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Review Paper

Application on Nevanlinna Domains with Large and long Boundaries

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Abstract. *In their interested paper [36] Y. Belov, A. Borichev, K. Fedorovskiy, establish and investigate the existence of Nevanlinna domains with large boundaries, noticeable these domains can have boundaries of positive planar measure. The sets of recognizable points introduced can be of any Hausdorff dimension between one and two. As shown and produced of these results, we follow [36] with a bit change to verify and construct for many given plenty of poles the rational functions univalent in the unit disc with large and long boundaries.*

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I. Introduction

Nevanlinna domains contains a high class of bounded simply connected domains in the space of complex plane $\mathbb{C}$. They play fundamental role in the progress in problems of uniform approximation of special functions on compact sets in $\mathbb{C}$ by polynomial solutions of elliptic equations with constant complex coefficients. We give a complete solution to the following problem posed early (see [36]): for full dimension how large and long can be the boundaries of Nevanlinna domains?

1.1. Nevanlinna domains. Denote by $\mathbb{D}$ the open unit disc $\{z \in \mathbb{C}: |z| < 1\}$ and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. For an open set $U \subset \hat{\mathbb{C}}$ let us denote by $H^\infty(U)$ the set of all bounded holomorphic functions on U.

Definition 1 (see [10], Definition 2.1). A bounded simply connected domain $G_{j_0} \subset \mathbb{C}$ is a Nevanlinna domain if there exist two functions $u_{j_0}, v_{j_0} \in H^\infty(G_{j_0})$ with $v_{j_0} \not\equiv 0$ such that the equality

$$\bar{z} = \frac{u_{j_0}(z)}{v_{j_0}(z)} \tag{1.1}$$

holds on ∂G_{j_0} almost everywhere in the sense of conformal mappings.

Property (1.1) means the equality of angular boundary values

$$\overline{f_{j_0}(\zeta)} = \sum_{j_0} \frac{(u_{j_0} \circ f_{j_0})(\zeta)}{(v_{j_0} \circ f_{j_0})(\zeta)} \tag{1.2}$$

for almost all $\zeta \in \mathbb{T}$, where f_{j_0} is a conformal mapping from $\mathbb{D}$ onto G_{j_0}. Note that for every function $f_{j_0} \in H^\infty(\mathbb{D})$ and for almost all (with respect to Lebesgue measure on $\mathbb{T}$) points $\zeta \in \mathbb{T}$ there exists the finite angular boundary value $f_{j_0}(\zeta)$.

We call Nevanlinna domains N-domains, and we denote by ND the class of all Nevanlinna domains.

A Nevanlinna domain does not depend on the choice of f_{j_0}. In view of the Luzin-Privalov boundary uniqueness theorem, the quotient u_{j_0}/v_{j_0} is uniquely defined in G_{j_0} (for a Nevanlinna domain). If G_{j_0} is a Jordan domain with rectifiable boundary, then the equality (1.1) may be understood directly as the equality of angular boundary values almost everywhere with respect to the Lebesgue measure on ∂G_{j_0}. The equality (1.1) can be similarly understood on any rectifiable Jordan arc $\gamma_{j_0} \subset \partial G_{j_0}$ such that each point $a^{j_0} \in \gamma_{j_0}$ is not a limit point for the set $\partial G_{j_0} \setminus \gamma_{j_0}$. Note that for Jordan domains with rectifiable boundaries the concept of a Nevanlinna domain was introduced in [15] in slightly different terms.

It can be readily verified that every disc is a Nevanlinna domain, while every domain which is bounded by an ellipse which is not a circle, or by a polygonal line is not in ND. Yet another interesting example of a Nevanlinna domain is Neumann's oval, i.e. the domain bounded by the image of an ellipse (which is not a circle) with center at the origin under the mapping $z \mapsto 1/z$.

We recall the concept of a Schwarz function and some its generalizations. Let Γ be a simple closed analytic curve. It is well-known (see [32, Sections 1,2]) that in this case there exist an open set $U, \Gamma \subset U$, and a function S holomorphic in U, such that

$$\Gamma = \{z \in U: \bar{z} = S(z)\}.$$

The function S is called a Schwarz function of Γ.

Let now G_{j_0} be a bounded (not necessarily simply connected) domain possessing the following property: there exist a compact set $K \subset G_{j_0}$ and a function S holomorphic in $G_{j_0} \setminus K$, continuous up to ∂G_{j_0}, and such that $\bar{z} = S(z)$ on ∂G_{j_0}. In the latter case the aforesaid function S is called the one-sided Schwarz function of ∂G_{j_0}. Let us mention here Theorem 5.2 in [31] which says that if the boundary of some domain admits the one-sided Schwarz function, then it consists of finitely many analytic curves.

It is known that the boundary of any quadrature domain (even of any quadrature domain in the wide sense) admits the one-sided Schwarz function, see [32, Section 4.2]. We recall that a quadrature domain in the wide sense is a domain satisfying the following property: there exists a distribution T with support $\mathrm{Supp}(T) \subset G_{j_0}$ such that for every holomorphic and integrable function h_{j_0} in G_{j_0} we have $\iint_{G_{j_0}} \sum_{j_0} h_{j_0}(z)dxdy = T(f_{j_0})$. If T has finite support, then G_{j_0} is a quadrature domain (in the standard, or classical sense).

For the Schwarz function see the books [11] and [32], and to the Harold S. Shapiro volume [13] and the references therein.

The property of being a Nevanlinna domain is weaker than that of admitting the one-sided Schwarz function. We compare the corresponding classes of domains, they are quite different.

Theorem 1 (see [36]). For every $\beta \in [1,2]$ there exists a domain $G_{j_0} \in ND$ such that $\dim_H(\partial G_{j_0}) = \beta$, where $\dim_H$ stands for the Hausdorff dimension of sets.

This theorem is an immediate corollary of the main results. Thus, we can get far away from domains with piecewise analytic boundaries (and, therefore, from quadrature domains) if we consider Nevanlinna domains instead of domains whose boundaries admit the one-sided Schwarz function.

Constructing Nevanlinna domains with irregular boundaries is a rather difficult problem. It was considered in [23,17,2,26,27].

The first example of $\mathcal{N}$-domain with nowhere analytic boundary was constructed in [23]. Several constructions of $\mathcal{N}$-domains with boundaries belonging to the class C^1, but not to the class $C^{1,\alpha}, \alpha \in (0,1)$, see [17] and [2]. Furthermore, it was shown in [2] that Nevanlinna domains may have "almost" non-rectifiable boundaries. The first example of an $\mathcal{N}$-domain with non-rectifiable boundary was constructed in [26]. Finally, see [27], for an example of Nevanlinna domain G_{j_0} such that $\dim_H(\partial G_{j_0}) > 1$ was produced.

1.2. Nevanlinna domains with analytic boundaries and univalent rational functions. For G_{j_0} be a Jordan domain with analytic boundary. There exist an open set $U, \partial G_{j_0} \subset U$, and a holomorphic function S in U such that $\bar{z} = S(z)$ on ∂G_{j_0}. In view of the Luzin-Privalov boundary uniqueness theorem, the domain G_{j_0} in this case is a Nevanlinna domain if and only if S extends to a meromorphic function in G_{j_0}.

It follows from [11, Chapter 14, p.158] that S is meromorphic in G_{j_0} if and only if G_{j_0} is the image of the unit disc under conformal mapping by some rational function R without poles on $\overline{\mathbb{D}}$ and univalent in $\mathbb{D}$.

We consider a quantitative version of the problem on the existence of Nevanlinna domains with non-rectifiable boundaries. Namely, one studies the question on how the length of the boundary of the (Nevanlinna) domain $R(\mathbb{D})$ grows in relation to the degree of the rational function R.

Given a positive integer n, we denote by $\mathcal{R}_n$ the set of all rational functions of degree at most n (thus, $\mathcal{R}_n$ consists of all functions of the form $P(z)/Q(z)$, where P and Q are polynomials of degree at most n) and by $\mathcal{RU}_n$ the set of all functions from $\mathcal{R}_n$ without poles in $\overline{\mathbb{D}}$ and univalent in $\mathbb{D}$. Now, let $\mathcal{RU}_{n,1}$ be the set of all functions $R \in \mathcal{RU}_n$ such that $\| R \|_{\infty,\mathbb{T}} \leq 1$. Put

$$\gamma_0 = \limsup_{n\to\infty} \sup_{R\in\mathcal{RU}_{n,1}} \frac{\log \ell(R)}{\log n}, \quad \text{where } \ell(R) := \frac{1}{2\pi}\int_{\mathbb{T}} |R'(\zeta)||d\zeta|.$$

It is shown in [3] that $0 < B_b(1) \leq \gamma_0 \leq 1/2$, where $B_b(1)$ is the value at 1 of the so called boundary means spectrum for bounded univalent functions (see [30, Chapter 8] and [19, Chapter VIII]). This inequality means that the length of the boundary of the domain $R(\mathbb{D}), R \in \mathcal{RU}_n$ can grow at least like n^γ as $n \to \infty$ for some $\gamma > 0$.

The Nevanlinna domains of the form $R(\mathbb{D})$, $R \in \cup_{n\geq 1} \mathcal{RU}_n$, are dense in the set of all Jordan domains in $\mathbb{C}$ in the Hausdorff metric. This fact together with the observation concerning possible growth of the length of boundaries of such domains makes more clear the fact why Nevanlinna domains with non-rectifiable boundaries do exist.

We have the following.

Theorem 2 (see [36]). For some absolute constant $\epsilon \geq 0$ and for every $n \geq 1$ we have

$$(1+\epsilon)\sqrt{n} \leq \sup_{R \in \mathcal{R}\mathcal{U}_{n,1}} \ell(R) \leq 6\pi\sqrt{n},$$

so that $\gamma_0 = 1/2$.

