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**Review Paper**

# The Inverse Remkan Distribution and its Applications to Uncensored Data

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***Abstract***
*In this study, we propose a new two-parameter lifetime distribution called the Inverse Remkan distribution. This distribution is derived using the inverse function to contribute to the growing need for upside-down bathtub distributions. Some important mathematical properties of the new distribution such as the density, mode, entropy, and reliability indices such as the stress-strength, existence measurement function and risk measurement function were derived and the model parameters estimated using the maximum likelihood estimate technique. Finally, the flexibility of the new inverse distribution was illustrated using two uncensored datasets and the results showed that the new inverse distribution was the best amongst other competing inverse distributions.*

***Keywords:*** *Inverse Remkan distribution; Lifetime distributions; Two-parameter distributions; Entropy; Stress-strength reliability; Risk measurement function.*

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## I. The Distribution

In recent years, new two-parameter distributions have emerged in the literature. These new two-parameter distributions have been shown to provide a better fit to complex real life datasets than the one-parameter distributions. Some of the recently developed two-parameter distributions include the Darna distribution (Shraa & Al-Omari, 2019), the Hamza distribution (Aijaz et al., 2020), the Samade distribution (Aderoju, 2021), and the Alzoubi distribution (Benrabia & Alzoubi, 2021).

It is important to note that these distributions are a mixture of the Exponential and Gamma distributions. These two distributions are known to have their weaknesses. The weakness of the Exponential distribution is that the hazard rate function is constant; hence, it cannot handle datasets with monotone non-decreasing hazard rates (Elechi et al., 2022; Epstein, 1958; Ronald et al., 2011; Shukla, 2018b; Shukla, 2018). Furthermore, the weakness of the Gamma distribution is that the survival rate function cannot be expressed in closed form (Elechi et al., 2022; Shanker, 2015a, 2015b). The weaknesses of these two distributions are what the aforementioned one-parameter and two-parameter distributions address, providing distributions whose survival rate function can be expressed in closed form and hazard rate functions capable of handling datasets with monotone non-decreasing hazard rates.

In contributing to this gap in the literature, Uwaeme and Akpan (2024) proposed a new two-parameter distribution called the Remkan distribution (Uwaeme & Akpan, 2024b). The Remkan distribution is a three-component density of an Exponential $(\eta)$, Gamma $(3,\eta)$, and Gamma $(4,\eta)$ distribution with mixing proportions $\pi_1 = \frac{\eta}{(\eta+2\phi+6)}$, $\pi_2 = \frac{2\phi}{(\eta+2\phi+6)}$, and $\pi_3 = \frac{6}{(\eta+2\phi+6)}$ such that

$$g(x;\eta,\phi) = \pi_1 g_1(x;\eta) + \pi_2 g_2(x;\eta) + \pi_3 g_3(x;\eta) \quad (1.1)$$

where $g_1(x;\eta) = \eta e^{-\eta x}$, $g_2(x;\eta) = \frac{\eta x^2 e^{-\eta x}}{\Gamma(3)}$, and $g_3(x;\eta) = \frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)}$

therefore,

$$g(x_k;\eta,\phi) = \eta e^{-\eta x} \cdot \frac{\eta}{\eta+2\phi+6} + \frac{\eta x^2 e^{-\eta x}}{\Gamma(3)} \cdot \frac{2\phi}{\eta+2\phi+6} + \frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)} \cdot \frac{6}{\eta+2\phi+6} \quad (1.2)$$

Solving equation 1.2 gives the probability density function (pdf) of the Remkan distribution

$$g(x;\eta,\phi) = \frac{\eta^2}{(\eta+2\phi+6)}[1 + \phi\eta x^2 + \eta^2 x^3]e^{-\eta x};\ x > 0, \eta > 0, \phi > 0 \quad (1.3)$$

The corresponding cumulative distribution function (cdf) of (1.3) is obtained as

$G(x;\eta,\phi)=1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}$ (1.4)

The authors introduced some of the statistical properties of the new distribution. They showed that the Remkan distribution exhibits shapes that are not bell-shaped, but positively skewed, unimodal, and right-tailed (Akpan & Uwaeme, 2024). One of the weaknesses of the Remkan distribution is that it does not have non-monotone hazard rates. One way of overcoming this weakness is to introduce an extension of the Remkan distribution using the inverse transformation technique. This technique produces a class of distribution known as Inverse distributions. Inverse distributions are known for their interesting advantages such as having the same parsimony as their corresponding parent distribution since no new parameter in required (Eliwa et al., 2018); and they are known to have upside-down bathtub risk measurement functions (Abouammoh & Alshingiti, 2009; Eliwa et al., 2018; E.W. Okereke et al., 2021; John et al., 2023; Lee et al., 2017; Uwaeme & Akpan, 2024a).

From the foregoing therefore, the motivation of this paper is to propose a new inverse distribution called the Inverse Remkan distribution and its statistical properties. The subsequent sections of the paper will be arranged as follows. Section 2 discusses the new inverse distribution with the derivation of the pdf, the cdf, and their plots, section 3 discusses the mathematical properties of the Inverse Remkan distribution as well as the plots of the risk measurement function to highlight the shape, section 4 looks at the application of the new distribution with real datasets alongside other competing distributions, and section 5 concludes the paper with some remarks.

## II. The Inverse Remkan distribution

This section will introduce the pdf and the cdf of the Inverse Remkan distribution and illustrate the different shapes of the Inverse Remkan distribution.

**Proposition 1:** If a random variable Y follows the Remkan distribution with parameters $\eta$ and $\phi$, then the random variable $X=\frac{1}{Y}$ has Inverse Remkan distribution with parameters $\eta$ and $\phi$ and its pdf and cdf are respectively given by

$g(y;\eta,\phi)=\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{y^3+\phi\eta y+\eta^2}{y^5}\right]e^{-\eta y^{-1}};\ y>0,\eta>0,\phi>0$ (2.1)

and

$G(y;\eta,\phi)=\left[1+\frac{\eta^2+(3+\phi)\eta y+(6+2\phi)\eta^2y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}};\ y>0,\eta>0,\phi>0$ (2.2)

