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Review Paper

Interior penalty discontinuous finite element method for second-order elliptic eigenvalue problems

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ABSTRACT: *We conduct a prior error analysis of eigenvalue problems for a class of second-order elliptic equations with variable coefficients, focusing on the Interior Penalty Discontinuous (IPDG) finite element method. Initially, we construct the IPDG scheme for the discrete problem using Green's formula, followed by stability estimation. Subsequently, numerical experiments demonstrate the attainment of optimal convergence order.*
KEYWORDS: *Second order elliptic eigenvalues; Interior penalty discontinuous finite element method; A prior error*

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I. INTRODUCTION

The eigenvalue problem has significant physical implications and finds widespread applications in quantum mechanics, fluid mechanics, modern science and technology, engineering, and other fields. Currently, there exist numerous methods for solving eigenvalue problems, such as the finite element method, finite difference method, and spectral method. In reference [1], a high-precision mixed-element method for the second-order elliptic eigenvalue problem in a new variational form is discussed. Reference [2] explores the penalty discontinuity finite element method on two grids for asymmetric or indefinite elliptic equations. Literature [3] presents an effective method for solving eigenvalue problems based on multigrid discretization with shift inverse iteration. Reference [4] discusses an HP-type symmetric interior penalty discontinuous Galerkin finite element method for fourth-order elliptic problems. The discontinuous finite element method serves as a fundamental discretization technique for numerically solving second-order elliptic eigenvalue problems. It typically exhibits local conservation, stability, and high-order accuracy, making it adept at resolving complex problems. For instance, it allows for handling changes in the types of eigenvalues across different solution regions, permits hanging nodes in triangulation meshes, and facilitates hp adaptivity. In recent years, the discontinuous finite element method has seen extensive application in elliptic eigenvalue problems. Compared to the continuous element method, the discrete linear algebraic system with the discontinuous finite element method boasts a higher degree of freedom. The interior penalty discontinuous Galerkin method was initially analyzed by Arnold, Wheeler, and others, garnering significant attention from researchers worldwide. Subsequently, Song, Yang, and others extended this method to nonlinear equation problems and furthered the Galerkin theory analysis of internal penalty discontinuity.

In this study, we focus on the interior penalty discontinuous (IPDG) finite element method for second-order elliptic eigenvalue problems with variable coefficients. We start by determining the HP-finite element space through solution region splitting. Subsequently, we construct the discrete IPDG formulation using Green's formula and establish its stability through theorem proving. Finally, we estimate the prior error and verify optimal convergence through numerical experiments.

II. BASIC THEORY PREPARATION

Let Ω be a bounded polygon region in R^2, and the boundary $\partial\Omega$ is Lipshitz continuous. Consider the Dirichlet boundary condition eigenvalue problem: find $\lambda \in C$ and $u \in H_0^1(\Omega)$, such that

$$\begin{cases} -\nabla\cdot(\alpha\nabla u)=\lambda u, & in\,\Omega. \\ u=0, & on\,\partial\Omega. \end{cases} \tag{2.1}$$

The coefficient $\alpha(x)$ satisfies $C_{\alpha_0}\le\alpha(x)\le C_{\alpha_1}, x\in\Omega$, where C_{α_0} and C_{α_1} are plus constant.
Define a bilinear form that is continuous

$$a(u,v)=(\alpha\nabla u,\nabla v),\ \ \forall u,v\in H_0^1(\Omega). \tag{2.2}$$

Where

$$(\alpha\nabla u,\nabla v)=\int_\Omega \alpha\,\nabla u\cdot\nabla v dx.$$

There exist two plus constants A and B that are independent of u and v, such that the bilinear form $a(\cdot,\cdot)$ is satisfied

$$\begin{aligned} |a(u,v)| &\le A\parallel u\parallel_{1,\Omega}\parallel v\parallel_{1,\Omega}, \quad \forall u,v\in H_0^1(\Omega), \\ |a(v,v)| &\ge B\parallel v\parallel_{1,\Omega}^2, \quad \forall v\in H_0^1(\Omega). \end{aligned} \tag{2.3}$$

