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Review Paper

Fixed Points using Suzuki-($\mathcal{Z}\varphi(\alpha,\beta)$)-type rational Contractive Conditions

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Abstract: In this attempt, based on a rational contractive condition, we build a common fixed point for two pairs of mappings using Sujuki-($\mathcal{Z}_\varphi(\alpha,\beta)$)-type rational contraction in TVS-valued Cone b-Metric Space. The results in this article is the extension of the results obtained by Dewangan et al. in which fixed point results were obtained using $\mathcal{Z}$- type contraction along with the use of simulation function for (α,β) admissible mappings in Metric-like spaces. Our findings extends, and generalize a variety of well-known conclusions from the existing literature. An illustration has been given to support the findings of this research.

Keywords: Fixed-point, TVS-valued cone b-Metric Space, $\mathcal{Z}$-contraction, Simulation function, Sujuki-($\mathcal{Z}_\varphi(\alpha,\beta)$)-type rational contraction.

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I. INTRODUCTION

The well known BCP (Banach,1922) established the presence and uniqueness of the fixed-point of a contraction on a Complete Metric Space. Following Banach's extraordinary finding, a good number of authors obtained fixed-points and common fixed-points in different spaces like Quasi Metric Space, Fuzzy Metric Space, Pseudo Metric Space, Menger Space, b-Metric Space etc. (Ali et al., 2021, Czerwik, 1993, Hamaizia and Aliouche, 2010).

By adding b-Metric Spaces, Bakhtin (Bakhtin,1989) and Czerwik (Czerwik, 1993) extended on the idea of Metric type Space. Aamri and Moutawakil, however, proposed the idea of (E.A)-property in Metric Spaces

(Aamri and Moutawakil, 2002). T. Sujuki's generalization theorem (Suzuki, 2008) for the BCP was published in 2008. This theory is also known as Sujuki type Contraction later. The α-contraction and α-admissible mappings were introduced in 2012 by Samet et al., (Samet et al., 2012) who also demonstrated a number of fixed-point solutions for this class of mappings.

Chandok (Chandok, 2015) introduced (α, β)-admissible mappings and Khojesteh et al.(Khojesteh et al., 2015) introduced the notion of simulation function and $\mathcal{Z}$-contraction with respect to φ which generalized the Banach Contraction. Following the direction, we use the concept of $\mathcal{Z}$-contraction to introduce Suzuki-$(\mathcal{Z}_{\varphi}(\alpha, \beta))$-type rational contractive mappings and establish fixed-point results for such mappings in TVS-valued CbMS .The results in this article is the extension of the results obtained by Dewangan et al.(Dewangan et al., 2022), in which fixed-point results were obtained using $\mathcal{Z}$-type contraction along with the use of simulation function for (α, β) admissible mappings in Metric-like Spaces. Our findings extends, and generalize a variety of well-known conclusions from the existing literature. An illustration has been given to support the findings of this research.

2. Preliminaries

The given definitions are required to establish the common FPt in this article.

Definition 2.1 (Kumar and Ansari, 2018) Let the TVS-valued CMS be $(\mathscr{X}_{\jmath}, d_{\gamma})$ and s be a real number such that $s \geqslant 1$.Suppose $d_{\gamma} : \mathscr{X}_{\jmath} \times \mathscr{X}_{\jmath} \to \mathscr{E}$ be a vector valued function which satisfies:

(i) $0 \leq d_{\gamma}(\vartheta, \eta) \ \forall \ \vartheta, \eta \in \mathscr{X}_{\jmath} \ and \ d_{\gamma}(\vartheta, \eta) = 0 \iff \vartheta = \eta$,

(ii) $d_{\gamma}(\vartheta, \eta) = d_{\gamma}(\vartheta, \eta) \ \forall \vartheta, \eta \in \mathscr{X}_{\jmath}$,

(iii) $d_{\gamma}(\vartheta, \eta) \leq s[d_{\gamma}(\vartheta, w) + d_{\gamma}(w, \eta)] \ \forall \ \vartheta, \eta, w \in \mathscr{X}_{\jmath}$.

