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Review Paper

A highlight sharp estimates for multilinear pseudo-differential operators

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Abstract
M. Cao, Q.Xue and K.Yabuta [52] investigate the boundedness of the multilinear pseudodifferential operator T_{σ_k}*.They establish the local exponential decay estimates for* T_{σ_k}*. In terms of the corresponding commutators* $T_{\sigma_k,\Sigma b}$*,they obtain the local subexponential decay estimates. They also derive the weighted mixed weak type inequality for* T_{σ_k}*, which parallels Sawyer's conjecture for Calderon-Zygmund operators and covers the endpoint weighted inequalities. In addition they present the sharp weighted estimates for* T_{σ_k} *and* $T_{\sigma_k,\Sigma b}$*.We follow [52] to do a slight application on the verified valid rare results.*
Keywords: *Pseudo-differential operators, Local decay estimates, Sharp weighted inequalities, Endpoint weak bounds, Sparse dominations.*

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I. Introduction

We study the boundedness of the multilinear pseudo- differential operators (ΨDOs). Introduce by [52]. Given a recognizable symbol σ_k, the m- linear ΨDOT_{σ_k} is defined by

$$T_{\sigma_k}(\overrightarrow{f_k})(x) = \int_{(\mathrm{R}^n)^m} \sum_k \sigma_k\left(x,\vec{\xi}\right) e^{2\pi i x\vec{\xi}}\widehat{(f_k)_1}(\xi_1)\cdots\widehat{(f_k)_m}(\xi_m)d\vec{\xi},$$

where the Fourier transform $\hat{f}_k$of the function f_k is given by

$$\hat{f}_k(\xi) = \int_{\mathrm{R}^n} \sum_k e^{-2\pi i x\cdot\xi} f_k(x)dx.$$

Definition 1.1.*Let*$0 < \epsilon \leq \infty, (1+\epsilon) \in$ R*and*$\rho,\delta \in [0,1]$*.Asmoothfunction*σ_k*isdefinedon*$\mathbb{R}^n \times \mathbb{R}^{mn}$
(a) *Wesay*$\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$*ifforeachtripleofmulti- indices*and$\beta_1^k,\ldots,\beta_m^k$*thereexistsaconstant*$C_{1+2\epsilon,\beta^k}$*suchthat*

$$\left|\sum_k \partial_x^{1+2\epsilon}\partial_{\xi_1}^{\mathcal{B}_1^k}\cdots\partial_{\xi_m}^{\mathcal{B}_m^k}\sigma_k\left(x,\vec{\xi}\right)\right| \leq \sum_k C_{(1+2\epsilon)\beta^k}(1+|\xi_1|+\cdots+|\xi_m|)^{1+\epsilon-\rho\Sigma_{j=1}^m\beta_j^k|+\delta|1+2\epsilon|}.$$

(b) *Wesay*$\sigma_k \in L^{1+\epsilon}S^{1+\epsilon}_{\rho}(n,m)$*ifforeachtripleofmulti- indices*$\beta_1^k,\ldots,\beta_m^k$*thereexistsaconstant*$C_{\beta^k}$*suchthat*

$$\left\|\left|\sum_k \partial_{\xi_1}^{\mathcal{B}_1^k}\ldots\partial_{\xi_m}^{\mathcal{B}_m^k}\sigma_k\left(.,\vec{\xi}\right)\right|\right\|_{L^{1+\epsilon}(\mathbb{R}^n)} \leq \sum_k C_{\beta^k}(1+|\xi_1|+\cdots|\xi_m|)^{1+\epsilon-\rho\Sigma_{j=1}^m|\beta_j^k|}.$$

Here $S^{1+\epsilon}_{\rho,\delta}(n,m)$ is called the Hörmander's class.

A collection of competent compact historical steps have been shown via [52], that is, the pseudo- differential operators (ΨDOs) have played important roles in Harmonic Analysis and PDE. In 1966, Hörmander [18] determined the most general boundary problems for elliptic systems such that the estimates involving a loss of 1/2 derivative are valid. The problem was reduced to certain estimates for systems of ΨDOs on the boundary. Moreover, suitably extended versions were applied to hypoelliptic equations. Indeed, [20] introduced a wide class of ΨDOs related to hypoelliptic equations in order to study the existence and regularity of solutions. Hörmander [19] gave a nearly complete characterization of hypoelliptic second order differential operators with real C^∞ coefficients. Kannai [26] boiled the hypoellipticity of a degenerate elliptic boundary problem that satisfies the root condition, down to a problem concerning hypoellipticity of a ΨDO on the boundary. Hölder regularity of subellipticΨDOs could be applied to oblique derivative problem with boundary. Which similarly methodically discussed by Guan [17]. The study of ΨDOs can be applied to other significant operators. For example, making use of microlocal analysis and ΨDOs on $\mathbb{R}^3$, Fefferman and Kohn [15] obtained optimal Hölder estimates for$\bar{\partial}$, the Bergman projection, and the Szegö projection on bounded- pseudoconvex domains of finite type in $\mathbb{C}^2$. They also proved corresponding estimates for $\bar{\partial}_b$, and the Szegö projection on 3- D CR manifolds of finite type.

In 1970's, the local and global L^2 boundedness of ΨDOs were given by Hörmander [21], Kumano- Go [28], Calderón and Vaillancourt [8]. Later, applying almost orthogonality principle, Hounie [22] established the equivalence between L^2 bounds for ΨDOs and the indexes of Hörmander class. However, these results were established under the conditions of the classical Hörmander's class. In 1978, Coifman and Meyer studied a class of symbols with Dini type conditions. They [12, Theorem 9] presented a sufficient and necessary condition for the $L^{1+\epsilon}$ boundedness of ΨDOs. After that, Nagase [40] and Bourdaud [6] respectively showed the $L^{1+\epsilon}$ bounds in terms of two types of general kernels, which improved the results in [12]. Additionally, basing on a mild regularity assumption, Yabuta [51] demonstrated the relationship between ΨDOs and Calderón- Zygmund operators (CZOs). Furthermore, the bilinear analogy was obtained in [37].

The study of bilinear ΨDOs originated in the work of Coifman and Meyer [11], where they used bilinear ΨDO as a model of Calderón commutator. Although they are formally the natural bilinear extension of the ΨDOs, the bilinear ΨDOs do not always mimic the mapping behavior of linear ΨDOs. In [2], Bényi and Torres showed T_{σ_k} with $\sigma_k \in S^0_{0,0}(n,2)$ does not satisfy $L^2 \times L^2 \to L^1$ boundedly in the bilinear case. The authors in [3] also proved that if $\sigma_k \in S^0_{1,1}(n,2)$ or $\sigma_k \in S^0_{1,0;\pi/4}(n,2)$, then T_{σ_k} is unbounded from $L^{1+\epsilon} \times L^{1+2\epsilon}$ to $L^{1+\epsilon}$ for any $0 < \epsilon < \infty$. Moreover, it was stated that T_{σ_k} with $\sigma_k \in S^0_{1,1}(n,2)$ maps neither $L^{1+\epsilon} \times L^\infty \to L^{1+\epsilon}$ nor $L^\infty \times L^\infty \to BMO$ in [49]. These facts distinguished the bilinear or the multilinear ΨDOs from the linear ΨDOs.

The local decay estimatesfor the m- linear ΨDOT_{σ_k} and its commutator $T_{\sigma_k,\Sigma \mathrm{b}}$ are formulated.The appropriate local estimates for CZOs and square functions were given in [43]. We get a weighted mixed weak type inequality with A_1 and A_∞weights, which yields an endpoint multilinear A_1 theorem. In this direction, Muckenhoupt and Wheeden [39] first formulated the mixed weak type estimates for Hardy- Littlewood maximal function and the Hilbert transform on the real line although Sawyer [50] considered a more singular case. Soon after, Cruz- Uribe, Martell and Pérez [13] extended them to higher dimensions. But the involved weights u and v still belong to A_1. Recently, a remarkable improvement to $u \in A_1$ and $v \in A_\infty$ was done by Li, Ombrosi and Pérez [34], which was extended to the generalized maximal operators with Young functions and commutators of CZOs in [4, 5] and to the multilinear case in [35]. [52] results are corresponding to those in [34] but for the multilinear pseudo- differential operators. We obtain the raised rare sharp weighted estimates for T_{σ_k} and $T_{\sigma_k,\Sigma \mathrm{b}}$. This improves the classical weighted inequalities [38]. The [52] method used is dyadic analysis, which is different from the known approach.

Now we state the main results.

Theorem 1.2. *Assumethat*$\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$*with*$\rho,\delta \in [0,1]$*and*$< mn(\rho-1)$. *Let*Q_k*beacubeand*$(f_k)_j \in L^\infty_{1+\epsilon}(\mathbb{R}^n)$*suchthat*$\mathrm{supp}\big((f_k)_j\big) \subset Q_k$*for*$1 \le j \le m$. *Thenthereexistconstants*$(1+2\epsilon),(1+\epsilon),\epsilon \ge 0$*suchthat*

$$|\{x \in Q_k : |T_{\sigma_k}(\overrightarrow{f_k})(x)| > (1+\epsilon)\mathcal{M}(\overrightarrow{f_k})(x)\}| \le (1+\epsilon)e^{-(1+2\epsilon)(1+\epsilon)}|Q_k|, \epsilon \ge 0.$$

For the operator T_{σ_k} and locally integrable functions $\mathrm{b} = (b_1,\dots,b_m)$, we define the m- linear commutator of T_{σ_k} as follows:

$$T_{\sigma_k,\Sigma \mathrm{b}}(\overrightarrow{f_k})(x) = \sum_{j=1}^{m}\sum_{k} \quad [b_j, T_{\sigma_k}]_j(\overrightarrow{f_k})(x) ,$$

where each term is the commutator of b_j and T_{σ_k} in the j- th entry of T_{σ_k} , that is

$$[b_j, T_{\sigma_k}]_j(\overrightarrow{f_k})(x) = b_j(x)T_{\sigma_k}(\overrightarrow{f_k})(x) - T_{\zeta\Gamma}\big((f_k)_1,\dots,b_j(f_k)_j,\dots,(f_k)_m\big)(x) .$$

This kind of commutators was introduced by Pérez and Torres [48] for the m- linear Calderón- Zygmund operators and the weighted strong and sharp weak- type estimates were obtained in [30] modeled by the approach in [45]. The more complicated iterated commutators were first formulated and studied in [46]. A pointwise estimate for commutators via the multilinear $L(\log L)$- maximal operators (see Section 2.2 below) is essential to show that the commutators are not weak type (1, 1). The original idea came from [45] and will be used again to establish Proposition 6.2.

In terms of commutators, we have the following subexponential decay estimates.

Theorem 1.3. *Assumethat*$\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$*with*$\rho,\delta \in [0,1]$*and*$(1+\epsilon) < mn(\rho-1)$. *Let*Q_k*beacubeand*$(f_k)_j \in L^\infty_{1+\epsilon}(\mathbb{R}^n)$*suchthat*$\mathrm{supp}\big((f_k)_j\big) \subset Q_k$*for*$1 \le j \le m$. *If***b** $\in BMO^m$, *thenthereareconstants*$(1+2\epsilon),(1+\epsilon),\epsilon \ge 0$*suchthat*

$$|\{x \in Q_k : |T_{\sigma_k,\Sigma \mathrm{b}}(\overrightarrow{f_k})(x)| > (1+\epsilon)\mathcal{M}(M(f_k)_1, \dots, M(f_k)_m)(x)\}| \leq (1+\epsilon)e^{-\sqrt{\frac{(1+2\epsilon)(1+\epsilon)}{||\mathrm{b}||_{BM^\circ}}}}|Q_k|, \quad \epsilon \geq 0,$$

where $||\mathrm{b}||_{BMO} := \sup_{1\leq j\leq m} ||b_j||_{BMO}$.

We obtain the weighted mixed weak type inequality as follows.

Theorem 1.4. *Assume that* $\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$ *with* $\rho, \delta \in [0,1]$ *and* $(1+\epsilon) < mn(\rho - 1)$. *Let* $\vec{\omega} = (\omega_1, \dots, \omega_m)$ *and* $\mu = \prod_{i=1}^m \omega_i^{1/m}$. *If* $\vec{\omega}$ *and* v *satisfy*

(1) $\vec{\omega} \in A_{\vec{1}}$ *and* $\mu v^{1/m} \in A_\infty$, *or*

(2) $\omega_1, \dots, \omega_m \in A_1$ *and* $v \in A_\infty$,

then there holds

$$\|\sum_k \frac{T_{\sigma_k}(\overrightarrow{f_k})}{v}\|_{L^{\frac{1}{m},\infty}\left(\mu v^{\frac{1}{m}}\right)} \leq (1+\epsilon)\sum_k \prod_{i=1}^m ||(f_k)_i||_{L^1(\omega_j)}.$$

Finally, we present the sharp weighted estimates for T_{σ_k} and $T_{\sigma_k,\Sigma \mathrm{b}}$.

Theorem 1.5. *Assume that* $\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$ *with* $\rho, \delta \in [0,1]$ *and* $(1+\epsilon) < mn(\rho - 1)$. *Let* $\frac{1}{1+\epsilon} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 < p_j < \infty, i = 1, \dots, m$. *If* $\vec{\omega} \in A_{\overrightarrow{1+\epsilon}}$ *and* $\mathrm{b} \in BMO^m$, *then we have*

$$||\sum_k T_{\sigma_k}||_{L^{p_1}(w_1)\times\cdots\times L^{p_m}(\omega_m)\to L^{1+\epsilon}(v_{\vec{\omega}})} \leq c_{n,\overrightarrow{1+\epsilon}} \sum_k \mathcal{N}_{weak}[\vec{\omega}]_{A_\beta}^{\beta^k(\overrightarrow{1+\epsilon})}, \qquad (1.1)$$

and

$$||\sum_k T_{\sigma_k,\Sigma \mathrm{b}}||_{L^{p_1}(\omega_1)\times\cdots\times U^m(w_m)\to L^{1+\epsilon}(v_{\vec{\omega}})} \leq c_{n,\overrightarrow{1+\epsilon}} \sum_k \mathcal{N}_{weak}||\mathrm{b}||_{BMO}[\vec{\omega}]_{A_{\beta^k}}^{2\beta^k(\overrightarrow{1+\epsilon})}, \qquad (1.2)$$

where $\mathcal{N}_{weak} = \|T_{\sigma_k}\|_{L^1\times\cdots\times L^1\to L^{\frac{1}{m},\infty}}$, *and* $\beta^k(\overrightarrow{1+\epsilon}) = \max_{1\leq i\leq m}\left\{1, \frac{p_i'}{1+\epsilon}\right\}$.

We present some definitions and lemmas that will be used later. The purpose of this paper is to study the unweighted endpoint weak type inequality, which will play a main role in the local decay estimates and sharp weighted estimates. Then the mixed weak type inequality is proved based on the sparse domination theorem. The proof of local decay estimates is presented. Finally, we discuss the sharp weighted inequalities.

II. Preliminaries

2.1. Multiple Weights

The multilinear maximal function is defined by

$$\mathcal{M}(\overrightarrow{f_k})(x) := \sup_{Q_k \ni x} \sum_k \prod_{i=1}^m \frac{1}{|Q_k|}\int_{Q_k} |(f_k)_i(y)|dy.$$

The related multiple weights are introduced in [30] as follows.

Definition 2.1. *Let* $1 \leq p_1, \dots, p_m < \infty$. *Given* $\vec{\omega} = (\omega_1, \dots, \omega_m)$, *where each* ω_j *is a nonnegative and measurable function on* $\mathbb{R}^n$, *we say that* $\vec{\omega} \in A_{\overrightarrow{1+\epsilon}}$ *if*

$$[\vec{\omega}]_{A_{\beta^k}} := \sup_{\mathrm{Q_k}} \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} v_{\vec{\omega}}\right) \prod_{i=1}^m \left(\frac{1}{|Q_k|}\int_{Q_k} \omega_i^{1-p_1'}\right)^{1+\epsilon/p_i'} < \infty, \qquad (2.1)$$

where $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{1+\epsilon/p_j}$. *When* $p_j = 1$, $\left(\frac{1}{|Q_k|}\int_{Q_k} \omega_i^{1-p_1'}\right)^{1+\epsilon/p_i'}$ *is understood as* $\left(\inf_{\mathrm{Q_k}} \omega_j\right)^{-1}$.

If $m = 1$, the multiple $A_{\overrightarrow{1+\epsilon}}$ weights coincide with the classical Muckenhoupt $A_{1+\epsilon}$ weights. In the linear case, the A_1 condition is given by

$$[\omega]_{A_1} := \sup_{\mathrm{x}\in\mathbb{R}^\mathrm{n}} \frac{M\omega(x)}{\omega(x)} < \infty.$$

Also we define the A_∞ constant by

$$[\omega]_{A_\infty} := \sup_{\mathrm{Q_k}} \sum_k \frac{1}{w(Q_k)}\int_{Q_k} M(\omega 1_{Q_k})dx.$$

Lemma 2.2([24]). *Let* $\omega \in A_\infty$. *Then there holds*

$$\sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} \omega(x)^{r_\omega}dx\right)^{1/r_\omega} \leq \sum_k \frac{2}{|Q_k|}\int_{Q_k} \omega(x)dx,$$

where $r_\omega = 1 + \frac{1}{v_n[\omega]_{A_\infty}}$ *and* $v_n = 2^{(n+11)}$.

The characterizations of multiple weights were given in [30] and [10].

Lemma 2.3. *Let* $\frac{1}{1+\epsilon}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ *with* $1\leq p_1$, , $p_m<\infty$, *and* $p_0=\min\{p_i\}_i$. *Then the following statements hold*:

(1) $A_{r_1\overrightarrow{1+\epsilon}}\subseteq A_{r_2\overrightarrow{1+\epsilon}}$, *for any* $1/p_0\leq r_1<r_2<\infty$.

(2) $A_{\overrightarrow{1+\epsilon}}=\bigcup_{1/p_0-1\leq\epsilon<0}A_{(1+\epsilon)\overrightarrow{(1+\epsilon)}}$.

(3) $\vec{\omega}\in A_{\overrightarrow{1+\epsilon}}$ *if and only if*

$v_{\vec{\omega}}\in A_{m(1+\epsilon)}$ *and* $w_i^{1-p_i{}'}\in A_{mp_i'}$, $i=1,\dots,m$,

where $\omega_i^{1-p_i{}'}\in A_{mp_i'}$ *is understood as* $\omega_i^{1/m}\in A_1$, *if* $p_i=1$.

2.2. Orlicz Spaces

A function $\phi_k:[0,\infty)\to[0,\infty)$ is said to be a Young function, if ϕ_k is continuous, convex, increasing function such that $\phi_k(0)=0$ and $\phi_k(1+\epsilon)/1+\epsilon\to\infty$ as $\epsilon\to\infty$.

The ϕ_k- norm of a function f_k over a set E with finite measure is defined by

$$||f_k||_{(\phi_k,E)}=\inf\{\lambda>0;\frac{1}{|E|}\int_E\sum_k\phi_k\left(\frac{|f_k(x)|}{\lambda}\right)dx\leq 1\}.$$

Then we define the multilinear $L(\log L)$- maximal operators:

$$\mathcal{M}_{L(\log L)}(\overrightarrow{f_k})(x):=\sup_{Q_k\ni x}\sum_k\prod_{i=1}^{m}\|(f_k)_i||_{L(\log L),Q_k}.$$

and

$$\mathcal{M}_{L(\log L)}^{j}(\overrightarrow{f_k})(x):=\sup_{Q_k\ni x}\sum_k||(f_k)_j||_{L(\log L),Q_k}\prod_{i\neq j}\langle|(f_k)_i|\rangle_{Q_k}.$$

The generalized Hölder's inequality is as follows.

Lemma 2.4([41]). *If* A, B *and* C *are Young functions satisfying*

$$A^{-1}(1+\epsilon)C^{-1}(1+\epsilon)\leq B^{-1}(1+\epsilon)\text{ , for any }\epsilon\geq 0,$$

then for all functions f_k,g_k *and any measurable set* $E\subset\mathbb{R}^n$, *it holds*

$$\|\sum_k f_kg_k\|_{B,E}\leq 2\sum_k||f_k||_{\mathrm{A,E}}||g_k||_{C,E}.$$

Here $A^{-1}(x)=\inf\{y\in\mathbb{R}:A(y)>x\}$ *for* $0\leq x\leq\infty$, *where* $\inf\ \emptyset=\infty$.

A fundamental fact about *BMO* function is John- Nirenberg theorem.

Lemma 2.5([16]). *For any* $b\in BMO$ *and cube* Q_k, *there holds that*

$$|\{x\in Q_k:|b(x)-b_{Q_k}|>(1+2\epsilon)\}|\leq e^{-\frac{A(1+2\epsilon)}{||b||_{BMO}}+1}|Q_k|,$$

where $A=(2^ne)^{-1}$.

Beyond that, some other inequalities involving *BMO* function are necessary for us.