The weak form of (2.1) is for $(\lambda,u)\in C\times H_0^1(\Omega),\ u\ne 0$,makes the following equation was established

$$a(u,v)=\lambda(u,v),\forall v\in H_0^1(\Omega). \tag{2.4}$$

Let $\mathcal{T}_h=\{\kappa\}$ be a grid of regular shape divided by region Ω, the length of the side in cell κ is represented by h_e, the diameter of cell κ is represented by h_κ, and $h=\max_{\kappa\in\mathcal{T}_h}h_\kappa$. $\Gamma_h=\Gamma_h^i\cup\Gamma_h^b$, where Γ_h^i represents the inner edge and Γ_h^b represents the edge on the boundary $\partial\Omega$. Define the mean and jump values of v over e:

$$\{v\}=\frac{1}{2}(v^++v^-),\quad [[v]]=v^+n_\kappa^++v^-n_\kappa^-,$$

Where $e=\partial\kappa^+\cap\partial\kappa^-,\ v^+=v|_{\kappa^+},\ v^-=v|_{\kappa^-}$, n is the unit external normal vector from κ^+ to κ^-. If $e\in\Gamma_h^b$, define the mean and jump values of v over e:

$$\{v\}=v,\quad [[v]]=vn.$$

The fragment function space on partition $\mathcal{T}_h$ is introduced:

$$H^s(\mathcal{T}_h)=\{v\in L^2(\Omega): v|_\kappa\in H^s(\kappa),\ \forall\kappa\in\mathcal{T}_h\},$$

Using $p_\kappa\ge 1$ to represent the degree of the polynomial in unit $\kappa\in\mathcal{T}_h$, denoted by $p=\{p_\kappa\}_{\kappa\in\mathcal{T}_h}$, the hp-finite element space is now defined as:

$$S^p(\mathcal{T}_h)=\{v\in L^2(\Omega): v|_\kappa\in S^{p_\kappa}(\kappa),\ \forall\kappa\in\mathcal{T}_h\}.$$

Introducing the functions h and p into the relevant local grid size and approximation order in $L^\infty(\Gamma_h)$ is:

$$h=h(x):=\begin{cases}\min\{h_\kappa,h'_\kappa\}, & x\in e_{\kappa\kappa'}, \\ h_\kappa, & x\in e_{\kappa\Omega},\end{cases} \qquad p=p(x):=\begin{cases}\max\{p_\kappa,p'_\kappa\}, & x\in e_{\kappa\kappa'}, \\ p_\kappa, & x\in e_{\kappa\Omega},\end{cases}$$

Where $e_{\kappa\kappa'}=int(\partial\kappa\cap\partial\kappa'), e_{\kappa\Omega}=int(\partial\kappa\cap\partial\Omega)$.
Multiply both sides of the first equation of formula (2.1) by v, which is obtained by Green's formula:

$$-\int_\kappa \nabla\cdot(\alpha\nabla u)\ \cdot\ v dx=\int_\kappa \alpha\,\nabla u\cdot\ \nabla v dx-\int_{\partial\kappa}\alpha\,\nabla u\cdot\ v\cdot n ds, \tag{2.5}$$

$$\int_\kappa \lambda\, u\cdot\ v dx=\int_\kappa \alpha\,\nabla u\cdot\ \nabla v dx-\int_{\partial\kappa}\alpha\,\nabla u\cdot\ v\cdot n ds, \tag{2.6}$$

And

$$\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \lambda\, u\cdot\ v dx=\sum_{\kappa\in\Gamma_h}\int_\kappa \alpha\,\nabla u\cdot\ \nabla v dx-\sum_{\kappa\in\Gamma_h}\int_{\partial\kappa}\alpha\,\nabla u\cdot\ v\cdot n ds, \tag{2.7}$$

And

$$\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}\alpha\,\nabla u\cdot v\cdot n ds=\sum_{e\in\Gamma_h}\int_e \alpha\,\nabla u\cdot v\cdot n ds=\sum_{e\in\Gamma_h}\int_e \{\alpha\nabla u\}\cdot[[v]]ds, \tag{2.8}$$

Then

$$\int_\Omega \lambda\, u\cdot v dx=\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \alpha\,\nabla u\cdot\nabla v dx-\sum_{e\in\Gamma_h}\int_e \{\alpha\nabla u\}\cdot[[v]]ds, \tag{2.9}$$