Then the pair $(\mathscr{X}_{\jmath}, d_{\gamma})$ is said to be a TVS-valued CbMS with the TVS-valued cone b-metric as d_{γ}.

Definition 2.2 (Samet et al., 2012) Let $(\mathscr{X}_{\jmath}, d_{\gamma})$ be a b-Metric Space and $\mathcal{F}'$and $\mathcal{G}'$ be two self mapped functions on $\mathscr{X}_{\jmath}$.

(i) If $\mathcal{F}'\vartheta_n$ and $\mathcal{G}'\eta_n$ in $\mathscr{X}_\jmath$ converges to some $\nu \in \mathscr{X}_\jmath$, then $\mathcal{F}'$ and $\mathcal{G}'$ are said to be compatible and

$lim_{n\to\infty} d_\gamma\ (\mathcal{F}'\mathcal{G}'\vartheta_n, \mathcal{F}'\eta_n) = 0.$

(ii) If there exists a sequence in $\mathscr{X}_\jmath$ such that $\mathcal{F}'$ and $\mathcal{G}'$ converge to some $\nu \in \mathscr{X}_\jmath$, but $lim_{n\to\infty} d_\gamma\ (\mathcal{F}'\mathcal{G}'\vartheta_n, \mathcal{F}'\eta_n)$ is either nonzero or it does not exist, then $\mathcal{F}'$ and $\mathcal{G}'$ are said to be noncompatible.

(iii)$\mathcal{F}'$ and $\mathcal{G}'$ satisfies the $b-(E.A)$-property if a sequence in $\mathscr{X}_\jmath$ is such that $lim_{n\to\infty}\mathcal{F}'\vartheta_n = \mathcal{G}'\vartheta_n = \nu$.

Definition 2.3 (Samet et al., 2012) A mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathscr{R}$ is said to be a simulation function if it holds the given conditions:

$(\zeta_1)\ \zeta(0,0) < 0;$

$(\zeta_2)\ \zeta(\vartheta,\eta) < \vartheta - \eta$, for all $\vartheta, \eta > 0$;

(ζ_3)if $\{\vartheta_n\}$and$\{\eta_n\}$ are sequences in $(0,\infty)$ in such a way that $lim_{n\to\infty}\vartheta_n = lim_{n\to\infty}\eta_n = \ell \in (0,\infty)$, then $lim_{n\to\infty} sup\, \zeta\,(\vartheta_n, \eta_n) < 0$.

The following fixed unique result is obtained in (**Samet et al., 2012**).

Definition 2.4 (Khojesteh et al., 2015) A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathscr{R}$ which satisfies the conditions stated:

(i) $\zeta(\vartheta,\eta) < \vartheta - \eta$, for all $\vartheta, \eta > 0$;

(ii) if $\{\vartheta_n\}$ and $\{\eta_n\}$are sequences in $(0,\infty)$ in such a way that $lim_{n\to\infty}\vartheta_n = lim_{n\to\infty}\eta_n = \ell \in (0,\infty)$;

then $lim_{n\to\infty} sup\, \zeta\,(\vartheta_n, \eta_n) < 0$ is a Simulation function.

Any Simulation function in the sense of Argoubi et al.(Argoubi et al., 2015) is obviously equivalent to any simulation function in the sense of Khojesteh et al. (**Khojesteh et al., 2015**). The opposite, however, is not always true.

3. Main Results

Definition 3.1: Let $(\mathscr{X}_\jmath, d_\gamma)$ be a TVS-valued cone b-Metric Space and let $\mathcal{F}', \mathcal{G}', \mathcal{Q}, and\, \mathscr{R} : \mathscr{X}_\jmath \times \mathscr{X}_\jmath \to \mathscr{X}_\jmath$ be mappings with $\mathcal{F}'(\mathscr{X}_\jmath) \subseteq \mathscr{R}(\mathscr{X}_\jmath)$ and $\mathcal{G}'(\mathscr{X}_\jmath) \subseteq \mathcal{Q}(\mathscr{X}_\jmath)$, such that the pairs $(\mathcal{F}', \mathcal{Q})$ and $(\mathcal{G}', \mathscr{R})$ are Sujuki $(\mathcal{Z}_\varphi(\alpha,\beta))$-type rational contraction if for every $\vartheta, \eta \in \mathscr{X}_\jmath$ and $s > 1$,