Lemma 2.6([16, 47]). *Let* $\epsilon\geq 0$ *and* $b\in BMO$. *Then there is a constant* $\epsilon\geq 0$ *independent of b such that the following inequalities hold*

$$(1+\epsilon)^{-1}(f_k)_{Q_k}|\sum_k f_k|\leq\sum_k\|f_k\|_{L(\log L),Q_k}\leq(1+\epsilon)\sum_k\left((f_k)_{Q_k}|f_k|^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}};\tag{2.2}$$

$$\sup_{Q_k}\sum_k\left((f_k)_{Q_k}|b-b_{Q_k}|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}}\leq(1+\epsilon)||b||_{BMO};\tag{2.3}$$

$$|||b-b_{Q_k}|^{1+\epsilon}||_{\exp L^{\frac{1}{1+\epsilon}},Q_k}\leq(1+\epsilon)||b||_{BMO}^{1+\epsilon};\tag{2.4}$$

$$(f_k)_{Q_k}\left|\sum_k(f_k)_1\dots(f_k)_kg_k\right|$$
$$\leq(1+\epsilon)\sum_k\prod_{i=1}^{k}\|(f_k)_i\left||_{\exp L^{s_i}Q_k}\right||g_k||_{L(\log L)^{\frac{1}{1+\epsilon}},Q_k},\tag{2.5}$$

where $\frac{1}{1+\epsilon}=\frac{1}{s_1}+\cdots+\frac{1}{s_{k_0}}$ *with* $s_1,\dots,s_{k_0}\geq 1$.

We will also employ several times the Kolmogorov's inequality

$$\sum_k\|f_k\|_{L\left(Q_k,\frac{dx}{|Q_k|}\right)}(1+\epsilon)\leq c_{1+\epsilon,1+2\epsilon}\sum_k\|f_k\|_{L^{1+2\epsilon,\infty}\left(Q_k,\frac{dx}{|Q_k|}\right)},\qquad 0\leq\epsilon<\infty.\tag{2.6}$$

2.3. Dyadic Lattics

Denote by $\ell(Q_k)$ the sidelength of the cube Q_k. Given a cube $(Q_k)_0 \subset \mathbb{R}^n$, let $D((Q_k)_0)$ denote the set of all dyadic cubes with respect to $(Q_k)_0$, that is, the cubes obtained by repeated subdivision of $(Q_k)_0$ and each of its descendants into 2^n congruent subcubes.

Definition 2.7. *Acollection, $\mathcal{D}$ofcubesissaidtobeadyadiclatticeifitsatisfies* (1) *If$Q_k \in D$, theneachchildof$Q_k$isin$D$aswell*;
(2) *Foreverycubes$Q_k', Q_k'' \in \mathcal{D}$, thereexists$Q_k \in \mathcal{D}$suchthat$Q_k', Q_k'' \in D(Q_k)$* ; (3) *Foreverycompactset$K \subset \mathbb{R}^n$, thereexistsacube$Q_k \in \mathcal{D}$containing$K$.*

Definition 2.8. *AsubsetSofadyadiclatticissaidtobeη- sparse,* $0 < \eta < 1$, *ifforevery$Q_k \in S$, thereexistsameasurableset$E_{Q_k} \subset Q_k$such*that $|E_{Q_k}| \geq \eta |Q_k|$, *andthesets$\{E_{Q_k}\}_{Q_k \in S}$arepairwisedisjoint.*

The following lemma given by Lerner et al. [31, Remark 2.2] enables us to clearly understand the structure of dyadic lattics.

Lemma 2.9. *Thereare3^ndyadiclattices$\mathcal{D}^{(j)}$suchthatforeverycube$Q_k \subset \mathbb{R}^n$, thereisacube$R \in D^{(J)}$forsome$j$, forwhich$3Q_k \subset R$* and $|R| \leq 9^n |Q_k|$.

An estimate by oscillations over a sparse family will play a main role in the proof of local estimates for commutators.

Lemma 2.10([31]). *Let$D$beadyadiclatticeandlet$S \subseteq \mathcal{D}$beaη- sparse*family*Assumethat$b \in L^1_{loc}$. Thenthereexistsa$\frac{\eta}{2(1+\eta)}$- sparsefamily$\tilde{S} \subset \mathcal{D}$suchthat$S \subset \tilde{S}$andforevery$Q_k \in \tilde{S}$,*

$$|\sum_k b(x) - b_{Q_k}| \leq 2^{(n+2)} \sum_{R \in \tilde{S}, R \subset Q_k} \sum_k \langle |b - b_{Q_k}| \rangle 1_R(x), \qquad a.e. x \in Q_k.$$

2.4. A Local Mean Oscillation Formula

By a median value of a measurable function f_k on a set Q_k we mean a possibly non- unique, real number $m_{f_k}(Q_k)$ such that

$$\max\{|\{x \in Q_k: f_k(x) > m_{f_k}(Q_k)\}|, |\{x \in Q_k: f_k(x) < m_{f_k}(Q_k)\}|\} \leq |Q_k|/2.$$

The decreasing rearrangement of a measurable function f_k on $\mathbb{R}^n$ is defined by

$$f_k^*(1+\epsilon) = \inf\{\epsilon > -\frac{1}{2} : |\{x \in 1\mathrm{R}^n : |f_k(x)| > 1 + 2\epsilon\}| < 1 + \epsilon\}(0 \leq \epsilon < \infty).$$

The local mean oscillation of f_k is

$$\omega_\lambda(f_k; Q_k) = \inf_{c \in \mathbb{R}} \left((f_k - (1+\epsilon))1_{Q_k}\right)^* (\lambda |Q_k|)(0 < \lambda < 1).$$

Given a cube $(Q_k)_0$, the local sharp maximal function is defined by

$$M^{\#}_{\lambda;(Q_k)_0} f_k(x) = \sup_{x \in Q_k \subset (Q_k)_0} \omega_\lambda(f_k; Q_k).$$

Observe that for any $\delta > 0$ and $0 < \lambda < 1$

$$\left|\sum_k m_{f_k}(Q_k)\right| \leq \sum_k \left(f_k 1_{Q_k}\right)^* \left(\frac{|Q_k|}{2}\right)$$

$$and \sum_k \left(f_k 1_{Q_k}\right)^* (\lambda |Q_k|) \leq \sum_k \left(\frac{1}{\lambda |Q_k|} \int_{Q_k} |f_k|^\delta dx\right)^{\frac{1}{\delta}} \qquad (2.7)$$

The following theorem was proved by Hytönen [23, Theorem 2.3] in order to improve Lerner's formula given in [29] by getting rid of the local sharp maximal function.

Lemma 2.11. *Let$f_k$beameasurablefunctiononR^nandlet$(Q_k)_0$beafixedcube. Thenthereexistsa* (*possiblyempty*) *sparsefamily$S$ofcubes$Q_k \in D((Q_k)_0)$*

$$\left|\sum_k f_k(x) - m_{f_k}((Q_k)_0)\right| \leq 2 \sum_{Q_k \in S} \sum_k \omega_{2^{-(n+2)}}(f_k; Q_k) 1_{Q_k}(x), \qquad a.e. x \in (Q_k)_0.$$

III. Endpoint Weak Type Estimates

We will prove Theorem 1.4 in the unweighted setting. Before doing it, we present an unweighted strong type inequality, which is a foundation in the proof of endpoint estimates.

Proposition 3.1 (see [52]). *Let$\frac{1}{1+\epsilon} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$with$0 \leq \epsilon \leq 1and1 < p_1, \ldots, p_m < \infty$. Let$0 \leq \delta, \rho \leq 1$. If$(1+\epsilon) < mn(\rho - 1)/1 + \epsilon$, theneach$m$- linearpseudo- differentialoperator$T_{\sigma_k}$with$\sigma_k \in L^\infty S^{1+\epsilon}_\rho(n,m)$isboundedfrom$L^{p_1}(R^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)toL^{1+\epsilon}(\mathbb{R}^n)$.*

Proof. Our strategy is to use a Littlewood- Paley decomposition of the symbol. Let $\{\psi_j\} \in C^\infty_{1+\epsilon}$ be a Littlewood- Paley partition of unity with

$$\mathrm{supp}(\psi_0) \subset \{\vec{\xi} \in \mathbb{R}^{mn}: |\vec{\xi}| \leq 2\},$$
$$\mathrm{supp}(\psi_j) \subset \{\vec{\xi} \in \mathbb{R}^{mn}: 2^{j-1} \leq |\vec{\xi}| \leq 2^{j+1}\},$$
$$\sum_{j=0}^{\infty} \psi_j\left(\vec{\xi}\right) = 1, \qquad \forall \vec{\xi} \in \mathbb{R}^{mn}.$$

One also has, for all multi- indices $(1+2\epsilon)$,

$$\left|\partial_{\vec{\xi}}^{1+2\epsilon}\psi_0(\vec{\xi})\right| \leq C_{1+2\epsilon,N}\left(1+|\vec{\xi}|\right)^{-N}, \qquad \forall N \geq 0, \tag{3.1}$$

and

$$\left|\partial_{\vec{\xi}}^{1+2\epsilon}\psi_j(\vec{\xi})\right| \leq C_{(1+2\epsilon)}2^{-j(|1+2\epsilon|)}, \qquad \forall j \geq 1. \tag{3.2}$$

Hence, we get

$$T_{o-}((f_k)_1, \dots, (f_k)_m)(x) = \sum_{j=0}^{\infty}\sum_k \quad T_{o_j^-}((f_k)_1, \dots, (f_k)_m)(x),$$

where $(\sigma_k)_j(x,\vec{\xi}) = \sigma_k(x,\vec{\xi})\psi_j(\vec{\xi})$. Furthermore, using the Leibniz rule, the definition of $L^\infty S_\rho^{1+\epsilon}(n,m)$, and the inequalities (3.1) and (3.2), we realize that

$$\left|\sum_k \quad \partial_{\vec{\xi}}^{\mathrm{B}}(\sigma_k)_0(\vec{\xi})\right| \leq C_{1+2\epsilon,N}\left(1+\sum_{i=1}^{m}|\xi_i|\right)^{-N} \quad , \qquad \forall N \geq 0, \tag{3.3}$$

and

$$\left|\sum_k \quad \partial_{\vec{\xi}}^{1+2\epsilon}(\sigma_k)_j(\vec{\xi})\right| \leq C_{(1+2\epsilon)}2^{j\left(1+\epsilon-\rho(|1+2\epsilon|)\right)}, \qquad \forall j \geq 1. \tag{3.4}$$

We rewrite

$$T_{(\sigma_k)_j}((f_k)_1, \dots, (f_k)_m)(x) = \int_{\mathrm{R}^{mn}}\sum_k \quad K_j(x,\vec{y})\prod_{j=1}^{m}(f_k)_j\left(x-y_j\right)d\vec{y},$$

where

$$K_j(x,f_k) = \int_{\mathrm{R}^{mn}}\sum_k \quad (\sigma_k)_j\left(x,\vec{\xi}\right)e^{2\pi i\vec{y}\vec{\xi}}d\vec{\xi} = \widehat{(\sigma_k)}_j(x,-\vec{y}).$$

Integrating by parts, we obtain that

$$|K_0(x,\vec{y})| \lesssim (1+|\vec{y}])^{-(1+\epsilon)}, \qquad \forall (1+\epsilon) \in \mathbb{N}_+.$$

Thus, it yields that for $\epsilon > mn-1$

$$\left|\sum_k \quad T_{(\sigma_k)_0}((f_k)_1, \dots, (f_k)_m)(x)\right| \leq \int_{\mathrm{R}^{mn}}\sum_k \quad \frac{\prod_{j=1}^{m}|(f_k)_j(x-y_j)|}{(1+|\vec{y}|)^{1+\epsilon}}d\vec{y}$$
$$= \int_{B(0,1)^m}\sum_k \quad + \sum_{k=0}^{\infty}\int_{B(0,2^{k+1})^m \setminus B(0,2^k)^m} \quad \sum_k \quad \dots \lesssim \mathcal{M}((f_k)_1, \dots, (f_k)_m)(x), \tag{3.5}$$

which implies that

$$||\sum_k \quad T_{(\sigma_k)_0}((f_k)_1, \dots, (f_k)_m)||_{L^{1+\epsilon}(\mathrm{R}^n)} \lesssim \sum_k \quad ||\mathcal{M}(\overrightarrow{f_k})||_{L^{1+\epsilon}(\mathrm{R}^n)} \leq \sum_k \quad \prod_{j=1}^{m}||(f_k)_j||_{L^{p_j}(\mathrm{R}^n)}.$$

In order to control $T_{(\sigma_k)_j}$, we introduce a class of weight functions by setting

$$\omega_j(\vec{y}) = \begin{cases} 2^{-mjn\rho/1+\epsilon}, & |\vec{y}| \leq 2^{-j\rho}, \\ 2^{-j\rho(mn/1+\epsilon-l)}|\vec{y}|^l, & |\vec{y}| > 2^{-j\rho}, \end{cases}$$

where l is a fixed constant satisfying $(1+\epsilon)l > mn$. Then we have

$$\int_{\mathrm{R}^{mn}}\omega_j(\vec{y})^{-(1+\epsilon)}d\vec{y} = \int_{|\vec{y}|\leq 2^{-j\rho}}2^{mjn\rho}\,d\vec{y} + \int_{|\vec{y}|>2^{-j\rho}}2^{j\rho(mn-(1+\epsilon)l)}\frac{d\vec{y}}{|\vec{y}|^{(1+\epsilon)l}} \lesssim 1.$$

Additionally, it follows from Hausdorff- Young's theorem and (3.4) that

$$\left(\int_{R^{mn}} |K_j(x,\vec{y})|^{p'}\omega_j(\vec{y})^{p'}d\vec{y}\right)^{1/p'}$$
$$\leq 2^{-mjn\rho/1+\epsilon}\left(\int_{R^{2n}} |K_j(x,\vec{y})|^{p'}d\vec{y}\right)^{1/p'} + 2^{-j\rho(mn/1+\epsilon-l)}\left(\int_{R^{mn}} |y^{\to\propto}K_j(x,\vec{y})|^{p'}d\vec{y}\right)^{1/p'}$$
$$\leq 2^{-mjn\rho/1+\epsilon}\left(\int_{R^{mn}}\sum_k |(\sigma_k)_j(x,\vec{\xi})|^{1+\epsilon}d\vec{\xi}\right)^{1/1+\epsilon}$$
$$+ 2^{-j\rho(mn/1+\epsilon-l)}\left(\int_{R^{mn}}\sum_k |\partial_{\vec{\xi}}^{1+2\epsilon}(\sigma_k)_j(x,\vec{\xi})|^{1+\epsilon}d\vec{\xi}\right)^{1/1+\epsilon}$$
$$\leq 2^{-mjn\rho/1+\epsilon}\left(\int_{|\vec{\xi}|\simeq 2^j} 2^{j(1+\epsilon)^2}d\vec{\xi}\right)^{1/1+\epsilon} + 2^{-j\rho(mn/1+\epsilon-l)}\left(\int_{|\vec{\xi}|\simeq 2^j} 2^{j(1+\epsilon-\rho l)(1+\epsilon)}d\vec{\xi}\right)^{1/1+\epsilon}$$
$$\simeq 2^{j(1+\epsilon-mn(J^3-1)/1+\epsilon)},$$

where the multi- index $(1+2\epsilon)\in\mathbb{N}^n\times\cdots\times\mathbb{N}^n$ with $|1+2\epsilon|=l$. Collecting the above estimates, we obtain that

$$\|T_{(\sigma_k)_j}((f_k)_1,\dots,(f_k)_m)\|_{L^{1+\epsilon}(R^n)}^{1+\epsilon} = \int_{R^n}\left|\int_{R^{mn}}\sum_k K_j(x,\vec{y})\prod_{j=1}^m (f_k)_j(x-y_j)d\vec{y}\right|^{1+\epsilon}dx$$
$$\leq \int_{R^n}\left(\int_{R^{mn}}|K_j(x,\vec{y})|^{p'}\omega_j(\vec{y})^{p'}d\vec{y}\right)^{1+\epsilon/p'}\left(\int_{R^{mn}}\sum_k \frac{\prod_{j=1}^m |(f_k)_j(x-y_j)|^{1+\epsilon}}{\omega_j(\vec{y})^{1+\epsilon}}d\vec{y}\right)dx$$
$$\leq 2^{j(1+\epsilon(1+\epsilon)-mn(\epsilon))}\int_{R^{mn}}\left(\int_{R^n}\sum_k\prod_{j=1}^m |(f_k)_j(x-y_j)|^{1+\epsilon}dx\right)\frac{d\vec{y}}{\omega_j(\vec{y})^{1+\epsilon}}$$
$$\leq 2^{j(1+\epsilon(1+\epsilon)-mn(\epsilon))}\sum_k\prod_{j=1}^m \|(f_k)_j\|_{L^{1+\epsilon}(R^n)}^{1+\epsilon}.$$

Finally, we deduce that

$$\|T_{\sigma_k}((f_k)_1,\dots,(f_k)_m)\|_{L^{1+\epsilon}(R^n)} \leq \sum_{j=0}^{\infty}\sum_k \|T_{(\sigma_k)_j}((f_k)_1,\dots,(f_k)_m)\|_{L^{\rho}(R^n)}$$
$$\lesssim \left(1+\sum_{j=1}^{\infty}2^{j(1+\epsilon-mn(\rho-1)/1+\epsilon)}\right)\sum_k\prod_{j=1}^m \|(f_k)_j\|_{L^{1+\epsilon}(R^n)} \lesssim \sum_k\prod_{j=1}^m \|(f_k)_j\|_{L^{1+\epsilon}(R^n)},$$

which is provided by $(1+\epsilon) < mn(\rho-1)/1+\epsilon$.

Proposition 3.2 (see [52]). *Let* $0\leq\rho\leq 1$. *If* $(1+\epsilon) < mn(\rho-1)/2$, *theneachm- linearpseudo- differentialoperator* T_{σ_k} *with* $\sigma_k \in L^\infty S_\rho^{1+\epsilon}(n,m)$ *isboundedfrom* $L^\infty(\mathbb{R}^n)\times\cdots\times L^\infty(\mathbb{R}^n)$ *to* $L^\infty(R^n)$.

Proof. The proof follows exactly the same scheme of that in Proposition 3.1 with slight modifications. We here only point out the difference. Without loss of generality, we may assume that $\|(f_k)_j\|_{L^\infty(R^n)} = 1, j = 1,\dots,m$. The inequality (3.5) indicates that

$$\left\|\sum_k T_{(\sigma_k)_0}((f_k)_1,\dots,(f_k)_m)\right\|_{L^\infty(R^n)} \leq \sum_k\prod_{j=1}^m \|(f_k)_j\|_{L^\infty(R^n)}.$$

Moreover, applying Cauchy- Schwartz inequality, Hausdorff- Young's theorem and (3.4), we obtain that

$$||\sum_k T_{(\sigma_k)_j}((f_k)_1,\dots,(f_k)_m)||_{L^\infty(\mathrm{R}^n)} \le \sum_k \prod_{j=1}^m ||(f_k)_j||_{L^\infty(\mathrm{R}^n)} \int_{\mathrm{R}^{mn}} |K_j(x,\vec{y})|d\vec{y}$$
$$= \int_{|\vec{y}|\le 2^{-j\rho}} + \int_{|\vec{y}|>2^{-j\rho}} |K_j(x,\vec{y})|d\vec{y}$$
$$\le \left(\int_{|\vec{y}|\le 2^{-j\rho}} d\,\vec{y}\right)^{1/2}\left(\int_{\mathrm{R}^{mn}} |K_j(x,\vec{y})|^2 d\vec{y}\right)^{1/2}$$
$$+\left(\int_{|\vec{y}|>2^{-j\rho}} |\vec{y}|^{-2l} d\vec{y}\right)^{1/2} (\int_{\mathrm{R}^{mn}} |y]^{2l}|K_j(x,y)|^2 d\vec{y})^{1/2}$$
$$\lesssim 2^{-j\rho mn/2}\cdot 2^{j(1+\epsilon+mn/2)} + 2^{-j\rho(mn/2-l)}\cdot 2^{j(mn/2+1+\epsilon-\rho l)} \simeq 2^{j(1+\epsilon-mn(\rho-1)/2)}.$$

The condition $(1+\epsilon) < mn(\rho-1)/2$ guarantees the series converges.
We restate Theorem 1.4 in the unweighted case as follows.
Theorem 3.3 (see [52]). *Assumethat* $\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$ *with* $\rho,\delta\in[0,1]$ *and* $(1+\epsilon) < mn(\rho-1)$. *Thenthereexistsaconstant* $\epsilon \ge 0$ *suchthatforany* $\lambda > 0$

$$|\{x\in\mathbb{R}^n: T_{\sigma_k}(\overrightarrow{f_k})(x) > \lambda^m\}| \le \frac{1+\epsilon}{\lambda}\sum_k \prod_{i=1}^m ||(f_k)_i||^{1/m}_{L^1(\mathrm{R}^n)}.$$

Proof. We may assume that each $||(f_k)_j||_{L^1(\mathrm{R}^n)} = 1$. Applying Calderón- Zygmund decomposition on each $(f_k)_j$ at height $\lambda^{1/m}$, we obtain a disjoint collection of cubes $\left\{(Q_k)_j^{((k_0)j)}\right\}$ and $(f_k)_j = (g_k)_j + b_j$ with $b_j = \sum_{(k_0)_j} b_j^{((k_0)j)}$, where

$$\int_{\mathrm{R}^n} b_j^{((k_0)j)}\,dx = 0, supp\; b_j^{((k_0)j)} \subset (Q_k)_j^{((k_0)j)}, and\; ||b_j^{((k_0)j)}||_{L^1} \lesssim \lambda^{1/m}|(Q_k)_j^{((k_0)j)}|.$$

Moreover, it holds

$$|\bigcup_{k_j}(Q_k)_j^{((k_0)j)}| \le \lambda^{-1/m}, and\; ||(g_k)_j||_{L^{1+\epsilon}} \le \lambda^{\overline{\frac{1}{m(1+\epsilon)'}}}, \qquad 0\le\epsilon\le\infty.$$

