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Review Paper

A New Generalization of the Copoun Distribution: Identifiability and Properties

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Abstract
This study proposes a new generalization of the two-parameter Copoun distribution known as the Alpha Power Transformed Copoun distribution. This distribution is derived using the inverse function to contribute to the growing need for upside-down bathtub distributions. Some important mathematical and statistical properties of the new distribution such as the density, moments, moment generating function, entropy, and reliability indices such as the existence measurement function and risk measurement function were derived and presented.
Keywords: *Alpha Power Transformed Copoun distribution; Lifetime distributions; Three-parameter distributions; Entropy; reliability; Risk measurement function.*

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I. Introduction

In recent years, new two-parameter distributions have emerged in the literature. These new two-parameter distributions have been shown to provide a better fit to complex real life datasets than the one-parameter distributions. Some of the recently developed two-parameter distributions include the Darna distribution (Shraa & Al-Omari, 2019), the Hamza distribution (Aijaz et al., 2020), the Samade distribution (Aderoju, 2021), and the Alzoubi distribution (Benrabia & Alzoubi, 2021).

It is important to note that these distributions are a mixture of the Exponential and Gamma distributions. These two distributions are known to have their weaknesses. The weakness of the Exponential distribution is that the hazard rate function is constant; hence, it cannot handle datasets with monotone non-decreasing hazard rates (Elechi et al., 2022; Epstein, 1958; Ronald et al., 2011; Shukla, 2018b; Shukla, 2018). Furthermore, the weakness of the Gamma distribution is that the survival rate function cannot be expressed in closed form (Elechi et al., 2022; Shanker, 2015a, 2015b). The weaknesses of these two distributions are what the aforementioned one-parameter and two-parameter distributions address, providing distributions whose survival rate function can be expressed in closed form and hazard rate functions capable of handling datasets with monotone non-decreasing hazard rates.

In contributing to this gap in the literature, Uwaeme et al. (2023) proposed a new two-parameter distribution called the Copoun distribution. The Copoun distribution is a two-component density of an Exponential (η) and Gamma $(4,\eta)$ distribution with mixing proportions π_1 and π_2 such that

$$g(x;\eta,\phi) = \pi_1 g_1(x;\eta) + \pi_2 g_2(x;\eta) \quad (1)$$

where $g_1(x;\eta) = \eta e^{-\eta x}$, $g_2(x;\eta) = \frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)}$, $\pi_1 = \frac{\eta}{(\phi+\eta)}$ and $\pi_2 = \frac{\phi}{(\phi+\eta)}$

therefore,

$$g(x_k;\eta,\phi) = \eta e^{-\eta x}\cdot\frac{\eta}{(\phi+\eta)} + \frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)}\cdot\frac{\phi}{(\phi+\eta)} \quad (2)$$

Solving equation 2 gives the probability density function (pdf) of the new distribution

$$g(x;\eta,\phi) = \frac{\eta^2}{(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x};\ x>0,\eta>0,\phi>0 \quad (3)$$

The corresponding cumulative distribution function (cdf) of (3) is obtained as

$$G(x;\eta,\phi) = 1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x} \quad (4)$$

The authors introduced some of the mathematical properties of the new distribution. They showed that the Copoun distribution exhibits shapes that are not bell-shaped, but positively skewed, unimodal, and right-tailed (Uwaeme et al., 2023). Furthermore, Uwaeme & Akpan (2024) addressed one of the weakness of the Copoun distribution which is its non-monotonic hazard rate function. They applied the inverse transformation techniques to develop the Inverse Copoun distribution which addressed this weakness and also showed an improved flexibility using some real lifetime datasets (Uwaeme & Akpan, 2024).

Another way of improving a baseline distribution is the introduction of an extra parameter to improve its overall flexibility. Over the years, several methods have been developed to improve the flexibility of a baseline distribution with the addition of an extra parameter. Some of these methods includes the skew normal distribution (Azzalini, 1985), the exponentiated family of distributions (Mudholkar & Srivastava, 1993), the Marshall-Olkin family of distributions (Marshall & Olkin, 1997), the beta family of distributions (Eugene et al., 2002), the transmuted family of distributions (Shaw & Buckley, 2009), and the T-X family of distributions (Alzaatreh et al., 2013) amongst others (Jones, 2015; Lee et al., 2013).

