Quest Journals
Journal of Research in Applied Mathematics
Volume 10 ~ Issue 2 (2024) pp: 13-22
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

The Inverse Copoun Distribution and its Applications

Uwaeme, O. R. [1*], Akpan, N. P.[2]
[1] *Department of Statistics, Kingsley Ozumba Mbadiwe University, Nigeria.*
[2] *Department of Mathematics and Statistics, University of Port Harcourt, Nigeria*

Abstract
We introduce in this study, a new two-parameter lifetime distribution called the inverse Copoun distribution. This distribution is derived using the inverse function to contribute to the growing need for upside-down bathtub distributions. Some important mathematical properties of the new distribution such as the density, mode, entropy, and reliability indices such as the stress-strength, existence measurement function and risk measurement function were derived and the model parameters estimated using the maximum likelihood estimate technique. Finally, the flexibility of the new inverse distribution was illustrated using a real life dataset and the results showed that the new inverse distribution was the best amongst other competing two parameter distributions.

Keywords: *Inverse Copoun distribution; Lifetime distributions; Two-parameter distributions; Entropy; Stress-strength reliability; Risk measurement function.*

Received 09 Feb., 2024; Revised 22 Feb., 2024; Accepted 24 Feb., 2024 © The author(s) 2024. Published with open access at ***www.questjournals.org***

I. The Distribution

In recent years, new two-parameter distributions have emerged in the literature. These new two-parameter distributions have been shown to provide a better fit to complex real life datasets than the one-parameter distributions. Some of the recently developed two-parameter distributions include the Darna distribution (Shraa & Al-Omari, 2019), the Hamza distribution (Aijaz et al., 2020), the Samade distribution (Aderoju, 2021), and the Alzoubi distribution (Benrabia & Alzoubi, 2021).

It is important to note that these distributions are a mixture of the Exponential and Gamma distributions. These two distributions are known to have their weaknesses. The weakness of the Exponential distribution is that the hazard rate function is constant; hence, it cannot handle datasets with monotone non-decreasing hazard rates (Elechi et al., 2022; Epstein, 1958; Ronald et al., 2011; Shukla, 2018b; Shukla, 2018). Furthermore, the weakness of the Gamma distribution is that the survival rate function cannot be expressed in closed form (Elechi et al., 2022; Shanker, 2015a, 2015b). The weaknesses of these two distributions are what the aforementioned one-parameter and two-parameter distributions address, providing distributions whose survival rate function can be expressed in closed form and hazard rate functions capable of handling datasets with monotone non-decreasing hazard rates.

In contributing to this gap in the literature, Uwaeme et al. (2023) proposed a new two-parameter distribution called the Copoun distribution. The Copoun distribution is a two-component density of an Exponential (η) and Gamma $(4,\eta)$ distribution with mixing proportions π_1 and π_2 such that

$$g(x;\eta,\phi)=\pi_1 g_1(x;\eta)+\pi_2 g_2(x;\eta) \tag{1.1}$$

where $g_1(x;\eta)=\eta e^{-\eta x}$, $g_2(x;\eta)=\frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)}$, $\pi_1=\frac{\eta}{(\phi+\eta)}$ and $\pi_2=\frac{\phi}{(\phi+\eta)}$

therefore,

$$g(x_k;\eta,\phi)=\eta e^{-\eta x}\cdot\frac{\eta}{(\phi+\eta)}+\frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)}\cdot\frac{\phi}{(\phi+\eta)} \tag{1.2}$$

Solving equation 2 gives the probability density function (pdf) of the new distribution

$$g(x;\eta,\phi)=\frac{\eta^2}{(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x};\ x>0,\eta>0,\phi>0 \tag{1.3}$$

The corresponding cumulative distribution function (cdf) of (3) is obtained as

$$G(x;\eta,\phi)=1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x} \tag{1.4}$$

The authors introduced some of the mathematical properties of the new distribution. They showed that the Copoun distribution exhibits shapes that are not bell-shaped, but positively skewed, unimodal, and right-tailed (Uwaeme et al., 2023). One of the weaknesses of the Copoun distribution is that it does not have non-monotone hazard rates. One way of overcoming this weakness is to introduce an extension of the Copoun distribution using the inverse transformation technique. This technique produces a class of distribution known as Inverse distributions. Inverse distributions are known for their interesting advantages such as having the same parsimony as their corresponding parent distribution since no new parameter in required (Eliwa et al., 2018); and they are known to have upside-down bathtub risk measurement functions (Abouammoh & Alshingiti, 2009; Eliwa et al., 2018; E.W. Okereke et al., 2021; John et al., 2023; Lee et al., 2017).

From the foregoing therefore, the motivation of this paper is to propose a new inverse distribution called the Inverse Copoun distribution and its statistical properties. The subsequent sections of the paper will be arranged as follows. Section 2 discusses the new inverse distribution with the derivation of the pdf, the cdf, and their plots, section 3 discusses the mathematical properties of the inverse Copoun distribution as well as the plots of the risk measurement function to highlight the shape, section 4 looks at the application of the new distribution with real datasets alongside other competing distributions, and section 5 concludes the paper with some remarks.

1. The Inverse Copoun distribution

This section will introduce the pdf and the cdf of the Inverse Copoun distribution and illustrate the different shapes of the Inverse Copoun distribution.

Proposition 1: If a random variable Y follows the Copoun distribution with parameters η and ϕ, then the random variable $X = \frac{1}{Y}$ has Inverse Copoun distribution with parameters η and ϕ and its pdf and cdf are respectively given by

$$g(y;\eta,\phi) = \frac{\eta^2}{6(\phi+\eta)}\left[\frac{\phi\eta^2+6y^3}{y^5}\right]e^{-\eta y^{-1}};\ y>0,\eta>0,\phi>0 \qquad (2.1)$$

and

$$G(y;\eta,\phi) = \left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}};\ y>0,\eta>0,\phi>0 \qquad (2.2)$$

Proof: If X follows the Copoun distribution with parameters η and ϕ, the pdf of X is given as

$$f(x;\eta,\phi) = \frac{\eta^2}{(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}$$

Let $y = x^{-1}, \frac{dx}{dy} = -y^{-2}$. Thus, the pdf of X is given by

$$g(y;\eta,\phi) = f(y^{-1})\left|\frac{dx}{dy}\right| \qquad (2.3)$$

$$= \frac{\eta^2}{(\phi+\eta)}\left[1+\frac{\phi\eta^2\left(\frac{1}{y}\right)^3}{6}\right]e^{-\eta y^{-1}}\left(\frac{1}{y^2}\right)$$

$$= \frac{\eta^2}{6(\phi+\eta)}\left[\frac{\phi\eta^2+6y^3}{y^5}\right]e^{-\eta y^{-1}};\ y>0,\eta>0,\phi>0.$$