Now, in order to show

$$|\{x \in]\mathrm{R}^n; T_{\sigma_k}((f_k)_1,\dots,(f_k)_m)(x) > \lambda\}| \le (1+\epsilon)\lambda^{1/m},$$

we see in virtue of symmetry that we have only to show

$$|\{x\in\mathrm{R}^n; T_{\sigma_k}((g_k)_1,\dots,(g_k)_m)(x) > \lambda/2^m\}| \lesssim \lambda^{1/m},$$
$$|\{x\in\mathrm{R}^n; T_{\sigma_k}(b_1,(g_k)_2,\dots,(g_k)_m)(x) > \lambda/2^m\}| \lesssim \lambda^{1/m},$$
$$|\{x\in\mathrm{R}^n; T_{\sigma_k}(b_1,\dots,b_\ell,(g_k)_{\ell+1},\dots,(g_k)_m)(x) > \lambda/2^m\}| \lesssim \lambda^{1/m}.(\ell = 2,\dots,m).$$

The first one is easy to deal with, since it follows from Proposition 3.1 that

$$\left|\left\{x\in\mathrm{R}^n; T_{\sigma_k}((g_k)_1,\dots,(g_k)_m)(x) > \frac{\lambda}{2^m}\right\}\right| \lesssim |\lambda^{-(1+2\epsilon)}||T_{\sigma_k}((g_k)_1,\dots,(g_k)_m)||^{1+2\epsilon}_{L^{1+2\epsilon}(\mathrm{R}^n)}$$
$$\lesssim \lambda^{-(1+2\epsilon)}\prod_{j=1}^m ||(g_k)_j||^{1+2\epsilon}_{L^{(1+2\epsilon)_j}(\mathrm{R}^n)} \lesssim \lambda^{-(1+2\epsilon)}\prod_{j=1}^m \lambda^{1+2\epsilon/m(1+2\epsilon)'_j} = \lambda^{-1/m}.$$

Since $|\cup_{j=1}^m \cup_{k_j} 2\,(Q_k)_j^{((k_0)j)}| \le \lambda^{-1/m}$, it suffices to show

$$|\{x \notin \bigcup_{k_1} 2\,(Q_k)_1^{((k_0)_1)}; T_{\sigma_k}(b_1,(g_k)_2,\dots,(g_k)_m)(x) > \lambda/2^m\}| \lesssim \lambda^{-1/m},$$

and

$$|\{x \notin \bigcup_{j=1}^l\bigcup_{k_j} 2\,(Q_k)_j^{((k_0)j)}; T_{\sigma_k}(b_1,\dots,b_{f_k},(g_k)_{l+1},\dots,(g_k)_m)(x) > \lambda/2^m\}| \lesssim \lambda^{-1/m}.$$

In view of the decomposition of b_j, it is enough to demonstrate that

$$\int_{\mathrm{R}^n\backslash 2(Q_k)_1^{((k_0)_1)}} \sum_k |T_{\sigma_k}\left(b_1^{((k_0)_1)},(g_k)_2,\dots,(g_k)_m\right)(x)|dx \lesssim \lambda\sum_k |(Q_k)_1^{((k_0)_1)}|, \qquad (3.6)$$

and

$$\int_{\mathrm{R}^n \setminus \cup_{j,(k_0)_j} 2(Q_k)_j^{(k_j)}} \sum_k \quad | T_{\sigma_k}(b_1, \dots, b_\ell, (g_k)_{\ell+1}, \dots, (g_k)_m)(x)|^{1/l} dx \lesssim \lambda^{\frac{1}{\ell}-\frac{1}{m}}, \qquad (3.7)$$

where the implied constants are independent of j and $(k_0)_j$. Indeed, the inequality (3.6) follow from Lemma 3.5below. Moreover, using Lemma 3.6 below, we obtain that

$$\int_{\mathrm{R}^n \setminus \cup_{j(k_0)_j} 2(Q_k)_j^{((k_0)_j)}} \sum_k \quad | T_{\sigma_k}(b_1, \dots, b_l, (g_k)_{l+1}, \dots, (g_k)_m)(x)|^{1/\ell dx}$$

$$\le \int_{\mathrm{R}^n \setminus \cup_{j,(k_0)_j} 2(Q_k)_j^{((k_0)_j)}} \sum_k \left(\sum_{(k_0)_1, \dots, (k_0)_{(1+\epsilon)}} | T_{\sigma_k}(b_1, \dots, b_\ell, (g_k)_{l+1}, \dots, (g_k)_m)(x)| \right)^{1/\ell} dx$$

$$\lesssim \int_{\mathrm{R}^n \setminus \cup_{j(k_0)_j} 2(Q_k)_j^{((k_0)_j)}} \sum_k \left(\prod_{j=1}^{\ell} \sum_{k_j} \lambda^{1/m} \frac{\ell\left((Q_k)_j^{((k_0)_j)}\right)^{n+\varepsilon}}{|x-(1+\epsilon)_{(Q_k)_j^{((k_0)_j)}}|^{n+\varepsilon}} \right)^{1/\ell} \left(\prod_{j=l+1}^{m} \lambda^{1/m} \right)^{1/\ell} dx$$

$$\le \lambda^{1/\ell} \prod_{j=1}^{\ell} \left(\sum_{(k_0)_j} \sum_k \quad | (Q_k)_j^{((k_0)_j)}| \int_{\mathrm{R}^n \setminus 2(Q_k)_j^{((k_0)_j)}} \frac{\ell\left((Q_k)_j^{((k_0)_j)}\right)^{\varepsilon}}{|x-(1+\epsilon)_{(Q_k)_j^{((k_0)_j}}, |^{n+\varepsilon}} dx \right)^{1/\ell}$$

$$\le \lambda^{1/l} \sum_k \quad \prod_{j=1}^{l} \left(\sum_{(k_0)_j} | (Q_k)_j^{((k_0)_j)}| \right)^{1/\ell} \le \lambda^{1/l-1/m}.$$

This shows (3.7). Therefore, we complete the proof of Theorem 3.3.

Lemma 3.4 (see [52]). *Let* $\sigma_k \in S_{\rho,\delta}^{1+\epsilon}(n,m)$ *with* $(1+\epsilon) \in \mathbb{R}$ *and* $0 \le \rho, \delta \le 1$, *and* $\phi_k, \psi_k \in C_c^\infty(\mathbb{R}^{mn})$ *with* $\operatorname{supp}(\phi_k) \subset \left\{\vec{\xi} \in \mathbb{R}^{mn} : \frac{1}{2} \le |\vec{\xi}| \le 2\right\}$ *and* $\operatorname{supp}(\psi_k) \subset \{\vec{\xi} \in \mathbb{R}^{mn} : |\vec{\xi}| \le 2\}$. *Then for any* $\epsilon \ge -1$ *there exists a constant* $c_{1+\epsilon} > 0$ *such that for* $j \ge 0$

$$\sup_{x,y_1,\dots,y_m \in R^n} (|\overrightarrow{y}|)^{1+\epsilon} | \int_{\mathrm{R}^{mn}} \sum_k \quad \sigma_k\left(x,\vec{\xi}\right)(\phi_k)_j(\vec{\xi}) e^{2\pi i \vec{y}\vec{\xi}} d\vec{\xi}| \le c_{(1+\epsilon)} 2^{j(mn+1+\epsilon-\rho(1+\epsilon))},$$

Proof. Applying Leibniz rule, we obtain that for $\sigma_k \in S_{\rho,1}^{1+\epsilon}$ and $j \ge 1$,
where $(\phi_k)_0 = \psi_k, (\phi_k)_j(\vec{\xi}) = \phi_k(2^{-j}\vec{\xi})$ *for* $j \ge 1$.

$$|\sum_k \quad \partial_{\vec{\xi}}^{1+2\epsilon}\left(\sigma_k(x,\vec{\xi})(\phi_k)_j(\vec{\xi})\right)| = |\sum_{\beta^k \le 1+2\epsilon} \sum_k \quad C_{1+2\epsilon,\beta^k} \partial_{\vec{\xi}}^{B} \sigma_k(x,\vec{\xi}) \partial_{\vec{\xi}}^{1+2\epsilon-\beta^k} \varphi(2^{-j}\vec{\xi})|$$

$$\le \sum_{\beta^k \le 1+2\epsilon} \sum_k \quad C_{1+2\epsilon,\beta^k} 2^{-j|1+2\epsilon-\beta^k|}(1+|\vec{\xi}|)^{1+\epsilon-\rho\beta^k|} 1_{\{2^{j-1} \le |\vec{\xi}| \le 2^{j+1}\}}$$

$$\simeq \sum_{\beta^k \le 1+2\epsilon} \sum_k \quad C_{1+2\epsilon,\beta^k} 2^{-j|1+2\epsilon-\beta^k|} 2^{j(1+\epsilon-\rho\beta^k|)} \le 2^{j(1+\epsilon-\rho|1+2\epsilon|)}.$$

If $(1+\epsilon) \in \mathbb{N}$, then it follows from integration by parts that

$$\sup_{x,y_1,\dots,y_m \in \mathbb{R}^n} (|\vec{y}|)^{(1+\epsilon)} \left| \int_{\mathrm{R}^{mn}} \sum_k \quad \sigma_k\left(x,\vec{\xi}\right)(\phi_k)_j(\vec{\xi}) e^{2\pi i \vec{y}\vec{\xi}} d\vec{\xi} \right|$$

$$\le \sup_{x,y_1,\dots,y_m \in \mathbb{R}^n} \sum_{|1+2\epsilon|=1+\epsilon} C_{1+2\epsilon,1+\epsilon} |\vec{y}^{(1+2\epsilon)}| \left| \int_{\mathrm{R}^{mn}} \sum_k \quad \sigma_k\left(x,\vec{\xi}\right)(\phi_k)_j(\vec{\xi}) e^{2\pi i \vec{y}\vec{\xi}} d\vec{\xi} \right|$$

$$\simeq \sup_{x,y_1,\dots,y_m \in \mathbb{R}^n} \sum_{|1+2\epsilon|=1+\epsilon} C_{1+2\epsilon,1+\epsilon} \left| \int_{\mathrm{R}^{mn}} \sum_k \quad \Gamma\left(x,\vec{\xi}\right)(\phi_k)_j(\vec{\xi}) \partial_{\vec{\xi}}^{1+2\epsilon}\left(e^{2\pi i \vec{y}\vec{\xi}}\right) d\vec{\xi} \right|$$

$$= \sup_{x,y_1,\dots,y_m \in \mathbb{R}^n} \sum_{|1+2\epsilon|=1+\epsilon} C_{1+2\epsilon,1+\epsilon} \left| \int_{2^{j-1} \le |\vec{\xi}| \le 2^{j+1}} \sum_k \quad \partial_{\vec{\xi}}^{1+2\epsilon}\left(\sigma_k(x,\vec{\xi})(\phi_k)_j(\vec{\xi})\right)) e^{2\pi i \vec{y}\vec{\xi}} d\vec{\xi} \right|$$

$$\le 2^{j(mn+1+\epsilon-\rho(1+\epsilon))}.$$

From this, the conclusion follows for any integer $(1+\epsilon)$. If $(1+\epsilon)$ is not an integer, then we can interpolate between the inequalities for k_0 and k_0+1, where $k_0 - 1 < \epsilon < k_0$.

The case $j = 0$ can be treated similarly.

Lemma 3.5 (see [52]). *Let* $\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$ *be as in Theorem* 3.3. *Let* Q_k *be a given cube. Suppose that* $\operatorname{supp}((f_k)_1) \subset Q_k, \int_{R^n} \sum_k (f_k)_1(x)dx = 0$ *and* $(f_k)_2, \dots, (f_k)_m \in L^\infty(\mathbb{R}^n)$. *Then there holds that*

$$\int_{\mathrm{R}^n \setminus 2Q_k} \sum_k |T_{\sigma_k}((f_k)_1, \dots, (f_k)_m)(x)|dx \le \sum_k ||(f_k)_1||_{L^1(\mathrm{R}^n)} \prod_{j=2}^m ||(f_k)_j||_{L^\infty(\mathrm{R}^n)}.$$

Proof. We begin with a Littlewood- Paley decomposition. Let $(\phi_k)_0 : (\mathbb{R}^n)^m \to \mathbb{R}$ be a nonnegative, radial, C^∞ function with compact support such that $(\phi_k)_0(\vec{\xi}) = 1$ for $|\vec{\xi}| \le 1$ and $(\phi_k)_0(\vec{\xi}) = 0$ for $|\vec{\xi}| \ge 2$. We define function $(\phi_k)_j$ by $(\phi_k)_j(\vec{\xi}) = \phi_k(2^{-j}\vec{\xi})$ and $(\vec{\xi}) = (\phi_k)_0(\vec{\xi}) - (\phi_k)_0(2\vec{\xi})$. Then we have the following partitions of unity

$$\sum_{j=0}^\infty \sum_k (\phi_k)_j(\vec{\xi}) = 1, \qquad for\ any\ \vec{\xi} = (\xi_1, \dots, \xi_m) \in \mathbb{R}^{mn}.$$

If we write $(\sigma_k)_j(x,\vec{\xi}) = \sigma_k(x,\vec{\xi})(\phi_k)_j(\vec{\xi})$, then there holds

$$T_{\sigma_k}((f_k)_1, \dots, (f_k)_m)(x) = \sum_{j=0}^\infty \sum_k T_{(\sigma_k)_j}((f_k)_1, \dots, (f_k)_m)(x) .$$

For convenience, denote $p_j(x,\vec{y},\vec{\xi}) = (\sigma_k)_j(x,\vec{\xi})e^{2\pi i(x-\vec{y})\vec{\xi}}$. Choose $\epsilon > -1$ so that $mn < 1+\epsilon < mn+1$ and $1+\epsilon-(\rho-1)(1+\epsilon) \le 0$. By Lemma 3.4 and change of variables, we have that

$$\begin{aligned}
&\int_{\mathrm{R}^{n(m-1)}} \sum_k \prod_{j=2}^m |\mathrm{f_j}(\mathrm{y_j})|| \int_{\mathrm{R}^{mn}} p_j(x,\vec{y},\vec{\xi})d\xi|dy_2 \dots dy_m \\
&\lesssim \sum_k \prod_{j=2}^m ||(f_k)_j||_{L^\infty(\mathbb{R}^n)} \int_{\mathrm{R}^{n(m-1)}} \frac{2^{j(mn+1+\epsilon-\rho(1+\epsilon))}}{\left(\sum_{j=1}^m |x-y_j|\right)^{1+\epsilon}} dy_2 \dots dy_m \\
&\lesssim \sum_k \frac{2^{j(mn+1+\epsilon-\rho(1+\epsilon))} \prod_{j=2}^m \|(f_k)_j\|_{L^\infty(\mathbb{R}^n)}}{|x-y_1|^{1+\epsilon-n(m-1)}} \int_{\mathrm{R}^{n(m-1)}} \frac{dy_2 \dots dy_m}{\left(1+\sum_{j=2}^m |x-y_j|\right)^{1+\epsilon}} \\
&\lesssim \sum_k \prod_{j=2}^m ||(f_k)_j||_{L^\infty(\mathrm{R}^n)} 2^{j(mn+1+\epsilon-\rho(1+\epsilon))} |x-y_1|^{n(m-1)-(1+\epsilon)}.
\end{aligned}$$

Note that

$$\int_{|y-y_0|>1+\epsilon} \frac{dy}{|y-y_0|^{n+\delta}} \le (1+\epsilon)^{-\delta}, \qquad \forall \epsilon \ge 0, \delta > 0. \tag{3.8}$$

Thus, we derive that

$$\begin{aligned}
&\int_{\mathrm{R}^n \setminus 2Q_k} \sum_k |T_{(\sigma_k)_j}((f_k)_1, \dots, (f_k)_m)(x)|\,dx \\
&\le \int_{Q_k} \sum_k |(f_k)_1(y_1)| \int_{\mathrm{R}^n \setminus 2Q_k} \Big(\int_{\mathrm{R}^{n(m-1)}} \prod_{j=2}^m |(f_k)_j(y_j)| \left| \int_{\mathrm{R}^{mn}} p_j(x,\vec{y},\vec{\xi})d\vec{\xi} \right| dy_2 \dots dy_m\Big) dx dy_1 \\
&\lesssim \sum_k \|(f_k)_1\|_{L^1(\mathrm{R}^n)} \prod_{j=2}^m ||(f_k)_j||_{L^\infty(\mathrm{R}^n)} 2^{j(mn+1+\epsilon-\rho(1+\epsilon))} (1+\epsilon)(Q_k)^{-((1+\epsilon)-mn)}.
\end{aligned}$$

On the other hand, in view of the cancelation of $(f_k)_1$, we deduce that

$$\int_{\mathrm{R}^n\backslash 2Q_k}\sum_k \left|T_{(\sigma_k)_j}\big((f_k)_1,\dots,(f_k)_j\big)(x)\right|dx$$
$$=\int_{\mathrm{R}^n\backslash 2Q_k}\sum_k \left|\int_{\mathrm{R}^m}\int_{\mathrm{R}}\ \big(p_j(x,\vec{y},\vec{\xi})\right.$$
$$\left.-\,p_j(x,y_1',y_2,\dots,y_m,\vec{\xi})\big)\right|(f_k)_1(y_1)\prod_{j=2}^m (f_k)_j\,(y_j)d\vec{y}d\vec{\xi}|dx$$
$$\le \int_{\mathrm{R}^n}\sum_k \left|(f_k)_1(y_1)\right|\int_{\mathrm{R}^n\backslash 2Q_k}\int_{\mathrm{R}^{n(m-1)}}\prod_{j=2}^m\left|(f_k)_j(y_j)\right|$$
$$\times\left|\int_{\mathrm{R}^{mn}}\Big(p_j(x,\vec{y},\vec{\xi})-p_j(x,y_1',y_2,\vec{\xi})\Big)d\vec{\xi}\right|dy_2\dots dy_m dx dy_1.$$

Write $y_l(1-\epsilon)=y_l+(1-\epsilon)(y_1'\text{-}\,y_1)$, for any $0<\epsilon\le 1$. It yields that

$$\left|\int_{\mathrm{R}^{mn}}\Big(p_j(x,\vec{y},\vec{\xi})-p_j(x,y_1',y_2,\dots,y_m,\vec{\xi})\Big)d\vec{\xi}\right|$$
$$=\left|\int_{\mathrm{R}^{mn}}\int_0^1 (y_1-y_1')\cdot\nabla_{y_1}\Big(p_j(x,y_1(1+\epsilon),y_2,\dots,y_m,\vec{\xi})\Big)d(1+\epsilon)d\vec{\xi}\right|$$
$$\le\sum_{k=1}^n \left|y_{1,k_0}-y_{1,k_0}'\right|\int_0^1\left|\int_{\mathrm{R}^{mn}}\partial_{y_{1,k_0}}\Big(p_j(x,y_1(1+\epsilon),y_2,\dots,y_m,\vec{\xi})\Big)d\vec{\xi}\right|d(1+\epsilon)$$
$$\le\sum_k \ell(Q_k)\sum_{k_0=1}^n\int_0^1(-\epsilon)\left|\int_{\mathrm{R}^{mn}}\Big(\partial_{y_{1,k_0}}p_j\Big)(x,y_1(1+\epsilon),y_2,\dots,y_m,\xi)d\vec{\xi}\right|d(1+\epsilon).$$

Applying Lemma 3.4 again, we have

$$\int_{\mathrm{R}^{n(m-1)}}\sum_k \prod_{j=2}^m\left|(f_k)_j(y_j)\right|\left|\int_{\mathrm{R}^{mn}}\Big(p_j(x,\vec{y},\vec{\xi})-p_j(x,y_1',y_2,\vec{\xi})\Big)d\vec{\xi}\right|dy_2\dots dy_m$$
$$\lesssim\sum_k \ell(Q_k)\sup_{1\le k_0\le n}\ \sup_{-1\le\epsilon\le 0}\int_{\mathrm{R}^{n(m-1)}}\prod_{j=2}^m\left|(f_k)_j(y_j)\right|$$
$$\times\left|\int_{\mathrm{R}^{mn}}\big(\partial_{y_{1,k}}p_j\big)\,(x,y_1(1+\epsilon),y_2,\dots,y_m,\vec{\xi})d\vec{\xi}\right|dy_2\dots dy_m$$
$$\lesssim\sum_k \ell(Q_k)\sup_{-1\le\epsilon\le 0}\int_{\mathrm{R}^{n(m-1)}}\frac{2^{j(mn+1+\epsilon-\rho(1+\epsilon))}(1+2^j)}{\big(|x-y_1(1+\epsilon)|+\sum_{j=2}^m|x-y_j|\big)^{1+\epsilon}}\prod_{j=2}^m\left|(f_k)_j(\mathrm{j}_j)\right|dy_2\dots dy_m$$
$$\lesssim\sum_k \ell(Q_k)2^{j(mn+2+\epsilon-\rho(1+\epsilon))}\sup_{-1\le\epsilon\le 1}\frac{\prod_{j=2}^m\|(f_k)_j\|_{L^\infty(\mathrm{R}^n)}}{|x-y_1(1+\epsilon)|^{(1+\epsilon)-n(m-1)}}$$
$$\simeq\sum_k \ell(Q_k)2^{j(mn+2+\epsilon-\rho(1+\epsilon))}\frac{\|(f_k)_2\|_{L^\infty(\mathrm{R}^n)}}{|x-c_{Q_k}|^{(1+\epsilon)-n(m-1)}}.$$