And because $[[u]]$ is on $e\in\Gamma_h$, there is

$$\int_\Omega \lambda\, u\cdot v dx=\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \alpha\,\nabla u\cdot\nabla v dx-\{\alpha\nabla u\}\cdot[[v]]ds-\sum_{e\in\Gamma_h}\int_e [[u]]\cdot\{\alpha\nabla v\}ds, \tag{2.10}$$

Finally written as

$$\int_\Omega \lambda\, u\cdot v dx=\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \alpha\,\nabla u\cdot\nabla v dx-\sum_{e\in\Gamma_h}\int_e \{\alpha\nabla u\}\cdot[[v]]\,ds-\sum_{e\in\Gamma_h}\int_e [[u]]\cdot\{\alpha\nabla v\}ds$$

$$+\sum_{e\in\Gamma_h}\int_e \eta\, p^2 h^{-1}[[u]]\cdot[[v]]ds, \tag{2.11}$$

Where η is the penalty parameter.
Define

$$a_h(u_h\,,v_h)=\sum_{\kappa\in\Gamma_h}\int_\kappa \alpha\,\nabla u_h\cdot\nabla v_h dx-\sum_{e\in\Gamma_h}\int_e\{\,\alpha\nabla u_h\}\cdot[[v_h]]ds$$
$$-\sum_{e\in\Gamma_h}\int_e[[u_h]]\cdot\{\alpha\nabla v_h\}ds+\sum_{e\in\Gamma_h}\eta\, p^2 h_e^{-1}\int_e[[u_h]]\cdot[[v_h]]ds. \tag{2.12}$$

The finite element approximation of (2.4) is to find $(\lambda_h\,,u_h)\in C\times S^p(\mathcal{T}_h), u_h\neq 0$, such that

$$a_h(u_h\,,v_h)=\lambda_h(u_h\,,v_h),\quad \forall v_h\in S^p(\mathcal{T}_h). \tag{2.13}$$

The source problem of (2.4) is: find $w\in H_0^1(\Omega)$, such that

$$a(w,v)=(f,v),\qquad \forall v\in H_0^1(\Omega). \tag{2.14}$$

The finite element approximation of (2.14) is: find $w_k\in S^p(\Gamma_h)$, such that

$$a_h(w_h\,,v_h)=(f,v_h)\,,\qquad \forall v_h\in S^p(\mathcal{T}_h). \tag{2.15}$$

Define linear bounded operator $T: L^2(\Omega)\to H_0^1(\Omega)$ satisfies

$$a(Tf,v)=(f,v),\quad \forall f\in L^2(\Omega),\ \ v\in H_0^1(\Omega), \tag{2.16}$$

Then (2.4) the equivalent operator form is:

$$Tu=\frac{1}{\lambda}u. \tag{2.17}$$

It is satisfied by (2.13) the corresponding discrete solution operator $T_h: L^2(\Omega)\to S^p(\Gamma_h)$ that can be defined

$$a_h(T_hf,v)=(f,v),\quad \forall f\in L^2(\Omega),\quad \forall v\in S^p(\mathcal{T}_h). \tag{2.18}$$

Then the equivalent operator form of (2.13) is:

$$T_hu_h=\frac{1}{\lambda_h}u_h. \tag{2.19}$$

Introduce a sum space $V(h)=S^p(\Gamma_h)+H_0^1(\Omega)$ endowed with a locally discontinuous finite element norm, where the discontinuous finite element norm is:

$$\|\, v_h\,\|_G^2=\sum_{\kappa\in\mathcal{T}_h}\|\,\alpha\nabla v_h\,\|_{0,\kappa}^2+\sum_{e\in\Gamma_h}p^2\,h_e^{-1}\|[[v_h]]\|_{0,e}^2\,, \tag{2.20}$$

And the h-norm is defined on the fragment function space $H^{1+s}(\mathcal{T}_h)\left(s>\frac{1}{2}\right)$ as:

$$\|\, v_h\,\|_h^2=\|\, v_h\,\|_G^2+\sum_{e\in\Gamma_h}p^{-2}\,h_e\|\{\alpha\nabla v_h\}\|_{0,e}^2. \tag{2.21}$$

Note that on a discontinuous finite element space $S^p(\mathcal{T}_h)$, $\|\cdot\|_G$ and $\|\cdot\|_h$ are equivalent.

