Quest Journals
Journal of Research in Applied Mathematics
Volume 10 ~ Issue 4 (2024) pp: 01-07
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

An application of extended Riemann Liouville Fractional Derivative Operator on modified Bessel Function

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ABSTRACT: *The main aim of the present paper is to apply extended Riemann Liouville Fractional Derivative of order υ on the modified Bessel Function. Two main results are obtained, which are presented in the form of two theorems. Some more results are obtained with the help of two main results.*
Keywords: *Gamma function, Beta function, Riemann-Liouville fractional derivative, Hypergeometric functions.*

Received 26 Mar., 2024; Revised 03 Apr., 2024; Accepted 05 Apr., 2024
Published with open access at ***www.questjournals.org***

I. INTRODUCTION

The Fractional Calculus now a days is one of the most rapidly growing subjects of mathematical analysis. The fractional integral operators involving various Special Functions have found significant importance and applications in various sub fields of applicable mathematical analysis. The applications of Fractional Calculus are also seen in various fields, including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermal nuclear fusion, non-linear control theory, image processing, nonlinear biological system, astrophysics etc..

In the last three decades, a number of workers like Love [16], Mc Bride [17], Kalla[18,19],Kalla and Saxena[20], Saigo[21,22],Kilbas [23],have studied the properties ,applications & different extensions of various operators of Fractional Calculus on a number of classical & non classical Special Functions & polynomials. A sufficient account of fractional calculus operators along with their properties and applications can be found in the research monographs by Miller and Ross [25], & Kiryakova[24]. The first application of fractional calculus was due to Abel [32] in the solution to the fractional problem. In Fractional Calculus, the fractional derivatives are defined via fractional integrals.

In recent years, certain extended fractional derivative operators associated with special functions have been actively investigated. Many authors [12,20], have introduced certain extended fractional derivative operators.

The Bessel functions are playing important role in wide range of problems of mathematical physics like problems of stochastic, radio physics, hydro dynamics, atomic & nuclear physics, which led to various research workers, who are working in the field of Special Functions to explore various extensions and applications of these Bessel Functions.

In the recent past, various generalizations, extensions of Bessel Functions have been given by many Researchers [9,11,12,13] who are working in the field of Special Functions and their applications.

Bessel Functions of the first kind of order υ defined by G.N. Watson [14] also occurs frequently in the problems like electrical engineering, finite elasticity, wave mechanics, Mathematical Physics and Chemistry, whereas the product of Bessel sand Modified Bessel Functions of first kind appear frequently in problems of statistical mechanics and plasma physics [15]. Very recently various researchers [14,15] are applying Fractional Calculus to the Bessel Functions of first kind and second kind and also on Modified Bessel Functions [33]. Also the product of Bessel Functions of first kind also appear frequently in the problems of statistical mechanics & plazma physics [34].

Motivated by the above recent applications of Fractional Calculus on various Classical Special Functions and polynomials and also by the works of various researchers [9,11,12,13], the authors in the present note have applied extended Riemann Liouville Fractional Derivative of order υ on the modified Bessel Function $I_\lambda(z)$.

Two main results are obtained, which are presented in the form of two theorems. Some more results are obtained with the help of two main results.

The Generalized Bessel Function of first kind $w_\lambda(x)$ is defined by Saigo [2], by the following series:

$$w_\lambda(z) = \sum_{l=0}^{\infty}(-1)^l \frac{c^l}{\Gamma\left(p+\frac{(b+1)}{2}+l\right)l!}\left(\frac{z}{2}\right)^{2l+\lambda} \quad ; \; z \in C, \qquad (1)$$

$$Re(p)>-1, \; (b, c, p) \in C,$$

where C denotes the set of complex numbers and Γa is the familiar Gamma Function.

Putting b=1 and c=1 in Eqn (1), we get the Modified Bessel Function of order p denoted by $I_\lambda(z)$ as [32]:

$$I_\lambda(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(p+1+l)l!}\left(\frac{z}{2}\right)^{2l+\lambda} \quad , z \in C \qquad (2)$$

1.1. Classical Riemann -Liouville fractional derivative of F(z) of order υ is defined by M.A. Ozarslan &E. Ozergin [37]:

$$D_z^{\upsilon} f(z) = \frac{1}{\Gamma-\upsilon}\int_0^z (z-t)^{-\upsilon-1} f(t)dt \qquad (Re(\upsilon) < 0), \qquad (3)$$

where the integration path is a line from o to z in the complex t- plane.