**Proof:** If X follows the Inverse Remkan distribution with parameters $\eta$ and $\phi$, the pdf of X is given as

$f(x;\eta,\phi)=\frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}$

Let $y=x^{-1},\frac{dx}{dy}=-y^{-2}$. Thus, the pdf of X is given by

$g(y;\eta,\phi)=f(y^{-1})\left|\frac{dx}{dy}\right|$ (2.3)

$$=\frac{\eta^2}{(\eta+2\phi+6)}\left[1+\phi\eta\left(\frac{1}{y}\right)^2+\eta^2\left(\frac{1}{y}\right)^3\right]e^{-\eta y^{-1}}\left(\frac{1}{y^2}\right)$$

$=\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{y^3+\phi\eta y+\eta^2}{y^5}\right]e^{-\eta y^{-1}};\ y>0,\eta>0,\phi>0.$

The corresponding cdf for the Inverse Remkan distribution can be expressed as

$G(y;\eta,\phi)=\frac{\eta^2}{(\eta+2\phi+6)}\int_0^y k^{-5}[k^3+\phi\eta k+\eta^2]e^{-\eta k^{-1}}dk$ (2.4)

$$=\frac{\eta^2}{(\eta+2\phi+6)}\left[\eta^2\int_0^y k^{-5}e^{-\eta k^{-1}}dk+\phi\eta\int_0^y k^{-4}e^{-\eta k^{-1}}dk+\int_0^y k^{-2}e^{-\eta k^{-1}}dk\right]$$

Let $t=\eta k^{-1},k=\eta t^{-1},and\ dk=-\eta t^{-2}dt.$

Applying integration by parts techniques, we have,

$$=\frac{\eta^2}{(\eta+2\phi+6)}[-\eta^{-2}k^3-(3+\phi)\eta^{-2}k^2-(6+2\phi)\eta^{-2}k-[\eta^{-1}+(6+2\phi)\eta^{-2}]e^{-k}]\Big|_{\eta y^{-1}}^{\infty}$$

Taking the limit,

$$G(y;\eta,\phi)=\left[1+\frac{\eta^2+(3+\phi)\eta y+(6+2\phi)\eta^2y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}$$

The Inverse Remkan distribution derived above is denoted by $IRD(\eta,\phi)$. The graphical plots of the theoretical density and distribution function (for some selected but different real points of $\eta$ and $\phi$) of the Inverse Remkan distribution are shown in the Figure 1 and Figure 2 below.

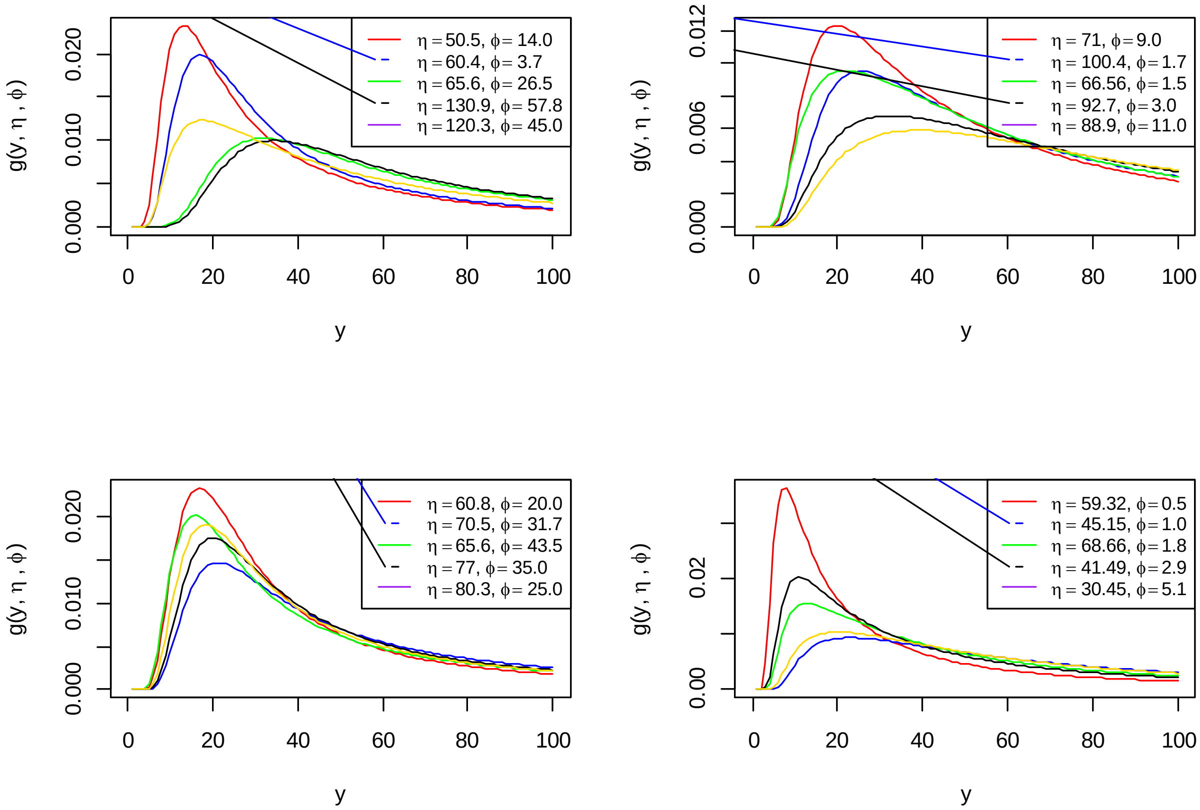


**Figure 1: The graphical plots of the probability density function (for some selected but different real points of $\eta$ and $\phi$) of an Inverse Remkan distribution**.