Then the inequality (3.8) gives that

$$\int_{\mathrm{R}^n\backslash 2Q_k}\sum_k |T_{(\sigma_k)_j}((f_k)_1,\dots,(f_k)_m)(x)|dx$$
$$\lesssim\sum_k \|(f_k)_1\|_{L^1(\mathrm{R}^n)}\prod_{j=2}^m\|(f_k)_j\|_{L^\infty(\mathrm{R}^n)}\ell(Q_k)2^{j(mn+2+\epsilon-\rho(1+\epsilon))}\int_{\mathrm{R}^n\backslash 2Q_k}\frac{dx}{|x-c_{Q_k}|^{(1+\epsilon)-n(m-1)}}$$
$$\lesssim\sum_k \|(f_k)_1\|_{L^1(\mathrm{R}^n)}\prod_{j=2}^m\|(f_k)_j\|_{L^\infty(\mathrm{R}^n)}(1+\epsilon)(Q_k)^{mn+1-(1+\epsilon)}2^{j(mn+2+\epsilon-\rho(1+\epsilon))}.$$

Summing up the above estimates, we obtain

$$\int_{\mathrm{R}^n\setminus 2Q_k}\sum_k |T_{\sigma_k}((f_k)_1,\dots,(f_k)_m)(x)|dx$$
$$\lesssim \sum_k \left(\sum_{2j\ell(Q_k)\geq 1}\left(2^j\ell(Q_k)\right)^{-((1+\epsilon)-mn)} + \sum_{2j\ell(Q_k)<1}\left(2^j\ell(Q_k)\right)^{(mn+1-(1+\epsilon))}\right)||(f_k)_1||_{L^1(\mathrm{R}^n)}\prod_{j=2}^m \|(f_k)_j\|_{L^\infty(\mathrm{R}^n)}$$
$$\simeq \sum_k ||(f_k)_1||_{L^1(\mathrm{R}^n)}\prod_{j=2}^m \|(f_k)_j\|_{L^\infty(\mathrm{R}^n)}.$$

Lemma 3.6 (see [52]). *Let* $(\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m) beasinTheorem 3.3. Let (Q_k)_1,\dots,(Q_k)_l begivencubes (2\leq l\leq m)$. *Supposethat* $(f_k)_j \in L^1(\mathbb{R}^n) with \operatorname{supp}\left((f_k)_j\right)\subset (Q_k)_j and \int_n (f_k)_j(x)dx=0, j = 1,\dots,\ell. Thenthereexistsapositiveconstant \varepsilon=\varepsilon(n,1+\epsilon,\rho) suchthat$

$$|\sum_k T_{\sigma_k}(\overrightarrow{f_k})(x)| \lesssim \sum_k \prod_{j=1}^l \frac{l\left((Q_k)_j\right)^\varepsilon}{|x-c_{(Q_k)_j}|^{n+\varepsilon}}\|(f_k)_j\|_{L^1(\mathrm{R}^n)}\prod_{j=l+1}^m ||(f_k)_j||_{L^\infty(\mathrm{R}^n)},\qquad x\notin\bigcup_{j=1}^l 2\,(Q_k)_j.$$

Proof. We still use the same notation as above. Choose $\epsilon > -1$ so that $mn < 1+\epsilon < mn+1$ and $(1+\epsilon)(2-\rho)\leq 0$. Let $x\notin \cup_{j=1}^l 2\,(Q_k)_j$. For any $z_j\in (Q_k)_j$, there holds $\ell'\left((Q_k)_j\right)\leq \left|x-c_{(Q_k)_j}\right|\simeq |x-z_j|, j=1,\dots,l$. We may assume that $\ell((Q_k)_1)=1\leq j\leq \min_{1\leq j\leq l}\ \ell'\left((Q_k)_j\right)$. A similar argument as that inLemma 3.5 yields that

$$\int_{\mathrm{R}^{n(m-l)}}|\int_{\mathrm{R}^{mn}} p_j\left(x,\vec{y},\vec{\xi}\right)d\vec{\xi}|dy_{l+1}\dots dy_m \lesssim \int_{\mathrm{R}^{n(m-l)}}\frac{2^{j(mn+1+\epsilon-\rho(1+\epsilon))}}{\left(\sum_{j=1}^m|x-y_j|\right)^{s_1}}dy_{l+1}\dots dy_m$$
$$\lesssim \frac{2^{j(mn+1+\epsilon-\rho(1+\epsilon))}}{\left(\sum_{j=1}^l|x-y_j|\right)^{s_1-n(m-l)}}\int_{\mathrm{R}^{n(m-l)}}\frac{dy_{l+1}\dots .d.y_m}{(1+|y_{l+1}|+.|y_m|)^{s_1}}$$
$$\lesssim \frac{2^{j(mn+1+\epsilon-\rho(1+\epsilon))}}{\prod_{j=1}^l|x-c_{(Q_k)_j}|^{((1+\epsilon)-n(m-l))/l}}$$
$$=\sum_k \left(2^j\ell((Q_k)_1)\right)^{mn-(1+\epsilon)}\prod_{j=1}^l\frac{\ell((Q_k)_1)^{((1+\epsilon)-mn)/l}}{|x-c_{(Q_k)_j}|^{((1+\epsilon)-n(m-l))/l}},$$

and letting $\vec{y}=(y_1',y_2,\dots,y_m)$ we get by using the mean value theorem

$$\int_{\mathrm{R}^{n(m-l)}}|\int_{\mathrm{R}^{mn}}\left(p_j(x,\vec{y},\vec{\xi})-p_j\left(x,\overrightarrow{y},\vec{\xi}\right)\right)d\vec{\xi}|dy_{l+1}\dots dy_m$$
$$\lesssim \int_{\mathrm{R}^{n(m-l)}}\int_0^1\sum_k \frac{l((Q_k)_1)2^{j(mn+2+\epsilon-\rho(1+\epsilon))}}{\left(|x-(y_1-y_1)(1+\epsilon)-y'_1|+\sum_{j=2}^m|x-y_j|\right)^{(1+\epsilon)}}d(1+\epsilon)dy_{l+1}\dots dy_m$$
$$\lesssim \sup_{-1\leq\epsilon\leq 0}\sum_k \frac{\ell((Q_k)_1)2^{j(mn+2+\epsilon-\rho(1+\epsilon))}}{\left(|x-(y_1-y_1)(1+\epsilon)-y_1|\sum_{j=2}^l|x-y_j|\right)^{(1+\epsilon)-n(m-l)}}\int_{\mathrm{R}^{n(m-l)}}\frac{dy_{l+1}\dots dy_m}{(1+|y_{l+1}|+\cdot+|y_m|)^{(1+\epsilon)}}$$
$$\lesssim \sum_k \frac{(1+\epsilon)((Q_k)_1)2^{j(mn+2+\epsilon-\rho(1+\epsilon))}}{\prod_{j=1}^l|x-c_{(Q_k)_j}|^{((1+\epsilon)-n(m-l))/l}}=\sum_k \left(2^j\ell((Q_k)_1)\right)^{mn-(1+\epsilon)}\prod_{j=1}^l\frac{l((Q_k)_1)^{((1+\epsilon)-mn)/i}}{|x-c_{(Q_k)_j}|^{((1+\epsilon)-n(m-l))/l}}.$$

Accordingly, setting $\varepsilon=\left((1+\epsilon)-mn\right)/l$. we deduce that

$$|\sum_k T_{(\sigma_k)_j}((f_k)_1, \dots, (f_k)_m)(x)| \le \int_{\mathrm{R}^{mn}} \sum_k \prod_{j=1}^{m} |(f_k)_j(y_j)|| \int_{\mathrm{R}^{mn}} p_j(x, \vec{y}, \vec{\xi}) d\vec{\xi}| d\vec{y}$$

$$\lesssim \sum_k \left(2^j(1 + \epsilon)'((Q_k)_1)\right)^{mn-(1+\epsilon)} \prod_{j=1}^{l} \frac{\ell((Q_k)_1)^{(1+\epsilon)/2-n}}{|x - c_{(Q_k)_j}|^{(1+\epsilon)/2}} ||(f_k)_j||_{L^1(\mathrm{R}^n)} \prod_{j=l+1}^{m} \|(f_k)_j\|_{L^\infty(\mathrm{R}^n)}$$

$$\le \sum_k \left(2^j \ell((Q_k)_1)\right)^{mn-(1+\epsilon)} \prod_{j=1}^{l} \frac{\ell((Q_k)_1)^{\varepsilon}}{|x - c_{(Q_k)_j}|^{n+\varepsilon}} \|(f_k)_j\|_{L^1(\mathrm{R}^n)} \prod_{j=l+1}^{m} ||(f_k)_j||_{L^\infty(\mathrm{R}^n)}.$$

Furthermore, it follows from the cancelation of $(f_k)_j$ for $j = 1, \dots , l$ that

$$|\sum_k T_{(\sigma_k)_j}((f_k)_1, \dots, (f_k)_j)(x)| \le \int_{\mathrm{R}^{mn}} \sum_k \prod_{j=1}^{m} |(f_k)_j(y_j)|| \int_{\mathrm{R}^{mn}} \left(p_j(x, \vec{y}, \vec{\xi}) - p_j(x, y_1', y_2, \vec{\xi})\right) d\vec{\xi}| d\vec{y}$$

$$= \sum_k \left(2^j \ell((Q_k)_1)\right)^{mn+1-(1+\epsilon)} \prod_{j=1}^{l} \frac{\ell((Q_k)_1)^{((1+\epsilon)-mn)/l}}{|x - c_{(Q_k)_j}|^{((1+\epsilon)-n(m-l))/l}} ||(f_k)_j||_{L^1(\mathrm{R}^n)} \prod_{j=l+1}^{m} ||(f_k)_j||_{L^\infty(\mathrm{R}^n)}$$

$$\lesssim \sum_k \left(2^j(1+\epsilon)((Q_k)_1)\right)^{mn+1-s_2} \prod_{j=1}^{l} \frac{\ell((Q_k)_1)^{\varepsilon}}{|x - c_{(Q_k)_j}|^{n+\varepsilon}} ||(f_k)_j||_{L^1(\mathrm{R}^n)} \prod_{j=l+1}^{m} \|(f_k)_j\|_{L^\infty(\mathrm{R}^n)}.$$

Therefore, the summation over j as before will give us the desired result.

Finally, we present an observation between the multilinear pseudo- differential operators and the multilinear singular integral operators.

Proposition 3.7. *Assumethat* $\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n, m)$ *with* $\rho, \delta \in [0,1]$ *and* $< mn(\rho - 1)$. *If* $1 + \epsilon + \delta < nm(\rho - 1)$ *forsome* $0 \le \delta \le 1$, *Then* T_{σ_k} *isamultilinearCalderón- Zygmundsingularintegraloperator*.

***Proof*.** Applying Proposition 3.1, we can show that T_{σ_k} is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$ to $L^{1+\epsilon}(\mathbb{R}^n)$ for any $1/1 + \epsilon = 1/p_1 + \cdots + 1/p_m$ with $0 \le \epsilon \le 2$ and $1 < p_1$, , $p_m < \infty$. Next, let $K_{\sigma_k}(x, \vec{x} - \vec{y})$ be the kernel function of T_{σ_k}, where $\vec{x} = (x, \dots, x) \in \mathbb{R}^{mn}$ and

$$K_{\sigma_k}(x, \vec{y}) = \int_{\mathrm{R}^{mn}} \sum_k \sigma_k(x, \vec{\xi}) e^{2\pi i \vec{y}\vec{\xi}} d\vec{\xi}.$$

So, for a multi- index $(1 + 2\epsilon)$

$$y^{\to\alpha} K_{\sigma_k}(x, \vec{y}) = (1 + \epsilon) \int_{\mathrm{R}^{mn}} \sum_k \partial^{1+2\epsilon}_{\vec{\xi}} \sigma_k(x, \vec{\xi}) e^{2\pi i \vec{y}\vec{\xi}} d\vec{\xi}.$$

From this and $(1 + \epsilon) < mn(\rho - 1)$, we get

$$|\vec{y}|^{mn} |\sum_k K_{\sigma_k}(x, f_k)| \le (1 + \epsilon) \int_{\mathrm{R}^{mn}} \left(1 + |\vec{\xi}|\right)^{1+\epsilon-\rho mn} d\vec{\xi} < \infty,$$

which implies that

$$|\sum_k K_{\sigma_k}(x, \vec{x} - \vec{y})| \le \frac{1 + \epsilon}{1\vec{x} - \vec{y}|mn}.$$

The following claim can be easily checked.

Claim 1. For nonzero $\vec{y} \in \mathbb{R}^{mn}$ there exist $j_0 \in \{1,2, \dots, m\}$ and $\ell_0 \in \{1,2, \dots, n\}$ such that

$$|y_{j_0}| \ge \frac{|\vec{y}|}{\sqrt{m}} \ and \ |y_{j_0,\ell_0}| \ge \frac{|\vec{y}|}{\sqrt{mn}}.$$

Furthermore, if $|y_{j_0} - y'_{j_0}| < |\vec{y}|/(m + 1)n$, then

$$|y'_{j_0,\ell_0}| \ge \left(\frac{1}{\sqrt{mn}} - \frac{1}{(m + 1)n}\right) |\vec{y}|.$$

In the following content, we take γ with $0 < \gamma < mn(\rho - 1) - (1 + \epsilon)$.

(a) Let $x \in \mathbb{R}^n$ and $0 \ne \vec{y} \in \mathbb{R}^{mn}$. Let $\vec{y}' = (y_1, y_j, \dots, y_m)$ and $|y_j - y_j'| < |\vec{y}|/(m + 1)n$. Let j_0, P_0 be given in Claim 1. We estimate $|K_{\sigma_k}(x, \vec{y}) - K_{\sigma_k}(x, \vec{y}')|$.

(a- 1) Let $1 \le j \le m$ with $j \ne j_0$. Then, it yields that

$$\left|\sum_k \left(K_{\sigma_k}(x,\vec{y}) - K_{\sigma_k}(x,\overrightarrow{y'})\right)\right| = \left|\frac{1+\epsilon}{y_{j_0\ell_0}^{mn}}\int_{\mathrm{R}^{mn}}\sum_k \partial_{\xi_{j_0(1+\epsilon)_0}}^{mn}\sigma_k(x,\vec{\xi})\left(e^{i\vec{y}\vec{\xi}} - e^{i\vec{y}\vec{\xi}}\right)d\vec{\xi}\right|$$
$$\lesssim \frac{1}{|\vec{y}|^{mn}}\int_{\mathrm{R}^{mn}}\left(1+|\vec{\xi}|\right)^{1+\epsilon-\rho mn}\min\{|y_j - y_j'||\xi|,1\}d\vec{\xi}$$
$$\lesssim \frac{|y_j - y_j'|^\gamma}{1y\neg^{mn}}\int_{\mathrm{R}^{mn}}\left(1+|\vec{\xi}|\right)^{1+\epsilon-\rho mn+\gamma}d\vec{\xi} \le \frac{|y_j - y_j'|^\gamma}{|j|^{mn}}. \tag{3.9}$$

Similarly, there holds

$$\left|\sum_k \left(K_{\sigma_k}(x,y^{\rightarrow}) - K_{\sigma_k}(x,\vec{y})\right)\right| \le \frac{|y_j - y_j'|^\gamma}{|\overrightarrow{y}|^{mn+1}}\int_{\mathrm{R}^{mn}}\left(1+|\vec{\xi}|\right)^{1+\epsilon-\rho(mn+1)+\gamma}d\vec{\xi}$$
$$\le \frac{|y_j - y_j'|^\gamma}{|\overrightarrow{y}|^{mn+1}}. \tag{3.10}$$

For $|\overrightarrow{y}| < 1$, we have

$$\frac{|y_j - y_j'|^\gamma}{|\overrightarrow{y}|^{mn}} \le \frac{1}{|\overrightarrow{y}|^{mn}}\left(\frac{|y_j - y_j'|}{|\overrightarrow{y}|}\right)^\gamma$$

And for $|\overrightarrow{y}| \ge 1$, we get

$$\frac{|y_j - y_j'|^\gamma}{|\overrightarrow{y}|^{mn+1}} \le \frac{1}{|\overrightarrow{y}|^{mn}}\left(\frac{|y_j - y_j'|}{|\overrightarrow{y}|}\right)^\gamma$$

Thus, using (3.9) and (3.10), we obtain

$$\left|\sum_k \left(K_{\sigma_k}(x,y^{\rightarrow}) - K_{\sigma_k}(x,\vec{y})\right)\right| \le \frac{1}{|\overrightarrow{y}|^{mn}}\left(\frac{|y_j - y_j'|}{|\overrightarrow{y}|}\right)^\gamma \tag{3.11}$$

(a- 2) For $\overrightarrow{y'} = (y_1, \dots, y_{j_0}', \dots, y_m)$, we have

$$\left|\sum_k \left(K_{\sigma_k}(x,\vec{y}) - K_{\sigma_k}(x,\vec{y}')\right)\right|$$
$$\le \left|\frac{1+\epsilon}{y_{j_0\ell_0}^{mn}}\int_{\mathrm{R}^{mn}}\sum_k \partial_{\xi_{j_0\ell_0}}^{mn}\sigma_k(x,\vec{\xi})\left(e^{i\vec{y}\vec{\xi}} - e^{i\vec{y}'\vec{\xi}}\right)d\vec{\xi}\right|$$
$$+ \left|\left(\frac{1+\epsilon}{y_{j_0\ell_0}^{mn}} - \frac{1+\epsilon}{y_{j_0\ell_0}^{\prime mn}}\right)\int_{\mathrm{R}^{mn}}\sum_k \partial_{\xi_{j_0(1+\epsilon)_0}}^{mn}\sigma_k(x,\vec{\xi})e^{i\vec{y}\vec{\xi}}d\vec{\xi}\right|$$
$$\lesssim \frac{1}{|\overrightarrow{y}|^{mn}}\int_{\mathrm{R}^{mn}}\left(1+|\vec{\xi}|\right)^{1+\epsilon-\rho mn+\gamma}|y_{j_0} - y_{j_0}'|^\gamma d\vec{\xi} + \frac{|y_{j_0\ell_0} - y_{j_0 l_0}'|}{|\overrightarrow{y}|^{mn+1}}\int_{\mathrm{R}^{mn}}\left(1+|\vec{\xi}|\right)^{1+\epsilon-\rho mn}d\vec{\xi}$$
$$\lesssim \frac{|y_{j_0} - y_{j_0}'|^\gamma}{|\overrightarrow{y}|^{mn}} + \frac{|y_{j_0} - y_{j_0}'|}{|\overrightarrow{y}|^{mn+1}}. \tag{3.12}$$

Similarly, we get

$$|\sum_k \left(K_{\sigma_k}(x,\vec{y}) - K_{\sigma_k}(x,\vec{y})\right)| \le \frac{|y_{j_0} - y_{j_0}'|^\gamma}{|\overrightarrow{y}|^{mn+1}} + \frac{|y_{j_0} - y_{j_0}'|}{|\overrightarrow{y}|^{mn+2}}. \tag{3.13}$$

For $|\overrightarrow{y}| < 1$, it holds

$$\frac{|y_{j_0} - y_{j_0}'|^\gamma}{|\overrightarrow{y}|^{mn}} + \frac{|y_{j_0} - y_{j_0}'|}{|\overrightarrow{y}|^{mn+1}} \le \frac{1}{|\overrightarrow{y}|^{mn}}\left(\frac{|y_{j_0} - y_{j_0}'|}{|\overrightarrow{y}|}\right)^\gamma$$

And for $|\overrightarrow{y}| \ge 1$, there holds

$$\frac{|y_{j_0} - y_{j_0}'|^\gamma}{|\overrightarrow{y}|^{mn+1}} + \frac{|y_{j_0} - y_{j_0}'|}{|\overrightarrow{y}|^{mn+2}} \le \frac{1}{|\overrightarrow{y}|^{mn}}\left(\frac{|y_{j_0} - y_{j_0}'|}{|\overrightarrow{y}|}\right)^\gamma$$

Hence, it follows from (3.12) and (3.13) that

$$\left|\sum_k \left(K_{\sigma_k}(x,\vec{y}) - K_{\sigma_k}(x,\vec{y}')\right)\right| \le \frac{1}{|\overrightarrow{y}|^{mn}}\left(\frac{|y_j - y_j'|}{|\overrightarrow{y}|}\right)^\gamma, \tag{3.14}$$

in the case $j = j_0$, too.