The corresponding cdf for the Inverse Copoun distribution can be expressed as

$$G(y;\eta,\phi) = \frac{\eta^2}{6(\phi+\eta)}\int_0^y\left[\frac{\phi\eta^2+6k^3}{k^5}\right]e^{-\eta k^{-1}}dk \qquad (2.4)$$

$$= \frac{\eta^2}{6(\phi+\eta)}\left[\phi\eta^2\int_0^y k^{-5}e^{-\eta k^{-1}}dk + 6\int_0^y k^{-2}e^{-\eta k^{-1}}dk\right]$$

Let $t = \eta k^{-1}, k = \eta t^{-1}, and\ dk = -\eta t^{-2}dt.$

Applying integration by parts techniques, we have,

$$G(y;\eta,\phi) = \frac{\eta^2}{6(\phi+\eta)}\left[\phi\eta^2\int_0^y k^{-5}e^{-\eta k^{-1}}dk + 6\int_0^y k^{-2}e^{-\eta k^{-1}}dk\right]$$

$$= \frac{\eta^2}{6(\phi+\eta)}\left[\phi\eta^{-2}\int_{\eta y^{-1}}^{\infty} t^3e^{-t}dt + 6\eta^{-1}\int_{\eta y^{-1}}^{\infty} e^{-t}dt\right]$$

$$G(y;\eta,\phi) = \left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}$$

The Inverse Copoun distribution derived above is denoted by $ICD(\eta,\phi)$. The graphical plots of the theoretical density and distribution function (for some selected but different real points of η and ϕ) of the Inverse Coupon distribution are shown in the Figure 1 and Figure 2 below.

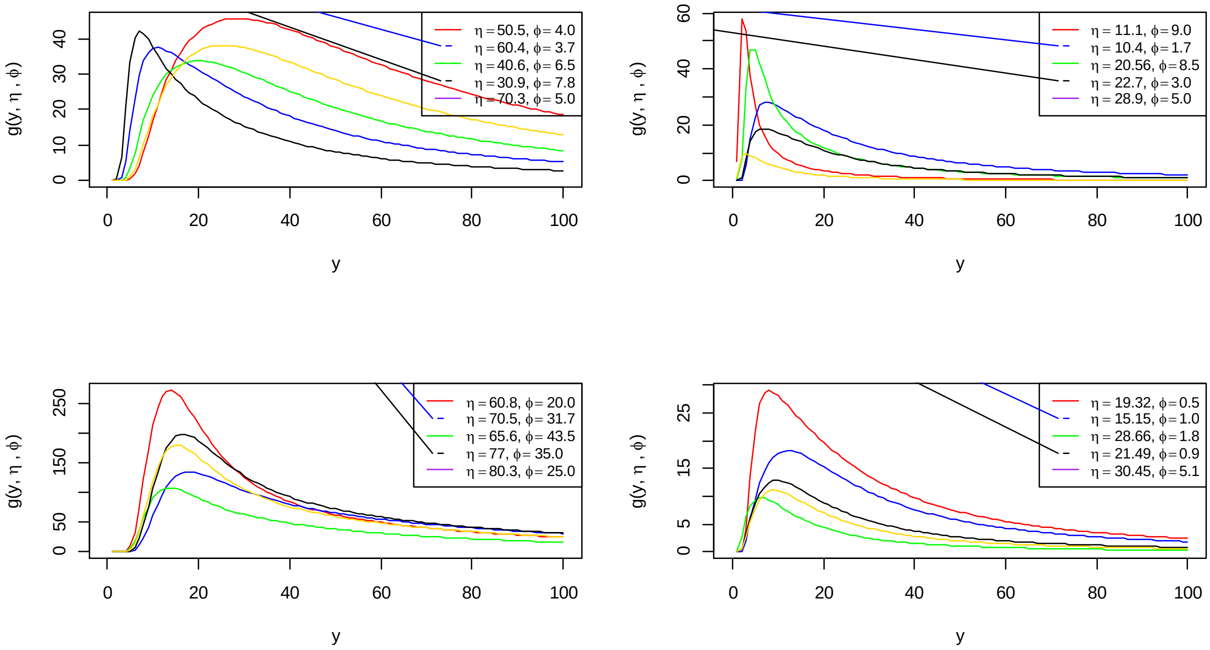

Figure 1: The graphical plots of the probability density function (for some selected but different real points of η and ϕ) of an Inverse Coupon distribution.

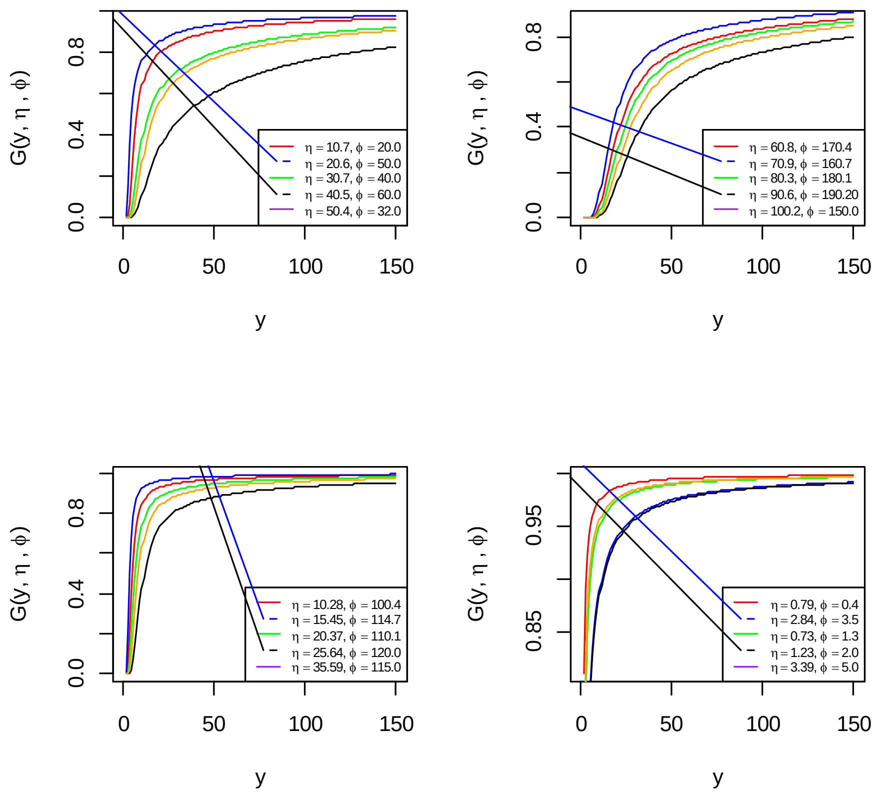

Figure 2: The graphical plots of the cumulative distribution function (for some selected but different real points of η and ϕ) of an Inverse Coupon distribution

The curves displayed in Figure 1 are not bell-shaped, but are positively skewed, unimodal, and right tailed. In addition, the curve shows that increasing the value of ϕ leads to a considerable increase in the peak of the curve. In addition, the curves displayed in Figure 2 shows that the cumulative distribution function converges to one.