Recently, Mahdavi and Kundu (2016) proposed a new family of distribution functions called the alpha power transformed (APT) family of distributions which adds an extra shape parameter to a baseline distribution (Mahdavi & Kundu, 2016). The pdf of the APT family of distributions is given by

$$f_{APT}(x) = \begin{cases} \frac{\log \alpha}{\alpha - 1} g(x) \alpha^{G(x)} & if\ \alpha > 0, \alpha \neq 1 \\ g(x) & if\ \ \alpha = 1 \end{cases} \tag{5}$$

The corresponding cdf is given as

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{G(x)} - 1}{\alpha - 1} & if\ \alpha > 0,\ \alpha \neq 1 \\ G(x) & if\ \alpha = 1 \end{cases} \tag{6}$$

where $g(x)$ and $G(x)$ are the pdf and cdf of the baseline distribution.

This new family of distribution was shown to bring more flexibility to the baseline distribution. Another important characteristic of this new family is that it incorporates skewness to the family of distribution functions (Mead et al., 2019). Since the introduction of the alpha power transformed family of distributions, several authors have adopted this approach to generalize some baseline distributions, improving flexibility, fits, and skewness. Some of these baseline distributions includes the extended exponential distribution (Hassan et al., 2018), Weibull distribution (Nassar et al., 2018), Lindley distribution (Dey et al., 2019), Power Lindley distribution (Hassan et al., 2019), Frechet distribution (Nasiru et al., 2019), Inverse Lomax distribution (ZeinEldin et al., 2020), Pareto distribution (Sakthivel, 2020), the Weibull-G family of distributions (Elbatal et al., 2021), the Garima distribution (Mohiuddin et al., 2021), the Kumaraswamy distribution (Hozaien et al., 2021), Rama distribution (Mohiuddin & Kannan, 2021a), the Aradhana distribution (Mohiuddin & Kannan, 2021b), Quasi Aradhana distribution (Mohiuddin et al., 2022), Pranav distribution (Mohiuddin et al., 2022), Sujatha distribution (Mohiuddin & Kannan, 2022), Power Ailamujia distribution (Gomaa et al., 2023)

From the foregoing therefore, the motivation of this paper is to propose a new generalization of the Copoun distribution using the alpha power transformed technique called the Alpha Power Transformed Copoun distribution and its statistical properties. The subsequent sections of the paper will be arranged as follows. Section 2 discusses the new alpha power transformed Copoun distribution with the derivation of the pdf, the cdf, and their plots, section 3 discusses the mathematical properties of the inverse Copoun distribution as well as the plots of the risk measurement function to highlight the shape, and section 4 concludes the paper with some remarks.

1. The Alpha Power Transformed Copoun distribution

This section will introduce the pdf and the cdf of the Alpha Power Transformed Copoun distribution and illustrate the different shapes of the Alpha Power Transformed Copoun distribution. The new Alpha Power Transformed Copoun distribution is obtained by putting equations 3 and 4 into equations 5 and 6 respectively.

Proposition 1: If a random variable X follows the Alpha Power Transformed Copoun distribution (APTC) with parameters η, ϕ, and α then the pdf and cdf are respectively given by

$$f_{APTC}(x; \eta, \phi, \alpha) = \frac{\eta^2 \log \alpha}{(\alpha - 1)(\phi + \eta)} \left[1 + \frac{\phi \eta^2 x^3}{6}\right] e^{-\eta x} \cdot \alpha^{1 - \left[1 + \frac{\phi \eta^3 x^3 + \phi \eta^2 x^3 + \phi \eta x}{6(\phi + \eta)}\right] e^{-\eta x}};\ x > 0, \eta > 0, \phi > 0, \alpha \neq 1 \tag{7}$$

and

$$F_{APTC}(x; \eta, \phi, \alpha) = \frac{\alpha^{1 - \left[1 + \frac{\phi \eta^3 x^3 + \phi \eta^2 x^3 + \phi \eta x}{6(\phi + \eta)}\right] e^{-\eta x}} - 1}{\alpha - 1};\ x > 0, \eta > 0, \phi > 0, \alpha > 0, \alpha \neq 1 \tag{8}$$

The Alpha Power Transformed Copoun distribution derived above is denoted by $APTC(\eta, \phi, \alpha)$. The graphical plots of the theoretical density and distribution function (for some selected but different real points of η, ϕ, and α) of the Alpha Power Transformed Coupon distribution are shown in the Figure 1 and Figure 2 below.