According to literature [8] and Green's formula, the consistency of the local discontinuous finite element method can be deduced. Combined with equation (2,15), the error formula can be obtained as follows:

$$a_h(w-w_h,v_h)=0,\ \forall v_h\in S^p(\mathcal{T}_h). \tag{2.22}$$

It is not difficult to see that the following continuity and ellipticity hold:

$$|a_h(u_h,v_h)|\lesssim\|\,u_h\,\|_h\|\,v_h\,\|_h,\quad \forall u_h,v_h\in S^p(\mathcal{T}_h)+H^{1+s}(\mathcal{T}_h)\left(s>\frac{1}{2}\right), \tag{2.23}$$

$$\|\,u_h\,\|_G^2\lesssim|a_h(u_h,u_h)|. \tag{2.24}$$

Let w be the solution to (2.14), and $f\in L^2(\Omega)$, assume that the following regularity estimates hold

$$\|w\|_{1+s}\lesssim\|f\|_{0,\Omega}\quad\left(\frac{1}{2}<s\leq 1\right).$$

Lemma 2.1 (Proposition 4.9 in [9]) Let $\kappa\in\mathcal{T}_h$ and $v\in H^{s_\kappa}(\kappa), s_\kappa\geq 1$ exist, then there is a polynomial $\Pi_{p_\kappa}^{h_\kappa}v\in S^{p_\kappa}, p_\kappa=1,2,\ldots$ that satisfies $(0\leq m\leq s_\kappa)$

$$\left\|v-\Pi_{p_\kappa}^{h_\kappa}v\right\|_{m,\kappa}\lesssim h_\kappa^{\min(p_\kappa+1,s_\kappa)-m}\,p_\kappa^{m-s_\kappa}\,\|v\|_{s_\kappa,\kappa}\,, \tag{2.25}$$

$$\left\|v-\Pi_{p_\kappa}^{h_\kappa}v\right\|_{0,\partial\kappa}\lesssim h_\kappa^{\min(p_\kappa+1,s_\kappa)-\frac{1}{2}}\,p_\kappa^{\frac{1}{2}-s_\kappa}\,\|v\|_{s_\kappa,\kappa}\,. \tag{2.26}$$

Now the global discontinuous interpolation operator $\Pi_p^h: H_0^1(\Omega)\to S^p(\mathcal{T}_h)$ is introduced so that $\Pi_p^h(u)|_\kappa=\Pi_{p_\kappa}^{h_\kappa}(u|_\kappa)$, for a vector-valued function $\mathbf{r}=(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_d)$, defines $\Pi_p^h(\mathbf{r})|_\kappa=\left(\Pi_p^h\mathbf{r}_1,\Pi_p^h\mathbf{r}_2,\cdots,\Pi_p^h\mathbf{r}_d\right)$.

Theorem 2.1 Let w and w_h be solutions to equations (2.14) and (2.15) respectively, w satisfies $w|_\kappa\in H^{s_\kappa}(\kappa)$, and for all $\kappa\in\mathcal{T}_h$ and $s_\kappa\geq 1$, the following inequalities are true

$$\|w-w_h\|_h\lesssim\inf_{v_h\in S_h}\|w-v_h\|_h. \tag{2.27}$$

$$\|w-w_h\|_h \lesssim \sum_{\kappa\in\mathcal{T}_h} h_\kappa^{\min(p_\kappa+1,s_\kappa)-1}\, p_\kappa^{\frac{3}{2}-s_\kappa}\|w\|_{s_\kappa,\kappa}, \tag{2.28}$$

Proof First, we prove that equation (2.27), by using equation (2.22), equation (2.23) and equation (2.24), can be derived

$$\begin{aligned}\|v_h-w_h\|_h^2 &\lesssim |a_h(v_h-w_h,v_h-w_h)| \\ &\lesssim a_h(v_h-w,v_h-w_h)+a_h(w-w_h,v_h-w_h) \\ &\lesssim \|v_h-w\|_h\|v_h-w_h\|_h,\end{aligned} \tag{2.29}$$

Using the triangle inequality

$$\| w-w_h \|_h \lesssim \| w-v_h \|_h + \| v_h-w \|_h. \tag{2.30}$$

From formula (2.29) and formula (2.30), formula (2.27) can be obtained.