2. Statistical Properties of the Inverse Copoun Distribution

In this section, we derive and present some statistical properties of the Inverse Copoun distribution. These includes the mode, survivorship or existence measurement function, risk measurement function, and average residual measurement life-time function, stochastic ordering of random variate, absolute deviations from average and mid points, bonferroni curve, lorenz curve, bonferroni and lorenz indices, entropy.

2.1 Mode

Theorem 1:

Given a continuous random variable Y which follows the Inverse Copoun distribution, the mode of Y, is given as

$$Mode = \begin{cases} -\frac{\eta^2(12y^4-6\eta y^3+5\phi\eta^2 y-\phi\eta^3)e^{-\eta y^{-1}}}{6(\eta+\phi)y^7}, & 0<\eta<\frac{1}{2} \\ 0, & otherwise \end{cases} \quad (3.1)$$

Proof:

From given a continuous random variable Y, the mode of Y, is obtained by

$$Mode = \frac{d}{dy}g(y;\eta,\phi) = 0 \quad (3.2)$$

$$\frac{d}{dy}\left(\frac{\eta^2}{6(\phi+\eta)}\left[\frac{\phi\eta^2+6y^3}{y^5}\right]e^{-\eta y^{-1}}\right) = 0 \quad (3.3)$$

$$\therefore \frac{d}{dy}g(y;\eta,\phi) = -\frac{\eta^2(12y^4-6\eta y^3+5\phi\eta^2 y-\phi\eta^3)e^{-\eta y^{-1}}}{6(\phi+\eta)y^7} = \quad (3.6)$$

Which completes the proof.

2.2 Order statistics

Theorem 2:

Given a continuous random variable X, pdf and cdf of the pth order statistics, say $X = X_{(p)}$, is given respectively by

$$g_p(y) = \frac{n!(\phi\eta^4+6\eta^2y^3)e^{-\eta y^{-1}}}{6y^5(\phi+\eta)(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[\left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{p+i-1} \quad (3.7)$$

and

$$G_p(y) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left[\left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{j+1} \quad (3.8)$$

Proof:

Given a continuous random variable Y, the pdf of the pth order statistics, say $Y = Y_{(p)}$, is obtained by

$$g_p(y) = \frac{n!g(x_k;\Phi)}{(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i G(x_k;\Phi)^{p+i-1} \quad (3.9)$$

$$g_p(y) = \frac{n!\left(\frac{\eta^2}{6(\phi+\eta)}\left[\frac{\phi\eta^2+6y^3}{y^5}\right]e^{-\eta y^{-1}}\right)}{(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[\left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{p+i-1} \quad (3.10)$$

$$g_p(y) = \frac{n!(\phi\eta^4+6\eta^2y^3)e^{-\eta y^{-1}}}{6y^5(\phi+\eta)(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[\left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{p+i-1} \quad (3.11)$$

Which completes the proof.

Correspondingly, given a continuous random variable Y, the cdf of the pth order statistics, say $Y = Y_{(p)}$, is obtained by

$$G_p(y) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i G(x_k;\Phi)^{j+1} \quad (3.12)$$

$$G_p(y) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left[\left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{j+1} \quad (3.13)$$

Which completes the proof.

2.3 Entropy

Entropy measures the uncertainties associated with a random variable of a probability distributions. Shannon (Shannon, 1951) and Rényi's entropy (Rényi, 1961)are widely used in the literature.

Theorem 3:

Given a random variable Y, which follows the Inverse Copoun distribution $g(y;\eta,\phi)$. The Rényi entropy is given by

$$T_R(\beta) = \frac{1}{1-\beta}\log\left[\frac{(\phi\eta^2)^{\beta-j}}{6^\beta(\phi+\eta)^\beta}\sum_{j=0}^{\beta}\binom{\beta}{j}\frac{6^j\Gamma(5\beta-3j-1)}{(\beta\eta)^{5\beta-3j-1}}\right] \quad (3.14)$$

Proof:
The Rényi entropy is given by

$$T_R(\beta) = \frac{1}{1-\beta}\log\left[\int_0^\infty g_n^\beta(y_k;\Phi)dy\right] \quad (3.15)$$

$$T_R(\beta) = \frac{1}{1-\beta}\log\left[\int_0^\infty\left(\frac{\eta^2}{6(\phi+\eta)}\left[\frac{\phi\eta^2+6y^3}{y^5}\right]e^{-\eta y^{-1}}\right)^\beta dy\right] \quad (3.16)$$

$$T_R(\beta) = \frac{1}{1-\beta}\log\left[\frac{\eta^{2\beta}}{6^\beta(\phi+\eta)^\beta}\int_0^\infty\left[\frac{\phi\eta^2+6y^3}{y^5}\right]^\beta e^{-\beta\eta y^{-1}}dy\right] \quad (3.17)$$

Recall that $(a+x)^\vartheta = \sum_{k=0}^{\vartheta}\binom{\vartheta}{k}x^k a^{\vartheta-k}$ and $\int_0^\infty z^{-w-1}e^{-\frac{\varrho}{z}}dz = \frac{\Gamma(w)}{\varrho^w}$ Substituting,

$$T_R(\beta) = \frac{1}{1-\beta}\log\left[\frac{\eta^{2\beta}}{6^\beta(\phi+\eta)^\beta}\sum_{j=0}^{\beta}\binom{\beta}{j}(\phi\eta^2)^{\beta-j}\left(6^j\int_0^\infty\frac{e^{-\beta\eta y^{-1}}}{y^{5\beta-3j}}dx\right)\right] \quad (3.18)$$

$$\therefore T_R(\beta) = \frac{1}{1-\beta}\log\left[\frac{(\phi\eta^2)^{\beta-j}}{6^\beta(\phi+\eta)^\beta}\sum_{j=0}^{\beta}\binom{\beta}{j}\frac{6^j\Gamma(5\beta-3j-1)}{(\beta\eta)^{5\beta-3j-1}}\right]$$

Which completes the proof.

2.4 Reliability Indices

Given any probability distribution, the reliability analysis is always considered based on the Existence Measurement Function and Risk Measurement Function. Hence, for the Inverse Copoun distribution, the Existence Measurement Function and Risk Measurement Function is given below.