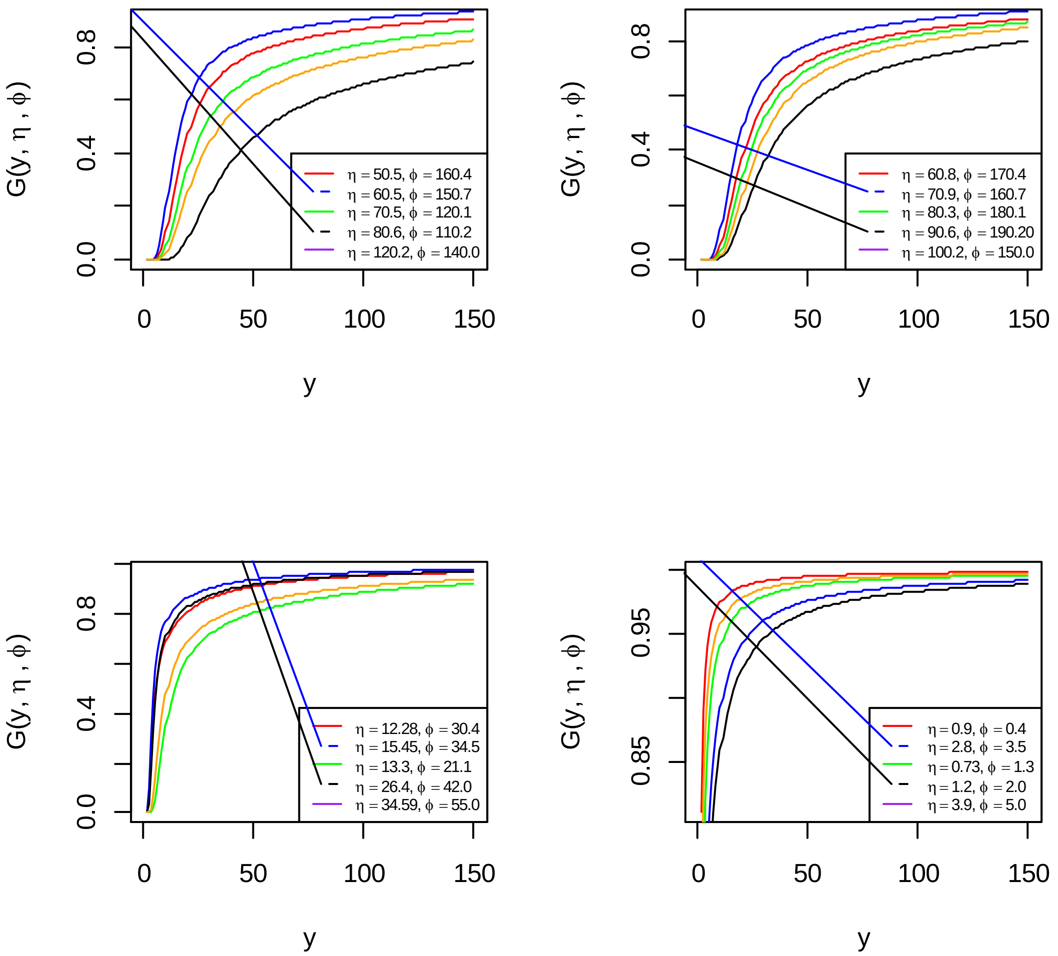


**Figure 2: The graphical plots of the cumulative distribution function (for some selected but different real points of $\eta$ and $\phi$) of an Inverse Remkan distribution**

The curves displayed in Figure 1 are not bell-shaped, but are positively skewed, unimodal, and right tailed. In addition, the curve shows that increasing the value of $\phi$ leads to a considerable increase in the peak of the curve. In addition, the curves displayed in Figure 2 shows that the cumulative distribution function converges to one.

## III. Statistical Properties of the Inverse Remkan Distribution

In this section, we derive and present some statistical properties of the Inverse Remkan distribution. These includes the mode, survivorship or existence measurement function, risk measurement function, stochastic ordering of random variate, stress-strength reliability, entropy, and order statistics.

### 3.1 Mode

**Theorem 1:**

Given a continuous random variable $Y$ which follows the Inverse Remkan distribution, the mode of $Y$, is given as

$$Mode = \begin{cases} -\frac{\eta^2(2y^4-\eta y^3+4\phi\eta y^2+(5-\phi)\eta^2 y-\eta^3)e^{-\eta y^{-1}}}{y^7(\eta+2\phi+6)}, & 0<\eta<\frac{1}{2} \\ 0, & otherwise \end{cases} \quad (3.1)$$

**Proof:**

From given a continuous random variable $Y$, the mode of $Y$, is obtained by

$$Mode = \frac{d}{dy}g(y;\eta,\phi) = 0 \quad (3.2)$$

$$\frac{d}{dy}\left(\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{y^3+\phi\eta y+\eta^2}{y^5}\right]e^{-\eta y^{-1}}\right) = 0 \quad (3.3)$$

$$\therefore \frac{d}{dy}g(y;\eta,\phi) = -\frac{\eta^2\left(2y^4-\eta y^3+4\phi\eta y^2+(5-\phi)\eta^2 y-\eta^3\right)e^{-\eta y^{-1}}}{y^7(\eta+2\phi+6)} = 0 \quad (3.6)$$

Which completes the proof.

### 3.2 Order statistics

**Theorem 2:**

Given a continuous random variable $X$, pdf and cdf of the pth order statistics, say $X = X_{(p)}$, is given respectively by

$$g_p(y) = \frac{n!\eta^2(y^3+\phi\eta y+\eta^2)e^{-\eta y^{-1}}}{y^5(\eta+2\phi+6)(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[\left[1+\frac{\eta^2+(3+\phi)\eta y+(6+2\phi)\eta^2y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{p+i-1} \quad (3.7)$$

and

$$G_p(y) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left[\left[1+\frac{\eta^2+(3+\phi)\eta y+(6+2\phi)\eta^2y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{j+1} \quad (3.8)$$

**Proof:**

Given a continuous random variable $Y$, the pdf of the pth order statistics, say $Y = Y_{(p)}$, is obtained by

$$g_p(y) = \frac{n!g(x_k;\Phi)}{(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i G(x_k;\Phi)^{p+i-1} \quad (3.9)$$

$$g_p(y) = \frac{n!\left(\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{y^3+\phi\eta y+\eta^2}{y^5}\right]e^{-\eta y^{-1}}\right)}{(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[\left[1+\frac{\eta^2+(3+\phi)\eta y+(6+2\phi)\eta^2y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{p+i-1} \quad (3.10)$$

$$g_p(y) = \frac{n!\eta^2(y^3+\phi\eta y+\eta^2)e^{-\eta y^{-1}}}{y^5(\eta+2\phi+6)(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[\left[1+\frac{\eta^2+(3+\phi)\eta y+(6+2\phi)\eta^2y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{p+i-1} \quad (3.11)$$

Which completes the proof.

Correspondingly, given a continuous random variable $Y$, the cdf of the pth order statistics, say $Y = Y_{(p)}$, is obtained by

$$G_p(y) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i G(x_k;\Phi)^{j+1} \quad (3.12)$$

$$G_p(y) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left[\left[1+\frac{\eta^2+(3+\phi)\eta y+(6+2\phi)\eta^2y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right]e^{-\eta y^{-1}}\right]^{j+1} \quad (3.13)$$

Which completes the proof.