1.2. Riemann -Liouville fractional derivative of F(z) of order υ is defined by M.A. Ozarslan & E. Ozergin [37]:

$$D_z^{\upsilon} f(z) = \frac{d^m}{dz^m} D_z^{\upsilon-m} f(z)$$

$$= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma m-\upsilon}\int_0^z (z-t)^{m-\upsilon-1} f(t)dt\right\} \qquad (Re(\upsilon) < 0), \qquad (4)$$

where, m-1≤ Re(υ)<m.

1.3. The extended Riemann -Liouville fractional derivative of F(z) of order υ is defined by M.A. Ozarslan & E. Ozergin [37]:

$$D_z^{\upsilon,p,k,\mu} f(z) = \frac{1}{\Gamma-\upsilon}\int_0^z (z-t)^{-\upsilon-1} f(t)dt \; {}_1F_1\left(\alpha; \beta; \frac{pz^{k+\mu}}{t^k(z-t)^\mu}\right)dt, \qquad (5)$$

$$D_z^{\upsilon,p,k,\mu} f(z) = \frac{d^m}{dz^m} D_z^{\upsilon-m;p;k,\mu} f(z)$$

$$= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma m-\upsilon}\int_0^z (z-t)^{m-\upsilon-1} f(t)dt\right.$$

$$\left.\times \; {}_1F_1\left(\alpha; \beta; -\frac{pz^{k+\mu}}{t^k(z-t)^\mu}\right)dt\right\} \qquad (6)$$

Where, $(m-1 \le Re(\upsilon) < m), (Re(\upsilon) < 0; Re(p) > 0; Re(k) > 0; \; Re(\mu) > 0)$,

The special cases of (5) & (6), when p=0 becomes the classical- Riemann Liouville fractional derivative.

II. Preliminaries

2.1. Srivastava [29] introduced the following extended Beta function:

$$B_p(x,y) = B_p^{\alpha,\alpha}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t(1-t)}\right)dt, \qquad (7)$$

2.2. The extended beta function $B_p^{(\alpha,\beta,\kappa,\mu)}(x, y)$ with Re(p)>0 is defined by [35,36]:

$$B_p^{(\alpha,\beta,\kappa,\mu)}(x,y) = \int_0^1 t^{(x-1)}(1-t)^{y-1} \; {}_1F_1\left(\alpha; \beta; -\frac{p}{t^k((1-t)^\mu}\right)dt, \qquad (8)$$

where, $\kappa \ge 0, \mu \ge 0$, min{Re(α), Re(β) ≥0 }>0, Re(x)>-Re(kα), Re(y)>-Re(μα).

2.3. The Gauss hypergeometric function $F_p^{(\alpha,\beta,\kappa,\mu)}$(a,b,c;z) is defined by [38]:

$$F_p^{(\alpha,\beta,\kappa,\mu)}(a,b,c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (9)$$

(|z|<1; min{Re(α), Re(β), Re(κ), Re(μ)}>0; Re(c)>Re(b)>0; Re(p)≥0),

where B(u,v) is the familiar Beta function defined by[4],

$$B(u,v) = \left\{\int_0^1 t^{u-1}(1-t)^{v-1}\right\}dt \; (Re(u)>; Re(v)>0)\}$$

$$= \frac{\Gamma u \Gamma v}{\Gamma v+v} \qquad (u, v \in C) \qquad (10)$$

here Γ denotes the Eulers Gamma function [4].