$\frac{1}{2}min\{d_\gamma(\mathcal{F}'\vartheta,\mathscr{R}\eta),d_\gamma(\mathcal{G}'\eta,\mathcal{Q}\vartheta)\}\leq d_\gamma(\mathcal{F}'\vartheta,\mathcal{G}'\eta)$

which implies

$\zeta(\alpha(\mathcal{F}'\vartheta,\mathcal{G}'\eta),\beta(\mathcal{F}'\vartheta,\mathcal{G}'\eta),s^4d_\gamma(\mathcal{F}'\vartheta,\mathcal{G}'\eta),\mathcal{M}(\vartheta,\eta))\geq 0$

for all $\vartheta,\eta\in\mathscr{X}_\jmath$ and $\zeta\in\mathcal{Z}_\varphi$ ...(1)

where $\mathcal{M}(\vartheta,\eta)=max\{d_\gamma(\mathcal{Q}\vartheta,\mathscr{R}\eta),d_\gamma(\mathcal{F}'\vartheta,\mathcal{Q}\vartheta),d_\gamma(\mathcal{G}'\eta,\mathscr{R}\eta),\frac{d_\gamma(\mathcal{F}'\vartheta,\mathscr{R}\eta)+d_\gamma(\mathcal{G}\eta,\mathcal{Q}\vartheta)}{2s}\}$.

Theorem 3.2: Let $(\mathscr{X}_\jmath,d_\gamma)$ be a TVS-valued CbMS with $s>1$ and let $\mathcal{F}',\mathcal{G}',\mathcal{Q}\,and\,\mathscr{R}:\mathscr{X}_\jmath\times\mathscr{X}_\jmath\to\mathscr{X}_\jmath$ be mappings with $\mathcal{F}'(\mathscr{X}_\jmath)\subseteq\mathscr{R}(\mathscr{X}_\jmath)$ and $\mathcal{G}(\mathscr{X}_\jmath)\subseteq\mathcal{Q}(\mathscr{X}_\jmath)$.

Let $\alpha,\beta:\mathscr{X}_\jmath\times\mathscr{X}_\jmath\to[0,\infty)$ be mappings such that the given conditions hold:

(i) the couples $(\mathcal{F}',\mathcal{Q})$ and $(\mathcal{G}',\mathscr{R})$ are $(\alpha.\beta)-$admissible mappings;

(ii) there exists $\vartheta_0'\in\mathscr{X}$ such that $\alpha(\vartheta_0',\mathcal{F}'\vartheta_0')\geq 1$ and $\beta(\vartheta_0',\mathcal{F}'\vartheta_0')\geq 1$;

(iii) the couples $(\mathcal{F}',\mathcal{Q})$ and $(\mathcal{G}',\mathscr{R})$ are Sujuki $(\mathcal{Z}_\varphi(\alpha,\beta))$-type rational contractions;

(iv) let one of the pairs satisfies the b-$(E.A)$-property and that one of the subspaces are closed in $\mathscr{X}_\jmath$, then the couples $(\mathcal{F}',\mathcal{Q})$ and $(\mathcal{G}',\mathscr{R})$ have a common fixed-point in $\mathscr{X}_\jmath$.

However if the pairs are weakly compatible , then $\mathcal{F}',\mathcal{G}',\mathcal{Q}\,and\,\mathscr{R}$ have a unique common fixed-point in $\mathscr{X}_\jmath$.

Proof: Let us assume that there exists $\vartheta_0'\in\mathscr{X}_\jmath$ in such a way that $\alpha(\vartheta_0',\mathcal{F}'\vartheta_0')\geq 1$.

Define a sequence $\{\vartheta_n'\}$ in $\mathscr{X}_\jmath$ by letting $\vartheta_1'=\mathcal{F}'\vartheta_0',\vartheta_2'=\mathcal{G}'\vartheta_1',\vartheta_3'=\mathcal{F}'\vartheta_2'$.