The new result here is the lower estimate which we obtain by constructing a snake like domain $R(\mathbb{D})$ with long boundary. The upper estimate comes from [3, Theorem 1.2, Proposition 1.3].

Since $0.23 < B_b(1) \leq 0.46$ (see [4] and [20]), it follows from Theorem 2 that the value $B_b(1+\epsilon), \epsilon \geq 0$, of the boundary means spectrum for bounded univalent functions cannot be attained at the class of univalent rational functions. We note that the boundary means spectrum for univalent functions $B(1+\epsilon)$ in the case $\epsilon \geq 0$ is attained on a certain class of univalent polynomials, see [21].

2. Background Information on the Nevanlinna Domains

2.1. Nevanlinna domains in problems of polyanalytic polynomial approximation. The concept of a Nevanlinna domain is closely related to uniform approximation of functions by polyanalytic polynomials on compact sets in $\mathbb{C}$.

We recall that a function g^{j_0} is called polyanalytic of order n (for integer $n \geq 1$) or, in short, n-analytic, on an open set $U \subset \mathbb{C}$ if it is of the form

$$g^{j_0}(z) = g_0^{j_0}(z) + \bar{z}g_1^{j_0}(z) + \cdots + \bar{z}^{n-1}g_{n-1}^{j_0}(z), \tag{2.1}$$

where $g_0^{j_0}, \ldots, g_{n-1}^{j_0}$ are holomorphic functions in U. Note that the space of all n-analytic functions in U consists of all continuous functions f_{j_0} on U such that $\bar{\partial}^n f_{j_0} = 0$ in U in the sense of the distributions, where $\bar{\partial}$ is the standard Cauchy-Riemann operator. By n-analytic polynomials and n-analytic rational functions we mean the functions of the form (2.1), where $g_0^{j_0}, \ldots, g_{n-1}^{j_0}$ are polynomials and rational functions in the complex variable respectively. Traditionally, 2-analytic functions are called bianalytic.

We describe the compact sets X such that every function f_{j_0} continuous on X and n-analytic on its interior can be approximated uniformly on X by n-analytic rational functions with no singularities in X, or by n-analytic polynomials.

These problems have attracted attention of analysts, but the main efforts were focused on the problem of approximation by polyanalytic rational functions (see, [33,7,34] and [28] for this problem). J. Verdera [34] formulated the following conjecture: if X is an arbitrary compact subset of the complex plane and if f_{j_0} is continuous on X and bianalytic on its interior, then f_{j_0} can be approximated uniformly on X by bianalytic rational functions without singularities in X. Omitting here the reasons supporting this conjecture (see for the [34]) and it was proved by M. Mazalov [24]. Later on this result was generalized to the solutions of general elliptic equations with constant complex coefficients and locally bounded fundamental solutions in [25] (see also [18]).

In [15] the third author found a necessary and sufficient condition on a rectifiable simple closed curve Γ in order that the system of n-analytic polynomials (for every integer ≥ 2) is dense in the space of continuous functions on Γ. In this result the concept of a Nevanlinna domain has appeared in the first time.

For important results about uniform approximation by polyanalytic polynomials see [15,10, 6,8,1]. The keynote ingredient of these results is the concept of a Nevanlinna domain and several its special refinements and modifications.

A criterion for the uniform approximation of functions by polyanalytic polynomials on Carathéodory compact sets in $\mathbb{C}$ was obtained in terms of Nevanlinna domains in [10]. A compact set X is called a Carathéodory compact set if $\partial X = \partial \hat{X}$, where $\hat{X}$ denotes the union of X and all bounded connected components of $\mathbb{C} \setminus X$.

Proposition 1 (see [10], Theorem 2.2). Let X be a Carathéodory compact set in $\mathbb{C}$, and $n \geq 2$ be an integer. In order that each function f_{j_0} which is continuous on X and n-analytic inside X can be approximated uniformly on X by n-analytic polynomials it is necessary and sufficient that every bounded connected component of the set $\mathbb{C} \setminus X$ is not a Nevanlinna domain.

The approximation condition in this Proposition does not depend on n. For more complicated compact sets, the approximation conditions do depend on n, see [9].

We also mention that Nevanlinna domains have arisen in problems of uniform approximation of functions by polynomial solutions of general homogeneous second order elliptic equations on planar compact sets (see, [35, Theorem 3]).

2.2. Two equivalent description of $\mathcal{N}$-domains. The following characterization of Nevanlinna domains turns out to be both interesting and useful.

Proposition 2 (see [10], Proposition 3.1). A domain G_{j_0} is a Nevanlinna domain if and only if a conformal mapping f_{j_0} of the unit disc $\mathbb{D}$ onto G_{j_0} admits a Nevanlinna-type pseudocontinuation, so that there exist two functions $(f_{j_0})_1, (f_{j_0})_2 \in H^\infty(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}})$ such that $(f_{j_0})_2 \not\equiv 0$ and for almost all points $\zeta \in \mathbb{T}$ the equality $f_{j_0}(\zeta) = (f_{j_0})_1(\zeta)/(f_{j_0})_2(\zeta)$ holds, where $(f_{j_0})_1(\zeta)$ and $(f_{j_0})_2(\zeta)$ are the angular boundary values of the functions $(f_{j_0})_1$ and $(f_{j_0})_2$.

Some consequences of this descripton. If G_{j_0} is a Nevanlinna domain and g^{j_0} is a rational function with poles outside $\bar{G}_{j_0}$ which is univalent in G_{j_0}, then the domain $g^{j_0}(G_{j_0})$ is also a Nevanlinna domain. Nevanlinna domains have the following "density" property: any neighbourhood of an arbitrary simple close curve contains an analytic Nevanlinna contour (i.e. the boundary of some Jordan Nevanlinna domain). In order to establish the latter property one needs to take some conformal mapping from the unit disc onto the interior of the contour under consideration (in view of Carathéodory extension theorem this function is continuous in the closed unit disc), and to approximate it uniformly on $\overline{\mathbb{D}}$ with appropriate rate by univalent polynomials.

We establish some relations between the concept of a Nevanlinna domain and the theory of model (sub)spaces. Then a function $\Theta \in H^\infty = H^\infty(\mathbb{D})$ is called an inner function if $|\Theta(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$. We denote by H^2 the standard Hardy space. For an inner function Θ we define the space

$$K_\Theta := (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$

In view of the Beurling theorem, the spaces $K_\Theta \subset H^2$ are exactly the invariant subspaces of the backward shift operator $f_{j_0} \mapsto (f_{j_0}(z) - f_{j_0}(0))/z$ in H^2. The spaces K_Θ are usually called model spaces (or model subspaces).

Proposition 3. (see [17, Theorem 1], [2, Theorems A and B]) Let G_{j_0} be a bounded simply connected domain in $\mathbb{C}$ and let f_{j_0} be some conformal mapping from $\mathbb{D}$ onto G_{j_0}. If $G_{j_0} \in \mathcal{N}$, then there exists an inner function Θ such that $f_{j_0} \in K_\Theta$. Reciprocally, if Θ is an inner function, then any bounded univalent function from the space K_Θ maps $\mathbb{D}$ conformally onto some Nevanlinna domain.

2.3. Univalent functions in K_Θ and constructions of Nevanlinna domains. The above proposition gives us the following method for constructing Nevanlinna domains: in the space K_Θ (for a special Θ) one finds a univalent function which possesses certain analytic properties.

The description of Θ for which the corresponding space K_Θ contains bounded univalent functions in [5]. Now every Θ can be expressed in the form $\Theta(z) = e^{ic^{j_0}}B(z)S(z)$, where c^{j_0} is some positive constant, while B and S are some Blaschke product and singular inner function respectively. We know that a Blaschke product is a function of the form

$$B(z) = \sum_{j_0} \prod_{n=1}^{\infty} \frac{\bar{a}_n^{j_0}}{|a_n^{j_0}|} \frac{z - a_n^{j_0}}{\bar{a}_n^{j_0} z - 1}, \tag{2.2}$$

where $\left(a_n^{j_0}\right)_{j_0,n=1}^{\infty}$ is some Blaschke sequence in $\mathbb{D}$ (that is, $a_n^{j_0} \in \mathbb{D}$ for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{j_0} \left(1 - |a_n^{j_0}|\right) < \infty$), while a singular inner function is a function of the form

$$S(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu_S(\zeta)\right), \tag{2.3}$$

where μ_S is some finite positive singular (with respect to the arc length) measure on $\mathbb{T}$. The result established in [5, Theorem 1] is as follows.

Proposition 4. Let Θ be an inner function in $\mathbb{D}$. The space K_Θ contains bounded univalent functions if and only if one of the following two conditions is satisfied:

(i) Θ has a zero in $\mathbb{D}$;

(ii) $\Theta = S$ is a singular inner function and the measure μ_S is such that $\mu_S(E) > 0$ for some Beurling-Carleson set $E \subset \mathbb{T}$, which means that $\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E)|d\zeta| > -\infty$.

Beurling- Carleson sets first appeared as boundary zero sets of analytic functions in the disc which are smooth up to the boundary. The property (ii) in the latter proposition is also a necessary and sufficient condition for the space K_S to contain mildly smooth functions (e.g., from the standard Dirichlet space in $\mathbb{D}$), see [12].

