_{BMO} \leq (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}.$$

Thus it is sufficient to prove that, for any ball $B = B(x_0, r_B)$ with $0 < r_B < 1$, the following inequality holds.

$$\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x) - \left(T((f_k)_1,(f_k)_2)\right)_B|dx \leq (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}. \quad (3.3)$$

We consider two cases, respectively.

Case 1: $4^{-1/(1-\epsilon)} \leq r_B < 1$.

Theorem 3.2 also implies that

$$\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x) - \left(T((f_k)_1,(f_k)_2)\right)_B|dx$$
$$\leq (1+\epsilon)\frac{1}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x) - \left(T((f_k)_1,(f_k)_2)\right)_B|dx$$
$$\leq (1+\epsilon)\sum_k \|T((f_k)_1,(f_k)_2)\|_{BMO} \leq (1+\epsilon)[(f_k)_1]_{LMO}[(f_k)_2]_{LMO}.$$

Case 2: $0 < r_B < 4^{-1/(1-\epsilon)}$.

Denote by $\tilde{B} = B(x_0, r_B^{1-\epsilon})$. Use the same decompositions as $(3.1)-(3.2)$ and choose the same $\tilde{d}_0$, then

$$\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x)-\left(T((f_k)_1,(f_k)_2)\right)_B|dx$$
$$\leq (1+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T((f_k)_1,(f_k)_2)(x)-\tilde{d}_0|dx$$
$$\leq (1+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T\left((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^2\right)(x)|dx+(1$$
$$+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T\left((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^3\right)(x)-T\left((\widetilde{f_k})_1^2,(\widetilde{f_k})_2^3\right)(x_0)|dx+(1$$
$$+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T\left((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^2\right)(x)-T\left((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^2\right)(x_0)|dx+(1$$
$$+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \sum_k |T\left((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^3\right)(x)-T\left((\widetilde{f_k})_1^3,(\widetilde{f_k})_2^3\right)(x_0)|dx:=\sum_{j=1}^4 H_j.$$

Notice that $0<2r_B^{1-\epsilon}<1/2, 0<\epsilon<\infty$ and $0<l/1+\epsilon\leq 1-\epsilon$, where l is given as in Definition 1.2. It follows from Lemma 2.1, (3) of Definition 1.2, Lemmas 3.3 and 3.5that

$$H_1 \leq (1+\epsilon)(1+|\ln r_B|)|B|^{-1}\sum_k \|T\left((f_k)_1^{\sim 2},(f_k)_2^{\sim 2}\right)\|_{L1(B)}$$
$$\leq (1+\epsilon)(1+|\ln r_B|)|B|^{-\frac{1}{1+\epsilon}}\sum_k \|T\left((f_k)_1^{\sim 2},(f_k)_2^{\sim 2}\right)\|_{L^{1+\epsilon,\infty}(B)}$$
$$\leq (1+\epsilon)(1+|\ln r_B|)|B|^{-\frac{1}{1+\epsilon}}|\tilde{B}|^{\frac{1}{l}}\left(\frac{1}{|2\tilde{B}|}\int_{2B^-}\sum_k |(f_k)_1(y_s)-((f_k)_1)_{2B^-}|^{l_1}dy_s\right)^{\frac{1}{l_1}}$$
$$\times\left(\frac{1}{|2\tilde{B}|}\int_{2B^-}|(f_k)_2(y_{s+1})-((f_k)_2)_{2B^-}|^{l_2}dy_{s+1}\right)^{\frac{1}{l_2}}$$
$$\leq (1+\epsilon)\sum_k (1+|\ln r_B|)|B|^{-\frac{1}{1+\epsilon}}|\tilde{B}|^{\frac{1}{l}}[(f_k)_1]_{LMO^{l_1}}(1+|\ln 2r_B^{1-\epsilon}|)^{-1}[(f_k)_2]_{LMO^{l_2}}(1+|\ln 2r_B^{1-\epsilon}|)^{-1}$$
$$\leq (1+\epsilon)\sum_k [(f_k)_l]_{LMO}[(f_k)_2]_{LMO}(1+|\ln r_B|)|B|^{-\frac{1}{1+\epsilon}}|\tilde{B}|^{\frac{1}{l}}(1+|\ln 2+(1-\epsilon)\ln r_B|)^{-1}$$
$$\leq (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}(1+|\ln r_B|)r_B^{2\left(\frac{1-\epsilon^2-l}{l(1-\epsilon)}\right)}(1+(1-\epsilon)|\ln r_B|)^{-1}(1+\ln 2)$$
$$\leq (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}$$