2.4.1 Existence Measurement Function

The existence measurement function (also known as survival function) is defined as the probability that an item does not fail prior to some time t (Elechi et al., 2022; Epstein, 1958; Ronald et al., 2011; Shanker & Shukla, 2020).
The existence measurement function of the Inverse Copoun distribution is given by

$$s(y) = 1 - G(y;\eta,\phi) \quad (3.19)$$

$$s(y) = 1 - \left[1 + \frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}; y>0, \eta>0, \phi>0 \quad (3.20)$$

2.4.2 Risk Measurement Function

The risk measurement function (also known as hazard rate function) on the other hand can be seen as the conditional probability of failure, given it has survived to the time t (Elechi et al., 2022; Ronald et al., 2011; Shanker, 2016b; Umeh & Ibenegbu, 2019). It is obtained as
The risk measurement function of the Inverse Copoun distribution is given by

$$h(y) = \frac{g(y_k;\Phi)}{1-G(y_k;\Phi)} \quad (3.21)$$

$$h(y) = \frac{\frac{\eta^2}{6(\phi+\eta)}\left[\frac{\phi\eta^2+6y^3}{y^5}\right]e^{-\eta y^{-1}}}{1-\left[1+\frac{\phi\eta^2+3\phi\eta y+6\phi y^2}{6(\phi+\eta)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}} \quad (3.22)$$

$$h(x) = \frac{\eta^2[\phi\eta^2+6y^3]}{y^2\left[6(\phi+\eta)y^3\left(e^{\eta y^{-1}}-1\right)-[\phi\eta^2+3\phi\eta y+6\phi y^2]\right]} \quad (3.23)$$

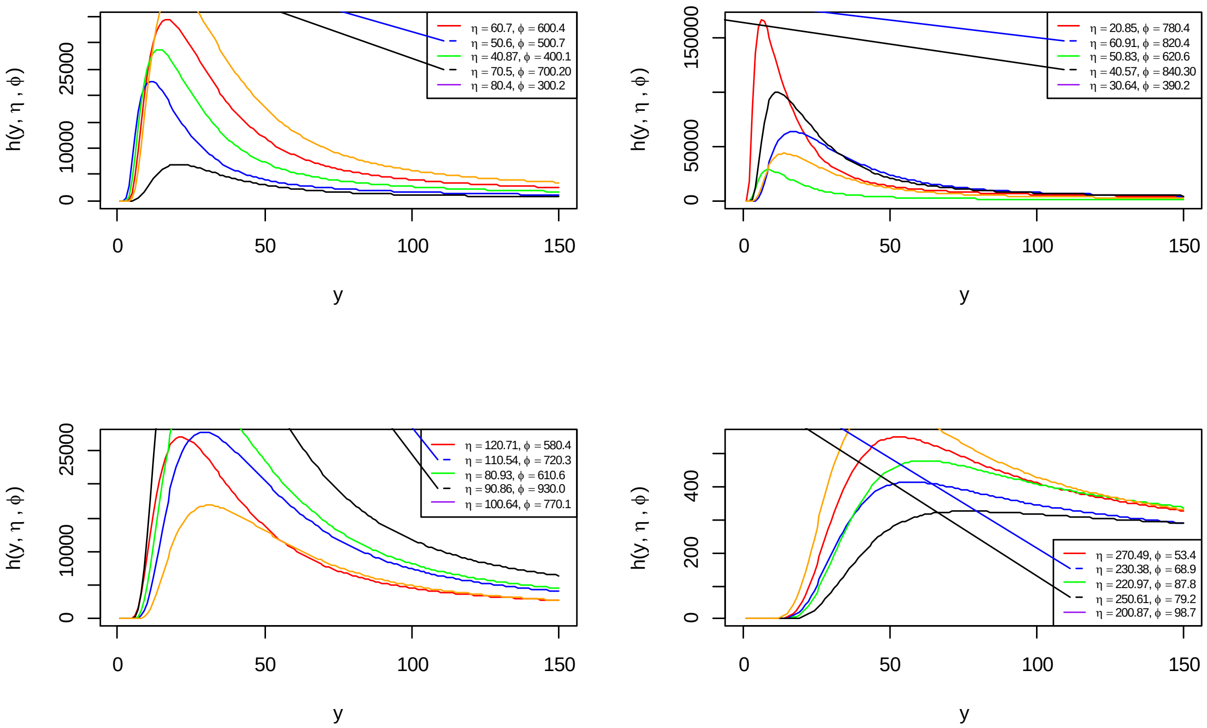

Figure 3: The graphical plots of the risk measurement function (for some selected but different real points of η and ϕ) of an Inverse Coupon distribution

2.5 Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behaviour. According to Shanker (2015), a random variable X is said to be smaller than a random variable Y in the

- Stochastic order ($S \leq_{stor} Q$) if $G_S(y) \geq G_Q(y)$ for all y.
- Hazard rate order ($S \leq_{Ror} Q$) if $h_S(y) \geq h_Q(y)$ for all y.
- Mean residual life order ($S \leq_{mrl} Q$) if $m_S(y) \geq m_Q(y)$ for all y.
- Likelihood ratio order ($S \leq_{lor} Q$) if $\left(\frac{g_S(y)}{g_Q(y)}\right)$ decreases in y.

Theorem 7:
Let $S \sim$ CD (η_1, ϕ_1) and Q~ CD (η_2, ϕ_2). If $\eta_1 = \eta_2$ and $\phi_1 = \phi_2$ and $\phi_1 \geq \phi_2$ (or if $\phi_2 = \phi_2$ and $\eta_1 = \eta_2$ and $\eta_1 \geq \eta_2$), then $S \leq_{stor} Q$ and hence $S \leq_{Ror} Q, S \leq_{mrl} Q \; and \; S \leq_{lor} Q$

Proof:
Let $S \sim$ CD (η_1, ϕ_1) and Q~ CD (η_2, ϕ_2). We obtain that

$$\frac{f_S(\varphi)}{f_Q(\varphi)} = \frac{\eta_1^2[6(\phi_2+\eta_2)]}{\eta_2^2[6(\phi_1+\eta_1)]} \cdot e^{-\frac{(\eta_1-\eta_2)}{\varphi}} \tag{3.24}$$

and

$$\log_\varphi \frac{f_S(\varphi)}{f_Q(\varphi)} = \log_\varphi \left[\frac{\eta_1^2[6(\phi_2+\eta_2)]}{\eta_2^2[6(\phi_1+\eta_1)]}\right] - \frac{(\eta_1-\eta_2)}{\varphi} \tag{3.25}$$

Hence,

$$\frac{d}{d\varphi} \log_\varphi \frac{f_S(\varphi)}{f_Q(\varphi)} = \frac{(\eta_1-\eta_2)}{\varphi^2} \tag{3.26}$$

Which completes the proof.