Below we prove that formula (2.28) is derived from formula (2.21), so that $E_h(w)=w-\Pi_p^h w$ has

$$\begin{aligned}\| E_h(w) \|_h^2 &\lesssim \sum_{\kappa\in\mathcal{T}_h}\|\alpha\nabla_h E_h(w)\|_{0,\Omega}^2+\sum_{e\in\Gamma_h}\left\|h_e^{-\frac{1}{2}}p[[E_h(w)]]\right\|_{0,e}^2+\sum_{e\in\Gamma_h}\left\|h_e^{\frac{1}{2}}p^{-1}\{\alpha\nabla_h E_h(w)\}\right\|_{0,e}^2 \\ &\lesssim \sum_{\kappa\in\mathcal{T}_h}\|\alpha\nabla_h E_h(w)\|_{0,\Omega}^2+\sum_{\kappa\in\mathcal{T}_h}\sum_{e\subset\partial\kappa}\left(h_e^{-\frac{1}{2}}p\|E_h(w)\|_{0,e}^2+h_e^{\frac{1}{2}}p^{-1}\| \alpha\nabla_h E_h(w) \|_{0,e}^2\right) \\ &:= I_1+I_2+I_3.\end{aligned} \tag{2.31}$$

I_1 is estimated, which can be obtained from equation (2.25)

$$\|\alpha\nabla_h E_h(w)\|_{0,\Omega}^2 \lesssim \left(h_\kappa^{\min(p_\kappa+1,s_\kappa)-1}p_\kappa^{1-s_\kappa}\|w\|_{s_\kappa,\kappa}\right)^2, \tag{2.32}$$

I_2 is estimated, which can be obtained from equation (2.26)

$$h^{-\frac{1}{2}}p\|E_h(w)\|_{0,e}^2 \lesssim \left(h_\kappa^{\min(p_\kappa+1,s_\kappa)-1}p_\kappa^{\frac{3}{2}-s_\kappa}\|w\|_{s_\kappa,\kappa}\right)^2, \tag{2.33}$$

I_3 is estimated, which can be obtained from equation (2.26)

$$h^{\frac{1}{2}}p^{-1}\| \alpha\nabla_h E_h(w) \|_{0,e}^2 \lesssim \left(h_\kappa^{\min(p_\kappa+1,s_\kappa)}p_\kappa^{-\frac{1}{2}-s_\kappa}\| w \|_{s_\kappa,\kappa}\right)^2, \tag{2.34}$$

It is obtained by formula (2.32), (2.33) and (2.34)

$$\| w-\Pi_p^h w \|_h \lesssim \sum_{\kappa\in\mathcal{T}_h} h_\kappa^{\min(p_\kappa+1,s_\kappa)-1}\, p_\kappa^{\frac{3}{2}-s_\kappa}\|w\|_{s_\kappa,\kappa}, \tag{2.35}$$

Error estimation formula and interpolation error formula are the following formula

$$\inf_{v_h\in V_h}\| w-v_h \| \lesssim \| w-\Pi_p^h w \|. \tag{2.36}$$

Formula (2.28) is obtained from formula (2.27), formula (2.35) and formula (2.36), and the proof is complete.

Theorem 2.2 Let w and w_h be solutions to equations (2.14) and (2.15) respectively, and let w satisfy $w|_\kappa \in H^{s_\kappa}(\kappa)$, and for all $\kappa\in\mathcal{T}_h$ and $s_\kappa\geq 1$, the following inequalities hold

$$\|w-w_h\|_{0,\Omega} \lesssim h^r p^{\frac{1}{2}-r}\| w-w_h \|_h. \tag{2.37}$$

$$\|w-w_h\|_{0,\Omega} \lesssim \sum_{\kappa\in\mathcal{T}_h} h^{\min(p+1,s)+r-1}p^{2-s-r}\|w\|_{s,\Omega}, \tag{2.38}$$

Where $s=\min_{\kappa\in\mathcal{T}_h} s_\kappa \geq 1, \frac{1}{2}<r\leq 1.$ (2.36)