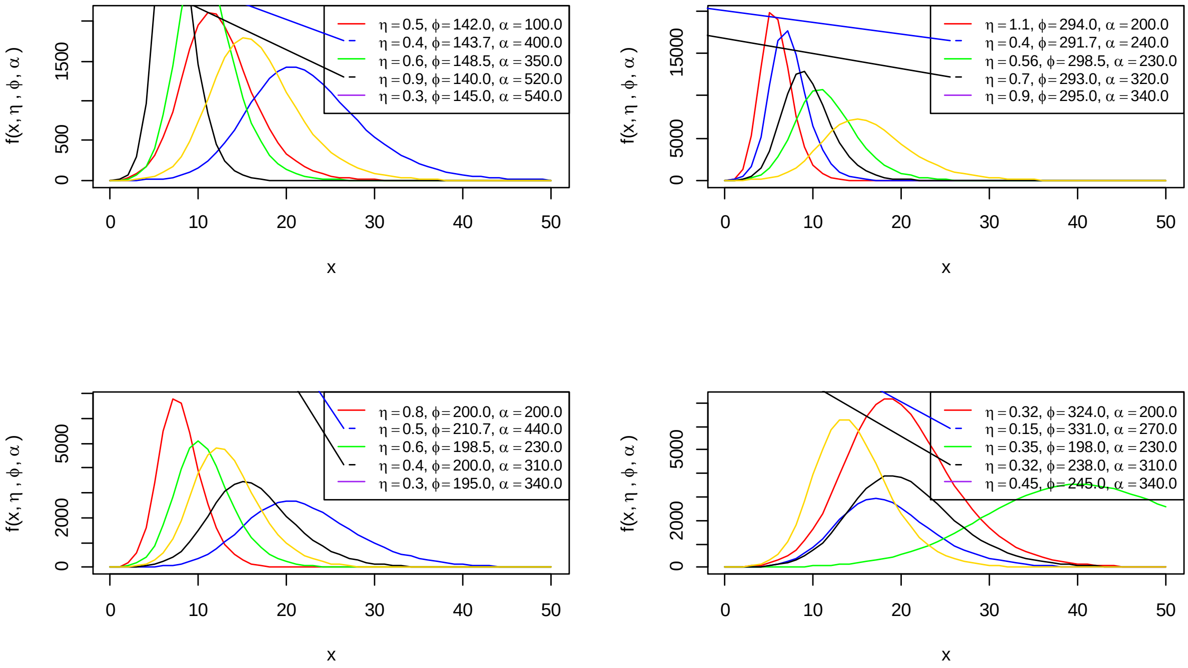

Figure 1: The graphical plots of the probability density function (for some selected but different real points of $\eta, \phi, and\, \alpha$) of an Alpha Power Transformed Coupon distribution.