Consequently, we have shown that if $\vec{x} = (x, \dots, x) \in \mathbb{R}^{mn}$ and $\vec{x} - \vec{y} \neq 0$, and $|y_j - y_j'| < |\vec{x} - \vec{y}]/(m+1)n,$

$$\left|\sum_{k} \left(K_{\sigma_k}(x,\vec{x}-\vec{y})-K_{\sigma_k}(x,\vec{x}-\vec{y})\right)\right| \le \frac{1}{|\vec{x}-\overrightarrow{y}|^{mn}}\left(\frac{|y_j-y_j'|}{|\vec{x}-\vec{y}|}\right)^{\gamma} \tag{3.15}$$

Let $x, \vec{y} = (y_1, \dots, y_m), \vec{y} = (y_1, \dots, y_{j-1}, y_j', y_{j+1}, \dots, y_m)$ satisfy $|y_j - y_j'| \le \frac{1}{2}\max|x-y_i|$.This implies that $|\vec{y}-\vec{y}'| \le \frac{1}{2}|\vec{x}-\vec{y}|$.On the joining $\vec{y}$ and $\vec{y}'$, we take $k_0 := (m+1)n+1$ points $\vec{y}_l (l = 1, \dots, k_0)$ with $\vec{y}_1 = \vec{y}, \vec{y}_{k_0} = \vec{y}'$ and

$$|\vec{y}_l - \vec{y}_{l+1}| \le \frac{1}{(m+1)n}|\vec{y}-\vec{y}'|.$$

Then, we see that

$$|\vec{x}-\vec{y}_l| \ge |\vec{x}-\vec{y}| - |\vec{y}-\vec{y}_l| \ge |\vec{x}-\vec{y}| - |\vec{y}-\vec{y}'| \ge \frac{1}{2}|\vec{x}-\vec{y}|$$

And

$$|\vec{x}-\vec{y}_l| \le \frac{3}{2}|\vec{x}-\vec{y}|.$$

Hence, we have

$$|\vec{y}_l - \vec{y}_{l+1}| \le \frac{1}{(m+1)n}|\vec{y}-\vec{y}'| \le \frac{1}{2(m+1)n}|\vec{x}-\vec{y}\quad| < \frac{1}{(m+1)n}|\vec{x}-\vec{y}_l|.$$

So, by (3.15) we get

$$|\sum_{k} \left(K_{\sigma_k}(x,\vec{x}-\vec{y}_l)-K_{\sigma_k}(x,\vec{x}-\vec{y}_{l+1})\right)| \le \frac{1+\epsilon}{|\vec{x}-\vec{y}_l|^{mn}}\left(\frac{|\vec{y}_l-\vec{y}_{l+1}|}{|\vec{x}-\vec{y}_l|}\right)^{\gamma} \le \frac{2^{mn+\gamma}(1+\epsilon)}{|\vec{x}-\vec{y}|mn}\left(\frac{|\vec{y}-\overrightarrow{y'}|}{|\vec{x}-y|}\right)^{\gamma}$$

Therefore, we deduce that

$$\begin{aligned}|\sum_{k} \left(K_{\sigma_k}(x,\vec{x}-\vec{y})-K_{\sigma_k}(x,\vec{x}-\vec{y})\right)| &\le \sum_{l=1}^{k_0-1}\sum_{k} \quad |K_{\sigma_k}(x,\vec{x}-\vec{y}_l)-K_{\sigma_k}(x,\vec{x}-\vec{y}_{l+1})| \\ &\le \frac{(1+\epsilon)'}{|\vec{x}-\vec{y}|^{mn}}\left(\frac{|\vec{y}-\vec{y}'|}{|\vec{x}-\vec{y}|}\right)^{\gamma},\end{aligned}$$

whenever $|y_j - y_j'| \le \frac{1}{2}\max|x-y_j|, j = 1, \dots, m$.

(b) Let $\vec{x} = (x, x) \in \mathbb{R}^{mn}, \vec{x}-\vec{y} \ne 0$, and $|x-x'| < |\vec{x}-\overrightarrow{y}|/(m+1)n$. Then by Claim 1 there exist $j_0 \in \{1,2,\dots,m\}$ and $\ell_0 \in \{1,2,,n\}$ such that

$$|x-y_{j_0}| \ge \frac{|\vec{x}-\vec{y}|}{\sqrt{m}} \; and \; |x_{\ell_0}-y_{j_0,\ell_0}| \ge \frac{|\vec{x}-\vec{y}|}{\sqrt{mn}}.$$

Using the mean value theorem and noting $1+\epsilon-mn(\rho-1)+\delta<0$, we obtain

$$\begin{aligned}&|\sum_{k} \left(K_{\sigma_k}(x,\vec{x}-\vec{y})-K_{\sigma_k}(x',\vec{x}-\vec{y})\right)| \\ &\le \sum_{k} \quad |K_{o^-}(x,\vec{x}-\vec{y})-K_{\sigma_k}(x',\vec{x}-\vec{y})| + \sum_{k} \quad |K_{\sigma_k}(x',\vec{x}-\vec{y})-K_{o^-}(x',\vec{x}-\vec{y})| \\ &\lesssim |\frac{1}{\left(x_{l_0}-y_{j_0l_0}\right)^{mn}}\int_{\mathrm{R}^{mn}}\sum_{k} \quad \left(\partial^{mn}_{\xi_{j_0(1+\epsilon)_0}}\sigma_k(x,\vec{\xi}) - \partial^{mn}_{\xi_{j_0(1+\epsilon)_0}}\sigma_k(x',\vec{\xi})\right)e^{i\left(\vec{x}-\vec{y}\quad\right)\vec{\xi}}d\vec{\xi}| \\ &+ |\left(\frac{1}{\left(x_{l_0}-y_{j_0l_0}\right)^{mn}} - \frac{1}{\left(x'_{l_0}-y_{j_0l_0}\right)^{mn}}\right)\int_{\mathrm{R}^{mn}}\sum_{k} \quad \partial^{mn}_{\xi_{j_0\ell_0}}\sigma_k(x',\vec{\xi})e^{i(\vec{x}-\vec{y})\vec{\xi}}d\vec{\xi}| \\ &+ |\frac{1}{\left(x'_{\ell_0}-y_{j_0\ell_0}\right)^{mn}}\int_{\mathrm{R}^{mn}}\partial^{mn}_{\xi_{j_0\ell_0}}\sigma_k(x',\vec{\xi}))\left(e^{i(\vec{x}-\vec{y})\vec{\xi}}-e^{i(\vec{x}-\vec{y})\vec{\xi}}\right)d\vec{\xi}| \\ &\lesssim |\frac{1}{\left(x\ell_0-y_{j_0\ell_0}\right)^{mn}}|\int_{\mathrm{R}^{mn}}\left(1*|\vec{\xi}|\right)^{1+\epsilon-mn+\delta}|x-x'|d\vec{\xi} + |\frac{1}{\left(x\ell_0-y_{j_0\ell_0}\right)^{mn}} \\ &-{}'\frac{1}{\left(x_{l_0}-y_{j_0c_0}\right)^{mn}}|\int_{\mathrm{R}^{mn}}\left(1*|\vec{\xi}|\right)^{1+\epsilon-mn}|x-x'|d\vec{\xi} \\ &+ |\frac{|x-x'|^{\gamma}}{\left(x'_{\ell_0}-y_{j_0\ell_0}\right)^{mn}}|\int_{\mathrm{R}^{mn}}\left(1*|\vec{\xi}|\right)^{1+\epsilon-mn+\gamma}d\vec{\xi} \lesssim \frac{|x-x'|}{|\vec{x}-\vec{y}|^{mn}} + \frac{|x-x'|}{|\overrightarrow{x}-\vec{y}|^{mn+1}} + \frac{|x-x'|^{\gamma}}{|\vec{x}-\vec{y}|^{mn}}.\end{aligned}$$

Similarly, there holds

$$|\sum_{k}\left(K_{\sigma_k}(x,\vec{x}-\vec{y})-K_{\sigma_k}(x',\vec{x}-\vec{y})\right)|\lesssim\frac{|x-x'|}{|\vec{x}-\vec{y}|^{mn+1}}+\frac{|x-x'|}{|\vec{x}-\vec{y}|^{mn+2}}+\frac{|x-x'|^{\gamma}}{|\vec{x}-\vec{y}|^{mn+1}}.$$

Then, by the similar arguments in the case (a), we obtain

$$|\sum_{k}\left(K_{\sigma_k}(x,\vec{x}-\vec{y})-K_{\sigma_k}(x',\vec{x}-\vec{y})\right)|\lesssim\frac{1}{|\vec{x}-\vec{y}|mn}\left(\frac{|x-x'|}{|\vec{x}-\vec{y}|}\right)^{\gamma},$$

whenever $|x-x'|\leq\frac{1}{2}\max_{1\leq i\leq m}|x-y_i|$.

Thus, (a) and (b) imply the desired result.

IV. Sparse Bounds

We establish the pointwise sparse bounds for the multilinear pseudo- differential operators.

Given a sparse family S, we define the dyadic sparse operators by setting

$$A_S(\overrightarrow{f_k})(x):=\sum_{k}\sum_{Q_k\in S}\prod_{i=1}^{m}\langle|(f_k)_i|\rangle_{Q_k}1_{Q_k}(x),$$

$$A_{S,b_j}(\overrightarrow{f_k})(x):=\sum_{k}\sum_{Q_k\in S}|b_j(x)-b_{j,Q_k}|\langle|(f_k)_j|\rangle_{Q_k}\prod_{i\neq j}\langle|(f_k)_i|\rangle_{Q_k}1_{Q_k}(x),$$

and

$$A^{*}_{S,b_j}(\overrightarrow{f_k})(x):=\sum_{k}\sum_{Q_k\in S}\langle|\left(b_j-b_{j,Q_k}\right)(f_k)_j|\rangle_{Q_k}\prod_{i\neq j}\langle|(f_k)_i|\rangle_{Q_k}1_{Q_k}(x).$$

Proposition 4.1.*Let*$\sigma_k\in S^{1+\epsilon}_{\rho,\delta}(n,m)$*with*$0\leq\rho,\delta\leq 1$*and*$(1+\epsilon)<mn(\epsilon)$. *Then, foreverycompactlysupportedfunctions*$(f_k)_i, i=1,\dots,m$, *thereexist*3^n+1*sparsecollections*S*and*$\{S_i\}_{i=1}^{3^n}$*suchthat*

$$|\sum_{k}T_{\sigma_k}(\overrightarrow{f_k})(x)|\lesssim\sum_{k}\mathrm{A}_S(f_k\rfloor(x),\qquad a.e.\,x\in\mathbb{R}^n,$$

and

$$|\sum_{k}T_{\sigma_k,\Sigma\mathrm{b}}(\overrightarrow{f_k})(x)|\lesssim\sum_{i=1}^{3^n}\sum_{j=1}^{m}\sum_{k}\left(A_{S_i,b_j}(\overrightarrow{f_k})(x)+\sum_{k}A^{*}_{S_j,b_j}\left(\overrightarrow{f_k}(x)\right),\qquad a.e.\,x\in\mathrm{R}^n.\right.$$

By a completely analogous argument to that in Theorems 1.4 and 1.5 [9] we will obtain Proposition 4.1 based on Proposition 4.2 below.

Now to show the sparse domination theorem, we need a multilinear analogue of grand maximal truncated operator. Given an operator T, define

$$\mathcal{M}_T(\overrightarrow{f_k})(x):=\sup_{Q_k\ni x}\ \operatorname*{ess\,sup}_{\xi\in Q_k}\sum_{k}|T(\overrightarrow{f_k})(\xi)-T\left(\overrightarrow{f_k}\cdot 1_{3Q_k}\right)(\xi)|.$$

Given a cube $(Q_k)_0$, for $x\in(Q_k)_0$, we also define a local version of $\mathcal{M}_T$ by

$$\mathcal{M}_{T,(Q_k)_0}(\overrightarrow{f_k})(x):=\sup_{Q_k\ni x,Q_k\subset(Q_k)_0}\ \operatorname*{ess\,sup}_{\xi\in Q_k}\sum_{k}|T\left(\overrightarrow{f_k}\cdot 1_{3(Q_k)_0}\right)(\xi)-T\left(\overrightarrow{f_k}\cdot 1_{3Q_k}\right)(\xi)|.$$

Proposition 4.2 (see [52]).*Let*$0\leq\rho,\delta\leq 1$*and*$(1+\epsilon)<mn(\rho-1)$. *If*$\sigma_k\in S^{1+\epsilon}_{\rho,\delta}(n,m)$, *thenthefollowingpointwiseestimateshold*:

(i) *for*$a.e.\,x\in(Q_k)_0$

$$|\sum_{k}T_{\sigma_k}\left(\overrightarrow{f_k}\cdot 1_{3(Q_k)_0}\right)(x)|\leq c_n\mathcal{N}_{weak}\sum_{k}\prod_{i=1}^{m}|(f_k)_i(x)|+\mathcal{M}_{T_{\sigma_k},(Q_k)_0}(\overrightarrow{f_k})(x),$$

where$\mathcal{N}_{weak}=\|T_{\sigma_k}\|_{L^1\times\cdots\times L^1\to L^{1/m,\infty}}$;

(ii) *forany*$x\in\mathbb{R}^n$*and*$0<\varepsilon<1/m$,

$$\mathcal{M}_{T_{\sigma_k}}(\overrightarrow{f_k})(x)\leq c_{n,1+\epsilon,\varepsilon}\sum_{k}\left(\mathcal{N}_{weak}\mathcal{M}(\overrightarrow{f_k})(x)+M_{\varepsilon}\left(T_{\sigma_k}(\overrightarrow{f_k})\right)(x)\right).$$

Inparticular, wehave

$$\mathcal{M}_{T_{\sigma_k}}:L^1(\mathbb{R}^n)\times\cdots\times L^1(\mathbb{R}^n)\to L^{1/m,\infty}(\mathbb{R}^n).$$

Proof. It suffices to show the pointwise control for $\mathcal{M}_{T_{\sigma_k}}(\overrightarrow{f_k})(x)$, since (i) can be obtained by a similar argument as that in [32, Lemma 2.1].

Let $x,z,z'\in Q_k$. We deduce that

$$\left|\sum_k \left(T_{\sigma_k}(\overrightarrow{f_k})(z) - T_{\sigma_k}(\overrightarrow{f_k}\cdot 1_{3Q_k})(z)\right)\right|$$
$$\leq \sum_k |T_{\sigma_k}(\overrightarrow{f_k})(z')| + \sum_k \left|T_{\sigma_k}(\overrightarrow{f_k}\cdot 1_{3Q_k})(z')\right| + \sum_k \Xi(\overrightarrow{f_k})(z,z'),$$

where

$$\Xi(\overrightarrow{f_k})(z,z') = \left|\int_{(\mathrm{R}^n)^m\setminus(3Q_k)^m}\sum_k \int_{\mathrm{R}^{mn}} A\left(z,z',\vec{y},\vec{\xi}\right)\prod_{i=1}^m (f_k)_i\, \mathrm{r_{J}}_i d\vec{\xi}d\vec{y}\right| \quad (4.1)$$

And

$$A\left(z,z',\vec{y},\vec{\xi}\right) = \sigma_k\left(z,\vec{\xi}\right)e^{2\pi i(z-\vec{y})\vec{\xi}} - \sigma_k\left(z',\vec{\xi}\right)e^{2\pi i\left(z'-\vec{y}\right)\vec{\xi}}$$

we claim that

$$\Xi(\overrightarrow{f_k})(z,z') \leq \mathcal{M}(\overrightarrow{f_k})(x)\,. \quad (4.2)$$

Once this inequality is obtained, we have by taking L^{ε} average over $z' \in Q_k$ that

$$|\sum_k \left(T_{\sigma_k}\left(\overrightarrow{f_k}\right)(z) - T_{\sigma_k}(\overrightarrow{f_k}\cdot 1_{3Q_k})(z)\right)|$$
$$\lesssim \sum_k \mathcal{M}(\overrightarrow{f_k})(x) + \sum_k ||T_{\sigma_k}(\overrightarrow{f_k}\cdot 1_{3Q_k})||_{L^{\varepsilon}\left(Q_k,\frac{dz'}{|Q_k|}\right)}$$
$$+\sum_k \left(\frac{1}{|Q_k|}\int_{Q_k}|\, T_{\sigma_k}(\overrightarrow{f_k})(z')|^{\varepsilon}dz'\right)^{1/\varepsilon}$$
$$\lesssim \sum_k \mathcal{M}(\overrightarrow{f_k})(x) + \sum_k ||T_{\sigma_k}(\overrightarrow{f_k}\cdot 1_{3Q_k})||_{L^{1/m,\infty}\left(Q_k,\frac{dz'}{|Q_k|}\right)} + \sum_k M_{\varepsilon}(T_{\sigma_k}(\overrightarrow{f_k}))(x)$$
$$\lesssim \sum_k \mathcal{M}(\overrightarrow{f_k})(x) + \mathcal{N}_{weak}\sum_k \prod_{i=1}^m\left(\frac{1}{|Q_k|}\int_{3Q_k}|\,(f_k)_i|dy\right) + \sum_k M_{\varepsilon}(T_{\sigma_k}(\overrightarrow{f_k}))(x)$$
$$\lesssim \sum_k \mathcal{M}(\overrightarrow{f_k})(x) + \sum_k M_{\varepsilon}(T_{\sigma_k}(\overrightarrow{f_k}))(x)\,,$$

where we have used Theorem 3.3.

Let us turn to the proof of (4.2). We introduce a Littlewood- Paley partition of unity. Let $(\phi_k)_0 : \mathbb{R}^{mn} \to \mathbb{R}$ be a nonnegative, radial, C^{∞} function with compact support such that $(\phi_k)_0(\xi_1,\ldots,\xi_m) = 1$ for $|\vec{\xi}| \leq 1$ and $(\phi_k)_0(\xi_1,\ldots,\xi_m) = 0$ for $|\vec{\xi}| \geq 2$. We define function ρ_j by $(\beta_j^k(\vec{\xi}) = \phi_k\left(2^{-j}\vec{\xi}\right)$ and $\left(\vec{\xi}\right) = (\phi_k)_0(\vec{\xi}) - (\phi_k)_0\left(2\vec{\xi}\right)$. Then we have the following partitions of unity

$$\sum_{j=0}^{\infty}\sum_k (\phi_k)_j\left(\vec{\xi}\right) = 1, for\ any\ \vec{\xi} = (\xi_1,\ldots,\xi_m) \in \mathrm{R}^{mn}.$$

Thus, we dominate $\Xi$by $\Xi \leq \sum_{j=0}^{\infty}\Xi_j$, where

$$\Xi_j(\overrightarrow{f_k})(z,z') = |\int_{(\mathrm{R}^n)^m\setminus(3Q_k)^m}\sum_k \int_{\mathrm{R}^{mn}}(\phi_k)_j\left(\vec{\xi}\right)A\left(z,z',\vec{y},\vec{\xi}\right)\prod_{i=1}^m (f_k)_i\,(\mathrm{y}_i)d\vec{\xi}d\vec{y}|.$$

Case 1:$\ell(Q_k) \geq 2^{-j}$. Let $s_1 > mn$ be chosen later. By Lemma 3.4, we have that

$$\Xi_j(\overrightarrow{f_k})(z,z') \lesssim \sum_{k_0=1}^{\infty}\int_{(3^{k_0+1})^m\setminus(3^{k_0}Q_k)^m}\sum_k \sup_{\mathrm{z}\in\mathrm{Q_k}}|\int_{\mathrm{R}^{mn}}(\phi_k)_j\left(\vec{\xi}\right)\sigma_k\left(z,\vec{\xi}\right)e^{2\pi i(z-\vec{y})\vec{\xi}}d\vec{\xi}|\prod_{i=1}^m|\,(f_k)_i(y_i)|d\vec{y}$$
$$\lesssim \sum_{k_0=1}^{\infty}\int_{(3^{k_0+1})^m\setminus(3^{k_0}Q_k)^m}\sum_k \sup_{\mathrm{z}\in\mathrm{Q_k}}\frac{2^{j(mn+1+\epsilon-\rho s_1)}}{(\sum_{i=1}^m|\,y_i - z|)^{s_1}}\prod_{i=1}^m|\,(f_k)_i(\mathrm{y_i})|d\vec{y}$$
$$\lesssim \sum_{k_0=1}^{\infty}\sum_k \frac{2^{j(mn+1+\epsilon-\rho s_1)}}{\left(3^{k_0}\ell(Q_k)\right)^{s_1-mn}}\prod_{i=1}^m\frac{1}{|3^{k_0+1}Q_k|}\int_{3^{k_0+1}Q_k}|\,(f_k)_i\ (\mathrm{y_i})|dy_i$$
$$\lesssim \sum_{k_0=1}^{\infty}\sum_k \frac{2^{j(mn+1+\epsilon-\rho s_1)}}{\left(3^{k_0}\ell(Q_k)\right)^{s_1-mn}}\mathcal{M}(f_k](x) \simeq \sum_k \left(2\ell(Q_k)\right)^{mn-s_1}2^{j(1+\epsilon-(\epsilon)s_1)}\mathcal{M}(\overrightarrow{f_k})(x)\,.$$