2.4. Extension of the Gauss hypergeometric function $F_{p;\kappa,\mu}(a,b,c;z;m)$ is defined by [38]:

$$F_{p;\kappa,\mu}(a,b,c;z;m) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,c-b+m)}{B(b+n,c-b+m)} \frac{z^n}{n!} \tag{11}$$

where, (p≥0; Re(κ)>0, Re(μ)}>0; Re(c)>Re(b)>m; Re(p)≥0),

2.5. The extended Appell hypergeometric function F_1 is defined by [39]:

$$F_{1;p;\kappa,\mu}(a,b,c,d,;x,y;m) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_n}{(d)_{n+k}} \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(a+n+k,d-a+m)}{B(a+n+k,d-a+m)} \frac{x^n}{n!}\frac{y^k}{k!} \tag{12}$$

where, (p≥0; Re(κ)>0, Re(μ)}>0; m<Re(d)>Re(a); Re(p)≥0; |x|<1; |y|<1)

2.6. The extended Appell hypergeometric function F_2 is defined by [39]:

$$F_{2;p;\kappa,\mu}(a,b,c,d,;x,y;m) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(d)_n(e)_k}$$

$$\times \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,d-a+m)}{B(b+n,d-b+m)} \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(c+k,e-c+m)}{B(c+k,e-c+m)} \frac{x^n}{n!}\frac{z^k}{k!} \tag{13}$$

where, (p≥0; Re(κ)>0, Re(μ)}>0; Re(d)>Re(b)>m; Re(e)>Re(c)>m; Re(p)≥0; |x|+|y|<1),

2.7. The extended Lauricella hypergeometric function F_D^3 is defined by [39]:

$$F_{D,p;k;\mu}^3(a,b,c,d,e;x,y,z;m) = \sum_{n,k,r=0}^{\infty} \frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(e)_{n+k+r}}$$

$$\times \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,d-a+m)}{B(b+n,d-b+m)} \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(c+k,e-c+m)}{B(c+k,e-c+m)} \frac{x^n}{n!}\frac{z^k}{k!}\frac{z^r}{r!} \tag{14}$$

where, (p≥0; Re(κ)>0, Re(μ)}>0; Re(e)>Re(a)>m; Re(p)≥0; |x|<0,|y|<1, |z|<1),

It is noted that the special cases of (11), (12), (13) & (14), when p=0 and m=0 reduce to the well -known Gauss hypergeometric function ${}_2F_1$, the Appell functions $F_1, F_2,$ & Lauricella functions F_D^3.

Also from Rainville [4] the following two results:

2.8. The generalized binomial theorem

$$(1-z)^{-\alpha} = \sum_{l=0}^{\infty} \frac{(\alpha)_l}{l!}, \ (|\, l|<1;\ \alpha\epsilon C) \tag{15}$$

2.9. $\Gamma(z+l) = \Gamma z\ (z)_l$ (16)

III. MAINS RESULTS

3.1. The extended fractional derivative of the modified Bessel Function $I_\lambda(z)$:

Theorem I. The following result holds:

$$D_z^{\upsilon,p,k,\mu}\{I_\lambda(z)\} = \sum_{l=0}^{\infty} \frac{\Gamma\lambda+2l+1\ B_p^{\alpha,\beta,\kappa,\mu}(\lambda+2l+1,m-\upsilon)}{(2)^{2l+1}\Gamma(\lambda+l+1)\ l!\ \Gamma\lambda+2l+1\ B(\lambda+2l+1,m-\upsilon)} z^{\lambda+2l-\upsilon} \tag{17}$$

where, m-1≤Re(υ)<m, for some mϵN & Re(v) < Re(λ), & S = $\frac{1}{(2)^{2l+1}\Gamma(\lambda+l+1)\, l!}$,

Proof: Applying Eqn (14) in definition (13) to the function $I_\lambda(z)$, we have:

$$D_z^{\upsilon,p,k,\mu}\{I_\lambda(z)\} = S \times \frac{d^m}{dz^m}\{\frac{1}{\Gamma m-\upsilon}\int_0^z (z-t)^{m-\upsilon-1}\, t^{\lambda+2l}\ {}_1F_1\ (\alpha;\beta;\frac{pz^{k+\mu}}{t^k(z-t)^\mu})dt\} \tag{18}$$

where, $S = \frac{1}{(2)^{2l+1}\Gamma(\lambda+l+1)\, l!}$

Putting t = zu in the above Equation we get:

$$D_z^{\upsilon,p,k,\mu}\{I_\lambda(z)\} = S \times (\frac{d^m}{dz^m} z^{m+\lambda+2l-\upsilon})$$

$$= \{\frac{1}{\Gamma m-\upsilon}\int_0^1 (1-u)^{m-\upsilon-1}\, u^{\lambda+2l}\ {}_1F_1\ (\alpha;\beta;\frac{p}{u^k(1-u)^\mu})du\},$$

$$= S \times (\frac{d^m}{dz^m} z^{m+\lambda+2l-\upsilon}) \times \frac{1}{\Gamma m-\upsilon}\{B_p^{\alpha,\beta,\kappa,\mu}(\lambda+2l+1,m-v)\},$$