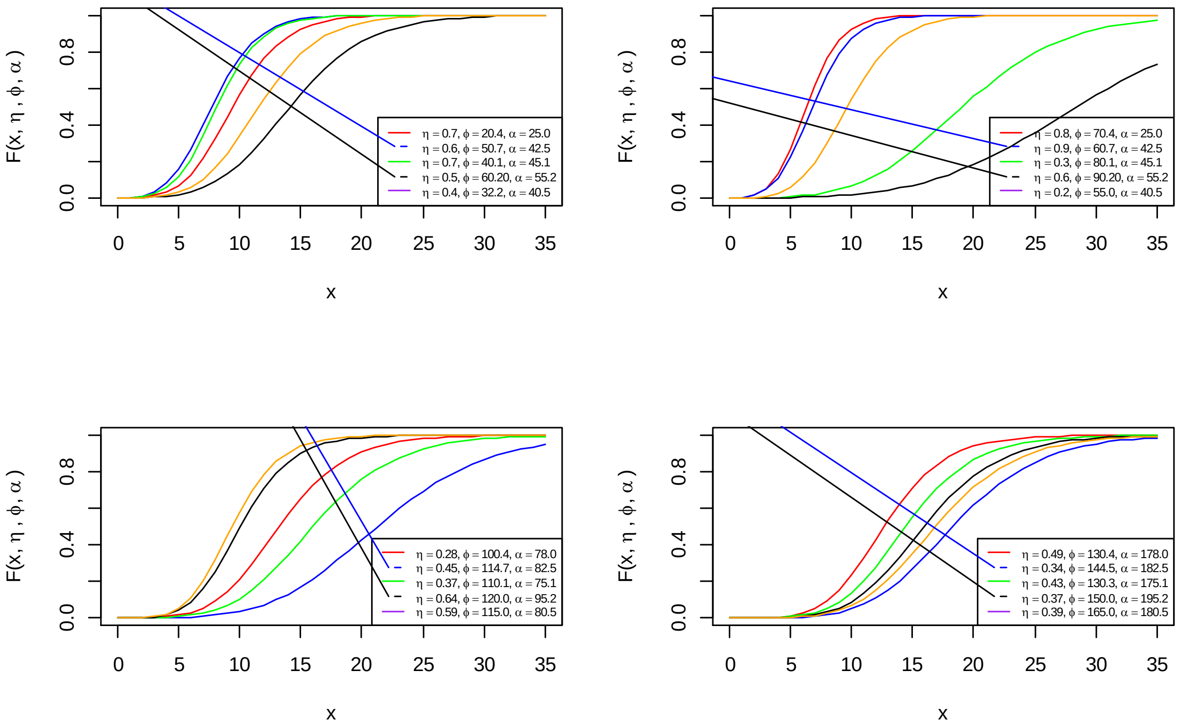

Figure 2: The graphical plots of the cumulative distribution function (for some selected but different real points of $\eta, \phi, and\, \alpha$) of an Alpha Power Transformed Coupon distribution

The curves displayed in Figure 1 are not bell-shaped, but are positively skewed, unimodal, and right tailed. In addition, the curve shows that increasing the value of ϕ leads to a considerable increase in the peak of the curve. In addition, the curves displayed in Figure 2 shows that the cumulative distribution function converges to one.

2. Mathematical Properties of the Alpha Power Transformed Copoun Distribution

In this section, we derive and present some of the mathematical properties of the Alpha Power Transformed Copoun distribution such as the moment, moment generating function, and characteristic function.

2.1 Moments of the Alpha Power Transformed Copoun Distribution

We derive the rth moment of the Alpha Power Transformed Copoun distribution in this subsection.

Theorem 1

Given a random variable X, following the Alpha Power Transformed Copoun distribution, the rth order moment about origin, $E(X^r)$ of the Alpha Power Transformed Copoun distribution is given by

$$\mu_r' = E(X^r) = P\left[\frac{\Gamma(r-j-l+3k+1)}{(\eta+\eta i)^{(r-j-l+3k+1)}} + \frac{\phi\eta^2\Gamma(r-j-l+3k+4)}{6(\eta+\eta i)^{(r-j-l+3k+4)}}\right] \quad (9)$$

where $P = \left(\frac{\alpha \log \alpha}{\alpha-1}\right)\frac{3^j 2^l \phi^k \eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i!}$

Proof:

The rth crude or uncorrected moments of a random variable and can be written as

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_{APTC}(x;\eta,\phi,\alpha)dx \quad (10)$$

$$= \int_0^\infty x^r \cdot \frac{\eta^2 \log\alpha}{(\alpha-1)(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}\cdot \alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}dx \quad (11)$$

$$= \frac{\eta^2\alpha\log\alpha}{(\alpha-1)(\phi+\eta)}\int_0^\infty x^r\cdot\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}\cdot\alpha^{-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} \quad (12)$$

Using the power series expansion $\alpha^{-z} = \sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i!}z^i$, we obtain

$$E(X^r) = \frac{\eta^2\alpha\log\alpha}{(\alpha-1)(\phi+\eta)}\sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i!}\int_0^\infty x^r\cdot\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}\cdot\left(\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}\right)^i dx \quad (13)$$

Using the binomial series expansion $(1+\alpha)^n = \sum_{k=0}^{n}\binom{n}{k}\alpha^k$, and after some simplification we have,

$$E(X^r) = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\frac{3^j2^l\phi^k\eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i!}\int_0^\infty x^{r-j+3k-l}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-(\eta+\eta i)x}dx \quad (14)$$

Let $P = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\frac{3^j2^l\phi^k\eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i!}$. Hence,

$$E(X^r) = P\left[\int_0^\infty x^{r-j+3k-l}e^{-(\eta+\eta i)x}dx + \frac{\phi\eta^2}{6}\int_0^\infty x^{r-j+3k-l+3}e^{-(\eta+\eta i)x}dx\right] \quad (15)$$

Recall $\int_0^\infty z^w e^{-\varrho z}dz = \frac{\Gamma(w+1)}{\varrho^{w+1}}$, then

$$\therefore E(X^r) = P\left[\frac{\Gamma(r-j-l+3k+1)}{(\eta+\eta i)^{(r-j-l+3k+1)}} + \frac{\phi\eta^2\Gamma(r-j-l+3k+4)}{6(\eta+\eta i)^{(r-j-l+3k+4)}}\right] \quad (16)$$

Which completes the proof.