Case 2:$\ell(Q_k) < 2^{-j}$. Let $s_2 > mn$ be chosen later. Note that

$$A\big(z,z',f_kf_k,\vec{\xi}\big)=\int_0^1\sum_k \nabla_z\Big(\sigma_k\big(z(1+\epsilon),\vec{\xi}\big)e^{2\pi i(z(1+\epsilon)-\vec{y})\vec{\xi}}\Big)d(1+\epsilon)$$
$$=\sum_{l=1}^{n}(z_l-z_l')\int_0^1\sum_k \Big(\big(\partial_{z_l}\sigma_k\big)\big(z(1+\epsilon),\vec{\xi}\big)+2\pi i\overrightarrow{\xi_{l}\sigma_k}\big(z(1+\epsilon),\vec{\xi}\big)\Big)e^{2\pi i(z(1+\epsilon)-\vec{y})\vec{\xi}}d(1+\epsilon),$$

where $(1+\epsilon)=z'+(1+\epsilon)(z-z')$. It is easy to verify that if σ_k is a symbol of order $(1+\epsilon)$, then $\partial_{x_l}\sigma_k$ and $\overrightarrow{\xi_l}\sigma_k$ are symbols of order $(1+\epsilon)+\delta$ and order $(2+\epsilon)$ respectively. Then for any $\vec{y}\in(3^{k+1})^m\backslash(3^kQ_k)^m$, we by Lemma 3.4 twice deduce that

$$|\int_{\mathrm{R}^{mn}}\sum_k \varphi_j\,(\vec{\xi})A\big(z,z',\vec{y},\vec{\xi}\big)d\vec{\xi}|$$
$$\le\sum_{l=1}^{n}|z_l-z_l'|\int_0^1\{|\int_{\mathrm{R}^{mn}}\sum_k \beta_j^k\,(\vec{\xi})\big(\partial_{z_l}\sigma_k\big)\big(z(1+\epsilon),\vec{\xi}\big)e^{2\pi i(z(1+\epsilon)-y)\vec{\xi}}d\vec{\xi}|$$
$$+2\pi|\int_{\mathrm{R}^{mn}}\sum_k \beta_j^k\,(\vec{\xi})\Big(\overrightarrow{\xi_l}\sigma_k\cdot\big(z(1+\epsilon),\vec{\xi}\big)\Big)e^{2\pi i(z(1+\epsilon)-\vec{y})\vec{\xi}}d\vec{\xi}|\}d(1+\epsilon)$$
$$\lesssim\sum_{l=1}^{n}|z_l-z_1'|\int_0^1\frac{2^{j(mn+1+\epsilon-\rho s_2)}\big(2^{j\delta}+2^j\big)}{(\sum_{i=1}^m|y_i-z(1+\epsilon)|)^{s_2}}d(1+\epsilon)$$
$$\lesssim 3^{-k_0s_2}\sum_k \ell(Q_k)^{1-s_2}2^{j(mn+1+\epsilon-\rho s_2+1)},$$

where we used the fact $\sum_{i=1}^m|y_i-z(1+\epsilon)|\simeq\sum_{i=1}^m|y_i-z|\simeq 3^{k_0}\ell(Q_k)$, for any $(1+\epsilon)\in(0,1)$. This immediately implies that

$$\Xi_{\mathrm{j}}(\overrightarrow{f_k})(z,z')\lesssim\sum_{k_0=1}^{\infty}\int_{(3^{k_0+1})^m\backslash(3^{k_0}Q_k)^m}\sum_k |\int_{\mathrm{R}^{mn}}(\beta_j^k(\vec{\xi})A\big(z,z',\vec{y},\vec{\xi}\big)d\vec{\xi}|\prod_{i=1}^{m}|(f_k)_i(y_i)|d\vec{y}$$
$$\lesssim\sum_{k_0=1}^{\infty}\sum_k \frac{2^{j(mn+2+\epsilon-\rho s_2)}}{3^{k_0(s_2-mn)}\ell(Q_k)^{s_2-mn-1}}\prod_{i=1}^{m}\frac{1}{|3^{k_0+1}Q_k|}\int_{3^{k_0+1}Q_k}|(f_k)_i(y_i)|dy_i$$
$$\lesssim\sum_{k_0=1}^{\infty}\frac{2^{j(mn+2+\epsilon-\rho s_2)}}{3^{k_0(s_2-mn)}\ell(Q_k)^{s_2-mn-1}}\mathcal{M}(\overrightarrow{f_k})(x)$$
$$\simeq\sum_k \Big(2^jl(Q_k)\Big)^{mn+1-s_2}2^{j(1+\epsilon-(\epsilon)s_2)}\mathcal{M}(\overrightarrow{f_k})(x)\,.$$

To ensure the convergence, we choose s_1 and s_2 as follows:

$$\begin{cases}s_1>mn\\1+\epsilon-(\rho-1)s_1\le 0\end{cases}\quad and\quad\begin{cases}mn<s_2<mn+1\\1+\epsilon-(\rho-1)s_2\le 0.\end{cases}\tag{4.3}$$

Consequently, it yields that

$$\Xi(\overrightarrow{f_k})(z,z')\le\sum_{j=0}^{\infty}\sum_k \Xi_{\mathrm{j}}\,(\overrightarrow{f_k})(z,z')$$
$$\lesssim\sum_k\left(\sum_{j:\ell(Q_k)\ge 2^{-j}}\Big(2^j\ell(Q_k)\Big)^{mn-s_1}+\sum_{j:\ell(Q_k)<2^{-j}}\big(2\ell(Q_k)\big)^{mn+1-s_2}\right)\mathcal{M}(\overrightarrow{f_k})(x)$$
$$\simeq\mathcal{M}(\overrightarrow{f_k})(x)\,.$$

This completes the proof.

5. Mixed Weak Type Estimates

We demonstrate Theorem 1.4. For this purpose, we first present a Coifman- Fefferman's inequality.

Proposition5.1._Let_$\sigma_k\in S_{\rho,\delta}^{1+\epsilon}(n,m)$_with_$0\le\rho,\delta\le 1$_and_$(1+\epsilon)<mn(\rho-1)$. _Then, forany_$0\le\epsilon<\infty$_and_$\omega\in A_\infty$, _wehave_

$$||\sum_k T_{\sigma_k}(\overrightarrow{f_k})||_{L^{1+\epsilon}(\omega)}\le(1+\epsilon)\sum_k ||\mathcal{M}(\overrightarrow{f_k})||_{L^{1+\epsilon}(\omega)}.\tag{5.1}$$

Proof. We will use Proposition 4.1 and notations in its proof. Let $\omega\in A_\infty$. Then, it is well known that for any $-\frac{1}{2}<\epsilon<\frac{1}{2}$ there exists $0<\beta^k<1$ such that for any cube Q_k and any measurable subset $A\subset Q_k$ with $|A|\le(1+2\epsilon)|Q_k|$ it holds $\omega(A)\le\beta^k\omega(Q_k)$. So, for the sparsity constant η of S there corresponds $0<\beta^k<1$

such that for $Q_k \in S$, we have

$$\omega(E_{Q_k}) = \omega(Q_k) - \omega(Q_k \backslash E_{Q_k}) \geq (1-\beta^k)\omega(Q_k)\,, \tag{5.2}$$

since $|Q_k \backslash E_{Q_k}| \leq (1-\eta)|Q_k|$. It follows from Proposition 4.1 and (5.2) that

$$||\sum_k T_{\sigma_k}(\overrightarrow{f_k})||_{L^1(\omega)} \leq \sum_k \sum_{Q_k \in S} \prod_{i=1}^{m} \langle (f_k)_i \rangle_{Q_k} \omega(\mathrm{Q_k}) \lesssim \sum_k \sum_{Q_k \in \mathcal{S}} \inf_{x \in Q_k} \mathcal{M}(\overrightarrow{f_k})(x)\omega(E_{Q_k})$$

$$\leq \sum_k \sum_{Q_k \in S} \int_{E_{Q_k}} \mathcal{M}\,(\overrightarrow{f_k})(x)\omega dx \leq \sum_k ||\mathcal{M}(\overrightarrow{f_k})||_{L^{1+\epsilon}(\omega)}^{1+\epsilon}. \tag{5.3}$$

This shows the inequality (5.1) for the case $\epsilon = 0$.

To obtain the result for every $0 \leq \epsilon < \infty$, we apply the A_∞ extrapolation theorem from [14, Corollary 3.15]. Let F be a family of pairs of functions. Suppose that for some $0 \leq \epsilon < \infty$ and for every weight $\omega_0 \in A_\infty$,

$$||\sum_k f_k||_{L^{1+\epsilon}(\omega_0)} \leq C_1 \sum_k \|g_k||_{L^{1+\epsilon}(\omega_0)}, \forall (f_k, g_k) \in F. \tag{5.4}$$

Then for all $0 \leq \epsilon < \infty$ and all $\omega \in A_\infty$,

$$\|\sum_k f_k||_{L^{1+\epsilon}(\omega)} \leq C_2 \sum_k \|g_k||_{L^{1+\epsilon}(\omega)}, \forall (f_k, g_k) \in \mathcal{F}'. \tag{5.5}$$

Note that (5.3) corresponds to (5.4) for $\epsilon = 0$. As a consequence, (5.1) follows from (5.5).

Proof of Theorem 1.4 (see [52]). We use a hybrid of the arguments in [13] and [35]. Define

$$Rh(x) = \sum_{j=0}^{\infty} \frac{S^j h(x)}{(2K)^j},$$

where $K > 0$ will be chosen later and

$$Sf_k(x) = \frac{M(f_k \mu)(x)}{\mu(x)}$$

if $\mu(x) \neq 0, Sf_k(x) = 0$ otherwise. It immediately yields that

$$h(x) \leq Rh(x)\,; \tag{5.6}$$

$$S(Rh)(x) \leq 2KRh(x)\,. \tag{5.7}$$

Moreover, we claim that for some $\epsilon > 0, Rh \cdot \mu v^{\overline{\frac{1}{m(1+\epsilon)'}}} \in A_\infty$ and

$$||Rh||_{L^{(1+\epsilon)',1}\left(\mu v^{\frac{1}{m}}\right)} \leq 2\|h||_{L^{(1+\epsilon)',1}\left(\mu v^{\frac{1}{m}}\right)}. \tag{5.8}$$

The proofs will be given at the end of this section.

Note that

$$||f_{\mathrm{k}}^{1+2\epsilon}||_{L^{1+\epsilon,\infty}(\omega)} = ||f_k||_{L^{(1+\epsilon)(1+2\epsilon).\infty}(\omega)}^{1+2\epsilon}, \qquad 0 \leq \epsilon < \infty. \tag{5.9}$$

This implies that

$$\|\frac{T_{\sigma_k}(\overrightarrow{f_k})}{v}\|_{L^{\frac{1}{m},\infty}\left(\mu v^{\frac{1}{m}}\right)}^{\frac{1}{m(1+\epsilon)}} = \|\left(\frac{T_{\sigma_k}(\overrightarrow{f_k})}{v}\right)^{\frac{1}{m(1+\epsilon)}}\|_{L^{1+\epsilon,\infty}\left(\mu v^{\frac{1}{m}}\right)}$$

$$= \sup_{||h||_{L^{(1+\epsilon)',1}}\left(\mu v^{\frac{1}{m}}\right)=1} |\int_{\mathrm{R}^n} \sum_k |T_{\sigma_k}(\overrightarrow{f_k})(x)|^{\frac{1}{m(1+\epsilon)}} h(x)\mu(x)v(x)^{\overline{\frac{1}{m(1+\epsilon)'}}}dx|$$

$$\leq \sup_{||h||_{L^{(1+\epsilon)',1}}\left(\mu v^{\frac{1}{m}}\right)=1} \int_{\mathrm{R}^n} \sum_k |T_{\sigma_k}(\overrightarrow{f_k})(x)|^{\frac{1}{m(1+\epsilon)}} Rh(x)\mu(x)v(x)^{\overline{\frac{1}{m(1+\epsilon)'}}}dx.$$

Invoking Proposition 5.1 and Hölder's inequality, we obtain

$$\int_{\mathrm{R}^n} \sum_k |T_{\sigma_k}(\overrightarrow{f_k})(x)|^{\frac{1}{m(1+\epsilon)}} Rh(x)\mu(x)v(x)^{\overline{\frac{1}{m(1+\epsilon)'}}}dx$$

$$\lesssim \int_{\mathrm{R}^n} \sum_k \mathcal{M}\,(\overrightarrow{f_k})(x)^{\frac{1}{m(1+\epsilon)}} Rh(x)\mu(x)v(x)^{\overline{\frac{1}{m(1+\epsilon)'}}}dx = \int_{\mathrm{R}^n} \sum_k \left(\frac{\mathcal{M}(\overrightarrow{f_k})(x)}{v(x)}\right)^{\frac{1}{m(1+\epsilon)}} Rh(x)\mu(x)v(x)^{\frac{1}{m}}dx$$

$$\leq \sum_k \|\left(\frac{\mathcal{M}(\overrightarrow{f_k})}{v}\right)^{\frac{1}{m(1+\epsilon)}}\|_{L^{(1+\epsilon),\infty}\left(\mu v^{\frac{1}{m}}\right)} ||Rh||_{L^{(1+\epsilon)',1}\left(\mu v^{\frac{1}{m}}\right)} \leq \sum_k \|\frac{\mathcal{M}(f_k)}{v}\|_{L^{\frac{1}{m},\infty}\left(\mu v^{\frac{1}{m}}\right)}^{\frac{1}{m(1+\epsilon)}} \|h||_{L^{(1+\epsilon)',1}\left(\mu v^{\frac{1}{m}}\right)},$$

where we used (5.9) and (5.8) in the last inequality. Here we need to apply the weighted mixed weak type estimates for $\mathcal{M}$ proved in Theorems 1.4 and 1.5 in [35]. Consequently, collecting the above estimates, we get the desired result

$$\|\sum_k \frac{T_{\sigma_k}(\overrightarrow{f_k})}{v}\|_{L^{\frac{1}{m},\infty}\left(\mu v^{\frac{1}{m}}\right)} \lesssim \sum_k \|\frac{\mathcal{M}(\overrightarrow{f_k})}{v}\|_{L^{\frac{1}{m},\infty}\left(\mu v^{\frac{1}{m}}\right)} \lesssim \sum_k \prod_{i=1}^{m} ||(f_k)_i\|_{L^1(\omega_i)}.$$

It remains to show our foregoing claim. The proof follows the same scheme of that in [13]. For the sake of completeness we here give the details. Together with **Lemma 2.3**, the hypothesis (1) or (2) indicates that $\mu \in A_1$ and $v^{\frac{1}{m}} \in A_\infty$. The former implies that

$$\|\sum_k Sf_k\|_{L^\infty\left(\mu v^{\frac{1}{m}}\right)} \le [\mu]_{A_1} \sum_k ||f_k||_{L^\infty\left(\mu v^{\frac{1}{m}}\right)}. \quad (5.10)$$

The latter yields that $v^{\frac{1}{m}} \in A_{1+\epsilon}$ for some $\epsilon > 1$. It follows from $A_{1+\epsilon}$ factorization theorem that there exist v_1 , $v_2 \in A_1$ such that $v^{\frac{1}{m}} = v_1 v_2^{-\epsilon}$.

Additionally, it follows from Lemma 2.3 in [13] that if u_1 , $u_2 \in A_1$, then there exists $\varepsilon_0 = \varepsilon_0\left([u_1]_{A_1}, [u_2]_{A_1}\right) \in (0,1)$ such that $u_1 v_1^{\varepsilon} \in A_{1+\epsilon}$ and $u_2 v_2^{\varepsilon} \in A_{1+2\epsilon}$ for any $0 < \varepsilon < \varepsilon_0, v_1 \in A_{1+\epsilon}$ and $v_2 \in A_{1+2\epsilon}$, $0 \le \epsilon < \infty$. Then $\mu v_2^{\frac{q_0 - 1}{\epsilon}} \in A_1$ if we set $(1+\epsilon) > 1 + \epsilon/\varepsilon_0$. Thus, we have

$$\mu^\epsilon v^{\frac{1}{m}} = v_1 \left(\mu v_2\right)^\epsilon \in A_{1+\epsilon}.$$

It immediately implies that

$$||\sum_k Sf_k\|_{L^{1+\epsilon}\left(\mu v^{\frac{1}{m}}\right)} = ||\sum_k M(f_k \mu)||_{L^{1+\epsilon}\left(\mu^{-\epsilon} v^{\frac{1}{m}}\right)} \le c_1 \sum_k ||f_k||_{L^{1+\epsilon}\left(\mu v^{\frac{1}{m}}\right)}. \quad (5.11)$$

By (5.10), (5.11) and Marcinkiewicz interpolation in [13, Proposition A.1], we have S is bounded on $L^{1+\epsilon,1}\left(\mu v^{\frac{1}{m}}\right), 1+\epsilon \in (1+\epsilon, \infty)$ with the constant

$$K(1+\epsilon) = 2^{\frac{1}{1+\epsilon}} \left(c_1 \left(\frac{1}{p_0} - \frac{1}{1+\epsilon} \right)^{-1} + c_2 \right),$$

and $c_2 := [v]_{A_1}$. Note that $K(1+\epsilon)$ is decreasing with respect to $(1+\epsilon)$. Hence, we obtain

$$||\sum_k Sf_k\|_{L^{1+\epsilon.1}\left(\mu v^{\frac{1}{m}}\right)} \le K \sum_k \|f_k\|_{L^{1+\epsilon.1}\left(\mu v^{\frac{1}{m}}\right)}, \qquad \epsilon \le -1 \quad (5.12)$$

where

$$K := 4(1+\epsilon)(c_1 + c_2) > K(2(1+\epsilon)) \ge K(1+\epsilon).$$

The inequality (5.7) indicates that $Rh \cdot \mu \in A_1$ with $[Rh \cdot \mu]_{A_1} \le 2K$. Let $0 < \varepsilon < min\left\{\varepsilon_0, \frac{1}{2(1+\epsilon)}\right\}$, and $(1+\epsilon) = \left(\frac{1}{\varepsilon}\right)'$. Then $(Rh \cdot \mu) v_1^{\varepsilon} \in A_1$, and the inequality (5.8) follows from (5.12). By $A_{1+\epsilon}$ factorization theorem again, we obtain

$$Rh \cdot \mu v^{\frac{1}{m(1+\epsilon)'}} = [(Rh \cdot \mu) v_1^{\varepsilon}] \cdot v_2^{1-[(\epsilon)\varepsilon+1]} \in A_{(\epsilon)\varepsilon+1} \subset A_\infty.$$

This completes the proof.

6. Local Estimates

We demonstrate Theorem 1.2 and Theorem 1.3. We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2 (see [52]). It follows from (2.7) and Theorem 3.3 that for any $0 < \gamma < 1/m$,

$$\sum_k m_{\tau_{\sigma_k}(\overrightarrow{f_k})}(Q_k) \le \sum_k \left(\frac{2}{|Q_k|}\int_{Q_k} |T_{\sigma_k}(\overrightarrow{f_k})(x)|^\gamma dx\right)^{\frac{1}{\gamma}} \le c_1 \sum_k ||T_{\sigma_k}(\overrightarrow{f_k})||_{L^{\frac{1}{m},\infty}\left(Q_k, \frac{dx}{|Q_k|}\right)}$$

$$\le c_1 \sum_k \prod_{i=1}^{m} \frac{1}{|Q_k|} \int_{Q_k} |(f_k)_i(x)| dx \le c_1 \mathcal{M}(\overrightarrow{f_k})(x), x \in Q_k.$$

Observe in [43] that given a cube $Q_k, \gamma > 0$ and $0 < \lambda < 1$, there exists a constant $(1+\epsilon) = c_\lambda$ such that

$$\sum_k M^{\#}_{\lambda;Q_k}\left((f_k)1_{Q_k}\right)(x) \le (1+\epsilon) M^{\#}_\gamma\left((f_k)1_{Q_k}\right)(x), x \in Q_k.$$

Here $M^{\#}_\gamma$ is the sharp maximal function of Fefferman and Stein,

$$M^{\#}_\gamma f_k(x) = \sum_k M^{\#}(|f_k|^\gamma)^{1/\gamma}(x),$$

where

$$M^{\#} f_k(x) = \sup_{x \in Q_k} \inf_{1+\epsilon} \sum_k \frac{1}{|Q_k|} \int_{Q_k} |f_k(y) - (1+\epsilon)| dy.$$

Thus, it yields that

$$\sum_k M^{\#}_{\lambda;Q_k}(T_{\sigma_k}(\overrightarrow{f_k}))(x) \le c_2 \sum_k M^{\#}_{\gamma}(T_{\sigma_k}(f_k])(x) \le c_2 \sum_k \mathcal{M}(f_k](x)\,,$$

where the latter is contained in (4.2). Additionally, the following Fefferman- Stein inequality was obtained in [43]:

$$\left|\{x \in Q_k; \left|f_k(x) - m_{f_k}(Q_k)\right| > (1+\epsilon)M^{\#}_{\lambda_n;Q_k}(f_k)(x)\}\right| \le c_3 d^{3(1+\epsilon)}|Q_k|, \epsilon \ge 0,$$

where $\lambda_n = 2^{-n-2}$. Consequently, we deduce that for $(1+\epsilon) > c_1$

$$|\{x \in Q_k; |T_{\sigma_k}(\overrightarrow{f_k})(x)| > (1+\epsilon)\mathcal{M}(\overrightarrow{f_k})(x)\}| \le |\{x \in Q_k; |T_{\sigma_k}(\overrightarrow{f_k})(x) - m_{T_{\sigma_k}(\overrightarrow{f_k})}(Q_k)|$$
$$> (1+\epsilon - c_1)\mathcal{M}(\overrightarrow{f_k})(x)\}| \le |\{x \in Q_k; |T_{\sigma_k}(\overrightarrow{f_k})(x) - m_{\tau_{\sigma_k}(\overrightarrow{f_k})}(Q_k)|$$
$$> c_2^{-1}(1+\epsilon - c_1)M^{\#}_{\lambda_n;Q_k}(T_{\sigma_k}(\overrightarrow{f_k}))(x)\}| \le c_3 e^{-\beta^k(1+\epsilon-c_1)/c_2}|Q_k|.$$

So, taking $(1+\epsilon) = \max\{1, c_3\}e^{\beta^k c_1/c_2}$ and $(1+2\epsilon) = \beta^k/c_2$, we obtain

$$|\{x \in Q_k; |T_{\sigma_k}(\overrightarrow{f_k})(x)| > (1+\epsilon)\mathcal{M}(\overrightarrow{f_k})(x)\}| \le (1+\epsilon)e^{-(1+2\epsilon)(1+\epsilon)}|Q_k| \, for \ \epsilon \ge 0.$$

This shows Theorem 1.2.

We devoted to the proof of Theorem 1.3. To this end, we first present some necessary estimates.