Continuing in this process, we get $\mathcal{T}\vartheta_n'=\mathcal{F}'\vartheta_{n+1}'=\vartheta_{n+1}'$and $\mathcal{Q}\vartheta_{n+1}'=\mathcal{G}'\vartheta_{n+2}'=\vartheta_{n+2}'$ where $n\geq 0$.

As the pairs $(\mathcal{F}',\mathcal{Q})$ and $(\mathcal{G}',\mathscr{R})$ are $(\alpha.\beta)-$admissible so $\alpha(\vartheta_0',\mathcal{F}'\vartheta_0')=\alpha(\vartheta_0',\vartheta_1')\geq 1,\alpha(\mathcal{F}'\vartheta_0',\mathcal{G}'\vartheta_1')=\alpha(\vartheta_1',\vartheta_2')\geq 1$ and $\alpha(\mathscr{R}\vartheta_0',\mathcal{Q}\vartheta_1')=\alpha(\vartheta_2',\vartheta_3')\geq 1$.

This is how we come to obtain $\alpha(\vartheta_n',\vartheta_{n+1}')\geq 1$ for all $n\geq 0$. ...(2)

In the same way, we can get $\beta(\vartheta_n',\vartheta_{n+1}')\geq 1$ for all $n\geq 0$. ...(3)

If for some $n\in\mathcal{N}$ there are $\vartheta_n'=\vartheta_{n+1}'$,then $\nu=\vartheta_n'$ is the coincidence fixed-point for the mappings $\mathcal{F}',\mathcal{G}',\mathcal{Q}\,and\,\mathscr{R}$ respectively.

Consequently let $\vartheta'_n \neq \vartheta'_{n+1}$ for all $n \in \mathcal{N}$.

Since $\frac{1}{2}min\{d_\gamma(\mathcal{F}'\vartheta'_{2n}, \mathscr{R}\vartheta'_{2n+1}), d_\gamma(\mathcal{G}'\vartheta'_{2n+1}, \mathcal{Q}\vartheta'_{2n})\} \leq d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})$;

from equation (1) we have

$0 \leq \xi(\alpha(\mathcal{F}'\vartheta'_{2n}, \mathcal{G}'\vartheta'_{2n+1}), \beta(\mathcal{F}'\vartheta'_{2n}, \mathcal{G}'\vartheta'_{2n+1}), s^4 d_\gamma(\mathcal{F}'\vartheta'_{2n}, \mathcal{G}'\vartheta'_{2n+1}), \mathcal{M}(\vartheta'_{2n}, \vartheta'_{2n+1}))$;

$< \xi(\alpha(\vartheta'_{2n}, \vartheta'_{2n+1}),\ \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), \mathcal{M}(\vartheta'_{2n}, \vartheta'_{2n+1}))$. ...(4)

since

$$\mathcal{M}(\vartheta'_{2n}, \vartheta'_{2n+1}) = max\{d_\gamma(\mathcal{Q}\vartheta'_{2n}, \mathscr{R}\vartheta'_{2n+1}), d_\gamma(\mathcal{F}'\vartheta'_{2n}, \mathcal{Q}\vartheta'_{2n}), d_\gamma(\mathcal{G}'\vartheta'_{2n+1}, \mathscr{R}\vartheta'_{2n+1}), \frac{d_\gamma(\mathcal{F}'\vartheta'_{2n}, \mathcal{T}\vartheta'_{2n+1}) + d_\gamma(\mathcal{G}'\vartheta'_{2n+1}, \mathcal{Q}\vartheta'_{2n})}{2s}\}$$

$$= max\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+2}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2}), \frac{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+2}) + d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+1})}{2s}\}$$

By triangular inequality, we have

$$\frac{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) + d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2}) + d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+1})}{2s} \leq max\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})\}$$

So we have

$$\mathcal{M}(\vartheta'_{2n}, \vartheta'_{2n+1}) = max\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})\}$$

Therefore from equation (4) we have

$0 \leq \xi(\alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), max\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})\})$

$< max\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})\} - (\alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})$

by (ξ_2)

Then

$\alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) < max\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})\}$ ...(5)

Necessarily, we have

$max\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}), d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})\} = d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})$ for all $n \geq 1$. ...(6)

Consequently, we get

$\alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) < d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})$

$for\ all\ \ n \geq 1.$ ...(7)

We know

$d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) \leq \alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})$ ...(8)

Since $\alpha(\vartheta'_n, \vartheta'_{n+1}) \geq 1$ and $\alpha(\vartheta'_n, \vartheta'_{n+1}) \geq 1$ for all $n \geq 1$.