In particular, the first four (4) moments about the origin of the Alpha Power Transformed Copoun distribution is obtained by substituting the values of $r = 1, 2, 3, 4$ as follows;

$$\mu_1' = E(X) = P\left[\frac{\Gamma(2-j-l+3k)}{(\eta+\eta i)^{(2-j-l+3k)}} + \frac{\phi\eta^2\Gamma(5-j-l+3k)}{6(\eta+\eta i)^{(5-j-l+3k)}}\right] \quad (17)$$

$$\mu_2' = E(X^2) = P\left[\frac{\Gamma(3-j-l+3k)}{(\eta+\eta i)^{(3-j-l+3k)}} + \frac{\phi\eta^2\Gamma(6-j-l+3k)}{6(\eta+\eta i)^{(6-j-l+3k)}}\right] \quad (18)$$

$$\mu_3' = E(X^3) = P\left[\frac{\Gamma(4-j-l+3k)}{(\eta+\eta i)^{(4-j-l+3k)}} + \frac{\phi\eta^2\Gamma(7-j-l+3k)}{6(\eta+\eta i)^{(7-j-l+3k)}}\right] \quad (19)$$

$$\mu_4' = E(X^4) = P\left[\frac{\Gamma(5-j-l+3k)}{(\eta+\eta i)^{(5-j-l+3k)}} + \frac{\phi\eta^2\Gamma(8-j-l+3k)}{6(\eta+\eta i)^{(8-j-l+3k)}}\right] \quad (20)$$

2.2 Moment generating function

Here, we propose the moment generating function for the Alpha Power Transformed Copoun distribution on this subsection.

Theorem 2

Given a random variable X, following the Alpha Power Transformed Copoun distribution, the moment generating function of X, $M_X(t)$ of the Alpha Power Transformed Copoun distribution is given by

$$M_X(t) = E(e^{tx}) = Q\left[\frac{\Gamma(r-j-l+3k+1)}{(\eta+\eta i)^{(r-j-l+3k+1)}} + \frac{\phi\eta^2\Gamma(r-j-l+3k+4)}{6(\eta+\eta i)^{(r-j-l+3k+4)}}\right] \quad (21)$$

where $Q = \left(\frac{\alpha \log \alpha}{\alpha-1}\right) \frac{3^j 2^l \phi^k \eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{i}{j}\binom{j}{k}\binom{k}{l} \frac{(-\log \alpha)^i}{i!} \frac{t^r}{r!}$

Proof:

The moment generating function of X, $M_X(t)$ can be written as

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{APTC}(x;\eta,\phi,\alpha)dx \qquad (22)$$

Recall that $e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$. Substituting, we obtain

$$M_X(t) = E(e^{tx}) = \int_0^\infty \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} f_{APTC}(x;\eta,\phi,\alpha)dx \qquad (23)$$

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \int_0^\infty x^r f_{APTC}(x;\eta,\phi,\alpha)dx \qquad (24)$$

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} E(X^r) \qquad (25)$$

$$= \left(\frac{\alpha \log \alpha}{\alpha-1}\right) \frac{3^j 2^l \phi^k \eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{i}{j}\binom{j}{k}\binom{k}{l} \frac{(-\log \alpha)^i}{i!} \frac{t^r}{r!} \left[\frac{\Gamma(r-j-l+3k+1)}{(\eta+\eta i)^{(r-j-l+3k+1)}} + \frac{\phi\eta^2 \Gamma(r-j-l+3k+4)}{6(\eta+\eta i)^{(r-j-l+3k+4)}}\right] \qquad (26)$$

$$\therefore M_X(t) = E(e^{tx}) = Q\left[\frac{\Gamma(r-j-l+3k+1)}{(\eta+\eta i)^{(r-j-l+3k+1)}} + \frac{\phi\eta^2 \Gamma(r-j-l+3k+4)}{6(\eta+\eta i)^{(r-j-l+3k+4)}}\right] \qquad (27)$$

Where $Q = \left(\frac{\alpha \log \alpha}{\alpha-1}\right) \frac{3^j 2^l \phi^k \eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{i}{j}\binom{j}{k}\binom{k}{l} \frac{(-\log \alpha)^i}{i!} \frac{t^r}{r!}$

Which completes the proof.