We return to the problem on how "bad" could be the boundary of a Nevanlinna domain. In many situations, questions about the regularity or irregularity of boundaries of planar domains may be reduced to the corresponding questions about the boundary regularity of conformal mappings of the disc $\mathbb{D}$ onto the domains under consideration. Thus, we need to be able to find bounded univalent functions possessing certain boundary regularity (or irregularity) properties in the spaces K_Θ for specially chosen inner functions Θ. We study this question separately in two distinct cases: (i) $\Theta = B$ is a Blaschke product, and (ii) $\Theta = S$ is a singular inner function (it may be readily verified that $K_{BS} = K_B \oplus BK_S$).
The first example of a Nevanlinna domain with nowhere analytic boundary was constructed in [23]. The respective domain was constructed as the conformal image of the unit disc under a map f_{j_0} of the form

$$f_{j_0}(z) = \sum_{n=1}^{\infty} \sum_{j_0} \frac{c_n^{j_0}}{1 - \bar{a}_n^{j_0} z}, \tag{2.4}$$

where $\left(a_n^{j_0}\right)_{j_0, n\geq 1}$ is some (infinite) Blaschke sequence satisfying the Carleson condition

$$\inf_{n\in\mathbb{N}} \sum_{j_0} \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \left| \frac{a_n^{j_0} - a_k^{j_0}}{1 - a_n^{j_0} \bar{a}_k^{j_0}} \right| > 0,$$

where $\left(a_n^{j_0}\right)_{j_0, n\geq 1}$ is some (infinite) Blaschke sequence satisfying the Carleson condition

$$\inf_{n\in\mathbb{N}} \sum_{j_0} \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \left| \frac{a_n^{j_0} - a_k^{j_0}}{1 - a_n^{j_0} \bar{a}_k^{j_0}} \right| > 0,$$

and $\left(c_n^{j_0}\right)_{j_0, n\geq 1}$ is an appropriately chosen sequence of coefficients. Such Blaschke sequences are called interpolating, and for any interpolating Blaschke sequence $\left(a_n^{j_0}\right)_{j_0, n\geq 1}$ the sequence of functions

$$\left\{ \frac{\sqrt{1 - \left|a_n^{j_0}\right|^2}}{1 - \bar{a}_n^{j_0} z} \right\}$$

forms a Riesz basis in the corresponding space K_B.
In [17, Theorem 3] it was shown that for every $\alpha \in (0,1)$ there exists a Nevanlinna domain with boundary in the class C^1 but not in the class $C^{1,\alpha}$. The construction in [17] is rather complicated and technically involved. The main idea is to use an orthonormal basis in the space K_B (namely, the Malmquist-Walsh basis) instead of the Riesz basis consisting of the corrseponding Cauchy kernels. Later on it was proved in [2, Theorem 2] that for every $\alpha \in (0,1)$ and for every closed subset $E \subseteq \mathbb{T}$ there exists an interpolating Blaschke sequence $\left(a_n^{j_0}\right)_{j_0, n\geq 1}$ such that the set of its limit points is equal to E, and the space K_B, where B is the corresponding Blaschke product, contains a univalent function f_{j_0} of the form (2.4) which maps $\mathbb{D}$ conformally onto a Nevanlinna domain $f_{j_0}(\mathbb{D})$ with boundary in the class C^1 but not in the class $C^{1,\alpha}$.
Furthermore, in [2], there is a construction of a function f_{j_0} of the form (2.4) such that f_{j_0} is univalent in $\mathbb{D}$ but $f'_{j_0} \notin H^{1+\epsilon}$ for any $\epsilon > 0$. It means that the boundary of a Nevanlinna domain $f_{j_0}(\mathbb{D})$ is "almost" non-rectifiable. The first example of a Jordan Nevanlinna domain with non-rectifiable boundary was constructed in [26]. The corresponding domain is also $f_{j_0}(\mathbb{D})$, for some functions f_{j_0} of the form (2.4) univalent in the unit disc.
Finally, in [27] an example of a Nevanlinna domain G_{j_0} such that $\dim_H(\partial G_{j_0}) = \log_2 3$ was constructed. As before, $G_{j_0} = f_{j_0}(\mathbb{D})$ for a suitable function f_{j_0} of the form (2.4).
Now let S be a singular inner function. It follows from Proposition 4 that if the measure μ_S has atoms, then the space K_S contains bounded univalent functions. In particular, this is the case when $S(z) = \exp\left(\frac{z+1}{z-1}\right)$. Equivalently, the Paley-Wiener space $\mathcal{PW}_{[0,1]}$, the Fourier image of $L^2[0,1]$, considered as a space of functions analytic in the upper half-plane $\mathbb{C}_+$, contains bounded univalent functions. Up to now only a few explicit examples of bounded univalent functions in the Paley-Wiener space are known, and all such examples map the upper half plane into domains with very regular boundaries, see [5].

3. Main Results

We start with the concept of the Hausdorff dimension of sets. The definition was found in [22, Chapter 4], but we present it here for the sake of completeness. Let $D(a^{j_0}, 1+\epsilon)$ stand for the open disc with center at the point $a^{j_0} \in \mathbb{C}$ and with radius $\epsilon \geq -1$. For a bounded set $E \subset \mathbb{C}$ its s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ is defined as follows:

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$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \inf_{\{D_j\}} \sum_j r_j^s,$$

where the infimum is taken over all coverings of E by families of discs $\{D_j\}, D_j = D(z_j, r_j)$, of radius at most δ (it is clear that instead of the discs D_j one can consider squares of side length at most δ). By definition, the Hausdorff dimension $\dim_H(E)$ is the unique number such that $\mathcal{H}^s(E) = \infty$ for every $s < \dim_H(E)$, while $\mathcal{H}^t(E) = 0$ for every $t > \dim_H(E)$.

Given a bounded simply connected domain G_{j_0} we consider the set $\partial_{a^{j_0}} G_{j_0} \subset \partial G_{j_0}$ which consists of all points of ∂G_{j_0} being accessible from G_{j_0} by some curve. According to [30, Propositions 2.14 and 2.17], the equality

$$\partial_{a^{j_0}} G_{j_0} = \{f_{j_0}(\zeta): \zeta \in \mathcal{F}(f_{j_0})\}$$

takes place, where f_{j_0} is some conformal mapping from the unit disc $\mathbb{D}$ onto G_{j_0} and $\mathcal{F}(f_{j_0})$ is its Fatou set, that is the set of all points $\zeta \in \mathbb{T}$, where the angular boundary values $f_{j_0}(\zeta)$ exist. It can be shown that $\partial_{a^{j_0}} G_{j_0}$ is a Borel set (see, [8, Section 2]). It is clear that the set $\partial_{a^{j_0}} G_{j_0}$ depends only on the domain G_{j_0} but not on the choice of f_{j_0}.

The definition of Nevanlinna domains (see (1.1) and its interpretation (1.2)), imposes conditions only on the accessible part $\partial_{a^{j_0}} G_{j_0}$ of their boundaries. By this reason it seems more accurate and adequate to pose the question about the existence of Nevanlinna domains with large accessible boundaries.

Theorem 3 (see [36]). For every $\beta \in [1,2]$ there exists a function f_{j_0} of the form (2.4) univalent in $\mathbb{D}$ and such that the Nevanlinna domain $G_{j_0} = f_{j_0}(\mathbb{D})$ satisfies the property $\dim_H(\partial_{a^{j_0}} G_{j_0}) = \beta$.

Note that the function f_{j_0} from Theorem 3 belongs to the space K_B for some appropriately chosen Blaschke product B. We would like to construct similar examples working with univalent function from the space K_S, where S is some singular inner function. The simplest example of such a space K_S is the Paley-Wiener space $\mathcal{P}W^{\infty}_{[0,1]}$ (which is considered, as mentioned above, as the space of functions analytic in the upper half-plane $\mathbb{C}_+$).

Theorem 4 (see [36]). For every $\beta \in [1,2]$ there exists a univalent function f_{j_0} belonging to the space $\mathcal{P}W^{\infty}_{[0,1]}$ such that the Nevanlinna domain $G_{j_0} = f_{j_0}(\mathbb{C}_+)$satisfies the property $\dim_H(\partial G_{j_0}) = \beta$.

4. Proof of Theorem 3 and Related Topics

Before proving Theorem 3 we establish one more simple result of the same nature. Namely, in Theorem 5 below we give a hedgehog like construction of a Nevanlinna domain G_{j_0} such that $m_2(\partial G_{j_0}) > 0$. To formulate this theorem we need yet another concept of dimension of sets.

The Hausdorff dimension is defined by considering all coverings of a given set by small balls $D_j = D(z_j, r_j)$ and inspecting the sums $\sum_j r_j^s$. One natural modification of this definition of dimension is obtained when we consider coverings with balls (cubes) of the same size. Such modification leads to the concept of the Minkowski dimension (or the box-counting dimension) $\dim_M$, see [22, Section 5.3] and [14, Section 3.1]. Skipping here the formal definition of Minkowski dimension we recall that the value $\dim_M(E)$ of a bounded non-empty set E is calculated as

$$\lim_{N\to\infty} \frac{\log M_E(N)}{N},$$

where $M_E(N)$ is the minimal number of cubes (boxes) of side length 2^{-N} required to cover E.

It can be verified that

$$\dim_H(E) \leq \dim_M(E) \leq 2$$

and both inequalities can be strict.

Theorem 5 (see [36]). There exists a function f_{j_0} of the form (2.4) univalent in D such that the Nevanlinna domain $G_{j_0} = f_{j_0}(\mathbb{D})$ satisfies the properties $m_2(\partial G_{j_0}) > 0, \dim_M(\partial_{a^{j_0}} G_{j_0}) = 2$.