(Using Eqn (3))

$$= S \times \frac{\Gamma 1+\lambda+2l-\upsilon+m}{\Gamma 1+\lambda+2l-\upsilon} z^{\lambda+2l-\upsilon} \times \frac{1}{\Gamma m-\upsilon} B_p^{\alpha,\beta,\kappa,\mu}(\lambda+2l+1,m-v),$$

where, $\frac{d^m}{dz^m} z^{m+\lambda+2l-\upsilon} = \frac{\Gamma 1+\lambda+2l-\upsilon+m}{\Gamma 1+\lambda+2l-\upsilon} z^{\lambda+2l-\upsilon}$,

or,

$$D_z^{\upsilon,p,k,\mu}\{I_\lambda(z)\} = \sum_{l=0}^{\infty} \frac{\Gamma\lambda+2l+1\ B_p^{\alpha,\beta,\kappa,\mu}(\lambda+2l+1,m-\upsilon)}{(2)^{2l+1}\Gamma(\lambda+l+1)\ l!\ \Gamma\lambda+2l+1\ B(\lambda+2l+1,m-\upsilon)} z^{\lambda+2l-\upsilon}, \tag{19}$$

(putting the value of S)

which proves the theorem I i.e., the desired result (17).

3.2. The extended fractional derivatives of the modified Bessel Function Using Maclaurin expansion:
In this Section we have obtained two different types of derivatives formula using Maclaurin expansion:

Theorem I : The following derivative holds:
Suppose that a function f(z) is analytic at origin with its Maclaurin expansion given by $f(z) = \sum_{l=0}^{\infty} a_l z^l$, (|z|<p) for some ρϵR.$^+$,
then

$$D_z^{\upsilon,p,\kappa,\mu}\{I_\lambda(z)\} = \sum_{l=0}^{\infty} S \times a_l D_z^{\upsilon,p,\kappa,\mu}\{z^{2l+\lambda}\}. \quad (20)$$

where, m-1≤Re(υ)<m for some mЄN & $S = \frac{1}{(2)^{2l+1}\Gamma(\lambda+l+1)\, l!}$,

Proof: Applying Eqn (14) in definition (13) to the function $I_\lambda(z)$, we have

$$D_z^{\upsilon,p,k,\mu} I_\lambda(z) = S \times \frac{d^m}{dz^m}\{\frac{1}{\Gamma m-\upsilon}\int_0^z (z-t)^{m-\upsilon-1} \times {}_1F_1\,(\alpha;\beta;\frac{pz^{k+\mu}}{t^k(z-t)^\mu}) \sum_{n=0}^{\infty} a_l\ t^{\lambda+2l}\, dt\},$$

$$= \sum_{n=0}^{\infty} a_l\, S \times \frac{d^m}{dz^m}\{\frac{1}{\Gamma m-\upsilon}\int_0^z (z-t)^{m-\upsilon-1} \times {}_1F_1\,(\alpha;\beta;\frac{pz^{k+\mu}}{t^k(z-t)^\mu})\ t^{\lambda+2l}\, dt\},$$

$$= \sum_{n=0}^{\infty} a_l\, S D_z^{\upsilon,p,\kappa,\mu} f(z),$$

{Using (14)}

$$= \sum_{l=0}^{\infty} S \times a_l D_z^{\upsilon,p,\kappa,\mu}\{z^{2l+\lambda}\},$$

or,

$$= \sum_{l=0}^{\infty} \frac{1}{(2)^{2l+1}\Gamma(\lambda+l+1)\, l!}\, a_l D_z^{\upsilon,p,\kappa,\mu}\{z^{2l+\lambda}\}, \quad (21)$$

(putting the value of S)

which proves the theorem I i.e., the desired result (20).