From equation (7) and equation (8) for all $n \geq 0$,

we have

$d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) \leq \alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) < d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})$ ...(9)

i.e $d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) < d_\gamma(\vartheta'_{2n+1}, \vartheta'_{2n+2})$. ...(10)

The sequence $\{d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})\}$ is a non-increasing.

Thus there exists $\varrho \geq 0$ in such a way that $lim_{n\to\infty} d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) = \varrho$.

We show that, $lim_{n\to\infty} d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1}) = 0$; ...(11)

Now on the contrary let us assume that $\varrho > 0$,

By equation (9) we have

$lim_{n\to\infty}\{\alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})\} = \varrho$;

Since $\varrho > 0$ and letting $\mathsf{t}_n = \alpha(\vartheta'_{2n}, \vartheta'_{2n+1}), \beta(\vartheta'_{2n}, \vartheta'_{2n+1}), s^4 d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})$

and

$\omega_n = d_\gamma(\vartheta'_{2n}, \vartheta'_{2n+1})$ such a way that

$lim_{n\to\infty}\mathsf{t}_n = lim_{n\to\infty}\omega_n = \varrho$

then by (ζ_3) $lim_{n\to\infty} sup\, \zeta(\mathsf{t}_n, \omega_n) < 0$.

Since $\zeta(\mathsf{t}_n, \omega_n) > 0$ so

$0 \leq lim_{n\to\infty} sup\, \zeta(\mathsf{t}_n, \omega_n) < 0$.

which contradicts itself. Thus, our presumption is incorrect. So $\varrho = 0$.

Now again,

we have $\{\vartheta'_n\}$ is a Cauchy sequence in $\mathscr{X}_\jmath$

i.e $lim_{n,m\to\infty} d_\gamma(\vartheta'_n, \vartheta'_m) = 0$. ...(12)

Let us assume the contradiction that $\{\vartheta'_n\}$ is not a Cauchy Sequence.

Thus there is $\epsilon > 0$ where we can assume the sequences $\{\vartheta'_{n_k}\}$and $\{\vartheta'_{m_k}\}$with $n(k) > m(k) > k$ in such a way that for each k

$d_\gamma(\vartheta'_{n_k}, \vartheta'_{m_k}) \geq \epsilon$; ...(13)

and $n(k)$ is the smallest number such that equation (13) holds.

From equation (13) we obtain,

$d_\gamma(\vartheta'_{n_k-1}, \vartheta'_{m_k}) < \epsilon$. ...(14)

using triangular inequality and equation (12), we get

$\epsilon \leq d_\gamma(\vartheta'_{n_k}, \vartheta'_{m_k}) \leq d_\gamma(\vartheta'_{n_k}, \vartheta'_{n_k-1}) + d_\gamma(\vartheta'_{n_k-1}, \vartheta'_{m_k})$

$< d_\gamma(\vartheta'_{n_k}, \vartheta'_{n_k-1}) + \epsilon.$

In the above equation, taking $lim_{n\to\infty}$ and applying equation (11), we get

$lim_{n\to\infty} d_\gamma(\vartheta'_{n_k}, \vartheta'_{m_k}) = \epsilon.$ ...(15)

With the help of triangular iequality , we have

$d_\gamma(\vartheta'_{n_k+1}, \vartheta'_{m_k}) \leq d_\gamma(\vartheta'_{n_k+1}, \vartheta'_{n_k}) + d_\gamma(\vartheta'_{n_k}, \vartheta'_{m_k})$

taking limit $n \to \infty$ and using quation (11), (13), and (15) we have

$lim_{n\to\infty} d_\gamma(\vartheta'_{n_k+1}, \vartheta'_{m_k}) = \epsilon.$ ...(16)