This implies that $S \leq_{lor} Q \Longrightarrow S \leq_{Ror} Q \Longrightarrow \begin{cases} S \leq_{sor} Q \\ S \leq_{mrl} Q \end{cases}$. This clearly indicates that the Inverse Copoun distribution is ordered in the likelihood ratio and consequently has risk measurement, average residual measurement life, and stochastic orderings. These results has been established in the literature for stochastic ordering of distributions (Shaked & Shanthikumar, 1994; Shanker, 2016a; Uwaeme et al., 2023).

2.6 Stress-Strength Reliability and Maximum Likelihood Estimations

Let Y and X be independent stress and strength random variables that follow Inverse Copoun distribution with parameter (η_1, ϕ_1) and (η_2, ϕ_2) respectively. Then, the stress-strength reliability (R) is given by

$$R_{SS} = P[X < Y] = \int_0^\infty [X < Y|Y = y]g_y(y)dy \tag{3.27}$$

$$= \int_0^\infty g(y; \eta_1, \phi_1)G(y; \eta_2, \phi_2)dy. \tag{3.28}$$

$$= \int_0^\infty \left(\frac{\eta_1^2}{6(\phi_1+\eta_1)}\left[\frac{\phi_1\eta_1^2+6y^3}{y^5}\right]e^{-\eta_1 y^{-1}}\right)\left[1 + \frac{\phi_2\eta_2^2+3\phi_2\eta_2 y+6\phi_2 y^2}{6(\phi_2+\eta_2)}\left(\frac{\eta_2}{y^3}\right)\right]e^{-\eta_2 y^{-1}}dy. \tag{3.29}$$

$$= \frac{\eta_1^2}{6(\phi_1+\eta_1)}\int_0^\infty \left[\frac{\phi_1\eta_1^2+6y^3}{y^5}\right]e^{-\frac{(\eta_1-\eta_2)}{y}}dy + \frac{\eta_1^2\eta_2}{36(\phi_1+\eta_1)(\phi_2+\eta_2)}\int_0^\infty \left[\frac{\phi_1\eta_1^2+6y^3}{y^5}\right]\left[\frac{\phi_2\eta_2^2+3\phi_2\eta_2 y+6\phi_2 y^2}{y^3}\right]e^{-\frac{(\eta_1+\eta_2)}{y}}dy. \tag{3.30}$$

$$= \frac{\eta_1^2}{6(\phi_1+\eta_1)}\left[\phi_1\eta_1^2\int_0^\infty y^{-5}e^{-\frac{(\eta_1+\eta_2)}{y}}dy + 6\int_0^\infty y^{-2}e^{-\frac{(\eta_1+\eta_2)}{y}}dy\right] + \frac{\eta_1^2\eta_2}{36(\phi_1+\eta_1)(\phi_2+\eta_2)}\left[\int_0^\infty \left[\frac{[\phi_1\eta_1^2+6y^3][\phi_2\eta_2^2+3\phi_2\eta_2 y+6\phi_2 y^2]}{y^8}\right]e^{-\frac{(\eta_1+\eta_2)}{y}}dy\right]. \tag{3.31}$$

Applying the gamma function, we obtain the expression for the Stress-strength reliability as

$$R_{SS} = \frac{\eta_1^2}{6(\phi_1+\eta_1)}\left[\frac{\phi_1\eta_1^2\Gamma(4)}{(\eta_1+\eta_2)^4} + \frac{6\Gamma(1)}{(\eta_1+\eta_2)}\right] + \frac{\eta_1^2\eta_2}{36(\phi_1+\eta_1)(\phi_2+\eta_2)}\left[\frac{\phi_1\phi_2\eta_1^2\eta_2^2\Gamma(7)}{(\eta_1+\eta_2)^7} + \frac{3\phi_1\phi_2\eta_1^2\eta_2\Gamma(6)}{(\eta_1+\eta_2)^6} + \frac{6\phi_1\phi_2\eta_1^2\Gamma(5)}{(\eta_1+\eta_2)^5} + \frac{6\phi_2\eta_2^2\Gamma(4)}{(\eta_1+\eta_2)^4} + \frac{18\phi_2\eta_2\Gamma(3)}{(\eta_1+\eta_2)^3} + \frac{32\phi_2\Gamma(2)}{(\eta_1+\eta_2)^2}\right] \tag{3.32}$$

and

$$R_{SS} = \frac{\eta_1^2\left\{[144\phi_1\eta_1^2+36(\eta_1+\eta_2)^3](\phi_2+\eta_2)(\eta_1+\eta_2)^3+\eta_2\left[\begin{matrix}5040\phi_1\phi_2\eta_1^2\eta_2^2+2160\phi_1\phi_2\eta_1^2\eta_2(\eta_1+\eta_2)+720\phi_1\phi_2\eta_1^2(\eta_1+\eta_2)^2+\\144\phi_2\eta_2^2(\eta_1+\eta_2)^3+108\phi_2\eta_2(\eta_1+\eta_2)^4+72\phi_2(\eta_1+\eta_2)^5\end{matrix}\right]\right\}}{36(\phi_1+\eta_1)(\phi_2+\eta_2)(\eta_1+\eta_2)^7} \tag{3.33}$$

Since R is the Stress-Strength Reliability function with parameters (η_1, ϕ_1) (η_2, ϕ_2), we need to obtain the maximum likelihood estimators (MLEs) of (η_1, ϕ_1) and (η_2, ϕ_2) to compute the maximum likelihood estimation R under Invariance property of the maximum likelihood estimation. Suppose $X_1, X_2, X_3, \dots, X_n$ is a Strength random variable sample from Inverse Copoun distribution (η_1, ϕ_1) and $Y_1, Y_2, Y_3, \dots, Y_m$ is a Stress random sample from Inverse Copoun distribution (η_2, ϕ_2). Thus, the likelihood function based on the observed sample is given by

$$L\left(\underline{\eta}, \underline{\phi}/\underline{x}, \underline{y}\right) = \prod_{i=1}^m \ln g(x_k; \Phi) \tag{3.34}$$

$$L\left(\underline{\eta}, \underline{\phi}/\underline{x}, \underline{y}\right) = \frac{\eta_1^{2n}\eta_2^{2m}}{6^n(\phi_1+\eta_1)^n 6^m(\phi_2+\eta_2)^m}\prod_{i=1}^n\left[\frac{\phi_1\eta_1^2+6x_i^3}{x_i^5}\right]\prod_{j=1}^m\left[\frac{\phi_2\eta_2^2+6y_j^3}{y_j^5}\right]e^{-(\eta_1 V_1+\eta_2 V_2)} \tag{3.35}$$

Where $V_1 = \frac{1}{\sum_{i=1}^n x_i}$, $\quad V_2 = \frac{1}{\sum_{j=1}^m y_i}$