Lemma 6.1([43]). *Let* $0 < \gamma < 1$, Q_k *be a fixed cube, and* f_k *be a measurable function such that* $\operatorname{supp}(f_k) \subset Q_k$. *If* $\omega \in A_{1+\epsilon}(0 \le \epsilon < \infty)$, *then there holds*

$$\|\sum_k \left(f_k - m_{f_k}(Q_k)\right)\|_{L^1(Q_k,\omega)} \le (1+\epsilon)\cdot 2^{(1+\epsilon)}[\omega]_{A_{1+\epsilon}} \sum_k \|M^{\#}_{\gamma}(f_k)\|_{L^1(Q_k,\omega)}.$$

Proposition 6.2. *Let* $0 < \gamma < \min\{\varepsilon, 1/m\}, 0 \le \rho, \delta \le 1, (1+\epsilon) < mn(\rho - 1)$ *and* $\in S^{1+\epsilon}_{\rho,\delta}(n,m)$. *Then there holds that*

$$\sum_k M^{\#}_{\gamma}(T_{\sigma_k,\Sigma b}(\overrightarrow{f_k}))(x) \le C_{\varepsilon,\gamma}||\mathrm{b}||_{BMO}(\sum_{j=1}^{m}\sum_k \mathcal{M}^{j}_{L(\log L)}(\overrightarrow{f_k})(x) + M_\varepsilon\left(T_{\sigma_k}(\overrightarrow{f_k})\right)(x)\right).$$

Proof. It suffices to prove the commutator with only one symbol:

$$T^j_{\sigma_k,b}(\overrightarrow{f_k})(x) := b(x)T_{\sigma_k}(\overrightarrow{f_k})(x) - T_{\sigma_k}\left((f_k)_1, \dots, b(f_k)_j, \dots, (f_k)_m\right)(x)\,.$$

since an iterative procedure will give the general case. For any constant λ, since T_{σ_k} is a bilinear operator, there holds that

$$T^j_{\sigma_k,b}(\overrightarrow{f_k})(x) = (b(x)-\lambda)T_{\sigma_k}(\overrightarrow{f_k})(x) - T_{\sigma_k}\left((f_k)_1, \dots, (b-\lambda)(f_k)_j, \dots, (f_k)_m\right)(x)\,.$$

For a fixed cube Q_k containing x and a constant c_{Q_k}, we have

$$\sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} \left| |T^j_{\sigma_k,b}(\overrightarrow{f_k})(z)|^\gamma - |c_{Q_k}|^\gamma\right| dz\right)^{\frac{1}{\gamma}}$$
$$\le \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} |T^j_{\sigma_k,b}(\overrightarrow{f_k})(z) - c_{Q_k}|^\gamma dz\right)^{\frac{1}{\gamma}} \le S_1 + S_2,$$

where

$$S_1 = \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} |(b(z)-\lambda)T_{\sigma_k}(\overrightarrow{f_k})(z)|^\gamma dz\right)^{\frac{1}{\gamma}}$$

and

$$S_2 = \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} |T_{\sigma_k}\left((f_k)_1, \dots, (b-\lambda)(f_k)_j, \dots, (f_k)_m\right)(z) - c_{Q_k}|^\gamma dz\right)^{\frac{1}{\gamma}}$$

Let $\lambda = (b)_{3Q_k}$ be the average of b on $3Q_k$. Select $1 < u < \varepsilon/\gamma$. Then it follows from Hölder's inequality and (2.3) that

$$S_1 \le \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} |b(z)-\lambda|^{\gamma u'} dz\right)^{\frac{1}{\gamma u'}}$$
$$\left(\frac{1}{|Q_k|}\int_{Q_k} |T_{\sigma_k}(f_k)|^{\gamma u} dz\right)^{\frac{1}{\gamma u}} \lesssim ||b||_{BMO} M_\varepsilon(T_{\sigma_k}(\overrightarrow{f_k}))(x)\,.$$

To analyze the contribution of S_2, we split $S_2 \le s_2' + s_2''$, where

$$s_2' = \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} |T_{\sigma_k}\left((f_k)_1 1_{3Q_k}, \dots, (b-\lambda)(f_k)_j 1_{3Q_k}, \dots, (f_k)_m 1_{3Q_k}\right)(z)|^\gamma dz\right)^{\frac{1}{\gamma}}$$

and

$$s_2'' = \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} |\,\varepsilon\big((f_k)_1, \ldots, (b-\lambda)(f_k)_j, \ldots, (f_k)_m\big)(z) - c_{Q_k}|^{\gamma} dz\right)^{\frac{1}{\gamma}}$$

where the bilinear operator ε is given by

$$\varepsilon(\overrightarrow{f_k})(z) = T_{\sigma_k}(\overrightarrow{f_k})(z) - T_{\sigma_k}(\overrightarrow{f_k} \cdot 1_{3Q_k})(z)\,.$$

Recall that $1 < \gamma < 1/m$. Applying Hölder's inequality, Kolmogorov's inequality, Theorem 3.3, the inequalities (2.5) and (2.4), we derive that

$$s_2' = \|\sum_k T_{\sigma_k}\big((f_k)_1 1_{3Q_k}, \ldots, (b-\lambda)(f_k)_j 1_{3Q_k} \ldots, (f_k)_m 1_{3Q_k}\big)\|_{L^{\gamma}\left(Q_k, \frac{dz}{|Q_k|}\right)}$$

$$\lesssim \sum_k \|T_{\sigma_k}\big((f_k)_1 1_{3Q_k}, \ldots, (b-\lambda)(f_k)_j 1_{3Q_k} \ldots, (f_k)_m 1_{3Q_k}\big)\|_{L^{1/m,\infty}\left(Q_k, \frac{dz}{|Q_k|}\right)}$$

$$\lesssim \sum_k \frac{1}{|Q_k|}\int_{Q_k} |\,b(y)-\lambda)(f_k)_j(y)|dy \times \prod_{i\neq j}\frac{1}{|Q_k|}\int_{Q_k} |\,(f_k)_i(y_i)|dy_j$$

$$\lesssim \sum_k ||b||_{BMO}||(f_k)_j||_{L(\log L),Q_k} \times \prod_{i\neq j}^{m}\langle\,|(f_k)_i|\rangle_{Q_k} \lesssim ||b||_{BMO}\mathcal{M}^j_{L(\log L)}(\overrightarrow{f_k})(x)\,.$$

It only remains to dominate S_2 If we choose

$$c_{Q_k} = \varepsilon\big((f_k)_1, \ldots, (b-\lambda)(f_k)_j, \ldots, (f_k)_m\big)(z')$$

for some given point $z' \in Q_k$, then there holds that

$$|\varepsilon\big((f_k)_1, \ldots, (b-\lambda)(f_k)_j, \ldots, (f_k)_m\big)(z) - c_{Q_k}| = \Xi\big((f_k)_1, \ldots, (b-\lambda)(f_k)_j, \ldots, (f_k)_m\big)(z,z')\,,$$

where $\Xi(\overrightarrow{f_k})(z,z')$ is defined in (4.1). A routine application of the method used in the proof of (4.2) gives us that

$$\Xi\big((f_k)_1, \ldots, (b-\lambda)(f_k)_j, \ldots, (f_k)_m\big)(z,z')$$

$$\lesssim \sum_k \left(\sum_{l:l(Q_k)\geq 2^{-l}}\sum_{k_0=1}^{\infty}\frac{2^{l(mn+1+\epsilon-\rho s_1)}}{\big(3^{k_0}\ell(Q_k)\big)^{s_1-mn}} + \sum_{l:l(Q_k)<2^{-l}}\sum_{k_0=1}^{\infty}\frac{2^{l(mn+2+\epsilon-\rho s_2)}}{3^{k_0(s_2-mn)}\ell(Q_k)^{s_2-mn-1}}\right)$$

$$\times \frac{1}{|3^{k_0+1}Q_k|}\int_{3^{k_0+1}Q_k} |\,b(y)-\lambda||(f_k)_j(y)|dy \times \prod_{i\neq j}\frac{1}{|3^{k_0+1}Q_k|}\int_{3^{k_0+1}Q_k} |\,(f_k)_i(y_i)|dy_i$$

$$\lesssim \sum_k \left(\sum_{l:l(Q_k)\geq 2^{-l}}\sum_{k_0=1}^{\infty}\frac{2^{l(mn+1+\epsilon-\rho s_1)}}{(3^{k}\ell)(Q_k))^{s_1-mn}} + \sum_{l:l(Q_k)<2^{-l}}\sum_{k_0=1}^{\infty}\frac{2^{l(mn+2+\epsilon-\rho s_2)}}{3^{k_0(s_2-mn)}\ell(Q_k)^{s_2-mn-1}}\right)$$

$$\times ||b||_{BMO}||(f_k)_j||_{L(\log L),3^{k_0+1}Q_k} \times \prod_{i\neq j}\langle\,|(f_k)_i|\rangle_{3^{k_0+1}Q_k} \lesssim ||b||_{BMO}\sum_k \mathcal{M}^j_{L(\log L)}(\overrightarrow{f_k})(x),$$

where s_1 and s_2 are defined in (4.3). Thus we complete the proof of this proposition. The following local Coifman- Fefferman inequality provides us a possibility to establish local estimates for the commutator $T_{\sigma_k,\Sigma \mathrm{b}}$.

Proposition 6.3.*Let*$\sigma_k \in S^{1+\epsilon}_{\rho,\delta}(n,m)$*with*$0 \leq \rho,\delta \leq 1$*and*$(1+\epsilon) < mn(\rho-1)$. *Let*$\omega \in A_{1+\epsilon}$*with*$0 \leq \epsilon < \infty$, *and*$\mathrm{b} \in BMO^m$. *Let*$\overrightarrow{f_k}$ be functions*suchthat*$\mathrm{supp}\big((f_k)_j\big) \subset Q_k, j = 1, \ldots, m$.*Thenwehave*

$$||\sum_k T_{\sigma_k,\Sigma \mathrm{b}}(\overrightarrow{f_k})||_{L^1(Q_k,\omega)} \leq c_n||\mathrm{b}||_{BMO}[\omega]^2_{A_{1+\epsilon}}\sum_k ||\mathcal{M}_{L(\log L)}(\overrightarrow{f_k})||_{L^1(Q_k,\omega)}.$$

Proof. By homogeneity, we may assume that $\langle|(f_k)_j|\rangle_{Q_k} = 1, j = 1, \ldots, m$. It follows from Lemma 6.1that

$$||\sum_k T_{\sigma_k,\Sigma \mathrm{b}}(\overrightarrow{f_k})||_{L^1(Q_k,\omega)} \leq [\omega]_{A_{1+\epsilon}}\sum_k ||M^{\#}_{\gamma}(T_{\sigma_k,\Sigma \mathrm{b}}\overrightarrow{f_k})||_{L^1(Q_k,\omega)}$$

$$+\sum_k m_{\tau_{(\sigma_k)j\Sigma \mathrm{b}(\overrightarrow{f_k})}}(Q_k)\omega(Q_k)\,. \tag{6.1}$$

In order to control the first term, we use Proposition 6.2. Thus, it immediately yields that

$$||\sum_k M^{\#}_{\gamma}(T_{\sigma_k},\Sigma \mathrm{b}\overrightarrow{f_k})||_{L^1(Q_k,\omega)} \lesssim ||\mathrm{b}||_{BMO}\sum_k (\|\mathcal{M}_{L(\ \mathrm{Log}L)}(\overrightarrow{f_k})||_{L^1(Q_k,\omega)} + \sum_k ||M_{\varepsilon}(T_{\sigma_k}(\overrightarrow{f_k}))||_{L^1(Q_k,\omega)}).$$

Applying Lemma 6.1again, we deduce that

$$\sum_k \|M_{\varepsilon}(\tau_{\sigma_k}(\overrightarrow{f_k}))||_{L^1(Q_k,\omega)} \leq [\omega]_{A_{1+\epsilon}}\sum_k ||M^{\#}_{\gamma}(M_{\varepsilon}(T_{\sigma_k}\overrightarrow{f_k}))||_{L^1(Q_k,\omega)} + \sum_k m_{M_{\varepsilon}(T_{\sigma_k}f_k 3^{(Q_k)\omega(Q_k)}}.$$

Note two facts:

$$M_{\gamma}^{\#,d}(M_{\varepsilon}^{d} f_k)(x) \le C_{\varepsilon,\delta} M!^{d} f_k(x), 0<\gamma<\varepsilon<1,$$

and

$$M_{\varepsilon}^{\#}(T_{\sigma_k}(\overrightarrow{f_k})(x) \le C_{\varepsilon}\mathcal{M}(f_k](x), 0<\varepsilon<1/m,$$

which were shown in [42] and in the proof of the inequality (4.2) respectively. Hence, we obtain

$$||\sum_k M_{\gamma}^{\#}(M_{\varepsilon}(T_{\sigma_k}\overrightarrow{f_k}))||_{L^1(Q_k,\omega)} \lesssim \sum_k \|M_{\varepsilon}^{\#}(T_{\sigma_k}\overrightarrow{f_k})||_{L^1(Q_k,\omega)} \lesssim \sum_k \|\mathcal{M}(\overrightarrow{f_k})||_{L^1(Q_k,w)}.$$

Moreover, we have by Kolmogorov's inequality twice that

$$\begin{aligned}\sum_k m_{M_{\varepsilon}(T_{\sigma_k}f_k]}(Q_k) &\lesssim \sum_k ||M_{\varepsilon}(T_{\sigma_k}\overrightarrow{f_k})||_{L^{\gamma}\left(Q_k,\frac{dx}{|Q_k|}\right)} = \sum_k ||M(|T_{\sigma_k}\overrightarrow{f_k}|^{\varepsilon})||^{1/\varepsilon}_{L^{\gamma/\varepsilon}\left(Q_k,\frac{dx}{|Q_k|}\right)} \\ &\lesssim \sum_k ||M(|T_{\sigma_k}\overrightarrow{f_k}|^{\varepsilon})\|^{1/\varepsilon}_{L^{1,\infty}\left(Q_k,\frac{dx}{|Q_k|}\right)} \lesssim \sum_k |||T_{\sigma_k}\overrightarrow{f_k}|^{\varepsilon}||^{1/\varepsilon}_{L^1\left(Q_k,\frac{dx}{|Q_k|}\right)} \\ &= \sum_k ||T_{\sigma_k}(\overrightarrow{f_k})||_{L^{\varepsilon}\left(Q_k,\frac{dx}{|Q_k|}\right)} \lesssim \sum_k ||T_{\sigma_k}(\overrightarrow{f_k})||_{L^{1/m,\infty}\left(Q_k,\frac{dx}{|Q_k|}\right)} \\ &\lesssim \sum_k \prod_{i=1}^{m}\frac{1}{|Q_k|}\int_{Q_k}|(f_k)_i|dx \le \inf\sum_k \mathcal{M}(\overrightarrow{f_k})(x),\end{aligned}$$

which implies that

$$m_{M_{\varepsilon}(T_{\sigma_k}\overrightarrow{f_k})}(Q_k)\omega(Q_k) \lesssim \int_{Q_k}\sum_k \mathcal{M}(\overrightarrow{f_k})(x)\omega(x)dx.$$

Collecting the above estimates, we have

$$||\sum_k M_{\varepsilon}(T_{\sigma_k}(\overrightarrow{f_k}))||_{L^1(Q_k,\omega)} \lesssim \sum_k \|\mathcal{M}(\overrightarrow{f_k})||_{L^1(Q_k,\omega)}.$$

Then we deduce that

$$||\sum_k M_{\gamma}^{\#}(T_{\sigma_k},\Sigma \mathrm{b}\overrightarrow{f_k})\|_{L^1(Q_k,\omega)} \le \sum_k ||\mathrm{b}||_{BMO}[\omega]_{A_{1+\epsilon}}\|\mathcal{M}_{L(\log L)}(\overrightarrow{f_k})||_{L^1(Q_k,\omega)}. \tag{6.2}$$

Next, we bound $m_{T_{(\sigma_k)\Sigma \mathrm{b}(\overrightarrow{f_k})}}(Q_k)\omega(Q_k)$. To this end, we will use the endpoint weak estimate for the iterated commutators. That is, for any $\vec{\omega} \in A_{\vec{1}}$, there holds

$$v_{\vec{\omega}}(\{x\in\mathbb{R}^n: T_{\sigma_k,\Sigma\mathrm{b}}(\overrightarrow{f_k})(x) > 1+\epsilon\}) \lesssim \left(\prod_{i=1}^{m}\int_{\mathrm{R}^n}\Phi_k\left(\frac{|(f_k)_i(x)|}{(1+\epsilon)^{1/m}}\right)\omega_i(x)dx\right)^{1/m} \tag{6.3}$$

where $1=(1+\log^{+}(1+\epsilon))$. In the proof of (6.3), the main ingredient is Proposition 6.2 and a weak type estimate for $\mathcal{M}^{j}_{L(\log L)}$. The latter is contained in [30, Theorem 3.17]. The rest of its proof is a routine application of Fefferman- Stein inequality. We omit the details. Accordingly, by (6.3), Hölder's inequality and the submultiplicativity of Φ_k, we derive that for any $\epsilon \ge 0$ and $A \ge 1$

$$m_{T_{(\sigma_k)j\Sigma b}(f_k 3^{(Q_k)})^{(1+\epsilon)}} \leq \sum_k \frac{1}{|Q_k|}\int_{Q_k} |T_{\sigma_k,\Sigma b}(\overrightarrow{f_k})(x)|^{1+\epsilon}dx = \sum_k \frac{1}{|Q_k|}\int_0^\infty (1+\epsilon)\,|\{x \in Q_k$$
$$: |T_{\sigma_k,\Sigma b}(\overrightarrow{f_k})(x)| > 1+\epsilon\}|d(1+\epsilon)$$
$$\lesssim A^{1+\epsilon} + \sum_k \frac{1}{|Q_k|}\int_A^\infty (1+\epsilon)\left(\prod_{i=1}^m \int_{Q_k} \Phi_k\left(\frac{|(f_k)_i(x)|}{(1+\epsilon)^{\frac{1}{m}}}\right)dx\right)^{\frac{1}{m}} d(1+\epsilon)$$
$$\leq A^{1+\epsilon} + \sum_k \prod_{i=1}^m \left(\frac{1}{|Q_k|}\int_{Q_k}\int_A^\infty (1+\epsilon)\,\Phi_k\left(\frac{|(f_k)_i(x)|}{(1+\epsilon)^{1/m}}\right)d(1+\epsilon)dx\right)^{1/m}$$
$$\leq A^{1+\epsilon}$$
$$+\sum_k \prod_{i=1}^m \left(\frac{1}{|Q_k|}\int_{Q_k}\Phi_k\,(|(f_k)_i(x)|)dx \int_A^\infty (1+\epsilon)^\epsilon\,\Phi_k\left(\frac{1}{(1+\epsilon)^{1/m}}\right)d(1+\epsilon)\right)^{1/m}$$
$$\lesssim A^{1+\epsilon}$$
$$+\sum_k \prod_{i=1}^m \left(\frac{1}{|Q_k|}\int_{Q_k}\Phi_k\,(|(f_k)_i(x)|)dx \int_A^\infty (1+\epsilon)^{2+\left(\epsilon-\frac{1}{m}\right)}\left(1+\log^+\frac{1}{(1+\epsilon)^{1/m}}\right)d(1$$
$$+\epsilon)\Bigg)^{1/m} \lesssim A^{1+\epsilon} + A^{(1+\epsilon)-\frac{1}{m}}\sum_k \prod_{i=1}^m \left(\frac{1}{|Q_k|}\int_{Q_k}\Phi_k\,(|(f_k)_i(x)|)dx\right)^{1/m}$$

Choosing

$$A = \sum_k \prod_{i=1}^m \frac{1}{|Q_k|}\int_{Q_k}\Phi_k\,(|(f_k)_i(x)|)dx \geq \sum_k \prod_{i=1}^m \langle\,|(f_k)_i|\rangle_{Q_k} = 1,$$

we obtain that

$$m_{T_{\sigma_k \Pi b}(f_k]}(Q_k) \lesssim A = \sum_k \prod_{i=1}^m \frac{1}{|Q_k|}\int_{Q_k}\Phi_k\,(|(f_k)_i(x)|)dx.$$

Note that for each $k \in \mathbb{N}_+$

$$\int_{Q_k}\sum_k |\,f_k(x)|(1+\log^+|f_k(x)|)^k dx \lesssim \sum_k \int_{Q_k} M\,f_k(x)\big(1+\log^+ Mf_k(x)\big)^{k-1}dx,$$

and

$$\sum_k \frac{1}{|Q_k|}\int_{Q_k} M\,f_k(x)dx \lesssim \sum_k ||f_k||_{L(\log L),Q_k}.$$

The former one is in [44] and the latter is in [45]. Therefore, it yields that

$$m_{\tau_{(\sigma_k)\Sigma b(\overrightarrow{f_k})}}(Q_k)\omega(Q_k) \lesssim \sum_k \prod_{i=1}^m \frac{1}{|Q_k|}\int_{Q_k} M\,(f_k)_i(x)dx\omega(Q_k) \leq \sum_k \prod_{i=1}^m ||(f_k)_i||_{L(\log L),Q_k}\omega(Q_k)$$
$$\leq \inf \mathcal{M}_{L(\log L)(\overrightarrow{f_k})(x)\omega(Q_k)\leq ||\mathcal{M}_{L(\log L)}(\overrightarrow{f_k})||_{L^1(Q_k,\omega)}}. \quad (6.4)$$

Our desired result follows from (6.1), (6.2) and (6.4).