For $x\in B$ and $y_{s+1}\in\left(2\tilde{B}\right)^c$, there is $|x-x_0|^{1-\epsilon}\leq\frac{1}{2}|y_{s+1}-x_0|$. By the condition of the kernel in Definition 1.2, the fact $0<2r_B^{1-\epsilon}<1/2$, Lemmas 3.4 and 3.5, we have

$$H_2 \le (1+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \int_{(2\tilde{B})^{1+\epsilon}}\int_{2\tilde{B}}\sum_k |K(x,y_s,y_{s+1})-K(x_0,y_s,y_{s+1})|\times|(f_k)_1(y_s) - ((f_k)_l)_{2B^-}||(f_k)_2(y_{s+1})-((f_k)_2)_{2B^-}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \int_{(2B^-)^c}\int_{2B^-}\sum_k \frac{|x-x_0|^\varepsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}\times|(f_k)_l(y_s) - ((f_k)_1)_{2\tilde{B}}||(f_k)_2(y_{s+1})-((f_k)_2)_{2B^-}|dy_s dy_{s+1}dx$$

$$\le (1+\epsilon)r_B^{\varepsilon}(1+|\ln r_B|)\left(\int_{2\tilde{B}}\sum_k |(f_k)_1(y_s) - ((f_k)_1)_{2B^-}|dy_s\right)\left(\int_{(2\tilde{B})^{1+\epsilon}}\sum_k \frac{|(f_k)_2(y_{s+1})-((f_k)_2)_{2B^-}|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{1-\epsilon}}}dy_{s+1}\right)$$

$$\le (1+\epsilon)\sum_k r_B^{\varepsilon}(1+|\ln r_B|)\|(f_k)_1\|_{BMO}|\tilde{B}|(r_B^{1-\epsilon})^{-\left(\frac{3\varepsilon+2}{1-\epsilon}\right)}[(f_k)_2]_{LMO}(1+|\ln 2r_B^{1-\epsilon}|)^{-1}$$

$$\le (1+\epsilon)\sum_k [(f_k)_l]_{LMO}[(f_k)_2]_{LMO}(1+|\ln r_B|)(1+|\ln 2+(1-\epsilon)\ln r_B|)^{-1}$$

$$\le (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}$$

Similarly we can get that

$$H_3 \le (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}$$

For $x\in B$ and $y_s, y_{s+1}\in (2\tilde{B})^c$, there are $|x-x_0|^{1-\epsilon}\le\frac{1}{2}|y_s-x_0|$ and $|x-x_0|^{1-\epsilon}\le\frac{1}{2}|y_{s+1}-x_0|$. It follows from (1) of Definition 1.2, Lemmas 3.4 and 3.5 that

$$H_4 \le (1+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \int_{(2B^-)^c}\int_{(2B^-)^c}\sum_k |K(x,y_s,y_{s+1})-K(x_0,y_s,y_{s+1})|\times|(f_k)_1(y_s) - ((f_k)_1)_{2\tilde{B}}||(f_k)_2(y_{s+1})-((f_k)_2)_{2\tilde{B}}|dy_s dy_{s+1}dx \le (1+\epsilon)\frac{1+|\ln r_B|}{|B|}\int_B \int_{(2B^-)^c}\int_{(2B^-)^c}\sum_k \frac{|x-x_0|^\varepsilon}{(|x_0-y_s|+|x_0-y_{s+1}|)^{\frac{4+5\varepsilon}{1-\epsilon}}}\times|(f_k)_1(y_s) - ((f_k)_1)_{2B^-}||(f_k)_2(y_{s+1}) - ((f_k)_2)_{2B^-}|dy_s dy_{s+1}dx \le (1+\epsilon)r_B^{\varepsilon}(1+|\ln r_B|)\left(\int_{(2B^-)^c}\sum_k \frac{|(f_k)_1(y_s)-((f_k)_1)_{2B^-}|}{|x_0-y_s|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_s\right)\left(\int_{(2B^-)^c}\sum_k \frac{|(f_k)_2(y_{s+1})-((f_k)_2)_{2B^-}|}{|x_0-y_{s+1}|^{\frac{4+5\varepsilon}{2(1-\epsilon)}}}dy_{s+1}\right) \le (1+\epsilon)\sum_k r_B^{\varepsilon}(1+|\ln r_B|)(r_B^{1-\epsilon})^{-\frac{\varepsilon}{2(1-\epsilon)}}[(f_k)_1]_{LMO}(1+|\ln 2r_B^{1-\epsilon}|)^{-1} \times (r_B^{1-\epsilon})^{-\frac{\varepsilon}{2(1-\epsilon)}}[(f_k)_2]_{LMO}(1+|\ln 2r_B^{1-\epsilon}|)^{-1} \le (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}(1+|\ln r_B|)(1+|\ln 2+(1-\epsilon)\ln r_B|)^{-1} \le (1+\epsilon)\sum_k [(f_k)_1]_{LMO}[(f_k)_2]_{LMO}.$$

Thus, combining the estimates in both cases, (3.3) holds, which completes the proof of the theorem.

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