Similarly, the characteristics function of the Alpha Power Transformed Copoun distribution is obtained by

$$\varphi_X(t) = M_X(it) \qquad (28)$$

$$= \left(\frac{\alpha \log \alpha}{\alpha-1}\right) \frac{3^j 2^l \phi^k \eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{i}{j}\binom{j}{k}\binom{k}{l} \frac{(-\log \alpha)^i}{i!} \frac{(it)^r}{r!} \left[\frac{\Gamma(r-j-l+3k+1)}{(\eta+\eta i)^{(r-j-l+3k+1)}} + \frac{\phi\eta^2 \Gamma(r-j-l+3k+4)}{6(\eta+\eta i)^{(r-j-l+3k+4)}}\right] \qquad (29)$$

$$\therefore \varphi_X(t) = M_X(it) = T\left[\frac{\Gamma(r-j-l+3k+1)}{(\eta+\eta i)^{(r-j-l+3k+1)}} + \frac{\phi\eta^2 \Gamma(r-j-l+3k+4)}{6(\eta+\eta i)^{(r-j-l+3k+4)}}\right] \qquad (30)$$

Where $T = \left(\frac{\alpha \log \alpha}{\alpha-1}\right) \frac{3^j 2^l \phi^k \eta^{2-j-l+3k}}{6^k(\phi+\eta)^{k+1}} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{i}{j}\binom{j}{k}\binom{k}{l} \frac{(-\log \alpha)^i}{i!} \frac{(it)^r}{r!}$

3. Statistical Properties of the Alpha Power Transformed Copoun Distribution

In this section, we derive and present some statistical properties of the Alpha Power Transformed Copoun distribution. These includes the survivorship or existence measurement function, risk measurement function, order statistics, and entropy.

3.1 Order statistics

Theorem 3

Given a continuous random variable X, pdf and cdf of the pth order statistics, say $X = X_{(p)}$, is given respectively by

$$f_{APTC}(x) = \frac{n!(\phi\eta^4x^3 + 6\eta^2)e^{-\eta x}(\alpha \log \alpha)}{6(\phi+\eta)(p-1)!(n-p)!(\alpha-1)} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i \alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} \left[\frac{\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} - 1}{\alpha-1}\right]^{p+i-1} \qquad (31)$$

and

$$F_{APTC}(x) = \sum_{j=p}^{n} \sum_{i=0}^{n-j} \binom{n}{j}\binom{n-j}{i} (-1)^i \left[\frac{\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} - 1}{\alpha-1}\right]^{j+1} \qquad (32)$$

Proof:

Given a continuous random variable X, the pdf of the pth order statistics, say $X = X_{(p)}$, is obtained by

$$f_{APTC}(x) = \frac{n!\,g(x_k;\Phi)}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i G(x_k;\Phi)^{p+i-1} \qquad (33)$$

$$f_{APTC}(x) = \frac{n!\left(\frac{\eta^2 \log \alpha}{(\alpha-1)(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x} \cdot \alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}\right)}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i \left[\frac{\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} - 1}{\alpha-1}\right]^{p+i-1}$$

(34)

$$f_{APTC}(x) = \frac{n!(\phi\eta^4x^3+6\eta^2)e^{-\eta x}(\alpha\log\alpha)}{6(\phi+\eta)(p-1)!(n-p)!(\alpha-1)}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\,\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}\left[\frac{\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}-1}{\alpha-1}\right]^{p+i-1} \quad (35)$$

Which completes the proof.

Correspondingly, given a continuous random variable Y, the cdf of the pth order statistics, say $Y = Y_{(p)}$, is obtained by

$$F_{APTC}(x) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i G(x_k;\Phi)^{j+1} \quad (36)$$

$$F_{APTC}(x) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left[\frac{\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}-1}{\alpha-1}\right]^{j+1} \quad (37)$$

Which completes the proof.

3.2 Entropy

Entropy measures the uncertainties associated with a random variable of a probability distributions. Shannon (Shannon, 1951) and Rényi's entropy (Rényi, 1961) are widely used in the literature.