Similarly it is easy to show that

$lim_{n\to\infty} d_\gamma(\vartheta'_{n_k+1}, \vartheta'_{m_k+1}) = \epsilon.$ ...(17)

As the pairs $(\mathcal{F}', \mathcal{Q})$ and $(\mathcal{G}', \mathscr{R})$ are are $(\alpha.\beta)-$admissible and by definition of $\mathcal{M}(\vartheta'_{2n}, \vartheta'_{2n+1})$

we have

$$lim_{n\to\infty}\mathcal{M}(\vartheta'_{n_k}, \vartheta'_{m_k}) = max\{d_\gamma(\mathcal{Q}\vartheta'_{n_k}, \mathscr{R}\vartheta'_{m_k}), d_\gamma(\mathcal{F}\vartheta'_{n_k}, \mathcal{Q}\vartheta'_{n_k}), d_\gamma(\mathcal{G}\vartheta'_{m_k}, \mathscr{R}\vartheta'_{m_k}), \frac{d_\gamma(\mathcal{F}\vartheta'_{n_k}, \mathscr{R}\vartheta'_{m_k}) + d_\gamma(\mathcal{G}\vartheta'_{m_k}, \mathcal{Q}\vartheta'_{n_k})}{2s}\}$$

$$lim_{n\to\infty}\mathcal{M}(\vartheta'_{n_k}, \vartheta'_{m_k}) = max\{d_\gamma(\vartheta'_{n_k+1}, \vartheta'_{m_k+1}), d_\gamma(\vartheta'_{n_k}, \vartheta'_{n_k+1}), d_\gamma(\vartheta'_{m_k}, \vartheta'_{m_k+1}), \frac{d_\gamma(\vartheta'_{n_k}, \vartheta'_{m_k+1}) + d_\gamma(\vartheta'_{m_k}, \vartheta'_{n_k+1})}{2s}\}$$

From equation (11), (15), (16) and (17) we get

$lim_{n\to\infty} d_\gamma(\vartheta'_{n_k+1}, \vartheta'_{m_k+1}) = lim_{n\to\infty}\mathcal{M}(\vartheta'_{n_k}, \vartheta'_{m_k}) = \epsilon.$ ...(18)

From equation (18), we have

$0 \leq lim_{n\to\infty} sup(\alpha(\mathcal{F}'\vartheta'_{n_k}, \mathcal{G}'\vartheta'_{m_k}), \beta(\mathcal{F}'\vartheta'_{n_k}, \mathcal{G}'\vartheta'_{m_k}), s^4 d_\gamma(\mathcal{F}'\vartheta'_{n_k}, \mathcal{G}'\vartheta'_{m_k}), \mathcal{M}(\vartheta'_{n_k}, \vartheta'_{m_k})) < 0;$

a contradiction due to our assumption. So $\{\vartheta'_n\}$ is a Cauchy Sequence.

Since $\mathscr{X}_\jmath$ is complete so there exists a $\nu \in \mathscr{X}_\jmath$ in such a way that $\{\vartheta'_n\} \to \nu$ as $n \to \infty$.

Now we claim that ν is the coincidence point for the pairs of mappings $(\mathcal{F}', \mathcal{Q})$ and $(\mathcal{G}', \mathscr{R})$ respectively. Since both the pairs satisfies $b-(E.A)$-property,

$\nu = lim_{n\to\infty}\vartheta'_{2n} = lim_{n\to\infty}\mathcal{F}'\vartheta'_{2n}$ ...(19) and

$\iota = lim_{n\to\infty}\vartheta'_{2n+1} = lim_{n\to\infty}\mathcal{G}'\vartheta'_{2n+1}$ ...(20)

Thus $\mathcal{F}'\nu = \mathcal{Q}\nu = \mathcal{G}'\iota = \mathscr{R}\iota = \rho_\hbar$.