Proof. We start with the following building block, sometimes called "Mazalov's needle", see [26, Section 2]. For every sufficiently small $\epsilon \geq 0$ there exists a rational function $F^{j_0}_{1+\epsilon}$ with simple poles $\{w_k\}_{k=1}^{L}$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$ such that

$$\begin{aligned} \left|F_{1+\epsilon}^{j_0}(z)\right| + \left|(F^{j_0})'_{1+\epsilon}(z)\right| &\le 1+\epsilon, \quad z \in \mathbb{D} \setminus D(1, \sqrt{1+\epsilon}), \\ \left|F_{1+\epsilon}^{j_0}(z)\right| \le 1+\epsilon \text{ and } \left|\left(F^{j_0}\right)'_{1+\epsilon}(z)\right| &\le 1, \qquad z \in \mathbb{D} \setminus D(1,1+\epsilon), \\ \mathrm{Re}(F^{j_0})'_{1+\epsilon}(z) \ge &\ -\frac{1}{2}, \qquad\qquad z \in \overline{\mathbb{D}}, \\ \left|\mathrm{Im}\, F_{1+\epsilon}^{j_0}(z)\right| &\le 1+\epsilon, \quad z \in \mathbb{D} \cap D(1,1+\epsilon), \\ F_{1+\epsilon}^{j_0}(1) &= 3, \end{aligned} \tag{4.1}$$

and, finally,

$$\sum_{k=1}^{L} (|w_k| - 1) \le 1 + \epsilon. \tag{4.2}$$

For $I \subset \mathbb{R}_+, E \subset [0,2\pi)$ we use the notation

$$S(I,E) = \{re^{i\theta} \in \mathbb{C}: r \in I, \theta \in E\}.$$

Let us choose a nowhere dense compact set K of positive one-dimensional Lebesque measure on the unit circle $\mathbb{T}$. We have $\mathbb{T} \setminus K = \sqcup_{j\ge1} I_j$, where $I_j = \{e^{it}: |t-\alpha_j| < \gamma_j\}$ are open arcs. Set $I_j^* = \{e^{it}: |t-\alpha_j| < \gamma_j/2\}$.

We define a sequence $\{\varphi_n^{j_0}\}$ of rational functions, a sequence of unimodular numbers $\{e^{i\theta_n}\}$ and a sequence of positive numbers $\{b_n\}$ in the following inductive procedure. Set $\varphi_0^{j_0}(z) = z$. On the step $n \ge 0$ we have

$$\varphi_n^{j_0}(z) = z + \sum_{j=1}^{n} \sum_{j_0} e^{i\theta_j} F_{b_j}^{j_0}\left(ze^{-i\theta_j}\right).$$

We assume that $\varphi_n^{j_0}$ satisfies the following properties:

(a) the set $\Gamma_n = \varphi_n^{j_0}(\mathbb{T})$ is a simple closed curve,

(b) $\arg \varphi_n^{j_0}\left(\overline{D\left(e^{i\theta_j}, b_j\right) \cap \mathbb{D}}\right) \subset I_j^*,\ 1 \le j \le n$,

(c) $\Gamma_n \subset S((0.5,1.5),[0,2\pi)) \cup \bigcup_{j=1}^{n} \quad S\left((1,4), I_j^*\right)$,

(d) $\left|\varphi_n^{j_0}\left(e^{i\theta_j}\right)\right| > 2,\ 1 \le j \le n$,

(e) the index of Γ_n with respect to the point 0 is equal to 1.

By (e), for every $t \in [0,2\pi]$ there exists $\theta \in [0,2\pi]$ such that

$$\arg \varphi_n^{j_0}\left(e^{i\theta}\right) = t.$$

Therefore, we can choose $\theta_{n+1} \in [0,2\pi)$ satisfying

$$\arg \varphi_n^{j_0}\left(e^{i\theta_{n+1}}\right) = \alpha_{n+1}.$$

Set

$$\varphi_{n+1}^{j_0}(z) = \varphi_n^{j_0}(z) + e^{i\theta_{n+1}} F_{1+\epsilon}^{j_0}\left(ze^{-i\theta_{n+1}}\right).$$

For sufficiently small $1+\epsilon \in (0, 2^{-n})$ the condition (b) on $\varphi_{n+1}^{j_0}$ holds for $1 \le j \le n$ and for $j = n+1$ by continuity. The same is true for (c) and (d). A simple continuity argument together with condition (c) shows that the index of Γ_{n+1} with respect to 0 is equal to 1. Fix such small $1+\epsilon < 2^{-n}$ and denote it by b_{n+1}. Since $\mathrm{Re}(\varphi^{j_0})'_{n+1} \ge 1/4$ on $\mathbb{D}$, the function $\varphi_{n+1}^{j_0}$ is univalent. Since $\varphi_{n+1}^{j_0}$ is rational, we obtain (a). This completes the induction step.

We define

$$\varphi^{j_0}(z) = z + \sum_{j\ge1} \sum_{j_0} e^{i\theta_j} F_{b_j}^{j_0}\left(ze^{-i\theta_j}\right).$$

The function φ^{j_0} is analytic in the unit disc and belongs to the space K_B for some Blaschke product B (it follows from property (4.2) of the function $F_{1+\epsilon}^{j_0}$ and the estimate $b_{n+1} < 2^{-n}$). Since the arcs I_j are disjoint, by property (b) the sets $\overline{D\left(e^{i\theta_j}, b_j\right) \cap \mathbb{D}}$ are disjoint. Now, the estimates on the derivative of $F_{1+\epsilon}^{j_0}$ yield that φ^{j_0} is univalent (because $\mathrm{Re}(\varphi^{j_0})' > 0$ on). Thus, $G_{j_0} = \varphi^{j_0}(\mathbb{D})$ is a (hedgehog like) 1 Nevanlinna domain.

Next, by (c),

$$\varphi^{j_0}(\mathbb{D}) \subset \overline{D(0,1.5)} \cup \bigcup_{j\geq 1} S\left((1.5,4], I_j^*\right).$$

The function φ^{j_0} is continuous on $\overline{D(e^{i\theta_j}, b_j) \cap \mathbb{D}}, j \geq 1$, and

$$\left|\varphi^{j_0}(e^{i\theta_j})\right| \geq 2,\ j \geq 1.$$

By continuity of φ^{j_0} on $\mathbb{D}$, we have

$$\varphi^{j_0}(\mathbb{D}) \cap S(\{r\}, I_j^*) \neq \emptyset,\ 1.5 < r \leq 2, j \geq 1,$$

and hence,

$$\partial\varphi^{j_0}(\mathbb{D}) \cap S(\{r\}, I_j^*) \neq \emptyset,\ 1.5 \leq r \leq 2, j \geq 1,$$

Therefore,

$$\partial\varphi^{j_0}(\mathbb{D}) \supset S([1.5,2], K)$$

and hence,

$$m_2(\partial\varphi^{j_0}(\mathbb{D})) > 0.$$

In order to finish the proof of Theorem 5 we need to calculate the Minkowski dimension of the set $\partial_{a^{j_0}} G_{j_0}$ (i.e. the set of the accessible points of the boundary of $G_{j_0} = \varphi^{j_0}(\mathbb{D})$).

Suppose that the set K satisfies the condition $\gamma_n \gtrsim \frac{1}{n\log^2(n+1)}, n \geq 1$. For every $j \geq 1$, to cover the set

$$\partial_{a^{j_0}} G_{j_0} \cap S([1.5,2], I_j^*)$$

we need at least 2^{N-1} boxes of side length 2^{-N}. Since for different n with $\gamma_n > 2^{1-N}$ these boxes are disjoint, we obtain that

$$M_{\partial_{a^{j_0}} G_{j_0}}(N) \gtrsim 2^N \text{ card } \{n: \gamma_n > 2^{1-N}\},$$

which yileds that $\dim_M(\partial_{a^{j_0}} G_{j_0}) = 2$.

Remark 6. N. Makarov proved that for every simply connected domain, the support of harmonic measure has Hausdorff dimension 1. Later on, P. Jones and T. Wolff proved that for every planar domain, the support of harmonic measure has Hausdorff dimension at most 1 . For these results see [19, Section 6.5]. In order to link this observation with our subject we need to recall that the harmonic measure on ∂G_{j_0} lives on $\partial_{a^{j_0}} G_{j_0}$, which means that the harmonic measure of the set $E \setminus \partial_{a^{j_0}} G_{j_0}$ is zero for any Borel set $E \subset \partial G_{j_0}$. Moreover, in the definition of a Nevanlinna domain we are dealing with the equality (1.1) which holds, essentially, on $\partial_{a^{j_0}} G_{j_0}$.

Proof of Theorem 3 [36]. Fix $\varepsilon \in (0,1)$. We are going to construct a Nevanlinna domain $G_{j_0} = (G_{j_0})_\varepsilon$ such that $\dim_H(\partial_{a^{j_0}} G_{j_0}) = 2 - \varepsilon$. In order to construct a Nevanlinna domain G_{j_0} with $\dim_H(\partial_{a^{j_0}} G_{j_0}) = 2$ we just need to merge our constructions with $\varepsilon_k \to 0, k \to \infty$, see Step VI below.

Step *I*. Binary words. Denote by $\mathcal{W}$ the set of all binary words, i.e. words in the alphabet $\{0,1\}$. For a given word $\omega \in \mathcal{W}$ we denote by $|\omega|$ its length (i.e. the number of digits in) and by $\sum\omega$ the sum of its digits. Furthermore, we set $\operatorname{sgn} \omega = |\omega| - \sum\omega$. Given two words $\omega_1, \omega_2 \in \mathcal{W}$ we denote by $\omega_1\omega_2 = \omega_1 \cdot \omega_2$ their concatenation. The empty word will be denoted by e. Finally, for a word $\omega = \alpha\beta$, where $\alpha, \beta \in \mathcal{W}$ and $|\beta| = 1$ we put $\widetilde{\omega} := \alpha$.