### 3.3 Entropy

Entropy measures the uncertainties associated with a random variable of a probability distributions. Shannon (Shannon, 1951) and Rényi's entropy (Rényi, 1961)are widely used in the literature.

**Theorem 3:**

Given a random variable $Y$, which follows the Inverse Copoun distribution $g(y;\eta,\phi)$. The Rényi entropy is given by

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left[\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^\lambda}\sum_{j=0}^{\infty}\binom{\lambda}{j}\left(\frac{(\phi\eta)^j\Gamma(2\lambda-4j-1)}{(\lambda\eta)^{2\lambda-4j-1}}+\frac{\eta^{2j}\Gamma(2\lambda-3j-1)}{(\lambda\eta)^{2\lambda-3j-1}}\right)\right] \quad (3.14)$$

**Proof:**

The Rényi entropy is given by

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left[\int_0^\infty g_n^\lambda(y_k;\Phi)dy\right] \quad (3.15)$$

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left[\int_0^\infty\left(\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{y^3+\phi\eta y+\eta^2}{y^5}\right]e^{-\eta y^{-1}}\right)^\lambda dy\right] \quad (3.16)$$

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left[\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^\lambda}\int_0^\infty\left[\frac{y^3+\phi\eta y+\eta^2}{y^5}\right]^\lambda e^{-\lambda\eta y^{-1}}dy\right] \quad (3.17)$$

Recall that $(1+\delta)^\vartheta = \sum_{k=0}^{\infty}\binom{\vartheta}{k}\delta^k$ and $\int_0^\infty z^{-w-1}e^{-\frac{\varrho}{z}}dz = \frac{\Gamma(w)}{\varrho^w}$ Substituting,

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left[\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^\lambda}\sum_{j=0}^{\infty}\binom{\lambda}{j}\left((\phi\eta)^j\int_0^\infty\frac{e^{-\lambda\eta y^{-1}}}{y^{-2\lambda-4j}}dy+\eta^{2j}\int_0^\infty\frac{e^{-\lambda\eta y^{-1}}}{y^{-2\lambda-3j}}dy\right)\right] \quad (3.18)$$

$$\therefore T_R(\lambda) = \frac{1}{1-\lambda}\log\left[\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^\lambda}\sum_{j=0}^{\infty}\binom{\lambda}{j}\left(\frac{(\phi\eta)^j\Gamma(2\lambda-4j-1)}{(\lambda\eta)^{2\lambda-4j-1}}+\frac{\eta^{2j}\Gamma(2\lambda-3j-1)}{(\lambda\eta)^{2\lambda-3j-1}}\right)\right]$$

Which completes the proof.

### 3.4 Reliability Indices

Given any probability distribution, the reliability analysis is always considered based on the Existence Measurement Function and Risk Measurement Function. Hence, for the Inverse Remkan distribution, the Existence Measurement Function and Risk Measurement Function is given below.

#### 3.4.1 Existence Measurement Function

The existence measurement function (also known as survival function) is defined as the probability that an item does not fail prior to some time t (Elechi et al., 2022; Epstein, 1958; Ronald et al., 2011; Shanker & Shukla, 2020; Uwaeme & Akpan, 2024a).

The existence measurement function of the Inverse Remkan distribution is given by

$$s(y) = 1 - G(y; \eta, \phi) \qquad (3.19)$$

$$s(y) = 1 - \left[1 + \frac{\eta^2 + (3+\phi)\eta y + (6+2\phi)\eta^2 y^2}{(\eta + 2\phi + 6)}\left(\frac{\eta}{y^3}\right)\right] e^{-\eta y^{-1}}; y > 0, \eta > 0, \phi > 0 \qquad (3.20)$$

#### 3.4.2 Risk Measurement Function

The risk measurement function (also known as hazard rate function) on the other hand can be seen as the conditional probability of failure, given it has survived to the time t (Elechi et al., 2022; Ronald et al., 2011; Shanker, 2016b; Umeh & Ibenegbu, 2019). It is obtained as

The risk measurement function of the Inverse Remkan distribution is given by

$$h(y) = \frac{g(y_k; \Phi)}{1 - G(y_k; \Phi)} \qquad (3.21)$$

$$h(y) = \frac{\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{y^3 + \phi\eta y + \eta^2}{y^5}\right] e^{-\eta y^{-1}}}{1 - \left[1 + \frac{\eta^2 + (3+\phi)\eta y + (6+2\phi)\eta^2 y^2}{(\eta+2\phi+6)}\left(\frac{\eta}{y^3}\right)\right] e^{-\eta y^{-1}}} \qquad (3.22)$$

$$h(x) = \frac{\eta^2 [y^3 + \phi\eta y + \eta^2]}{y^2 \left[y^3(\eta + 2\phi + 6)\left(e^{\eta y^{-1}} - 1\right) - [\eta^3 + \eta^2(3+\phi)y + \eta^3(6+2y)y^2]\right]} \qquad (3.23)$$