The log-likelihood function is given by;

$$LL\left(\underline{\eta}, \underline{\phi}/\underline{x}, \underline{y}\right) = 2n\ln(\eta_1) + 2m\ln(\eta_2) - n\ln\big(6(\phi_1+\eta_1)\big) - m\ln\big(6(\phi_2+\eta_2)\big) - \eta_1 V_1 - \eta_2 V_2 + \sum_{i=1}^m \ln\left(\frac{\phi_1\eta_1^2+6x_i^3}{x_i^5}\right) + \sum_{j=1}^m \ln\left(\frac{\phi_2\eta_2^2+6y_i^3}{y_j^5}\right) \tag{3.36}$$

In order to maximize the log-likelihood, we solve the nonlinear likelihood equations obtained from the partial differentiation of (3.36) w.r.t η and ϕ as shown below;

$$\frac{\partial LL\left(\underline{\eta}, \underline{\phi}/\underline{x}, \underline{y}\right)}{\partial\eta_1} = \frac{2n}{\eta_1} - \frac{n}{(\phi_1+\eta_1)} - V_1 + \sum_{i=1}^n\left[\frac{2\phi_1\eta_1}{\phi_1\eta_1^2+6x_i^3}\right] \tag{3.37}$$

$$\frac{\partial LL\left(\underline{\eta}, \underline{\phi}/\underline{x}, \underline{y}\right)}{\partial\eta_2} = \frac{2m}{\eta_2} - \frac{m}{(\phi_2+\eta_2)} - V_2 + \sum_{j=1}^m\left[\frac{2\phi_2\eta_2}{\phi_2\eta_2^2+6y_j^3}\right] \tag{3.38}$$

$$\frac{\partial LL\left(\underline{\eta}, \underline{\phi}/\underline{x}, \underline{y}\right)}{\partial\phi_1} = -\frac{n}{(\phi_1+\eta_1)} + \sum_{i=1}^n\left[\frac{\eta_1^2}{\phi_1\eta_1^2+6x_i^3}\right] \tag{3.39}$$

$$\frac{\partial LL\left(\underline{\eta}, \underline{\phi}/\underline{x}, \underline{y}\right)}{\partial\phi_2} = -\frac{m}{(\phi_2+\eta_2)} + \sum_{j=1}^m\left[\frac{\eta_2^2}{\phi_2\eta_2^2+6y_j^3}\right] \tag{3.40}$$

We obtain the Maximum Likelihood Estimators (MLE) of ϕ_1, η_1 and ϕ_2, η_2 say $\widehat{\phi_1}, \widehat{\eta_1}$ and $\widehat{\phi_2}, \widehat{\eta_2}$ respectively as the solution of the equations above as

$$\eta_1^2 V_1\phi_1 - n\eta_1 + \phi_1\eta_1 V_1 - 2\phi_1 n + (\phi_1\eta_1 + \eta_1^2)\sum_{i=1}^n\left[\frac{2\phi_1\eta_1}{\phi_1\eta_1^2+6x_i^3}\right] = 0 \tag{3.41}$$

Similarly,

$$\eta_2^2 V_2 \phi_2 - m\eta_2 + \phi_2\eta_2 V_2 - 2\phi_2 m + (\phi_2\eta_2 + \eta_2^2)\sum_{j=1}^{m}\left[\frac{2\phi_2\eta_2}{\phi_2\eta_2^2 + 6y_j^3}\right] = 0 \qquad (3.42)$$

Also,

$$-\frac{n}{(\phi_1+\eta_1)} + \sum_{i=1}^{n}\left[\frac{\eta_1^2}{\phi_1\eta_1^2 + 6x_i^3}\right] = 0 \qquad (3.43)$$

Similarly,

$$-\frac{m}{(\phi_2+\eta_2)} + \sum_{j=1}^{m}\left[\frac{\eta_2^2}{\phi_2\eta_2^2 + 6y_j^3}\right] = 0 \qquad (3.44)$$

Hence, using the invariance property of the MLE, the maximum likelihood estimator $\hat{R}_{mle}$ of R_{SS} can be obtained by substituting $\hat{\eta}_k$ in place of η_k and $\hat{\phi}_k$ in place of ϕ_k for $k = 1, 2$

$$\hat{R}_{mle} = \left.\frac{\eta_1^2\left\{\begin{array}{c}[144\phi_1\eta_1^2+36(\eta_1+\eta_2)^3](\phi_2+\eta_2)(\eta_1+\eta_2)^3 \\ +\eta_2\left[\begin{array}{c}5040\phi_1\phi_2\eta_1^2\eta_2^2+2160\phi_1\phi_2\eta_1^2\eta_2(\eta_1+\eta_2)+720\phi_1\phi_2\eta_1^2(\eta_1+\eta_2)^2+ \\ 144\phi_2\eta_2^2(\eta_1+\eta_2)^3+108\phi_2\eta_2(\eta_1+\eta_2)^4+72\phi_2(\eta_1+\eta_2)^5\end{array}\right]\end{array}\right\}}{36(\phi_1+\eta_1)(\phi_2+\eta_2)(\eta_1+\eta_2)^7}\right|_{\eta_k=\hat{\eta}_k,\phi_k=\hat{\phi}_k;k=1,2} \qquad (3.45)$$

2.7 Parameter Estimation

Let $Y_1, Y_2, Y_3, \ldots, Y_m$ be a random sample of size m from the Inverse Copoun distribution $g(y_k; \eta, \phi)$. Then the log-likelihood function of parameters is given by

$$L(y_1, y_2, y_3, \ldots, y_m; \Phi) = \prod_{i=1}^{m} \ln g(y_k; \Phi) \qquad (3.46)$$

$$L(y; \eta, \phi) = \frac{\eta^{2m}}{[6(\phi+\eta)^m]}\prod_{i=1}^{m}\left[\frac{\phi\eta^2+6y_i^3}{y_i^5}\right]e^{-\eta V} \qquad (3.47)$$

Where $V = \frac{1}{\sum_{i=1}^{m} y_i}$

The log-likelihood function is given by;

$$LL(y; \eta, \phi) = 2m\ln(\eta) - m\ln\big(6(\phi+\eta)\big) - \eta V + \sum_{i=1}^{m}\ln\left(\frac{\phi\eta^2+6y_i^3}{y_i^5}\right) \qquad (3.48)$$

In order to maximize the log-likelihood, we solve the nonlinear likelihood equations obtained from the partial differentiation of (3.48) w.r.t $\eta \; and \; \phi$ as shown below;

$$\frac{\partial LL}{\partial \eta} = \frac{2m}{\eta} - \frac{m}{(\phi+\eta)} - V + \sum_{i=1}^{m}\left[\frac{2\phi\eta}{\phi\eta^2+6y_i^3}\right] \qquad (3.49)$$

$$\frac{\partial LL}{\partial \phi} = -\frac{m}{(\phi+\eta)} + \sum_{i=1}^{m}\frac{\eta^2}{\phi\eta^2+6y_i^3} \qquad (3.50)$$

In order to obtain the estimates of the parameters using the nonlinear equations above, we equate equations to zero and solve simultaneously. The solutions cannot be solved analytically. Hence, we solve numerically using the MaxLik package of in the R software (Toomet et al., 2015) with "BFGS" algorithm.