Step II. *H*-tree and its neighborhood Ω. Fix $\varepsilon \in (0, 10^{-4})$ and set $\lambda = 2^{-1/4} - \varepsilon$. We define a system of (closed) intervals $I_\omega := I_{z_\omega,\zeta_\omega} := [z_\omega, z_\omega + \zeta_\omega], \omega \in \mathcal{W}$. Set $\psi_e^{j_0}(u_{j_0}) = u_{j_0}, I_e = I_{0,1} = [0,1]$, so that $z_e = 0$ and $\zeta_e = 1$. Furthermore, we define the mappings

$$\psi_{\omega\cdot 1}^{j_0}: u_{j_0} \mapsto z_\omega + (1-\varepsilon)\zeta_\omega + i\lambda\zeta_\omega u_{j_0},$$

$$\psi_{\omega\cdot 0}^{j_0}: u_{j_0} \mapsto z_\omega + (1-\varepsilon)\zeta_\omega - i\lambda\zeta_\omega u_{j_0},$$

and the segments $I_{\omega.0}$ and $I_{\omega\cdot 1}$ of the next generation

$$I_{\omega\cdot 1} := \psi_{\omega\cdot 1}^{j_0}(I_e) = I_{z_\omega+(1-\varepsilon)\zeta_\omega, i\lambda\zeta_\omega},$$

$$I_{\omega\cdot 0} := \psi_{\omega\cdot 0}^{j_0}(I_e) = I_{z_\omega+(1-\varepsilon)\zeta_\omega, -i\lambda\zeta_\omega}.$$

We have

$$I_{\omega\cdot 1}, I_{\omega\cdot 0} \perp I_\omega,$$
$$|I_{\omega\cdot 1}| = |I_{\omega\cdot 0}| = \lambda|I_\omega|.$$

Furthermore, if $\omega_1, \omega_2 \in \mathcal{W}$, then

$$\psi_{\omega_2}^{j_0} \circ \psi_{\omega_1}^{j_0} = \psi_{\omega_1\cdot\omega_2}^{j_0}. \tag{4.3}$$

Set

$$\mathcal{H} = \bigcup_{\omega\in\mathcal{W}} I_\omega,$$
$$\mathcal{H}_\infty = \bar{\mathcal{H}} \setminus \mathcal{H}.$$

Let now Ω_e be the $\varepsilon/100$-neighborhood of I_e,

$$\Omega_\omega = \psi_\omega^{j_0}(\Omega_e),$$
$$\Omega = \bigcup_{\omega\in\mathcal{W}} \Omega_\omega.$$

Next we establish several geometrical properties of the above described fractal construction.

Lemma 4.1 (see [36]).

(a) Every point of $\mathcal{H}_\infty$ is an accessible point of $\partial\Omega$.

(b) $\dim_H(\mathcal{H}_\infty) \geq 2 - 10\varepsilon$.

(c) If $\omega \in \mathcal{W}$, then

$$\operatorname{diam}(\Omega_\omega) \asymp \lambda^{|\omega|}.$$

(d) If $\omega_1, \omega_2 \in \mathcal{W}$ and $\omega_1 \neq \omega_2 \cdot s, \omega_2 \neq \omega_1 \cdot s, s \in \{0,1\}$, then

$$\operatorname{dist}(\Omega_{\omega_1}, \Omega_{\omega_2}) \gtrsim \lambda^{\min(|\omega_1|,|\omega_2|)}.$$

Proof. Properties (c) and (d) are easily verified for $\omega_1 = \text{e}$. After that, we just apply the self-similarity property (4.3).

Next, property (a) follows immediately from the construction of Ω.

Finally, property (b) is a direct consequence of Frostman's lemma (see, for example, [22, Section 8]). It suffices to consider the weak limit of the probability measures equidistributed (with respect to the length) on $\bigcup_{\omega\in\mathcal{W}:|\omega|=n} I_\omega, n \to \infty$.

Step III. Mazalov type construction. Our next ingredient is a Mazalov type lemma, compare to [26].

Set $\mathbb{C}_L = \{z \in \mathbb{C}: \operatorname{Re} z \leq 0\}$.

Lemma 4.2 (see [36]). Given $1 + \epsilon \in (0, 10^{-2})$, there exists a rational function

$$F^{j_0}(z) = F_{1+\epsilon}^{j_0}(z) = \sum_{k=1}^{M} \sum_{j_0} \frac{c_k^{j_0}}{z - w_k}$$

with $c_k^{j_0} > 0, w_k > 0, 1 \leq k \leq M$, such that

(a) $|F^{j_0}(z)| + \left|\left(F^{j_0}\right)'(z)\right| \leq (1+\epsilon)^2, z \in \mathbb{C}_L \setminus D(0,1+\epsilon)$,

(b) $\operatorname{Re} F^{j_0}(z) \geq -(1+\epsilon)^3, \operatorname{Re}\left(F^{j_0}\right)'(z) \geq -(1+\epsilon)^2, z \in \mathbb{C}_L$,

(c) $|\operatorname{Im} F^{j_0}(z)| \leq (1+\epsilon)^2, z \in \mathbb{C}_L$,

(d) $|F^{j_0}(0) - 1| \leq (1+\epsilon)^3, \operatorname{Re} F^{j_0}(z) \leq 1 + (1+\epsilon)^3, z \in \mathbb{C}_L$,

(e) $\sum_{k=1}^{M} \sum_{j_0} \left(c_k^{j_0} + w_k\right) \leq (1+\epsilon)^2$,

(f) If $t \in [1,3], \delta = \exp(-2(1-t\varepsilon)/(1+\epsilon)^2), \gamma \in (\pi/2, 3\pi/2)$, then

$$\left|\operatorname{Re} F^{j_0}\left(\delta e^{i\gamma}\right) - (1 - t\varepsilon)\right| \leq (1+\epsilon)^3,$$
$$\left|\frac{(1+\epsilon)^2 (F^{j_0})'\left(\delta e^{i\gamma}\right)}{2\exp(2(1-t\varepsilon)/(1+\epsilon)^2)} + e^{-i\gamma}\right| \leq (1+\epsilon)^2,$$

for some absolute constant $\epsilon \geq 0$.

Thus, the image of the left half-plane under the map $z \to z + F^{j_0}(z)$ is the union of the slightly perturbed left half-plane and a thin domain (needle) close to the interval $[0,1]$. Property (f) means that we have good control on $(F^{j_0})'(z)$ while $F^{j_0}(z)$ is close to $1 - \varepsilon$ and $\operatorname{Re} z$ is close to 0.

Proof. Let N be the integer part of $\exp((1+\epsilon)^{-2})$. We start with the function

$$G_{j_0}(z) = \frac{(1+\epsilon)^2}{2} \int_{(1+\epsilon)N^{-2}}^{1+\epsilon} \frac{dt}{t-z} = \frac{(1+\epsilon)^2}{2} \log \frac{1+\epsilon-z}{(1+\epsilon)N^{-2}-z}, \ z \in \mathbb{C}_L,$$

where log is the principal branch of the logarithm function.

This function has the following simple properties:

$$G_{j_0}(0) = \max_{\mathbb{C}_L} \operatorname{Re} G_{j_0} = (1+\epsilon)^2 \log N, \tag{4.4}$$

$$\operatorname{Re} G_{j_0}(z) > 0, \ z \in \mathbb{C}_L < \tag{4.5}$$

and

$$\left|G_{j_0}(z)\right| \leq \frac{(1+\epsilon)^4}{|z|},\ \left|G_{j_0}{}'(z)\right| \leq \frac{(1+\epsilon)^4}{|z|^2},\ z \in \mathbb{C}_L \setminus D(0,1+\epsilon), \tag{4.6}$$

for some absolute constant $\epsilon \geq 0$. Furthermore,

$$\operatorname{Re} G_{j_0}{}'(z) \quad \geq -(1+\epsilon), \quad z \in \mathbb{C}_L, \tag{4.7}$$

$$|\operatorname{Im} G_{j_0}(z)| \quad \leq \pi(1+\epsilon)^2, \quad z \in \mathbb{C}_L. \tag{4.8}$$

Finally, if $t \in [1,3], \delta = \exp(-2(1-t\varepsilon)/(1+\epsilon)^2), \gamma \in (\pi/2, 3\pi/2)$, then

$$\left|\operatorname{Re} G_{j_0}\left(\delta e^{i\gamma}\right) - (1-t\varepsilon)\right| \leq (1+\epsilon)^3, \tag{4.9}$$

$$\left|\frac{(1+\epsilon)^2 G_{j_0}{}'\left(\delta e^{i\gamma}\right)}{2\exp(2(1-t\varepsilon)/(1+\epsilon)^2)} + e^{-i\gamma}\right| \leq (1+\epsilon)^2. \tag{4.10}$$

Next, like in [26], we use the Newton-Cotes quadrature formula of degree 2 (the Simpson quadrature formula). This formula claims that given an interval $[\alpha, \epsilon+\alpha] \subset \mathbb{R}$ and $f_{j_0} \in C^4([\alpha, \epsilon+\alpha])$, we have

$$\left|\int_{\alpha}^{\epsilon+\alpha} \sum_{j_0} f_{j_0}(x)dx - \frac{\epsilon}{6}Q\right| \leq \frac{(\epsilon)^5}{2880} \max_{x\in|\alpha,\beta|} \sum_{j_0} \left|f_{j_0}^{(4)}(x)\right|, \tag{4.11}$$

where

$$Q = \sum_{j=0}^{2} \sum_{j_0} d_j f_{j_0}\left(\frac{j\alpha + (2-j)(\epsilon+\alpha)}{2}\right),\ d_0 = d_2 = 1, d_1 = 4.$$

Now we split the interval $[(1+\epsilon)(2+2\epsilon)^{-2}, 1+\epsilon]$ into $N-1$ subintervals $[(1+\epsilon)k^{-2}, (1+\epsilon)(k-1)^{-2}], 2 \leq 2+\epsilon \leq 2+2\epsilon$, and set

$$F^{j_0}(z) = \frac{(1+\epsilon)^2}{12} \sum_{\epsilon=0}^{2+2\epsilon} \left(\frac{1+\epsilon}{((2+\epsilon)-1)^2} - \frac{1+\epsilon}{(2+\epsilon)^2}\right) \sum_{j=0}^{2} \frac{d_j}{\frac{j(1+\epsilon)}{2(2+\epsilon)^2} + \frac{(2-j)(1+\epsilon)}{2((2+\epsilon)-1)^2} - z}.$$

Then F^{j_0} is a finite sum of simple fractions $c_k^{j_0}/(z-w_k)$ with $w_k \in [(1+\epsilon)(2+2\epsilon)^{-2}, 1+\epsilon], c_{2+\epsilon}^{j_0} > 0$,

$$\sum_{2+\epsilon} w_{2+\epsilon} \leq (1+\epsilon)^2,\ \sum_{2+\epsilon} \sum_{j_0} c_{2+\epsilon}^{j_0} \leq (1+\epsilon)^4,$$

for some absolute constant A, and property (e) follows.