Proof of Theorem 1.3 (see [52]). Define

$$R(h) = \sum_{k_0=0}^\infty \frac{1}{2^{k_0}}\frac{M^{k_0}h}{||M||^{k_0}_{L^{1+\epsilon}(\mathrm{R}^n)\to L^{1+\epsilon}(\mathrm{R}^n)}}, \epsilon > 1.$$

Then it is obvious that $h \leq R(h)$ and $||Rh||_{L^{1+\epsilon}(\mathrm{R}^n)} \leq 2||h||_{L^{1+\epsilon}(\mathrm{R}^n)}$. Moreover, for any nonnegative$h \in L^{1+\epsilon}(\mathbb{R}^n)$, we have that $Rh \in A_1$ with

$$[Rh]_{A_1} \leq 2||M||_{L^{1+\epsilon}(\mathrm{R}^n)\to L^{1+\epsilon}(\mathrm{R}^n)} \leq c_n(1+\epsilon)'.$$

Let $\epsilon > 1$ be chosen later. By Riesz theorem, we obtain that for some nonnegative function h with $||h||_{L^{(1+2\epsilon)'}(Q_k)} = 1$,

$$\mathcal{A}:=|\{x\in Q_k: T_{\sigma_k,\Sigma b}(\overrightarrow{f_k})(x)>(1+\epsilon)\mathcal{M}(M(f_k)_1, M(f_k)_m)(x)\}|^{1/1+2\epsilon}$$
$$\le \frac{1}{1+\epsilon}\sum_k \|\frac{T_{\sigma_k,\Sigma b},(\overrightarrow{f_k})}{\mathcal{M}(M(f_k)_1,..,M(f_k)_m)}\|_{L^{1+2\epsilon}(Q_k)}$$
$$\le \frac{1}{1+\epsilon}\int_{Q_k}\sum_k |T_{\sigma_k,\Sigma b}(\overrightarrow{f_k})(x)|\frac{h(x)}{\mathcal{M}(M(f_k)_1,\dots,M(f_k)_m)(x)}dx$$
$$\le (1+\epsilon)^{-1}\sum_k \|T_{\sigma_k,\Sigma b}(\overrightarrow{f_k})\|_{L^1(Q_k,\omega)},$$

where $\omega=\omega_1\omega_2^{\epsilon},\omega_1(x)=Rh(x)$ and $\omega_2(x)=\mathcal{M}(M(f_k)_1,\dots,M(f_k)_m)(x)^{p'-1}$. Moreover, ω_1 , $\omega_2\in A_1$ implies that $\omega=\omega_1\omega_2^{\epsilon}\in A_{1+\epsilon}$ and

$$[\omega]_{A_{1+\epsilon}}\le[\omega_1]_{A_1}[\omega_2]_{A_1}^{\epsilon}\le c_{n,m}(1+2\epsilon),$$

which was provided by $\epsilon>m$ and the multilinear extension of Coifmann- Rochberg theorem in [43, Lemma 1]:

$$[(\mathcal{M}(\overrightarrow{f_k}))^{\delta}]_{A_1}\le\frac{c_n}{1-m\delta}, for\ any\ 0<\delta<\frac{1}{m}.$$

Note that for any cube Q_k' and locally integral function f_k, there holds that

$$\|f_k\|_{L(\ \mathrm{Log}L),Q_k'}\le\sum_k \frac{1}{|Q_k'|}\int_{Q_k'}M\left(f_k1_{Q_k'}\right)(x)dx.$$

Thus, we by Proposition 6.3obtain

$$\mathcal{A}\le c_n(1+\epsilon)^{-1}\sum_k \|b\|_{BMO}[\omega]_{A_{1+\epsilon}}^2\|\mathcal{M}_{L(\log L)}(\overrightarrow{f_k})\|_{L^1(Q_k,\omega)}$$
$$\le c_n(1+\epsilon)^{-1}\|b\|_{BMO}[\omega]_{A_{1+\epsilon}}^2\sum_k \|\mathcal{M}(M(f_k)_1,\dots,M(f_k)_m)\|_{L^1(Q_k,\omega)}$$
$$=c_n(1+\epsilon)^{-1}\sum_k \|b\|_{BMO}[\omega]_{A_{1+\epsilon}}^2\|R(h)\|_{L^1(Q_k)}$$
$$\le c_n(1+\epsilon)^{-1}\|b\|_{BMO}[\omega]_{A_{1+\epsilon}}^2\|R(h)\|_{L^{(1+2\epsilon)'}(Q_k)}|Q_k|^{1/1+2\epsilon}$$
$$\le 2c_nc_{n,m}^2(1+\epsilon)^{-1}\sum_k \|b\|_{BMO}(1+2\epsilon)^2|Q_k|^{1/1+2\epsilon}.$$

Consequently, for $(1+\epsilon)>(1+\epsilon)_0:=2c_nc_{n,m}^2e\|b\|_{BMO}$, choosing $\epsilon>1$ so that $(1+\epsilon)/e=2c_nc_{n,m}^2(1+2\epsilon)^2\|b\|_{BMO}$, we have

$$\mathcal{A}^{1+2\epsilon}\le\left(2c_nc_{n,m}^2(1+2\epsilon)^2\|b\|_{BMO}(1+\epsilon)^{-1}\right)^{1+2\epsilon}\sum_k |Q_k|=\sum_k e^{-(1+2\epsilon)}|Q_k|$$
$$=\sum_k e^{-\sqrt{\frac{(1+2\epsilon)(1+\epsilon)}{\|b\|_{BMo}}}}|Q_k|,$$

where $(1+2\epsilon)=1/\left(2c_nc_{n,m}^2e\right)$ only depends on n and m. Since $e^{-\sqrt{\frac{(1+2\epsilon)(1+\epsilon)_0}{\|b\|_{BMo}}}}=e^{-1}$, we get

$|\{x\in Q_k:\ T_{\sigma_k,\Sigma b}(\overrightarrow{f_k})(x)>(1+\epsilon)\mathcal{M}(M(f_k)_1, M(f_k)_m)(x)\}|\le ee^{-\sqrt{\frac{(1+2\epsilon)(1+\epsilon)}{\|b\|_{BMo}}}}|Q_k|$, for $\epsilon\ge 0$.
This completes the proof.

7. Sharp Weighted Estimates

We shall prove Theorem 1.5. The inequality (1.1) follows from Proposition 4.1 and the weighted inequality for A_S given in [7, Theorem 2.1].

To show (1.2), we first estimate A_{S,b_j} and A_{S,b_j}^*. Let $(\sigma_k)_i=\omega_i^{1-p_i'}$, $i=1,\dots,m$. Let us begin with showing the case $0<\epsilon\le 1$. We recall two useful inequalities in [9,Lemma 3.1] as follows. Let $\epsilon>0,\epsilon\ge 0$, and $\omega\in A_\infty$. Then there holds

$$\|\sum_k \omega^{\frac{1}{1+\epsilon}}\|_{L^{1+\epsilon}(\log L)^{(1+\epsilon)^2},Q_k}\lesssim\sum_k [\omega]_{A_\infty}^{1+\epsilon}\langle\omega\rangle_{Q_k}^{\frac{1}{1+\epsilon}},\tag{7.1}$$

and

$$\|\sum_k f_k\omega\|_{L(\log L)^{1+\epsilon},Q_k}\lesssim[\omega]_{A_\infty}^{1+\epsilon}\sum_k \frac{1}{|Q_k|}\int_{Q_k}M_\omega(|f_k|^{1+\epsilon})(x)^{1/1+\epsilon}\omega dx.\tag{7.2}$$

Using (2.5), (2.4) and (7.1), we have

$$
\begin{aligned}
||\sum_{k} A_{S,b_j}(\overrightarrow{f_k})||_{L^{1+\epsilon}(v_{\vec{\omega}})}^{1+\epsilon} &\leq \sum_{k} \sum_{Q_k\in S} |Q_k|\langle |b_j-b_{j,Q_k}|^{1+\epsilon} v_{\vec{\omega}}\rangle_{Q_k} \prod_{i=1}^{m} \langle |(f_k)_i|\rangle_{Q_k}^{1+\epsilon} \\
&\leq \sum_{k} \sum_{Q_k\in S} \left(|Q_k|\|v_{\vec{\omega}}\|_{L(\log L)^{1+\epsilon},Q_k}\right) \|\left(b_j-b_{j,Q_k}\right)^{1+\epsilon}\|_{\exp L^{\frac{1}{1+\epsilon}},Q_k} \prod_{i=1}^{m} \langle |(f_k)_i|\rangle_{Q_k}^{1+\epsilon} \\
&\lesssim ||b_j||_{BMO}^{1+\epsilon} [v_{\vec{\omega}}]_{A_\infty}^{1+\epsilon} \sum_{k} \sum_{Q_k\in S} \prod_{i=1}^{m} \langle |(f_k)_i|\rangle_{Q_k}^{1+\epsilon} v_{\vec{\omega}}(Q_k).
\end{aligned}
$$

Moreover, it follows from (2.5), (2.4) and (7.2) that

$$
\begin{aligned}
\|\sum_{k} \mathrm{A}_{S,b_j}^{*}(\overrightarrow{f_k})||_{L^{1+\epsilon}(v_{\vec{\omega}})}^{(1+\epsilon)} &\leq \sum_{k} \sum_{Q_k\in \mathcal{S}} \langle |\left(b_j-b_{j,Q_k}\right)(f_k)_j|\rangle_{Q_k}^{1+\epsilon} \prod_{i\neq j} \langle |(f_k)_i|\rangle_{Q_k}^{1+\epsilon} v_{\vec{\omega}}(Q_k) \\
&\lesssim \sum_{k} \sum_{Q_k\in \mathcal{S}} ||b_j-b_{j,Q_k}||_{\exp L,Q_k}^{1+\epsilon} ||(f_k)_j||_{L(\log L),Q_k}^{1+\epsilon} \prod_{i\neq j} \langle |(f_k)_i|\rangle_{Q_k}^{1+\epsilon} v_{\vec{\omega}}(Q_k) \\
&\lesssim ||b_j||_{BMO}^{1+\epsilon} \sum_{k} \left[(\sigma_k)_j\right]_{A_\infty}^{1+\epsilon} \sum_{Q_k\in S} \langle M_{(\sigma_k)_j}\left(|(f_k)_j(\sigma_k)_j^{-1}|^{1+\epsilon}\right)^{1/1+\epsilon} \\
&\quad \cdot (\sigma_k)_j\rangle_{Q_k}^{1+\epsilon} \prod_{i\neq j} \langle |(f_k)_i|\rangle_{Q_k}^{1+\epsilon} v_{\vec{\omega}}(Q_k)\,.
\end{aligned}
$$

Since $(\sigma_k)_j=\omega_j^{1-p_j'} \in A_{mp_j'}$ is a doubling measure, we see that

$$
\begin{aligned}
&\sum_{k} \left(\int_{Q_k} \left(M_{(\sigma_k)_j}\left(|(f_k)_j(\sigma_k)_j^{-1}|^{1+\epsilon}\right)^{1/1+\epsilon} (\sigma_k)_j\right)^{p_j} \omega_j dx\right)^{1/p_j} \\
&\quad = \sum_{k} \left(\int_{Q_k} M_{(\sigma_k)_j}\left(|(f_k)_j(\sigma_k)_j^{-1}|^{1+\epsilon}\right)^{p_j/1+\epsilon} (\sigma_k)_j dx\right)^{1/p_j} \\
&\quad \lesssim \sum_{k} \left(\int_{Q_k} |(f_k)_j(\sigma_k)_j^{-1}|^{p_j} (\sigma_k)_j dx\right)^{1/p_j} = \sum_{k} \left(\int_{Q_k} |(f_k)_j|^{p_j} \omega_j dx\right)^{1/p_j}
\end{aligned}
$$

So, we see, we obtain the same type upper bound for A_{S,b_j} and A_{S,b_j}^{*}. Thus, following the same technique used in the proof of Theorem 1.2 [33], we deduce that

$$
\|A_{S,b_j}\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L^{1+\epsilon}(v_{\vec{\omega}})} \lesssim \sum_{k} \|b_j\|_{BMO} [v_{\vec{\omega}}]_{A_\infty} [\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{1\leq i\leq m}\left\{\frac{p_i'}{1+\epsilon}\right\}} \tag{7.3}
$$

and

$$
\|A_{S,b_j}^{*}\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L^{1+\epsilon}(v_{\vec{\omega}})} \lesssim \sum_{k} \|b_j\|_{BMO} \left[(\sigma_k)_j\right]_{A_\infty} [\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{1\leq i\leq m}\left\{\frac{p_i'}{1+\epsilon}\right\}} \tag{7.4}
$$

Next, we turn our attention to the case $\epsilon \geq 0$. Let $g_k \in L^{p'}(v_{\vec{\omega}})$ be a nonnegative function satisfying $\|g_k\|_{L^{p'}(v_{\vec{\omega}})}=1$. Applying (2.5) and (2.4) again, we have

$$\int_{\mathrm{R}^n}\sum_k A_{S,b_j}(\overrightarrow{f_k})(x)g_k(x)v_{\vec{w}}dx \leq \sum_k \sum_{Q_k\in S} |Q_k|\langle|(b_j-b_{j,Q_k})g_k v_{\vec{\omega}}|\rangle_{Q_k}\prod_{i=1}^m\langle|(f_k)_i|\rangle_{Q_k}$$

$$\lesssim \sum_k \sum_{Q_k\in\mathcal{S}}(|Q_k|\|g_k v_{\vec{\omega}}\|_{L(\log L),Q_k})\,\|(b_j-b_{j,Q_k})\|_{\exp L,Q_k}\prod_{i=1}^m\langle|(f_k)_i|\rangle_{Q_k}$$

$$\lesssim \|b_j\|_{BMO}[v_{\vec{\omega}}]_{A_\infty}\int_{\mathrm{R}^n}\sum_k A_S(\overrightarrow{f_k})(x)M_{v_{\vec{\omega}}}(|g_k|^{1+\epsilon})(x)^{1/1+\epsilon}v_{\vec{\omega}}dx$$

$$\leq \|b_j\|_{BMO}[v_{\vec{\omega}}]_{A_\infty}\sum_k \|A_{\mathcal{S}}(\overrightarrow{f_k})\|_{L^{1+\epsilon}(v_{\vec{\omega}})}\left\|\left(M_{v_{\vec{\omega}}}(|g_k|^{1+\epsilon})\right)^{1/1+\epsilon}\right\|_{U'(v_{\vec{\omega}})}$$

$$\lesssim \|b_j\|_{BMO}[v_{\vec{\omega}}]_{A_\infty}[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{1\leq i\leq m}\left\{\frac{p_i'}{1+\epsilon}\right\}}$$

This implies that

$$\|\mathrm{A}_{S,b_j}\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L(v_{\vec{\omega}})}(1-\epsilon)\leq\sum_k \|b_j\|_{BMO}[v_{\vec{\omega}}]_{A_\infty}[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{1\leq i\leq m}\left\{\frac{p_i'}{1+\epsilon}\right\}} \tag{7.5}$$

In order to bound $A^*_{\mathcal{S},b_j}$ in the case $\epsilon\geq 0$, we set

$$(1+\epsilon)=p_j',(1+2\epsilon)_j=p',(1+2\epsilon)_i=p_j, i\neq j,$$

and

$$\vec{\mu}=\left(\omega_1,\dots,\omega_{j-1},v_{\vec{\omega}}^{1-p'},\omega_{j+1},\dots,\omega_m\right).$$

Then there holds that

$$v_{\vec{\mu}}:=\prod_{i=1}^m \mu_i^{1+\epsilon/q_j}=\omega_j^{1-p_j'}=(\sigma_k)_j$$

And

$$\frac{1}{1+\epsilon}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}.$$

Additionally, we have

$$[\vec{u}]_{A_{\overrightarrow{1+\epsilon}}}=\sup_{Q_k}\sum_k\left(\frac{1}{|Q_k|}\int_{Q_k}v_{\vec{\mu}}\,dx\right)\prod_{i=1}^m\left(\frac{1}{|Q_k|}\int_{Q_k}\mu_{i^{1-q_i'}}\,dx\right)^{\frac{1+\epsilon}{q_i'}}$$

$$=\sup_{Q_k}\sum_k\left(\frac{1}{|Q_k|}\int_{Q_k}\omega_j^{1-p_j'}\,dx\right)\prod_{i\neq j}\left(\frac{1}{|Q_k|}\int_{Q_k}\omega_i^{1-p_i'}\,dx\right)^{\frac{p_j'}{p'i}}$$

$$\times\left(\frac{1}{|Q_k|}\int_{Q_k}\left(v_{\vec{\omega}}^{1-p'}\right)^{\epsilon}dx\right)^{\frac{p_j'}{1+\epsilon}}$$

$$=\sup_{Q_k}\sum_k\left\{\left(\frac{1}{|Q_k|}\int_{Q_k}v_{\vec{\omega}}\,dx\right)\prod_{i=1}^m\left(\frac{1}{|Q_k|}\int_{Q_k}\omega_{i^{1-p_i'}}\,dx\right)^{1+\epsilon/p_i'}\right\}^{p_j'/1+\epsilon}=[\vec{\omega}]_{A_{jJ}}^{p_j'/1+\epsilon},$$

and hence,

$$\varpi]_{A_{\vec{q}}}^{\max_{1i\leq m}\{1,q_i'/1+\epsilon\}}=[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\frac{p_j'}{1+\epsilon}\max_{i\neq j}\left\{\frac{1+\epsilon}{p_j'},\frac{p_i'}{p_j'}\right\}}=\sum_k[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{i\neq j}\left\{1,\frac{p_i'}{1+\epsilon}\right\}}$$

Now, let $0\leq g_k\in L^{p'}(v_{\vec{\omega}})$ with $\|g_k\|_{L^{\rho'}(v_{\vec{\omega}})}=1$. Then

$$\int_{\mathrm{R}^n}\sum_{k} A_{S,b_j}^{*}(f_k)](x)g_k(x)v_{\vec{\omega}}dx=\int_{\mathrm{R}^n}\sum_{k}\sum_{Q_k\in S}\langle|(b_j-b_{j,Q_k})(f_k)_j|\rangle_{Q_k}\prod_{i\neq j}\langle|(f_k)_i|\rangle_{Q_k}1_{Q_k}(x)g_k(x)v_{\vec{\omega}}dx$$

$$=\int_{\mathrm{R}^n}\sum_{k}\sum_{Q_k\in S}|b_j(\mathrm{J})$$

$$-b_{j,Q_k}|\langle|g_kv_{\vec{\omega}}|\rangle_{Q_k}\prod_{i\neq j}\langle|(f_k)_i|\rangle_{Q_k}1_{Q_k}(y)|(f_k)_j(y)(\sigma_k)_j(y)^{-1}|(\sigma_k)_j(y)dy$$

$$=\int_{\mathrm{R}^n}\sum_{k} A_{S,b_j}\left(((f_k)_1,\cdots,g_kv_{\vec{\omega}},\cdots,(f_k)_m)\right)(y)|(f_k)_j(y)(\sigma_k)_j(y)^{-1}|\omega_j(y)^{1-p_j'}dy.$$

Therefore, since $||g_kv_{\vec{\omega}}||_{L^{p'}\left(v_{\vec{\omega}}^{1-p'}\right)}=||g_k||_{L(v_{\vec{\omega}})}p'$, it yields that

$$\|A_{S,b_j}^{*}||_{L^{p_1}(\omega_1)\times\cdots\times L^{pm}(\omega_m)\to L^{1+\epsilon}(v_{\vec{\omega}})}=\left\|A_{S,b_j}\right\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p'}\left(v_{\vec{\omega}}^{1-p'}\right)\times\cdots\times L^{\rho m}(\omega_m)\to L^{p'}_{j}\left(\omega_j^{1-p_j'}\right)}$$

$$=||A_{1+\epsilon,b_j}||_{L^{q_1}(\mu_1)\times\cdots\times L^{qm}(\mu_m)\to L^{q}\left(v_{\vec{\mu}}\right)}\leq||b_j||_{BMO}[v_{\vec{\mu}}]_{A_\infty}|\vec{\mu}|_{A_{\vec{q}}}^{\max_{1\leq i\leq m}\left\{1,\frac{q_i'}{q}\right\}}$$

$$=\sum_{k}\|b_j||_{BMO}\left[(\sigma_k)_j\right]_{A_\infty}[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{i\neq j}\left\{1,\frac{p_i'}{1+\epsilon}\right\}}\qquad(7.6)$$

Note that if $\vec{\omega}\in A_{\overrightarrow{1+\epsilon}}$, then there holds that

$$[v_{\vec{\omega}}]_{A_{m(1+\epsilon)}}\leq\sum_{k}[\vec{\omega}]_{A_\beta}\ and\ [(\sigma_k)_i]_{A_{mp_i'}}\leq[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{p_i'/1+\epsilon},i=1,\dots,m,\qquad(7.7)$$

which was essentially contained in the proof of [30, Theorem 3.6]. Hence, invoking Proposition 4.1 and inequalities (7.3)-(7.7), we deduce that

$$||\sum_{k}T_{\sigma_k,\Sigma\mathrm{b}}||_{L^{p_1}(\omega_1)\times\cdots\times L^{pm}(\omega_m)\to L^{1+\epsilon}(v_{\vec{\omega}})}$$

$$\lesssim\sum_{j=1}^{m}\sum_{k}||b_j||_{BMO}\left([v_{\vec{\omega}}]_{A_\infty}[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{1\leq i\leq m}\left\{\frac{p_i'}{1+\epsilon}\right\}}+\left[(\sigma_k)_j\right]_{A_\infty}[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{\max_{1\leq i\leq m}\left\{\frac{p_i'}{1+\epsilon}\right\}}\right)$$

$$\leq\sum_{k}\|\mathrm{b}||_{BMO}[\vec{\omega}]_{A_{\overrightarrow{1+\epsilon}}}^{2\max_{1\leq i\leq m}\left\{\frac{p_i'}{1+\epsilon}\right\}}$$

So far, we have proved Theorem 1.5.

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