Proof Consider the source problem $a(v,w^*)=(v,g),\ \forall v\in H_0^1(\Omega)$, for any fixed $g\in L^2(\Omega)$, let $w_h^*=\Pi_p^h w^*$, derived using Galerkin orthogonality and equation (2.23) for the duality problem of formula (2.4).

$$\begin{aligned}(w-w_h,g) &= a_h(w-w_h,w^*)=a_h(w-w_h,w^*-w_h^*) \\ &\lesssim \| w-w_h \|_h \| w^*-w_h^* \|_h.\end{aligned} \tag{2.39}$$

Estimated by equation (2.28) and the assumption of elliptic regularity, let $g=w-w_h$ be obtained

$$|w^*-w_h^* \|_h \lesssim h^r p^{\frac{1}{2}-r}\| w^* \|_{1+r,\Omega} \lesssim h^r p^{\frac{1}{2}-r}\| w-w_h \|_{0,\Omega}. \tag{2.40}$$

It is obtained by formula (2.39) and formula (2.40)

$$\| w-w_h \|_{0,\Omega}=\sup_{g\in L^2(\Omega)}\frac{|(w-w_h,g)|}{\| g \|_{0,\Omega}} \lesssim h^r p^{\frac{1}{2}-r}\| w-w_h \|_h.$$

Formula (2.37) can be obtained.

Below we prove formula (2.38), obtained from formula (2.28) and formula

$$\| w-w_h \|_{0,\Omega} \lesssim h^r p^{\frac{1}{2}-r}\| w-w_h \|_h \lesssim h^{\min(p+1,s)+r-1}p^{2-s-r}\|w\|_{s,\Omega}.$$

Thus formula (2.38) can be obtained, and the proof is complete. (2.41)

Taking $s_\kappa = 1 + r \quad \left(\frac{1}{2} < r \le 1\right)$ from equation (2.28), the following stable estimation can be obtained by regularization estimation

$$\begin{aligned} \|T_h f\|_h &\lesssim \|T_h f - Tf\|_h + \|Tf\|_h \\ &\lesssim \| T_h f - Tf \|_h + \| Tf \|_1 \\ &\lesssim h^{\min(p,r)} p^{\frac{1}{2}-r} \| Tf \|_{1+r} + \| Tf \|_1 \\ &\lesssim \|f\|_{0,\Omega}. \end{aligned} \tag{2.41}$$

III. A PRIORI ERROR ANALYSIS

3.1 A Priori Error Analysis For Eigenvalue Problems

Let λ be the j eigenvalue of (2.4) with algebraic multiplicative q, where $\lambda_j = \lambda_{j+1} = \ldots = \lambda_{j+q-1}$. When $\|T_h - T\|_{0,\Omega} \to 0$, the q eigenvalues $\lambda_{j,h}, \ldots \lambda_{j+q-1,h}$ of (2.13) converge to λ. Let $M(\lambda)$ be the generalized eigenvector space of formula (2.4) related to λ, $M_h(\lambda)$ the direct sum of the generalized eigenvector space of formula (2.13) related to λ_h, and λ_h converges to λ.

Given two closed subspaces of *V* and *U*, the gap between these two subspaces is expressed as

$$\delta(U,V) = \sup_{u \in V, \|u\|_{0,\Omega}=1} \inf_{v \in U} \| u - v \|_{0,\Omega}, \quad \hat{\delta}(U,V) = \max\{\delta(U,V), \delta(V,U)\}.$$

$\hat{\lambda}_h = \frac{1}{q}\sum_{i=j}^{j+q-1} \lambda_{i,h}$ stands for arithmetic average. (3.1)

Theorem 3.1 Set $M(\lambda) \subset H^{1+r}(\Omega) \left(\frac{1}{2} < r \le 1\right)$, then the following inequality is true

$$|\lambda_h - \lambda| \lesssim h^{2r} p^{1-2r}. \tag{3.1}$$

Let $u_h \in M_h(\lambda)$ be the direct sum of the generalized eigenvector space of (2.13), then there exists an eigenvalue function u of (2.4) that makes

$$\| u - u_h \|_h \lesssim h^{r-1} p^{\frac{3}{2}-r}. \tag{3.2}$$

$$\| u - u_h \|_{0,\Omega} \lesssim h^{2r-2} p^{3-2r} \| u - u_h \|_h. \tag{3.3}$$