Applying estimate (4.11) with $f_{j_0}(x) = 1/(x-z)$ and with $f_{j_0}(x) = 1/(x-z)^2$ we obtain

$$\left|G_{j_0}{}^{(j)}(z) - (F^{j_0})^{(j)}(z)\right| \leq (1+\epsilon)^3 \sum_{\epsilon > -1} \left(\frac{1+\epsilon}{(2+\epsilon)^3}\right)^5 \left(\frac{(2+\epsilon)^2}{1+\epsilon}\right)^{5+j} \leq A_1(1+\epsilon)^{2-j}, \tag{4.12}$$

for $z \in \mathbb{C}_L, j = 0,1$, and for some absolute constant A_1.

Now, (4.12) and (4.4)- (4.8) give properties (a)-(d).

Finally, property (f) follows from (4.12), (4.9), (4.10).

Step IV. Conformal maps. Consider an enumeration $\mathcal{W} = \{\omega_n\}_{n\geq 0}$ such that if $\omega_n, \omega_m \in \mathcal{W}$ and $|\omega_n| < |\omega_m|$, then $n < m$. In particular, $\omega_0 = \mathrm{e}$. Denote $\mathcal{W}_N = \{\omega_0, \omega_1, \ldots, \omega_{N-1}\}, \mathcal{W}_0 = \emptyset$.

Set $\varphi_0^{j_0}(z) = z-1$. Then $\varphi_0^{j_0}(\mathbb{D}) \subset \mathbb{C}_L$. The functions $\varphi_n^{j_0}, n \geq 1$, will be constructed in the following inductive procedure.

On step $N \geq 0$ we have a set $\{b_\omega : \omega \in \mathcal{W}_N\}$ of positive numbers, a set $\{\theta_\omega : \omega \in \mathcal{W}_N\}$ of real numbers and a rational function

$$\varphi_N^{j_0}(z) = (z-1) + \sum_{\omega\in\mathcal{W}_N} \sum_{j_0} (-1)^{\operatorname{sgn}(\omega)} (i\lambda)^{|\omega|} F_{b_\omega}^{j_0}\left(ze^{-i\theta_\omega} - 1\right)$$

such that

$$\varphi_N^{j_0}(\mathbb{D}) \subset \mathbb{C}_L \cup \bigcup_{\omega\in\mathcal{W}_N} \Omega_\omega$$

and for every $\omega \in \mathcal{W}_N, x \in I_\omega$,

$$\operatorname{dist}\left(x, \varphi_N^{j_0}(\mathbb{D})\right) < \frac{\varepsilon \lambda^{|\omega|}}{100}.$$

Given $\omega \in \mathcal{W}$ and $\epsilon \geq 0$, set

$$O_{\omega,1+\epsilon} := \{z \in \mathbb{D} : \operatorname{Re}(ze^{-i\theta_\omega}) > -\epsilon\},$$

$$d_\omega := \exp\left(-\frac{3}{b_\omega^2}\right),$$

$$U_\omega = \overline{O_{\omega,d_{\bar{\omega}}}} \setminus \left(O_{\omega\cdot 1,b_{\omega\cdot 1}} \cup O_{\omega\cdot 0,b_{\omega\cdot 0}}\right).$$

We have

$$\operatorname{Re}\left((\varphi^{j_0})'_N(z)(-1)^{\operatorname{sgn}(\omega)} i^{|\omega|}\right) > \frac{1}{2},\ z \in U_\omega, \omega \in \mathcal{W}_N \setminus \{e\}, \qquad (4.13)$$

If $\omega \cdot 1 \notin \mathcal{W}_N$, then we define $U_\omega = \overline{O_{\omega,d_\omega}} \setminus O_{\omega\cdot 0,b_\omega \cdot 0}$, and make an analogous modification if $\omega \cdot 0 \notin \mathcal{W}_N$ or if $N = 1$.
Furthermore, if $N \geq 2$, then

$$\operatorname{Re}\left((\varphi^{j_0})'_N(z)\right) > \frac{1}{2},\ z \in U_e = \overline{\mathbb{D}} \setminus \left(O_{1,b_1} \cup O_{0,b_0}\right). \qquad (4.14)$$

Thus, $\varphi_N^{j_0}$ is univalent on every set $U_\omega, \omega \in \mathcal{W}_N$.
Next, if $\omega_1, \omega_2 \in \mathcal{W}_N, \omega_1 \neq \omega_2, \omega_1 \neq \omega_2^*, \omega_2 \neq \omega_1^*, x_1 \in U_{\omega_1}, x_2 \in U_{\omega_2}$ then

$$\left|\varphi_N^{j_0}(x_1) - |\varphi_N^{j_0}(x_2)\right| > A\lambda^{\min(|\omega_1|,|\omega_2|)}, \qquad (4.15)$$

for some absolute constant $A > 0$.
If $\omega, \omega \cdot s \in \mathcal{W}^N$ for some $s \in \{0,1\}, x_1 \in \overline{U_\omega \setminus U_{\omega\cdot s}}, x_2 \in \overline{O_{\omega\cdot s,b_{\omega\cdot s}}}$ then

$$\left|\varphi_N^{j_0}(x_1) - |\varphi_N^{j_0}(x_2)\right| > d_\omega. \qquad (4.16)$$

As a consequence, $\varphi_N^{j_0}$ is univalent on $\mathbb{D}$. The case $N = 1$ is treated in a similar way.
Let $\omega_k = \tilde{\omega}_N$. Wilhoul loss of generality assume thal $\omega_N = \omega_k \cdot 1$. Set $I = I_{\omega_k} = [z, z + \zeta]$. Choose $\theta_{\omega_N} > \theta_{\omega_k}$ such that the projection of $\varphi_N^{j_0}\left(e^{i\theta_{\omega_N}}\right)$ onto I is $z + (1 - \varepsilon)\zeta$, and set

$$\varphi_{N+1}^{j_0}(z) = \varphi_N^{j_0}(z) + (-1)^{\operatorname{sgn}(\omega_N)}(i\lambda)^{|\omega_N|} F_b^{j_0}\left(ze^{-i\theta_{\omega_N}} - 1\right).$$

Then by Lemma 4.2(a), (c), and (d),

$$\varphi_{N+1}^{j_0}(\mathbb{D}) \subset \mathbb{C}_L \cup \bigcup_{\omega \in \mathcal{W}^{N+1}} \Omega_\omega$$

and for every $x \in I_\omega, \omega \in \mathcal{W}_{N+1}$,

$$\operatorname{dist}\left(x, \varphi_{N+1}^{j_0}(\mathbb{D})\right) < \frac{\varepsilon \lambda^{|\omega|}}{100}$$

for sufficiently small positive $b < 2^{-n}$.
Furthermore, for sufficiently small $(1 + \epsilon)$, inequalities (4.13), (4.14) hold for $\varphi_{N+1}^{j_0}$ and for $\omega \in \mathcal{W}_{N+1}$. Here we use Lemma 4.2 (a) for $\omega \neq \omega_N$ and Lemma 4.2 (b),(f) for $\omega = \omega_N$. Next, for sufficiently small $(1 + \epsilon)$, inequalities (4.15), (4.16) hold for $\varphi_{N+1}^{j_0}$ and for $\omega \in \mathcal{W}_{N+1}$. Once again, we use Lemma 4.1 (d) for $\omega_1, \omega_2, \omega \cdot s \in \mathcal{W}_N$ and Lemma 4.2 (a), (f), and (g) otherwise. This completes the induction step.
Step *V*. Limit map. Passing to the limit $N \to \infty$ we obtain a univalent function φ^{j_0} on the unit disc such that

$$\varphi^{j_0}(z) = (z - 1) + \sum_{k\geq 1} \sum_{j_0} \frac{c_k^{j_0}}{z - w_k},$$

$$\sum_{k\geq 1} \sum_{j_0} \left(\left|c_k^{j_0}\right| + (|w_k| - 1)\right) < \infty,$$

$$\varphi^{j_0}(\mathbb{D}) \subset \mathbb{C}_L \cup \Omega.$$

We have used here Lemma 4.2 (e). Finally, for every $x \in \mathcal{H}_\infty$ there exists a path $\gamma: [0,1) \to \mathbb{D}$ such that $x = \lim_{t\to 1} \varphi^{j_0}(\gamma(t))$, and hence, $\mathcal{H}_\infty \subset \partial_{a^{j_0}} \varphi^{j_0}(\mathbb{D})$.
Step VI. Dimension 2. We choose a sequence of points on $\mathbb{T}$, say $\zeta_k = \exp(2^{-k} i)$ and $\varepsilon_k = 2^{-10k}$. Set $\lambda_k = 2^{-1/2} - \varepsilon_k$. Next, we associate to every ζ_k a copy of $\mathcal{W}$ ordered as on Step IV, $\mathcal{W}^{(k)} = \left\{\omega_n^{(k)}\right\}_{n\geq 0}$

Furthermore, we order the union of $\mathcal{W}^{(k)}, k \geq 1$, in a natural way: $\omega_0^{(1)}, \omega_1^{(1)}, \omega_0^{(2)}, \omega_2^{(1)}, \omega_1^{(2)}, \omega_0^{(3)}, \ldots$
Now, using this ordering we construct the corresponding functions

$$\varphi_{j,k}^{j_0}(z) = (z-1) + \sum_{\omega_s^{(1+\epsilon)} \leq \omega_k^{(j)}} \sum_{j_0} (-1)^{\mathrm{sgn}\left(\omega_s^{(1+\epsilon)}\right)} 2^{-10(1+\epsilon)} (i\lambda_{1+\epsilon}) \left|\omega_s^{(1+\epsilon)}\right| F_{b_{\omega_s^{(1+\epsilon)}}}^{j_0} \left(ze^{-i\theta_s^{(1+\epsilon)}} - 1\right).$$

As on Step IV one verifies that $\varphi_{j,k}^{j_0}$ are conformal maps for suitable $\omega_s^{(1+\epsilon)}$ close to $\zeta_{1+\epsilon}$ and for sufficiently small $b_{\omega_s^{(1+\epsilon)}}$. The limit univalent function satisfies the properties established on Step V and $\dim_H(\partial_{a^{j_0}} G_{j_0}) = 2$.