3. Application

This section discusses the flexibility and superiority of the Inverse Copoun distribution (ICD) to some competing distributions using two real life data sets. The first dataset represents lifetime data relating to times (in months) of 200 patients who were diagnosed with Hepatitis B where second visit is the event of interest.

The dataset is shown below

27, 32, 8, 30, 34, 23, 41, 36, 28,16, 30, 3, 24, 77, 30, 33, 17, 38, 36, 29, 17, 14, 7, 27, 12, 32, 32, 26, 15, 31, 34, 28, 27, 6, 7, 28, 44, 31, 27, 32, 7, 32, 35, 26, 16, 3, 8, 28, 35, 32, 29, 28, 27, 32, 33, 10, 14, 10, 1, 26, 23, 32, 29, 27, 31, 32, 36, 28, 39, 10, 2, 26, 23, 31, 29, 30, 31, 28, 34, 23, 39, 10, 9, 26 ,18, 31, 46, 30, 39, 33, 34, 31, 19, 3, 6, 27, 23, 30, 46, 38, 39, 33, 37, 30, 29, 3, 6, 32, 26, 30, 33, 48, 40, 32, 30, 25, 30, 3, 7.

The second dataset employed represents the survival times of a group of patients suffering from Head and Neck cancer disease and were treated using radiotherapy (RT). It was reported by (Umeh & Ibenegbu, 2019) and used by (Enogwe et al., 2020).

The data is shown below.

6.53,7,10.42,14.48,16.10,22.70,34,41.55,42,45.28,49.40,53.62,63,64,83,84,91,108,112,129,13
3,133,139,140,140,146,149,154,157,160,160,165,146,149,154,157,160,160,165,173,176,218,
225,241,248,273,277,297,405,417,420,440,523,583,594,1101,1146,1417.

This dataset is then fitted with the Inverse Copoun distribution (ICD) and compared with the Inverse Akash distribution (IAD) (Okereke et al., 2021), Inverse Suja distribution (ISD) (John et al., 2023), and Inverse Lindley distribution (ILD) (Sharma et al., 2015) with corresponding pdfs.

$$g_{ILD}(x; \eta) = \frac{\eta^2}{1+\eta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\eta}{x}} \qquad (4.1)$$

$$g_{IAD}(x; \eta) = \frac{\eta^3}{\eta^2+2}\left(\frac{1+x^2}{x^4}\right)e^{-\frac{\eta}{x}} \qquad (4.2)$$

$$g_{ISD}(x;\Phi) = \frac{\eta^5}{\eta^4+24}\left(\frac{1+x^4}{x^6}\right)e^{-\frac{\eta}{x}} \quad (4.3)$$

This comparison is done using some measures for testing the goodness of fit of a distribution. The measures used are the parameter estimates, the log likelihood, the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC) -2lnL, Akaike Information Criterion (AIC) (Club, 2016), Bayesian Information Criterion (BIC) (Pollock et al., 1999), Consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC). In general, the smaller the values of AIC, BIC, CAIC, and HQIC the better the fit to the data.

$$AIC = 2k - 2\ln L \quad (4.4)$$

$$BIC = k\ln n - 2\ln L \quad (4.5)$$

$$CAIC = AIC + \frac{2k(k+1)}{(n-k-1)} \quad (4.6)$$

$$HQIC = 2k\ln(\ln(n)) - 2\ln L \quad (4.7)$$

where k is the number of parameters, n is the sample size of the dataset, and L is the likelihood function.

Table 1: Goodness of fit for the Hypertention Data

Distribution	MLE's	S.E	-2ln L	AIC	BIC	CAIC	HQIC
IC	**$\hat{\eta}$ =16.1872** **$\hat{\phi}$ =0.8882**	**1.5470** **0.5687**	**532.2033**	**1068.407**	**1073.965**	**1068.51**	**1067.535**
IA	$\hat{\eta}$ =14.275	1.326	535.3027	1072.605	1080.164	1072.709	1073.734
IS	$\hat{\eta}$ =14.037	1.211	535.1981	1072.396	1079.954	1072.5	1073.525
IL	$\hat{\eta}$ =14.884	1.326	542.0826	1086.165	1093.723	1086.269	1087.294

The parameter estimates and their goodness of fit of the different models for the first dataset are presented in Table 1. From the results, the Inverse Copoun distribution (ICD) performed better than the competing distributions.

Table 2: Goodness of fit for the Hypertention Data

Distribution	MLE's	S.E	-2ln L	AIC	BIC	CAIC	HQIC
IC	**$\hat{\eta}$ =74.431** **$\hat{\phi}$ =7.029**	**4.194** **2.966**	**381.3487**	**766.6974**	**770.8183**	**766.9156**	**765.5**
IA	$\hat{\eta}$ =59.193	2.966	385.6517	773.3033	779.4242	773.5215	774.1059
IS	$\hat{\eta}$ =59.126	2.422	385.6861	773.3722	779.4931	773.5904	774.1748
IL	$\hat{\eta}$ =60.094	4.194	386.5834	775.1669	781.2878	775.3851	775.9695

The parameter estimates and their goodness of fit of the different models for the second dataset are presented in Table 2. From the results, the Inverse Copoun distribution (ICD) performed better than the competing distributions.

II. Conclusion

This paper proposed a new two-parameter distribution known as the Inverse Copoun distribution (ICD). The statistical properties of the Inverse Copoun distribution such as the mode, order statistics, entropy, stochastic ordering, stress-strength reliability, and reliability indices was derived and presented. The properties of the new Inverse Copoun distribution showed that the Copoun distribution can be used to model lifetime datasets with unimodal, positively skewed, and right tailed properties. Furthermore, the risk measurement function of the Inverse Copoun distribution can model datasets with upside-down bathtub shape in survival analysis. In addition, the method of maximum likelihood estimate was adopted to derive the estimates of the parameters. The flexibility of the new Inverse Copoun distribution was compared with other competing distributions using two different real life datasets. The results obtained showed that the new Inverse Copoun distribution gave the best fit to the data based on some model selection criteria. Hence, the new Inverse Copoun distribution is therefore recommended as an alternative to other existing distributions.