Proof $Tf = w$ and $T_h f = w_h$, combined with the operator form, regularity estimation and (2.38) formula, can be obtained

$$\begin{aligned} \| T - T_h \|_{0,\Omega} &= \sup_{0 \ne f \in L^2(\Omega)} \frac{\| Tf - T_h f \|_{0,\Omega}}{\| f \|_{0,\Omega}} = \sup_{0 \ne f \in L^2(\Omega)} \frac{\| w - w_h \|_{0,\Omega}}{\| f \|_{0,\Omega}} \\ &\lesssim \sup_{0 \ne f \in L^2(\Omega)} \frac{h^{2r-2} p^{3-2r} \| f \|_{0,\Omega}}{\| f \|_{0,\Omega}} \lesssim h^{2r-2} p^{3-2r} \to 0, (h \to 0, p \to \infty). \end{aligned}$$

Theorems (7.1), theorems (7.2) and theorems (7.4) in [10], there are

$$\hat{\delta}\big(M(\lambda), M_h(\lambda)\big) \lesssim \|(T - T_h)|_{M(\lambda)}\|_{0,\Omega}, \tag{3.4}$$

$$|\lambda - \lambda_h| \lesssim \sum_{i,l=j}^{j+q-1} |(T - T_h)\varphi_i, \varphi_l| + \|(T - T_h)|_{M(\lambda)}\|_{0,\Omega}^2, \tag{3.5}$$

$$|u - u_h|_{0,\Omega} \lesssim \|(T - T_h)|_{M(\lambda)}\|_{0,\Omega}. \tag{3.6}$$

Where $\{\varphi_i\}_{i=j}^{j+q-1}$ and $\{\varphi_l\}_{l=j}^{j+q-1}$ constitutes a basis for $M(\lambda)$.

From theorem 2.1 and theorem 2.2, it can be inferred

$$\begin{aligned} \|(T - T_h)|_{M(\lambda)}\|_{0,\Omega} &= \sup_{f \in M(\lambda), \|f\|_{0,\Omega}=1} \| Tf - T_h f \|_{0,\Omega} \\ &\lesssim \sup_{f \in M(\lambda), \|f\|_{0,\Omega}=1} h^{2r-2} p^{3-2r} \| Tf \|_{r+1,\Omega}. \end{aligned} \tag{3.7}$$

Using operator properties, the regularity estimate can be obtained from Galerkin orthogonality and the (2.23) formula

$$\begin{aligned} \big((T - T_h)\varphi_i, \varphi_l\big) &= a_h(T\varphi_i - T_h\varphi_i, T\varphi_l) \\ &= a_h(T\varphi_i - T_h\varphi_i, T\varphi_l - T_h\varphi_l) \\ &\lesssim \| T\varphi_i - T_h\varphi_i \|_h \| T\varphi_l - T_h\varphi_l \|_h \\ &\lesssim h^{r-1} p^{\frac{3}{2}-r} \| T\varphi_i \|_{r+1} \, h^{r-1} p^{\frac{3}{2}-r} \| T\varphi_l \|_{r+1} \\ &\lesssim h^{2r-2} p^{3-2r}. \end{aligned} \tag{3.8}$$

Formula (3.1) is obtained by substituting formula (3.7) and formula (3.8) into formula (3.5).

Because of $u = \lambda Tu$ and $u_h = \lambda_h T_h u_h$, using the triangle inequality, (2.41), (3.1), (3.6) and (3.7) can be derived

$$\left|\|u - u_h\|_h - \|u - \lambda T_h u\|_h\right| \lesssim \|u_h - \lambda T_h u\|_h = \|T_h(\lambda_h u_h - \lambda u)\|_h \lesssim \|\lambda_h u_h - \lambda u\|_{0,\Omega} \lesssim h^{2r-2} p^{3-2} \tag{3.9}$$

It is obtained by formula (2.27) and formula (2.28)

$$\|u-\lambda T_h u\|_h = \|\lambda Tu - \lambda T_h u\|_h \leq \lambda \|Tu - T_h u\|_h \lesssim \inf_{v_h \in V_h} \|Tu - v_h\|_h \lesssim \|u-u_h\|_h \lesssim h^{r-1} p^{\frac{3}{2}-r} \tag{3.10}$$