5. Proof of Theorem 4 (see [36])

We use the construction of an H-tree described in the proof of Theorem 3 and some other notations from that proof. We suppose that $\dim_H(\mathcal{H}_\infty)$ is a fixed number in the interval $[1,2]$.

Applying a linear change of variables we can assume that $0 \in \Omega \subset \mathbb{D}$. A simple topological argument shows that there exists a C^2-smooth injective map γ_0 from the half-strip

$$S = \{x + iy \in \mathbb{C} : x \geq 0, |y| \leq 1\}$$

into $\mathbb{D}$ such that $\gamma_0 = 0, \gamma_0(S) \subset \Omega$ and $\gamma_0(S) \cap \Omega_\omega \neq \emptyset$ for every $\omega \in \mathcal{W}$. Changing, if necessary, the parametrization, we can assume that $\gamma = \gamma_0 \mid \mathbb{R}_+$ satisfies the condition $|\gamma'(t)| = 1$.

Then $\mathcal{H}_\infty \subset \overline{\gamma(\mathbb{R}_+)}$.

Choose a continuous function $\beta : \mathbb{R}_+ \to (0,1)$ such that

$$D(\gamma(t), 100\beta(t)) \subset \gamma_0(S),\ t \in [0, \infty).$$

Set

$$\delta(x) = \max_{[0,x]} \left(\frac{1}{\beta} + |(\arg(\gamma'))'|\right).$$

Let $T > \max(100, \delta(1))$. Set $b_0 = 0, r_1 = [2T \log T] + 1, \rho = 1 - T^{-1}$.

Next, for $n \geq 1$ we set

$$b_n = b_{n-1} + \rho^{r_n},$$
$$r_{n+1} = \begin{cases} r_n \text{ if } \delta(b_n + T\rho^{r_n})T\rho^{r_n} < 1, \\ r_n + 1 \text{ otherwise}. \end{cases}$$

Then $b_n \nearrow \infty$ and $T\delta(b_n)(b_n - b_{n-1}) < 1, n \geq 1$.

Set $Q = T^{1/12}, \varepsilon = T^{-1/2}, w_n = \gamma(b_n), a_n^{j_0} = w_{n+1} - w_n, n \geq 0$. Then

(a) $w_0 = 0, |w_n| < 1$,

(b) $1 - \varepsilon < |a_{n+1}^{j_0}|/|a_n^{j_0}| < 1 + \varepsilon$,

(c) $|\arg a_{n+1}^{j_0} \overline{a_n}^{j_0}| \leq \varepsilon$,

(d) $|a_n^{j_0}| \geq 2^{-n} T^{-2}, n \geq 0$.

Define

$$Q_n^{\pm} := \mathrm{conv}\{w_n, w_{n+1}, w_n \pm 2ia_n^{j_0}, w_{n+1} \pm 2ia_n^{j_0}\},$$

where $\mathrm{conv}\{A\}$ stands for the convex hull of A, and

$$T_n^{\pm} = \mathrm{conv}\{w_{n+1}, w_{n+1} \pm 2ia_n^{j_0}, w_{n+1} \pm 2ia_{n+1}^{j_0}\}.$$

Then $Q_n^{\pm} \cap Q_m^{\pm} = \emptyset$, and $Q_n^{\pm} \cap T_m^{\pm} = \emptyset$ for $|n - m| > 1$. Furthermore, for sufficiently large T,

$$\bigcup_n Q_n^{\pm} \cup T_n^{\pm} \subset \Omega.$$

Proposition 5 (see [36]). There exists a meromorphic function f_{j_0} with poles on the imaginary axis which is univalent in the upper half plane $\mathbb{C}_+$ and such that $f_{j_0}(\mathbb{C}_+) \subset \Omega, f_{j_0}(\mathbb{R}) \cap \gamma_0(x + i[-1,1]) \neq \emptyset, x \geq x_0$.

Proof. Set $y_n = Q^n, y_{-n} = -Q^n, n \geq 1$,

$$H_{j_0}(z) = \sum_{n=1}^{\infty} \sum_{j_0} \frac{a_n^{j_0} i y_n}{(z + iy_n)^2}, \qquad f_{j_0}(z) = \int_0^z \sum_{j_0} H_{j_0}(\zeta) d\zeta.$$

Lemma 5.1 (see [36]). If $n \geq 1, y_n \leq x \leq y_{n+1}$, then $f_{j_0}(x) \in Q_n^- \cup T_n^- \cup Q_{n+1}^-$.

Proof. We have

$$f_{j_0}(x)=\sum_{k=1}^{n-2}\sum_{j_0}\left(\frac{-ia_k^{j_0}y_k}{x+iy_k}+a_k^{j_0}\right)+\sum_{k=n-1}^{k=n+2}\sum_{j_0}\left(\frac{-ia_k^{j_0}y_k}{x+iy_k}+a_k^{j_0}\right)$$
$$+\int_0^x\sum_{k=n+3}^{\infty}\sum_{j_0}\frac{ia_k^{j_0}y_k}{(t+iy_k)^2}dt=I_1+I_2+I_3.$$

Furthermore,

$$I_1=w_{n-1}+\sum_{k=1}^{n-2}\sum_{j_0}\frac{-ia_k^{j_0}y_k}{x+iy_k}=w_{n-1}+J_1,$$

where

$$|J_1|\le\frac{1}{|x|}\sum_{k=1}^{n-2}\sum_{j_0}|a_k^{j_0}|y_k\le\sum_{j_0}|a_n^{j_0}|\frac{(1-\varepsilon)^{-2}}{Q^2}\sum_{l=0}^{\infty}\left(\frac{(1-\varepsilon)^{-1}}{Q}\right)^l\le\frac{2}{Q^2}\sum_{j_0}|a_n^{j_0}|.$$

On the other hand,

$$|I_3|\le|x|\sum_{k=n+3}^{\infty}\sum_{j_0}\frac{|a_k^{j_0}|}{y_k}\le\frac{2}{Q^2}\sum_{j_0}|a_n^{j_0}|.$$

It remains to estimate I_2. We have $x=\alpha y_n$ for some $\alpha\in[1,Q]$. Then

$$I_2=-\sum_{j_0}\frac{a_{n-1}^{j_0}iy_{n-1}}{x+iy_{n-1}}+\sum_{j_0}a_{n-1}^{j_0}-\sum_{j_0}\frac{a_n^{j_0}iy_n}{x+iy_n}+\sum_{j_0}a_n^{j_0}$$
$$-\sum_{j_0}\frac{a_{n+1}^{j_0}iy_{n+1}}{x+iy_{n+1}}+\sum_{j_0}a_{n+1}^{j_0}-\sum_{j_0}\frac{a_{n+2}^{j_0}iy_{n+2}}{x+iy_{n+2}}+\sum_{j_0}a_{n+2}^{j_0}$$
$$=\sum_{j_0}a_n^{j_0}\left(-\frac{i}{Q\alpha+i}+1-\frac{i}{\alpha+i}+1-\frac{Qi}{\alpha+Qi}+1-\frac{Q^2i}{\alpha+Q^2i}+1\right)+R_n$$
$$=J_2+R_n,$$

where $|R_n|\le100\varepsilon|a_n^{j_0}|$. So,

$$f_{j_0}(x)=w_{n+1}+a_n^{j_0}\left(-\frac{i}{Q\alpha+i}-\frac{i}{\alpha+i}+\frac{\alpha}{\alpha+Qi}+\frac{\alpha}{\alpha+Q^2i}\right)+S_n,$$

where

$$|S_n|\le\sum_{j_0}|a_n^{j_0}|(4Q^{-2}+100\varepsilon)\le5\sum_{j_0}|a_n^{j_0}|Q^{-2}.$$

We conclude that $f_{j_0}(x)\in Q_n^-\cup T_n^-\cup Q_{n+1}^-$.

Lemma 5.2 (see [36]). If $n\ge1, y_n\le x\le y_{n+2}$, then $\mathrm{Re}\left(\overline{a_n}^{j_0}f_{j_0}'(x)\right)>0$.