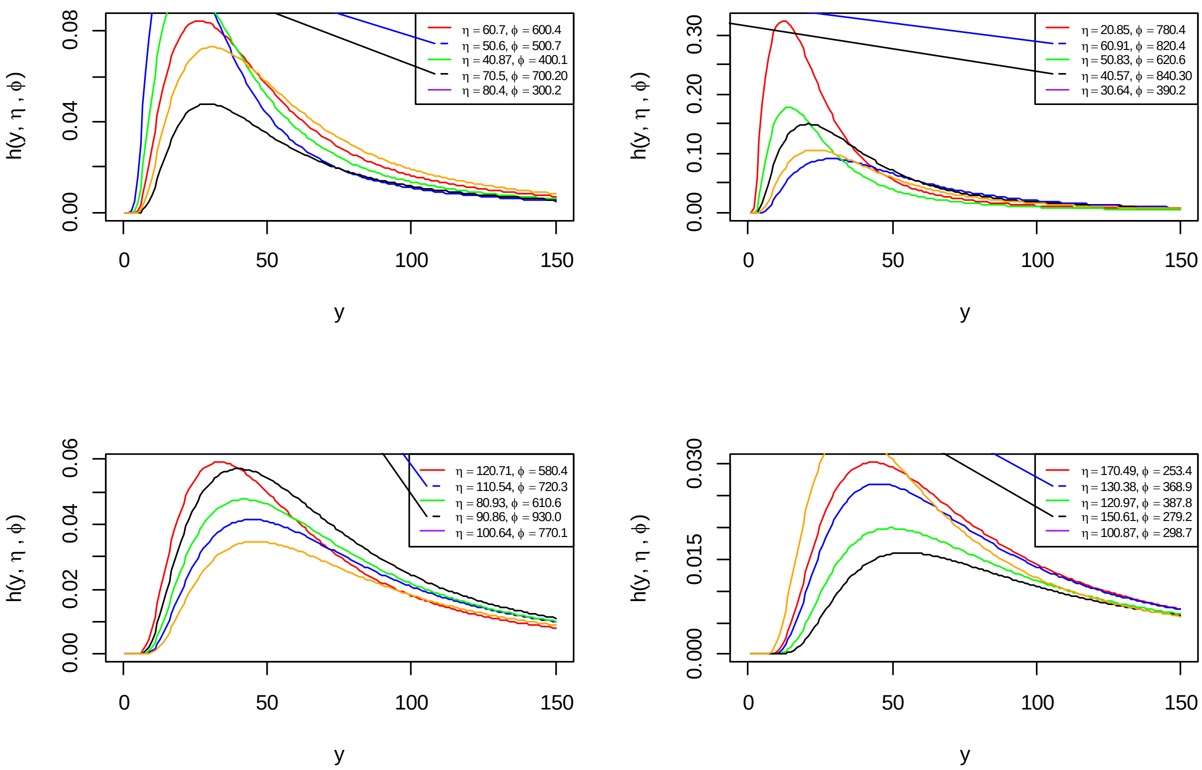


**Figure 3: The graphical plots of the risk measurement function (for some selected but different real points of $\eta$ and $\phi$) of an Inverse Coupon distribution**

### 3.5 Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behaviour. According to Shanker (2015), a random variable X is said to be smaller than a random variable Y in the

- Stochastic order ($S \leq_{stor} Q$) if $G_S(y) \geq G_Q(y)$ for all y.
- Hazard rate order ($S \leq_{Ror} Q$) if $h_S(y) \geq h_Q(y)$ for all y.
- Mean residual life order ($S \leq_{mrl} Q$) if $m_S(y) \geq m_Q(y)$ for all y.
- Likelihood ratio order ($S \leq_{lor} Q$) if $\left(\frac{g_S(y)}{g_Q(y)}\right)$ decreases in y.

**Theorem 7:**

Let $S\sim$ CD $(\eta_1, \phi_1)$ and Q~ CD $(\eta_2, \phi_2)$. If $\eta_1 = \eta_2$ and $\phi_1 = \phi_2$ and $\phi_1 \geq \phi_2$ (or if $\phi_2 = \phi_2$ and $\eta_1 = \eta_2$ and $\eta_1 \geq \eta_2$), then $S \leq_{stor} Q$ and hence $S \leq_{Ror} Q, S \leq_{mrl} Q \; and \; S \leq_{lor} Q$

**Proof:**

Let $S\sim$ CD $(\eta_1, \phi_1)$ and Q~ CD $(\eta_2, \phi_2)$. We obtain that

$$\frac{f_S(\varphi)}{f_Q(\varphi)} = \frac{\eta_1^2[\eta+2\phi+6]}{\eta_2^2[\eta+2\phi+6]} \cdot e^{-\frac{(\eta_1-\eta_2)}{\varphi}} \quad (3.24)$$

and

$$\log_\varphi \frac{f_S(\varphi)}{f_Q(\varphi)} = \log_\varphi \left[\frac{\eta_1^2[\eta+2\phi+6]}{\eta_2^2[\eta+2\phi+6]}\right] - \frac{(\eta_1-\eta_2)}{\varphi} \quad (3.25)$$

Hence,

$$\frac{d}{d\varphi}\log_\varphi \frac{f_S(\varphi)}{f_Q(\varphi)} = \frac{(\eta_1-\eta_2)}{\varphi^2} \quad (3.26)$$

Which completes the proof.

This implies that $S \leq_{lor} Q \Rightarrow S \leq_{Ror} Q \Rightarrow \begin{cases} S \leq_{sor} Q \\ S \leq_{mrl} Q \end{cases}$. This clearly indicates that the Inverse Remkan distribution is ordered in the likelihood ratio and consequently has risk measurement, average residual measurement life, and stochastic orderings. These results has been established in the literature for stochastic ordering of distributions (Shaked & Shanthikumar, 1994; Shanker, 2016a; Uwaeme et al., 2023; Uwaeme & Akpan, 2024a).

### 3.6 Stress-Strength Reliability and Maximum Likelihood Estimations

Let Y and X be independent stress and strength random variables that follow Inverse Remkan distribution with parameter $(\eta_1, \phi_1) \; and \; (\eta_2, \phi_2)$ respectively. Then, the stress-strength reliability (R) is given by

$$R_{SS} = P[X < Y] = \int_0^\infty [X < Y | Y = y] g_y(y) dy \quad (3.27)$$

$$= \int_0^\infty g(y; \eta_1, \phi_1) G(y; \eta_2, \phi_2) dy. \quad (3.28)$$

$$= \int_0^\infty \left(\frac{\eta_1^2}{(\eta_1+2\phi_1+6)}\left[\frac{y^3+y\phi_1\eta_1+\eta_1^2}{y^5}\right] e^{-\eta_1 y^{-1}}\right)\left[1 + \frac{\eta_2^2+y\phi_2(3+\phi_2)+y^2\eta_2^2(6+2y)}{(\eta_2+\phi_2+6)}\left(\frac{\eta_2}{y^3}\right)\right] e^{-\eta_2 y^{-1}} dy. \quad (3.29)$$

$$= \frac{\eta_1^2}{(\eta_1+2\phi_1+6)}\int_0^\infty \left[\frac{y^3+y\phi_1\eta_1+\eta_1^2}{y^5}\right] e^{-\frac{(\eta_1-\eta_2)}{y}} dy + \frac{\eta_1^2\eta_2}{(\eta_1+2\phi_1+6)(\eta_2+\phi_2+6)}\int_0^\infty \left[\frac{y^3+y\phi_1\eta_1+\eta_1^2}{y^5}\right]\left[\frac{\eta_2^2+y\phi_2(3+\phi_2)+y^2\eta_2^2(6+2y)}{y^3}\right] e^{-\frac{(\eta_1+\eta_2)}{y}} dy. \quad (3.30)$$

$$= \frac{\eta_1^2}{(\eta_1+2\phi_1+6)}\left[\int_0^\infty y^{-2} e^{-\frac{(\eta_1+\eta_2)}{y}} dy + \phi_1\eta_1 \int_0^\infty y^{-4} e^{-\frac{(\eta_1+\eta_2)}{y}} dy + \eta_1^2 \int_0^\infty y^{-5} e^{-\frac{(\eta_1+\eta_2)}{y}} dy\right] + \frac{\eta_1^2\eta_2}{(\eta_1+2\phi_1+6)(\eta_2+\phi_2+6)}\left[\int_0^\infty \left[\frac{[y^3+y\phi_1\eta_1+\eta_1^2]\eta_2[2\eta_2 y^2+6\eta_2 y^2+y\phi_2+\eta_2+3y]}{y^8}\right] e^{-\frac{(\eta_1+\eta_2)}{y}} dy\right]. \quad (3.31)$$