Theorem II. The following derivatives holds:
Suppose that a function f(z) is analytic at origin with its Maclaurin expansion given by $f(z) = \sum_{l=0}^{\infty} a_l z^l$, (|z|<p) for some ρϵR.$^+$,
then

$$D_z^{\upsilon,p,k,\mu} I_\lambda(z) = \sum_{l=0}^{\infty} \frac{\Gamma\lambda+n-1+2l+1\ B_p^{\alpha,\beta,\kappa,\mu}(\lambda+n-1+2l+1,m-\upsilon)}{(2)^{2l+1}\Gamma(\lambda+n+l)\, l!\ \Gamma\lambda+n-1+2l+1\ B(\lambda+n-1+2l+1,m-\upsilon)} z^{\lambda+2l-\upsilon}, \quad (22)$$

where, m-1≤Re(υ)<m, for some mϵN & Re(v) < Re(λ), & $S = \frac{1}{(2)^{2l+1}\Gamma(\lambda+n-1+l+1)\, l!}$,

Proof : Applying Eqn (14) in definition (13) to the function $I_\lambda(z)$, we have:

$$D_z^{\upsilon,p,k,\mu}\{I_\lambda(z)\} = S\frac{d^m}{dz^m}\{\frac{1}{\Gamma m-\upsilon}\int_0^z (z-t)^{m-\upsilon-1}\, t^{\lambda+n-1+2l}\ {}_1F_1\,(\alpha;\beta;\frac{pz^{k+\mu}}{t^k(z-t)^\mu})dt\} \quad (23)$$

Putting t = zu, in the above Equation, we get:

$$= S \times (\frac{d^m}{dz^m} z^{m+\lambda+n-1+2l-\upsilon})$$

$$= \{\frac{1}{\Gamma m-\upsilon}\int_0^1 (1-u)^{m-\upsilon-1}\, u^{\lambda+n-1+2l}\ {}_1F_1\,(\alpha;\beta;\frac{p}{u^k(1-u)^\mu})du\}$$

$$= S \times (\frac{d^m}{dz^m} z^{m+\lambda+n-1+2l-\upsilon}) \times \frac{1}{\Gamma m-\upsilon}\{B_p^{\alpha,\beta,\kappa,\mu}(\lambda+n-1+2l+1, m-v)\}$$

(Using Eqn (3))

$$= S \times \frac{\Gamma 1+\lambda+2l-\upsilon+m}{\Gamma 1+\lambda+2l-\upsilon} z^{\lambda+2l-\upsilon} \times \frac{1}{\Gamma m-\upsilon} B_p^{\alpha,\beta,\kappa,\mu}(\lambda+n-1+2l+1, m-v)$$

where, $\frac{d^m}{dz^m} z^{m+\lambda+2l-\upsilon} = \frac{\Gamma\lambda+n+2l-\upsilon+m}{\Gamma 1+\lambda+2l-\upsilon} z^{\lambda+n-1+2l-\upsilon}$

or,

$$= \sum_{l=0}^{\infty} \frac{\Gamma\lambda+n+2l\ B_p^{\alpha,\beta,\kappa,\mu}(\lambda+n+2l, m-\upsilon)}{(2)^{2l+1}\Gamma(\lambda+n+l)\, l!\ \Gamma\lambda+2l+1\ B(\lambda+n+2l, m-\upsilon)} z^{\lambda+n-1+2l-\upsilon} \quad (24)$$

(putting value the of S)

which proves the theorem II i.e., the desired result (22)

3.3. The Extended Fractional derivative of modified Bessel Function, using Binomial expansion:
Under this section three different types of derivatives are obtained for modified Bessel Function, using Binomial expansions:

Theorem I. The following derivatives holds :

$$\mathbf{D_z^{\lambda+2l-\upsilon,p,k,\mu}\{I_\lambda(z)\times(1-z)^{-\alpha}\}}$$
$$=\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}\sum_{l=0}^{\infty}\frac{(\alpha)_n(\lambda)_n}{(2)^{2l+1}(\upsilon)_n\Gamma(\lambda+l+1).l!}\;\frac{B_p^{\alpha,\beta,k,\mu}(\lambda+3l,\upsilon-\lambda-2l+m)}{B(\lambda+3l,\upsilon-\lambda-2l+m)}\;\frac{z^l}{l!}$$
$$=\mathbf{S}.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}\mathbf{F_{p,k,\mu}(\alpha,\lambda+2l;\upsilon;z;m)} \qquad \textbf{(25)}$$

where, m-1≤Re(υ)<m, for some m∈N & Re(v) < Re(λ), & S = $\frac{1}{(2)^{2l+1}\Gamma(\lambda+n-1+l+1)\,l!}$ **,**