Theorem 4:

Given a random variable Y, which follows the Alpha Power Transformed Copoun distribution $f_{APTC}(x;\eta,\phi,\alpha)$. The Rényi entropy is given by

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\gamma}\frac{3^k\phi^{j+m}\eta^{2\gamma-l-k+3j+2m}}{2^l(6[\phi+\eta])^{\gamma+j}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\gamma}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\gamma}{m}\frac{\gamma^i(-\log\alpha)^i}{i!}\left[\frac{\Gamma(3j-k-l+3m+1)}{\eta(i+\gamma)^{(3j-k-l+3m+1)}}\right]\right] \quad (38)$$

Proof:

The Rényi entropy is given by

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^{\infty} f_{APTC}^{\gamma}(x_k;\Phi)dx\right] \quad (39)$$

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^{\infty}\left(\frac{\eta^2\log\alpha}{(\alpha-1)(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}\cdot\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}\right)^{\gamma}dx\right] \quad (40)$$

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\gamma}\frac{\eta^{2\gamma}}{(\phi+\eta)^{\gamma}}\int_0^{\infty}\left[1+\frac{\phi\eta^2x^3}{6}\right]^{\gamma}e^{-\gamma\eta x}\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]\gamma e^{-\eta x}}dx\right] \quad (41)$$

Applying the power series expansion $\alpha^{-z} = \sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i!}z^i$, we obtain

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\gamma}\frac{\eta^{2\gamma}}{(\phi+\eta)^{\gamma}}\sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i!}\int_0^{\infty}\left[1+\frac{\phi\eta^2x^3}{6}\right]^{\gamma}e^{-\gamma\eta x}\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]^{i}\gamma^i e^{-\eta i x}dx\right] \quad (42)$$

Using the binomial series expansion $(1+\alpha)^n = \sum_{k=0}^{n}\binom{n}{k}\alpha^k$, and after some simplification we have,

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\begin{matrix}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\gamma}\frac{3^k\phi^{j+m}\eta^{2\gamma-l-k+3j+2m}}{2^l(6[\phi+\eta])^{\gamma+j}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\gamma}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\gamma}{m}\frac{\gamma^i(-\log\alpha)^i}{i!}\\ \int_0^{\infty}3^{3j-k-l+3m}e^{-\eta(i+\gamma)x}dx\end{matrix}\right] \quad (43)$$

$$\therefore T_R(\beta) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\gamma}\frac{3^k\phi^{j+m}\eta^{2\gamma-l-k+3j+2m}}{2^l(6[\phi+\eta])^{\gamma+j}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\gamma}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\gamma}{m}\frac{\gamma^i(-\log\alpha)^i}{i!}\left[\frac{\Gamma(3j-k-l+3m+1)}{\eta(i+\gamma)^{(3j-k-l+3m+1)}}\right]\right] \quad (44)$$

Which completes the proof.

3.3 Reliability Indices

Given any probability distribution, the reliability analysis is always considered based on the Existence Measurement Function and Risk Measurement Function. Hence, for the Inverse Copoun distribution, the Existence Measurement Function and Risk Measurement Function is given below.

3.3.1 Existence Measurement Function

The existence measurement function (also known as survival function) is defined as the probability that an item does not fail prior to some time t (Elechi et al., 2022; Epstein, 1958; Ronald et al., 2011; Shanker & Shukla, 2020).

The existence measurement function of the Alpha Power Transformed Copoun distribution is given by

$$S_{APTC}(x) = 1 - F_{APTC}(x;\eta,\phi,\alpha) \tag{45}$$

$$S_{APTC}(x) = 1 - \left[\frac{\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} - 1}{\alpha - 1}\right] \tag{46}$$

$$S_{APTC}(x) = \frac{\alpha - \alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}}{\alpha - 1}; x > 0, \eta > 0, \phi > 0, \alpha > 0, \alpha \neq 1 \tag{47}$$

3.3.2 Risk Measurement Function

The risk measurement function (also known as hazard rate function) on the other hand can be seen as the conditional probability of failure, given it has survived to the time t (Elechi et al., 2022; Ronald et al., 2011; Shanker, 2016; Umeh & Ibenegbu, 2019). It is obtained as

The risk measurement function of the Alpha Power Transformed Copoun distribution is given by

$$h_{APTC}(x) = \frac{f_{APTC}(x_k;\Phi)}{1 - F_{APTC}(x_k;\Phi)} \tag{48}$$

$$h_{APTC}(x) = \frac{\frac{\eta^2 \log\alpha}{(\alpha-1)(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}\cdot\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}}{1-\left[\frac{\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} - 1}{\alpha-1}\right]} \tag{49}$$

$$h_{APTC}(x) = \frac{\frac{\eta^2 \log\alpha}{(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}\cdot\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}}{\alpha - \alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}}}; x > 0, \eta > 0, \phi > 0, \alpha > 0, \alpha \neq 1 \tag{50}$$