Applying the gamma function, we obtain the expression for the Stress-strength reliability as

$$R_{SS}=\frac{\eta_1^2}{(\eta_1+2\phi_1+6)}\left[\frac{\Gamma(1)}{(\eta_1+\eta_2)}+\frac{\phi_1\eta_1\Gamma(3)}{(\eta_1+\eta_2)^3}+\frac{\eta_1^2\Gamma(4)}{(\eta_1+\eta_2)^3}\right]+\frac{\eta_1^2\eta_2}{(\eta_1+2\phi_1+6)(\eta_2+\phi_2+6)}\left[9\phi_1\eta_1\eta_2^2\left(\frac{\Gamma(3)}{(\eta_1+\eta_2)^3}+\frac{\Gamma(4)}{(\eta_1+\eta_2)^4}+\frac{\Gamma(6)}{(\eta_1+\eta_2)^6}\right)+9\eta_2^2\left(\frac{\Gamma(1)}{(\eta_1+\eta_2)}+\frac{\Gamma(2)}{(\eta_1+\eta_2)^2}+\frac{\Gamma(4)}{(\eta_1+\eta_2)^4}\right)+4\phi_1\phi_2\eta_1\eta_2\left(\frac{\Gamma(5)}{(\eta_1+\eta_2)^5}\right)+7\phi_1^2\eta_2^2\left(\frac{\Gamma(5)}{(\eta_1+\eta_2)^5}+\frac{\Gamma(7)}{(\eta_1+\eta_2)^7}\right)+\phi_2\eta_2\left(\frac{\Gamma(3)}{(\eta_1+\eta_2)^3}\right)+\phi_1^2\phi_2\eta_2\left(\frac{\Gamma(6)}{(\eta_1+\eta_2)^6}\right)+3\phi_1^2\eta_2\left(\frac{\Gamma(6)}{(\eta_1+\eta_2)^6}\right)+3\eta_2\left(\frac{\Gamma(3)}{(\eta_1+\eta_2)^3}\right)\right] \quad (3.32)$$

and

$$R_{SS}=\frac{\eta_1^2\left\{\left[2\phi_1\eta_1^2+2\phi_1\eta_1\eta_2+6\eta_1^2+(\eta_1+\eta_2)\right](2\phi_2+\eta_2+6)(\eta_1+\eta_2)^3+\eta_2\begin{bmatrix}5040\phi_1^2\eta_2^2+[120\phi_1^2\phi_2\eta_2+360\phi_1^2\eta_2+1080\phi_1\eta_1\eta_2^2](\eta_1+\eta_2)+\\ [168\phi_1^2\eta_2^2+96\phi_1\phi_2\eta_1\eta_2](\eta_1+\eta_2)^2+[54\eta_2^2+54\phi_1\eta_1\eta_2^2](\eta_1+\eta_2)^3+\\ [18\phi_1\eta_1\eta_2^2+6\eta_2+2\phi_2\eta_2](\eta_1+\eta_2)^4+9\eta_2(\eta_1+\eta_2)^5+\\ 9\eta_2^2(\eta_1+\eta_2)^5\end{bmatrix}\right\}}{(\eta_1+2\phi_1+6)(\eta_2+\phi_2+6)(\eta_1+\eta_2)^7} \quad (3.33)$$

Since R is the Stress-Strength Reliability function with parameters $(\eta_1,\phi_1)$ $(\eta_2,\phi_2)$, we need to obtain the maximum likelihood estimators (MLEs) of $(\eta_1,\phi_1)$ *and* $(\eta_2,\phi_2)$ to compute the maximum likelihood estimation R under Invariance property of the maximum likelihood estimation. Suppose $X_1,X_2,X_3,\dots,X_n$ is a Strength random variable sample from Inverse Remkan distribution $(\eta_1,\phi_1)$ and $Y_1,Y_2,Y_3,\dots,Y_m$ is a Stress random sample from Inverse Remkan distribution $(\eta_2,\phi_2)$. Thus, the likelihood function based on the observed sample is given by

$$L\left(\underline{\eta},\underline{\phi}/\underline{x},\underline{y}\right)=\prod_{i=1}^{m}\ln g(x_k;\Phi) \quad (3.34)$$

$$L\left(\underline{\eta},\underline{\phi}/\underline{x},\underline{y}\right)=\frac{\eta_1^{2n}\eta_2^{2m}}{(\eta_1+2\phi_1+6)^n(\eta_2+\phi_2+6)^m}\prod_{i=1}^{n}\left[\frac{x_i^3+\phi_1\eta_1x_i+\eta_1^2}{x_i^5}\right]\prod_{j=1}^{m}\left[\frac{y_j^3+\phi_2\eta_2y_j+\eta_2^2}{y_j^5}\right]e^{-(\eta_1V_1+\eta_2V_2)} \quad (3.35)$$

Where $V_1=\frac{1}{\sum_{i=1}^{n}x_i}$, $\quad V_2=\frac{1}{\sum_{j=1}^{m}y_i}$

The log-likelihood function is given by;

$$LL\left(\underline{\eta},\underline{\phi}/\underline{x},\underline{y}\right)=2n\ln(\eta_1)+2m\ln(\eta_2)-n\ln(\eta_1+2\phi_1+6)-m\ln(\eta_2+\phi_2+6)-\eta_1V_1-\eta_2V_2+\sum_{i=1}^{m}\ln\left(\frac{x_i^3+\phi_1\eta_1x_i+\eta_1^2}{x_i^5}\right)+\sum_{j=1}^{m}\ln\left(\frac{y_j^3+\phi_2\eta_2y_j+\eta_2^2}{y_j^5}\right) \quad (3.36)$$