Proof: Using Eqn (15) & applying results of Eqn (17) & (20), we get:

$$D_z^{\lambda+2l-\upsilon,p,k,\mu}\{I_\lambda(z)\times(1-z)^{-\alpha}\}=S.\,D_z^{\lambda+2l-\upsilon,p,k,\mu}\{z^{\lambda+2l-1}\sum_{l=0}^{\infty}(\alpha)_l\frac{z^l}{l!}\}$$
$$=S.\sum_{l=0}^{\infty}\frac{(\alpha)_n}{n!}D_z^{\lambda+2l-\upsilon,p,k,\mu}z^{\lambda+3l-1}$$
$$=S.\sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!}\;\frac{\Gamma\lambda+2l+l}{\Gamma\upsilon+l}\;\frac{B_p^{\alpha,\beta,\kappa,\mu}(\lambda+2l+n,m-\lambda-2l+\upsilon)}{B(\lambda+2l+n,m-\lambda-2l+\upsilon)}z^{\upsilon+l-1}$$
(Using Eqn (3))
$$=S.\frac{\Gamma\lambda+2l}{\Gamma\upsilon}z^{\upsilon-1}\sum_{l=0}^{\infty}\frac{(\alpha)_l(\lambda+2l)_l}{(\upsilon)_l}\;\frac{B_p^{\alpha,\beta,\kappa,\mu}(\lambda+3l,m-\lambda-2l+\upsilon)}{B(\lambda+3l,m-\lambda-2l+\upsilon)}\;\frac{z^l}{l!}$$
(Using Eqn (16))
$$=S.\frac{\Gamma\lambda+2l}{\Gamma\upsilon}z^{\upsilon-1}F_{p,\kappa,\mu}(\alpha,\lambda+2l;\upsilon;z;m),$$
(Using Eqn (6))

or,

$$=\frac{1}{(2)^{2l+1}\Gamma(\lambda+n-1+l+1)\,l!}\,.\frac{\Gamma\lambda+2l}{\Gamma\upsilon}z^{\upsilon-1}F_{p,\kappa,\mu}(\alpha,\lambda+2l;\upsilon;z;m), \quad (26)$$
(Putting the value of S)

which proves the theorem I i.e., the desired result. (25).

Theorem II. The following derivatives holds:

$$\mathbf{D_z^{\lambda+2l-\upsilon,p,k,\mu}\{I_\lambda(z)\times(1-az)^{-\alpha}(1-az)^{-\beta}\}=}$$
$$\mathbf{S}.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}\sum_{n,k=0}^{\infty}\frac{(\alpha)_n(\lambda+2l)_{n+k}(\beta)_k}{(\upsilon)_{n+k}}\;\frac{B_p^{\alpha,\beta,k,\mu}(\lambda+2l+n+k,\upsilon-\lambda-2l+m)}{B(\lambda+2l+n+k,\upsilon-\lambda-2l+m)}\;\frac{(az)^n}{n!}\frac{(bz)^k}{k!}$$
$$=\mathbf{S}.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}\mathbf{F_{1,p,k,\mu}(\alpha,\beta,\lambda+2l;\upsilon;az;bz;m)}, \qquad \textbf{(27)}$$

where, m-1≤Re(υ)<m, for some m∈N & Re(v) < Re(λ), & S = $\frac{1}{(2)^{2l+1}\Gamma(\lambda+n-1+l+1)\,l!}$

Proof : Using Eqn(15) & applying results of Eqn (17) & (20), we get:

$$D_z^{\lambda+2l-\upsilon,p,k,\mu}\{I_\lambda(z)\times(1-az)^{-\alpha}(1-az)^{-\beta}\}=D_z^{\lambda+2l-\upsilon,p,k,\mu}\{z^{\lambda+2l-1}\sum_{n,k}^{\infty}\frac{(az)^n}{n!}\;\frac{(bz)^k}{k!}\}$$
$$=S.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}\sum_{n,k=0}^{\infty}\frac{(\alpha)_n(\lambda+2l)_{n+k}(\beta)_k}{(\upsilon)_{n+k}}$$
$$\times\frac{B_p^{\alpha,\beta,k,\mu}(\lambda+2l+n+k,\upsilon-\lambda-2l+m)}{B(\lambda+2l+n+k,\upsilon-\lambda-2l+m)}\;\frac{(az)^n}{n!}\frac{(bz)^k}{k!}$$
(Using Eqn(3) & Eqn(15))
$$=S.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{1,p,k,\mu}(\alpha,\beta,\lambda+2l;\upsilon;az;bz;m), \quad (28)$$
(Using Eqn (8))