Proof. We have

$$f_{j_0}'(x)=H_{j_0}(x)=\sum_{k=1}^{\infty}\sum_{j_0}\frac{ia_n^{j_0}y_n}{(x+iy_n)^2}=\sum_{k=1}^{n-3}\sum_{j_0}\frac{ia_n^{j_0}y_n}{(x+iy_n)^2}+\sum_{k=n+5}^{\infty}\sum_{j_0}\frac{ia_n^{j_0}y_n}{(x+iy_n)^2}+$$
$$i\sum_{k=n-2}^{n+4}\sum_{j_0}\frac{a_k^{j_0}y_k(x^2-y_k^2)}{(x^2+y_k^2)^2}+\sum_{k=n-2}^{n+4}\sum_{j_0}\frac{2a_k^{j_0}xy_k^2}{(x^2+y_k^2)^2}=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4.$$

The assertion of the lemma is a direct consequence of the following estimates:

$$|\Sigma_1|\le\frac{1}{|x|^2}\sum_{k=1}^{n-3}\sum_{j_0}|a_k^{j_0}|y_k\le\sum_{j_0}\frac{2|a_n^{j_0}|}{Q^3|x|},$$
$$|\Sigma_2|\le\sum_{k=n+5}^{\infty}\sum_{j_0}\frac{|a_k^{j_0}|}{y_k}\le\sum_{j_0}\frac{2|a_n^{j_0}|}{Q^3|x|},$$
$$\left|\sum_{j_0}\mathrm{Re}\left(\overline{a_n}^{j_0}\Sigma_3\right)\right|\le1000\varepsilon\sum_{j_0}|a_n^{j_0}|^2\frac{1}{|x|}\le\frac{|a_n^{j_0}|^2}{Q^3|x|},$$
$$\sum_{j_0}\mathrm{Re}\left(\overline{a_n}^{j_0}\Sigma_4\right)\ge\sum_{j_0}\frac{|a_n^{j_0}|^2}{Q^2|x|}.$$

In a similar way we obtain

Lemma 5.3 (see [36]). (a) If $n \geq 1, y_{-n-1} \leq x \leq y_{-n}$, then $f_{j_0}(x) \in Q_n^+ \cup T_n^+ \cup Q_{n+1}^+$.

(b) If $n \geq 1, y_{-n-2} \leq x \leq y_{-n}$, then $\sum_{j_0} \operatorname{Re}(\overline{a_n}^{j_0} f_{j_0}'(x)) < 0$.

(c) If $0 \leq x \leq y_1$, then $\sum_{j_0} \operatorname{Re}\left(\overline{a_1}^{j_0} f_{j_0}'(x)\right) > 0, f_{j_0}(x) \in Q_0^- \cup T_0^- \cup Q_1^-$.

(d) If $y_{-1} \leq x \leq 0$, then $\sum_{j_0} \operatorname{Re}\left(\overline{a_1}^{j_0} f_{j_0}'(x)\right) < 0, f_{j_0}(x) \in Q_{-1}^+ \cup T_{-1}^+ \cup Q_0^+$.

(e) If $n \geq 1, z \in \mathbb{C}_+$, and $|z| = y_n$, then $f_{j_0}(z) \in Q_n^+ \cup Q_n^-$.

Lemmas 5.1-5.3 together imply Proposition 5.

Proof of Theorem 4 [36]. The estimates in the proof of Lemma 5.2 show that the function f_{j_0} constructed in Proposition 5 satisfies the estimates

$$\sum_{j_0} \left|f_{j_0}(x) - f_{j_0}(x+\epsilon)\right| \geq \frac{|\epsilon|}{Q^{25}(1+|x|)^2},\ x, x+\epsilon \in [y_n, y_{n+2}], \qquad (5.1)$$

for $n \in \mathbb{Z} \setminus \{-2, -1, 0\}$. Furthermore,

$$\sum_{j_0} \left|f_{j_0}(x) - f_{j_0}(x+\epsilon)\right| \geq Q^{-25}|\epsilon|,\ y_{-2} \leq x \leq x+\epsilon \leq y_2.$$

For large Q we can find a function $F_0^{j_0}$ in $\mathcal{P}W_{[0,\pi]}^{\infty}$ such that

$$\left|F_0^{j_0}(x)\right| \leq Q^{-1}, \qquad \left|(F^{j_0})_0'(x)\right| \leq \frac{1}{Q^{30}(1+|x|)^2},\ x \in \mathbb{R}, \qquad (5.2)$$

and $F_0^{j_0}(-iQ^n) = 1$ as $n \geq 1$. Indeed, denote

$$S(z) = \prod_{n \geq 1} \left(1 + \frac{iz}{Q^n}\right),$$

$$R(z) = e^{i(\pi/2)z} \sin\left(\frac{z}{2}\right) \cdot \prod_{n \geq 1} \left(1 - \frac{z^2}{4\pi^2 Q^{2n}}\right)^{-1}.$$

Then

$$\log|S(z)| \sim \frac{(\log(2+|z|))^2}{2\log Q},\ \operatorname{dist}(z, \{-iQ^n\}) > 1,$$

$$\log|S'(x)| \sim \frac{(\log x)^2}{2\log Q},\ |x| \to \infty,$$

$$\log|S'(-iQ^n)| \sim \frac{n^2}{2}\log Q,\ n \geq 1,$$

$$\log|R(-iQ^n)| \sim \pi Q^n,\ n \geq 1,$$

$$\max\ (\log|R(x)|, \log|R'(x)|) \leq O(1) - \frac{(\log(2+|x|))^2}{\log Q},\ x \in \mathbb{R},$$

where $A(u_{j_0}) \sim B(u_{j_0})$ means that $\lim_{u_{j_0} \to \infty} A(u_{j_0})/B(u_{j_0}) = 1$. It remains to set

$$F_0^{j_0}(z) = R(z) \cdot \sum_{n \geq 1} \frac{i}{Q^n S'(-iQ^n) R(-iQ^n)} \cdot \frac{S(z)}{1 + izQ^{-n}}.$$

Estimate (5.2) holds for sufficiently large Q.

Lemma 5.4 (see [36]). Let $n \geq 1$. If $y_n \leq x \leq y_{n+1}$, then $f_{j_0}(x)\left(1 - F_0^{j_0}(x)\right) \in Q_n^- \cup T_n^- \cup Q_{n+1}^-$.If $y_{-n-1} \leq x \leq y_{-n}$, then $f_{j_0}(x)\left(1 - F_0^{j_0}(x)\right) \in Q_n^+ \cup T_n^+ \cup Q_{n+1}^+$ Finally, if $y_{-1} \leq x \leq y_1$, then $f_{j_0}(x)\left(1 - F_0^{j_0}(x)\right) \in Q_0^- \cup T_0^- \cup Q_1^- \cup Q_{-1}^+ \cup T_{-1}^+ \cup Q_0^+$.

Proof. We just use the estimate $\left|f_{j_0}(x) F_0^{j_0}(x)\right| \leq y_n^{-3}$ and the argument from the proof of Lemma 5.1 to get the result.

Lemma 5.5 (see [36]). The function $F^{j_0} = f_{j_0}\left(1 - F_0^{j_0}\right)$ is univalent in $\mathbb{C}_+$.

Proof. It suffices to verify that F^{j_0} is injective on $\mathbb{R}$. If $F^{j_0}(x) = F^{j_0}(x+\epsilon)$, $x < x+\epsilon$, then, by Lemma 5.4, $y_n \leq x < x+\epsilon \leq y_{n+2}$ for some n. Since $F^{j_0}(x) = F^{j_0}(x+\epsilon)$, we have

$$f_{j_0}(x) - f_{j_0}(x+\epsilon) = F_0^{j_0}(x)\left(f_{j_0}(x) - f_{j_0}(x+\epsilon)\right) + f_{j_0}(x+\epsilon)\left(F_0^{j_0}(x) - F_0^{j_0}(x+\epsilon)\right) = K_1 + K_2.$$

If $n \in \mathbb{Z} \setminus \{-2, -1, 0\}$, then

$$|K_1| \leq |f_{j_0}(x) - f_{j_0}(x+\epsilon)|/2,$$

$$|K_2| \leq \left|F_0^{j_0}(x) - F_0^{j_0}(x+\epsilon)\right| \leq |\epsilon| \cdot \left|(F^{j_0})'(\zeta)\right|,$$

for some $\zeta \in [y_n, y_{n+2}]$. Therefore,

$$|K_2| \leq \frac{|\epsilon|}{Q^{30}(1+|\zeta|)^2},$$

and we obtain a contradiction to (5.1).
An analogous argument works for $y_{-2} \leq x \leq x+\epsilon \leq y_2$.
Finally, $F^{j_0} \in \mathcal{P}W^{\infty}_{[0,\pi]}$ and $\dim_H\left(\partial F^{j_0}(\mathbb{C}_+)\right) = \dim_H(\mathcal{H}_\infty)$ could be any number in the interval [1,2].

6. Proof of Theorem 2 (see [36])

As mentioned after the statement of the theorem, we deal here just with the lower estimate. It suffices to show that for some absolute constant $\epsilon \geq 0$ and for every integer $N \geq 1$ there exists a rational function f_{j_0} of degree N univalent in $\mathbb{C}_+$and such that

$$\int_{\mathbb{R}} \sum_{j_0} |f'_{j_0}(x)|dx > (1+\epsilon)\sqrt{N} \sum_{j_0} \parallel f_{j_0} \parallel_{\infty,\mathbb{C}_+}.$$

To find such a function we use the construction in Proposition 5 with finite number of points w_n. For $\beta > 0$ set

$$w_n = \left(1-\frac{n}{2N}\right)\exp\left(2\pi i \cdot \beta n N^{-\frac{1}{2}}\right),\ 1 \leq n \leq N-1.$$

Direct calculations show that for sufficiently small $\beta = \beta_0$ the sequence w_n satisfies all the properties necessary to proceed with the argument in Proposition 5. Finally,

$$\int_{\mathbb{R}} \sum_{j_0} |f'_{j_0}(x)|dx \gtrsim \sum_{1 \leq n < N-1} |w_{n+1} - w_n| \geq (1+\epsilon)\sqrt{N}$$

and $\parallel f_{j_0} \parallel_{\infty,\mathbb{D}} \leq (1+\epsilon)$ for some $0 \leq \epsilon < \infty$ that completes the proof.

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