In order to maximize the log-likelihood, we solve the nonlinear likelihood equations obtained from the partial differentiation of (3.36) w.r.t $\eta$ *and* $\phi$ as shown below;

$$\frac{\partial LL\left(\underline{\eta},\underline{\phi}/\underline{x},\underline{y}\right)}{\partial\eta_1}=\frac{2n}{\eta_1}-\frac{n}{(\eta_1+2\phi_1+6)}-V_1+\sum_{i=1}^{n}\left[\frac{2\eta_1+\phi_1x_i}{\eta_1^2+\phi_1\eta_1x_i+x_i^3}\right] \quad (3.37)$$

$$\frac{\partial LL\left(\underline{\eta},\underline{\phi}/\underline{x},\underline{y}\right)}{\partial\eta_2}=\frac{2m}{\eta_2}-\frac{m}{(\eta_2+\phi_2+6)}-V_2+\sum_{j=1}^{m}\left[\frac{2\eta_2+\phi_2y_j}{\eta_2^2+\phi_2\eta_2y_j+y_j^3}\right] \quad (3.38)$$

$$\frac{\partial LL\left(\underline{\eta},\underline{\phi}/\underline{x},\underline{y}\right)}{\partial\phi_1}=-\frac{2n}{(\eta_1+2\phi_1+6)}+\sum_{i=1}^{n}\left[\frac{\eta_1x_i}{\eta_1^2+\phi_1\eta_1x_i+x_i^3}\right] \quad (3.39)$$

$$\frac{\partial LL\left(\underline{\eta},\underline{\phi}/\underline{x},\underline{y}\right)}{\partial\phi_2}=-\frac{2m}{(\eta_2+\phi_2+6)}+\sum_{j=1}^{m}\left[\frac{\eta_2y_j}{\eta_2^2+\phi_2\eta_2y_j+y_j^3}\right] \quad (3.40)$$

We obtain the Maximum Likelihood Estimators (MLE) of $\phi_1,\eta_1$ and $\phi_2,\eta_2$ say $\widehat{\phi_1},\widehat{\eta_1}$ and $\widehat{\phi_2},\widehat{\eta_2}$ respectively as the solution of the equations above as

$$\eta_1^2V_1-n\eta_1+2\phi_1\eta_1V_1-2\phi_1n+6\eta_1V_1+(6\eta_1+2\phi_1\eta_1+\eta_1^2)\sum_{i=1}^{n}\left[\frac{2\eta_1+\phi_1x_i}{\eta_1^2+\phi_1\eta_1x_i+x_i^3}\right]=0 \quad (3.41)$$

Similarly,

$$\eta_2^2V_2-m\eta_2+2\phi_2\eta_2V_2-2\phi_2m+6\eta_2V_2+(6\eta_2+\phi_2\eta_2+\eta_2^2)\sum_{j=1}^{m}\left[\frac{2\eta_2+\phi_2y_j}{\eta_2^2+\phi_2\eta_2y_j+y_j^3}\right]=0 \quad (3.42)$$

Also,

$$-\frac{2n}{(\eta_1+2\phi_1+6)}+\sum_{i=1}^{n}\left[\frac{\eta_1x_i}{\eta_1^2+\phi_1\eta_1x_i+x_i^3}\right]=0 \quad (3.43)$$

Similarly,

$$-\frac{2m}{(\eta_2+\phi_2+6)}+\sum_{j=1}^{m}\left[\frac{\eta_2y_j}{\eta_2^2+\phi_2\eta_2y_j+y_j^3}\right]=0 \quad (3.44)$$

Hence, using the invariance property of the MLE, the maximum likelihood estimator $\hat{R}_{mle}$ of $R_{SS}$ can be obtained by substituting $\hat{\eta}_k$ in place of $\eta_k$ and $\hat{\phi}_k$ in place of $\phi_k$ for $k=1,2$

$$\hat{R}_{mle} = \left. \frac{\eta_1^2 \left\{ [2\phi_1\eta_1^2+2\phi_1\eta_1\eta_2+6\eta_1^2+(\eta_1+\eta_2)](2\phi_2+\eta_2+6)(\eta_1+\eta_2)^3+\eta_2 \begin{bmatrix} 5040\phi_1^2\eta_2^2+[120\phi_1^2\phi_2\eta_2+360\phi_1^2\eta_2+1080\phi_1\eta_1\eta_2^2](\eta_1+\eta_2)+ \\ [168\phi_1^2\eta_2^2+96\phi_1\phi_2\eta_1\eta_2](\eta_1+\eta_2)^2+[54\eta_2^2+54\phi_1\eta_1\eta_2^2](\eta_1+\eta_2)^3+ \\ [18\phi_1\eta_1\eta_2^2+6\eta_2+2\phi_2\eta_2](\eta_1+\eta_2)^4+9\eta_2(\eta_1+\eta_2)^5+ \\ 9\eta_2^2(\eta_1+\eta_2)^5 \end{bmatrix} \right\}}{(\eta_1+2\phi_1+6)(\eta_2+\phi_2+6)(\eta_1+\eta_2)^7} \right|_{\eta_k=\hat{\eta}_k,\phi_k=\hat{\phi}_k;k=1,2}$$

(3.45)

### 3.7 Parameter Estimation

Let $Y_1, Y_2, Y_3, \dots, Y_m$ be a random sample of size $m$ from the Inverse Remkan distribution $g(y_k; \eta, \phi)$. Then the log-likelihood function of parameters is given by

$$L(y_1, y_2, y_3, \dots, y_m; \Phi) = \prod_{i=1}^{m} \ln g(y_k; \Phi) \quad (3.46)$$

$$L(y; \eta, \phi) = \frac{\eta^{2m}}{(\eta+2\phi+6)^m} \prod_{i=1}^{m} \left[\frac{y_i^3+\phi\eta y+\eta^2}{y_i^5}\right] e^{-\eta V} \quad (3.47)$$

Where $V = \frac{1}{\sum_{i=1}^{m} y_i}$

The log-likelihood function is given by;

$$LL(y; \eta, \phi) = 2m\ln(\eta) - m\ln(\eta+2\phi+6) - \eta V + \sum_{i=1}^{m} \ln\left(\frac{y_i^3+\phi\eta y_i+\eta^2}{y_i^5}\right) \quad (3.48)$$