or,

$$=\frac{1}{(2)^{2l+1}\Gamma(\lambda+n-1+l+1)\,l!}$$
$$\times\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{1,p,k,\mu}(\alpha,\beta,\lambda+2l;\upsilon;az;bz;m),$$
(Putting the value of S)

which proves the theorem II i.e., the desired result (27).

Theorem III. The following derivatives holds:

$$D_z^{\lambda+2l-\upsilon,p,k,\mu}\{I_\lambda(z)\times(1-az)^{-\alpha}(1-az)^{-\beta}(1-az)^{-\gamma}\}=$$

$$S.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}\sum_{n,k,r=0}^{\infty}\frac{(\alpha)_n(\lambda+2l)_{n+k}(\beta)_k(\gamma)_r}{(\upsilon)_{n+k+r}}\frac{B_p^{\alpha,\beta,k,\mu}(\lambda+2l+n+k+r,\upsilon-\lambda-2l+m)}{B(\lambda+2l+n+k+r,\upsilon-\lambda-2l+m)}\frac{(az)^n}{n!}\frac{(bz)^k}{k!}\frac{(cz)^r}{r!}$$

$$=S.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{p,k,\mu}^{3}(\alpha,\beta,\lambda+2l;\upsilon;az;bz;m), \qquad \textbf{(29)}$$

where, m-1≤Re(υ)<m, for some m∈N & Re(v) < Re(λ), & $S=\frac{1}{(2)^{2l+1}\Gamma(\lambda+n-1+l+1)\,l!}$

Proof: Using Eqn (15) & applying results of Eqn (17) & (20), we get:

$$D_z^{\lambda+2l-\upsilon,p,k,\mu}\{I_\lambda(z)\times(1-az)^{-\alpha}(1-az)^{-\beta}(1-az)^{-\gamma}\}$$

$$D_z^{\lambda+2l-\upsilon,p,k,\mu}\left\{z^{\lambda+2l-1}\sum_{n,k}^{\infty}\frac{(az)^n}{n!}\frac{(bz)^k}{k!}\frac{(cz)^r}{r!}\right\}=$$

$$S.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}\sum_{n,k,r=0}^{\infty}\frac{(\alpha)_n(\lambda+2l)_{n+k}(\beta)_k(\gamma)_r}{(\upsilon)_{n+k+r}}\frac{B_p^{\alpha,\beta,k,\mu}(\lambda+2l+n+k+r,\upsilon-\lambda-2l+m)}{B(\lambda+2l+n+k+r,\upsilon-\lambda-2l+m)}\frac{(az)^n}{n!}\frac{(bz)^k}{k!}\frac{(cz)^r}{r!}$$

((Using Eqn (3) & (15))

$$=S.\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{p,k,\mu}^{3}(\alpha,\beta,\lambda+2l;\upsilon;az;bz;cz;m),$$

((Using Eqn (10))

or,

$$=\frac{1}{(2)^{2l+1}\Gamma(\lambda+n-1+l+1)\,l!}$$

$$\times\frac{\Gamma(\lambda+2l)z^{(\upsilon-1)}}{\Gamma\upsilon}F_{p,k,\mu}^{3}(\alpha,\beta,\lambda+2l;\upsilon;az;bz;cz;m), \qquad (30)$$

(Putting the value of S)

which proves the theorem III i.e., the desired result. (29).

IV. Concluding Remarks:

Extended Riemann Liouville fractional derivative of order υ has been applied on the modified Bessel function to obtain different type of derivatives, for this function. While obtaining these derivatives, we have also used Maclaurin & Binomial expansion. Results obtained seems to be new & interesting. While deriving the main results, we have used the extended Beta function & extended Gauss hypergeometric functions.

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