References

[1]. Abouammoh, A. M., & Alshingiti, A. M. (2009). Reliability estimation of generalized inverted exponential distribution. Journal of Statistical Computation and Simulation, 79(11), 1301–1315. https://doi.org/10.1080/00949650802261095

[2]. Aderoju, S. (2021). Samade Probability Distribution: Its Properties and Application to Real Lifetime Data. Asian Journal of Probability and Statistics, 14(1), 1–11. https://doi.org/10.9734/ajpas/2021/v14i130317

[3]. Aijaz, A., Jallal, M., Ain, S. Q. U., & Tripathi, R. (2020). The Hamza Distribution with Statistical Properties and Applications. Asian Journal of Probability and Statistics, 8(1), 28–42. https://doi.org/10.9734/AJPAS/2020/v8i130196

[4]. Benrabia, M., & Alzoubi, L. M. A. (2021). Alzoubi distribution: Properties and applications. An International Journal of Statistics Applications & Probability, 11(2), 1–16. https://doi.org/10.18576/jsap/ALzoubi dist nsp

[5]. Club, T. B. (2016). Akaike, H. 1973. Information theory and an extension of the maximum likelihood principle. Pp. 267-281 in 2nd International Symposium on Information Theory, Tsahkadsor, Armenia, USSR, September 2-8, 1971, eds. BN Petrov and F. Csáki. Budapest: Akadémiai Kiadó. Bankevich, A., S. Nurk, D. Antipov, AA Gurevich, M. Dvorkin, AS Kulikov, VM Lesin, et al. 2012. SPAdes: A new genome assembly algorithm and its applications to single. Systematics and Evolution of the Arundinoideae and Micrairoideae (Poaceae), 18(1), 139.

[6]. Elechi, O., Okereke, E. W., Chukwudi, I. H., Chizoba, K. L., & Wale, O. T. (2022). Iwueze's Distribution and Its Application. Journal of Applied Mathematics and Physics, 10(12), Article 12. https://doi.org/10.4236/jamp.2022.1012251

[7]. Eliwa, M. S., Altun, E., Alhussain, Z. A., Ahmed, E. A., Salah, M. M., Ahmed, H. H., & El-Morshedy, M. (2021). A new one-parameter lifetime distribution and its regression model with applications. PLOS ONE, 6(2), 1–19. https://doi.org/10.1371/journal.pone.0246969

[8]. Eliwa, M. S., El-Morshedy, M., & Ibrahim, M. (2018). Inverse Gompertz Distribution: Properties and Different Estimation Methods with Application to Complete and Censored Data. 6, 321–339.

[9]. Enogwe, S. U., Nwosu, D. F., Ngome, E. C., Onyekwere, C. K., & Omeje, I. L. (2020). Two-parameter Odoma distribution with applications. Journal of Xidian University, 14(8), 740–764.

[10]. Epstein, N. (1958). EXPONENTIAL DISTRIBUTION AND ITS ROLE IN LIFE TESTING. 1–21.

[11]. E.W. Okereke, S.N. Gideon, & J. Ohakwe. (2021). Inverse Akash distribution and its applications. Scientific African, 20(2), 61–72. https://doi.org/10.4314/sa.v20i2.6

[12]. John, C., Pokalas, T., Umezurike, C., & Ohakwe, J. (2023). INVERSE SUJA DISTRIBUTION AND ITS APPLICATIONS. Journal of the Nigerian Association of Mathematical Physics, 17(October 2021), 151–158.

[13]. Lee, S., Noh, Y., & Chung, Y. (2017). Inverted exponentiated Weibull distribution with applications to lifetime data. Communications for Statistical Applications and Methods, 24(3), 227–240.

[14]. Pollock, D. S. G., Green, R. C., & Nguyen, T. (1999). Handbook of time series analysis, signal processing, and dynamics. Elsevier.

[15]. Rényi, A. (1961). On measures of entropy and information. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1, 547–562.

[16]. Ronald, W. E., Myers, R. H., Myers, S. L., & Ye, K. E. (2011). Probability and Statistics for Engineers and Scientists (9th Edition) (9th ed.). Prentice Hall. http://library.lol/main/09C6713DAA3BF89644499A5EB315D12E

[17]. Shaked, M., & Shanthikumar, J. (1994). Stochastic orders and their applications. Academic Press. https://www.semanticscholar.org/paper/Stochastic-orders-and-their-applications-Shaked-Shanthikumar/a29a9228cdb7a7dcb205ba6ee2bf64bc0b63fa90

[18]. Shanker, R. (2015a). Akash Distribution and Its Applications. International Journal of Probability and Statistics, 4(3), 65–75. https://doi.org/10.5923/j.ijps.20150403.01

[19]. Shanker, R. (2015b). Shanker Distribution and Its Applications. International Journal of Statistics and Applications, 5, 338–348.

[20]. Shanker, R. (2016a). Amarendra Distribution and Its Applications. American Journal of Mathematics and Statistics, 6(1), 44–56. https://doi.org/10.5923/j.ajms.20160601.05

[21]. Shanker, R. (2016b). Sujatha Distribution and its Applications. Statistics in Transition New Series, 17(3), 391–410. https://doi.org/10.21307/stattrans-2016-029

[22]. Shanker, R., & Shukla, K. K. (2020). A New Quasi Sujatha Distribution. Statistics in Transition New Series, 21(3), 53–71.

[23]. Shannon, C. E. (1951). Prediction and entropy of printed English. Bell System Technical Journal, 30(1), 50–64.

[24]. Sharma, V. K., Sanjay Kumar Singh, Singh, U., Varun Agiwal, & Agiwal, V. (2015). The inverse Lindley distribution: A stress-strength reliability model with application to head and neck cancer data. Journal of Industrial and Production Engineering, 32(3), 162–173. https://doi.org/10.1080/21681015.2015.1025901

[25]. Shraa, D. H., & Al-Omari, A. I. (2019). Darna distribution: Properties and application. Electronic Journal of Applied Statistical Analysis, 12(02), 520–541. https://doi.org/10.1285/i20705948v12n2p520

[26]. Shukla, K. K. (2018a). Prakaamy distribution with properties and applications. JAQM, 13(3), 30–38.

[27]. Shukla, K. K. (2018b). Ram Awadh distribution with properties and applications. Biometrics & Biostatistics International Journal, 7(6), 515–523. https://doi.org/10.15406/bbij.2018.07.00254

[28]. Toomet, O., Henningsen, A., & Toomet, M. O. (2015). Package 'maxLik.'

[29]. Umeh, E., & Ibenegbu, A. (2019). A Two-Parameter Pranav Distribution with Properties and Its Application. Journal of Biostatistics and Epidemiology, 5(1), 74–90. https://doi.org/10.18502/jbe.v5i1.1909

[30]. Uwaeme, O. R., Akpan, N. P., & Orumie, U. C. (2023). The Copoun Distribution and Its Mathematical Properties. Asian Journal of Probability and Statistics, 24(1), 37–44.