In order to maximize the log-likelihood, we solve the nonlinear likelihood equations obtained from the partial differentiation of (3.48) w.r.t $\eta$ $and$ $\phi$ as shown below;

$$\frac{\partial LL}{\partial \eta} = \frac{2m}{\eta} - \frac{m}{(\eta+2\phi+6)} - V + \sum_{i=1}^{m}\left[\frac{2\eta+\phi y_i}{\eta+\phi\eta y_i+y_i^3}\right] \quad (3.49)$$

$$\frac{\partial LL}{\partial \phi} = -\frac{2m}{(\eta+2\phi+6)} + \sum_{i=1}^{m}\frac{\eta y_i}{y_i^3+\phi\eta y_i+\eta^2} \quad (3.50)$$

In order to obtain the estimates of the parameters using the nonlinear equations above, we equate equations to zero and solve simultaneously. The solutions cannot be solved analytically. Hence, we solve numerically using the MaxLik package of in the R software (Toomet et al., 2015) with "BFGS" algorithm.

## IV. Application

This section discusses the flexibility and superiority of the Inverse Remkan distribution (IRD) to some competing distributions using two real life uncensored data sets. The datasets were first reported by (Mahmoud & Mandouh, 2013) and used by (Enogwe et al., 2020).

The first dataset represents the Uncensored strengths of glass fibres.

The dataset is shown below

1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.91, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, 4.121.

The second dataset the Uncensored breaking stress of carbon fibres in (Gba).

The dataset is shown below.

0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.83, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306.

These dataset are then fitted with the Inverse Remkan distribution (IRD) and compared with the Inverse Akash distribution (IAD) (E.W. Okereke et al., 2021), Inverse Suja distribution (ISD) (John et al., 2023), and Inverse Lindley distribution (ILD) (Sharma et al., 2015) with corresponding pdfs.

$$g_{ILD}(x; \eta) = \frac{\eta^2}{1+\eta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\eta}{x}} \quad (4.1)$$

$$g_{IAD}(x; \eta) = \frac{\eta^3}{\eta^2+2}\left(\frac{1+x^2}{x^4}\right)e^{-\frac{\eta}{x}} \quad (4.2)$$

$$g_{ISD}(x; \Phi) = \frac{\eta^5}{\eta^4+24}\left(\frac{1+x^4}{x^6}\right)e^{-\frac{\eta}{x}} \quad (4.3)$$

This comparison is done using some measures for testing the goodness of fit of a distribution. The measures used are the parameter estimates, the log likelihood, the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC) -2lnL, Akaike Information Criterion (AIC) (Club, 2016), Bayesian Information Criterion (BIC) (Pollock et al., 1999), Consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC). In general, the smaller the values of AIC, BIC, CAIC, and HQIC the better the fit to the data.

$$AIC = 2k - 2\ln L \tag{4.4}$$

$$BIC = k\ln n - 2\ln L \tag{4.5}$$

$$CAIC = AIC + \frac{2k(k+1)}{(n-k-1)} \tag{4.6}$$

$$HQIC = 2k\ln(\ln(n)) - 2\ln L \tag{4.7}$$

where $k$ is the number of parameters, $n$ is the sample size of the dataset, and $L$ is the likelihood function.

**Table 1: Goodness of fit for the Uncensored strengths of glass fibres**

| Distribution | MLE's | S.E | -2ln $L$ | AIC | BIC | CAIC | HQIC |
|---|---|---|---|---|---|---|---|
| **IR** | $\hat{\eta}$ **=4.5655**<br>$\hat{\phi}$ **=121.449** | **0.3316**<br>**2.9705** | **117.6972** | **121.6972** | **125.9835** | **121.8972** | **120.5401** |
| IA | $\hat{\eta}$ =2.342 | 0.194 | 192.4501 | 194.4501 | 200.7364 | 194.6501 | 195.293 |
| IS | $\hat{\eta}$ =2.9680 | 0.1699 | 229.7694 | 231.7694 | 238.0557 | 231.9694 | 232.6123 |
| IL | $\hat{\eta}$ =2.0297 | 0.2054 | 195.9979 | 197.9979 | 204.2841 | 198.1979 | 198.8408 |

The parameter estimates and their goodness of fit of the different models for the first dataset are presented in Table 1. From the results, the Inverse Remkan distribution (IRD) performed better than the competing distributions.

**Table 2: Goodness of fit for the Uncensored breaking stress of carbon fibres in (Gba)**

| Distribution | MLE's | S.E | -2ln $L$ | AIC | BIC | CAIC | HQIC |
|---|---|---|---|---|---|---|---|
| **IC** | $\hat{\eta}$ **=4.601**<br>$\hat{\phi}$ **=114.748** | **0.268**<br>**4.228** | **195.6066** | **199.6066** | **204.817** | **199.7304** | **198.661** |
| IA | $\hat{\eta}$ =2.345 | 0.154 | 309.6631 | 311.6631 | 318.8735 | 311.7868 | 312.7175 |
| IS | $\hat{\eta}$ =2.9706 | 0.1354 | 366.6776 | 368.6776 | 375.888 | 368.8013 | 369.732 |
| IL | $\hat{\eta}$ =2.0335 | 0.1634 | 315.3832 | 317.3832 | 324.5936 | 317.5069 | 318.4376 |

The parameter estimates and their goodness of fit of the different models for the second dataset are presented in Table 2. From the results, the Inverse Remkan distribution (IRD) performed better than the competing distributions.

## V. Conclusion

This paper proposed a new two-parameter distribution known as the Inverse Remkan distribution (IRD). The statistical properties of the Inverse Remkan distribution such as the mode, order statistics, entropy, stochastic ordering, stress-strength reliability, and reliability indices was derived and presented. The properties of the new Inverse Remkan distribution showed that the Remkan distribution can be used to model lifetime datasets with unimodal, positively skewed, and right tailed properties. Furthermore, the risk measurement function of the Inverse Remkan distribution can model datasets with upside-down bathtub shape in survival analysis. In addition, the method of maximum likelihood estimate was adopted to derive the estimates of the parameters. The flexibility of the new Inverse Remkan distribution was compared with other competing distributions using two different real life datasets. The results obtained showed that the new Inverse Remkan distribution gave the best fit to the data based on some model selection criteria. Hence, the new Inverse Remkan distribution is therefore recommended as an alternative to other existing distributions.

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