Formula (3.2) is obtained from formula (3.9) and Formula (3.10).
It is obtained by formula (2.27) and formula (2.37)

$$\|u-u_h\|_{0,\Omega} \leq \|Tu - T_h u\|_{0,\Omega} \lesssim h^{r-1} p^{\frac{3}{2}-r} \|Tu - T_h u\|_h$$
$$\lesssim h^{r-1} p^{\frac{3}{2}-r} \inf_{v_h \in V_h} \|Tu - v_h\|_h \lesssim h^{r-1} p^{\frac{3}{2}-r} \|u-u_h\|_h$$

Formula (3.3) is obtained, and the proof is complete.

3.2 Numerical Experiment

In this section, we present several numerical experiments to demonstrate the effectiveness of our approach in tackling problem (2.1), where $\alpha = 1$ Our program is implemented using the iFEM package, and we employ the IPDG method ($\eta = 10$) for the computations. We consider two distinct test domains: the L-shaped domain $\Omega_L = (-1,1)^2 \backslash ([0,1] \times (-1,0])$, the square domain Ω_S, and the vertex to $(0,-1),(1,0),(0,1),(-1,0)$. As the exact eigenvalue is unknown, we utilize a reference eigenvalue $\lambda_1 = |r|^2/4 + 9.63972384472$ for the L-shaped domain and $\lambda_1 = |r|^2/4 + 19.7392088022$ for the Square domain. Analysis of the results in Table 1 and Table 2 reveals that the algorithm achieves optimal convergence rates.

Table 1 When $\alpha = 1$, the numerical solution results of primary eigenvalues for region Ω_L

Domin	*P*	h=1/2		h=1/4	
		dof	λ_1	*dof*	λ_1
	4	360	9.654223671970552	1440	9.645448483439346
	5	504	9.648715340175682	2016	9.643290842030046
	6	672	9.645712947540130	2688	9.642101993127229
Ω_L	7	864	9.643927734108718	3456	9.641393750545754
	8	1080	9.642797464972297	4320	9.640944521432180
	9	1320	9.642050421427307	5280	9.640669450151195

Table 2 When $\alpha = 1$, the numerical solution results of primary eigenvalues for region Ω_S

Domin	*P*	h=1/2		h=1/4	
		dof	λ_1	*dof*	λ_1
	4	120	19.740215197598424	480	19.739213395401787
	5	168	19.739230145337821	672	19.739208824379499
	6	224	19.739209161483238	896	19.739208805789204
Ω_S	8	360	19.739208781824644	1440	19.739208930658755
	9	440	19.739208300464178	1760	19.739209092565559
	10	624	19.739208792357033	2496	19.739209099626390

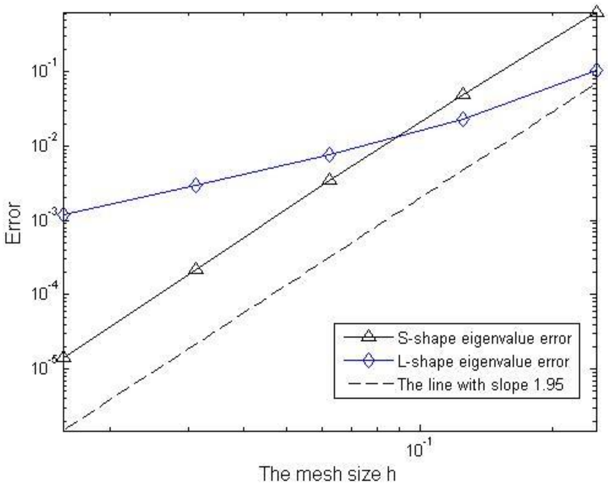

Figure1: When $\alpha = 1$, the error curve of the primary eigenvalues

In Table 1 and Table 2, we present the numerical solutions of eigenvalues computed using the IPDG method. Additionally, we depict the eigenvalue error curve of primary elements in the figure. It is evident from both the numerical results in the figure and the tables that our method attains the optimal convergence order of eigenvalues and provides the optimal order error estimation of eigenvalue functions. Numerical experiments corroborate the effectiveness of the proposed algorithm.

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