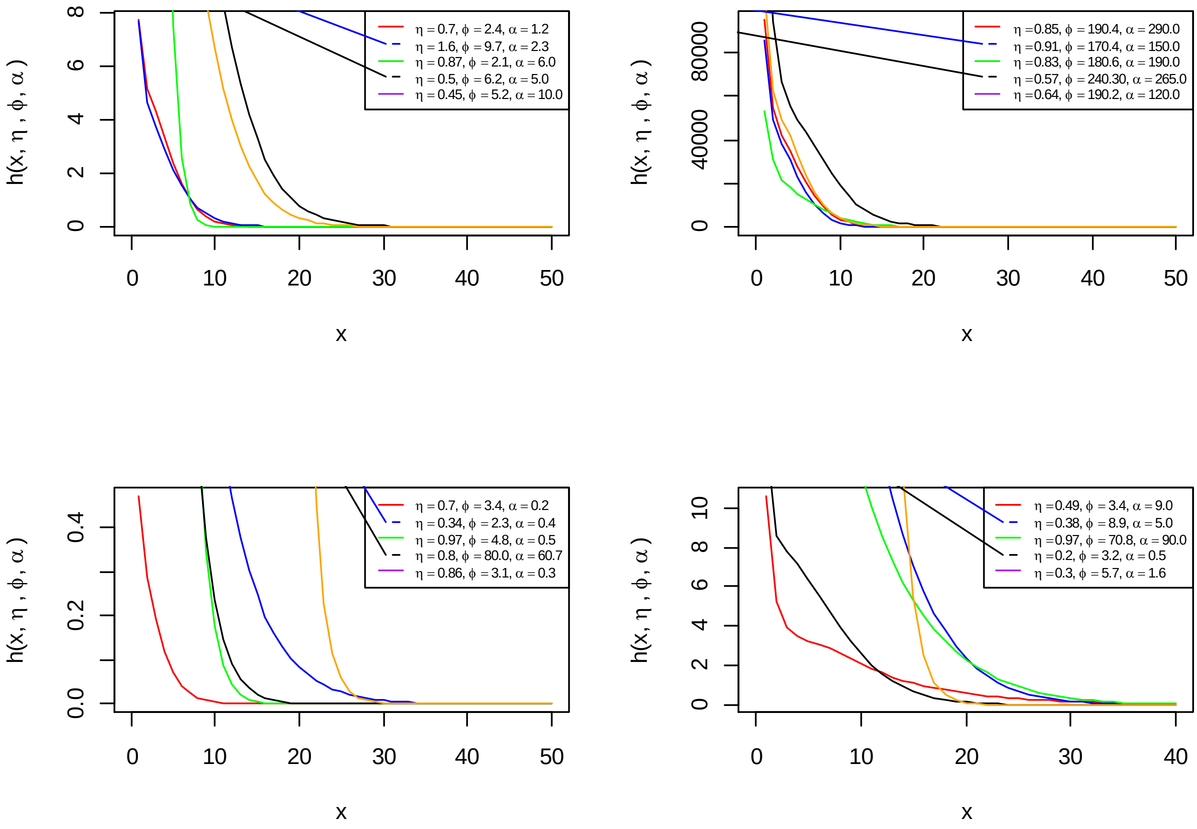

Figure 3: The graphical plots of the risk measurement function (for some selected but different real points of $\eta, \phi, and\,\alpha$) of an Alpha Power Transformed Coupon distribution

II. Conclusion

This paper proposed a new three-parameter distribution known as the Alpha Power Transformed Copoun (APTC) distribution. The new distribution is a generalization of the Copoun distribution using the Alpha Power Transformed technique. The mathematical and statistical properties of the Alpha Power Transformed Copoun distribution such as the moments, moment generating functions, order statistics, entropy, and reliability indices was derived and presented. The properties of the new Alpha Power Transformed Copoun distribution showed that the Alpha Power Transformed Copoun distribution can be used to model lifetime datasets with unimodal, positively skewed, and right tailed properties. Furthermore, the risk measurement function of the Alpha Power Transformed Copoun distribution can model datasets with decreasing failure rate (DFR) and L-Shaped hazard rate in survival analysis.

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