By using the weak compatibility of both the pairs $(\mathcal{F}', \mathcal{Q})$ and $(\mathcal{G}', \mathcal{T})$, we conclude that $\mathcal{F}'\rho_\hbar = \mathcal{Q}\rho_\hbar$ and $\mathcal{G}'\rho_\hbar = \mathscr{R}\rho_\hbar$.

Now we are required to show that $\rho_\hbar$ is the fixed point of $\mathcal{F}', \mathcal{Q}, \mathcal{G}'$ and $\mathscr{R}$.

On contrary let us assume that $\mathcal{F}'\nu = \mathcal{Q}\nu \neq \rho_\hbar$;

By equation (1), we have

$\frac{1}{2}min\{d_\gamma(\mathcal{F}'\vartheta'_{2n_k}, \mathscr{R}\rho_\hbar), d_\gamma(\mathcal{G}'\rho_\hbar, \mathcal{Q}\vartheta'_{2n_k})\} \leq d_\gamma(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar)$

$0 \leq \xi(\alpha(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar), \beta(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar), s^4 d_\gamma(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar), \mathcal{M}(\vartheta'_{2n_k}, \rho_\hbar)) <$
0

$< \mathcal{M}(\vartheta'_{2n_k}, \rho_\hbar) - (\alpha(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar), \beta(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar), s^4 d_\gamma(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar))$

so we get

$\alpha(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar), \beta(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar), s^4 d_\gamma(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{G}'\rho_\hbar) < \mathcal{M}(\vartheta'_{2n_k}, \rho_\hbar)$ ...(21)

On the other hand , we have

$$\mathcal{M}(\vartheta'_{2n_k}, \rho_\hbar) = max\{d_\gamma(\mathcal{Q}\vartheta'_{2n_k}, \mathscr{R}\rho_\hbar), d_\gamma(\mathcal{F}'\vartheta'_{2n_k}, \mathcal{Q}\vartheta'_{2n_k}), d_\gamma(\mathcal{G}'\rho_\hbar, \mathscr{R}\rho_\hbar), \frac{d_\gamma(\mathcal{F}'\vartheta'_{2n_k}, \mathscr{R}\rho_\hbar) + d_\gamma(\mathcal{G}'\rho_\hbar, \mathcal{Q}\vartheta'_{2n_k})}{2s}\}$$

Taking limit $k \to \infty$ we get

$lim_{k\to\infty}\mathcal{M}(\vartheta'_{2n_k}, \rho_\hbar) \leq d_\gamma(\rho_\hbar, \mathscr{R}\rho_\hbar)$

from equation (21) we have

$d_\gamma(\vartheta'_{2n_k}, \rho_\hbar) \leq \alpha(\vartheta'_{2n_k}, \rho_\hbar), \beta(\vartheta'_{2n_k}, \rho_\hbar), s^4 d_\gamma(\vartheta'_{2n_k}, \rho_\hbar) < \mathcal{M}(\vartheta'_{2n_k}, \rho_\hbar)$; ...(22)

Taking limit $k \to \infty$ we get, in equation (22)

$d_\gamma(\mathcal{F}'\rho_\hbar, \rho_\hbar) < d_\gamma(\rho_\hbar, \mathcal{F}'\rho_\hbar)$ a contradiction. Hence $\rho_\hbar = \mathcal{F}\rho_\hbar = \mathcal{Q}\rho_\hbar$.

Similarly we can prove $\rho_\hbar = \mathcal{G}'\rho_\hbar = \mathcal{G}'\rho_\hbar$.

In order to show the uniqueness of the fixed-point, let another fixed-point $\sigma_\hbar$ of $\mathcal{F}', \mathcal{Q}, \mathcal{G}'$ and $\mathcal{G}'$.

By equation (1) we have

$\frac{1}{2}min\{d_\gamma(\mathcal{F}'\sigma_\hbar, \mathcal{G}'\rho_\hbar), d_\gamma(\mathcal{G}'\rho_\hbar, \mathcal{Q}\sigma_\hbar)\} \leq \frac{1}{2}min\{0, 0\} < d_\gamma(\mathcal{F}'\sigma_\hbar, \mathcal{G}'\rho_\hbar)$

$0 \leq \xi(\alpha(\mathcal{F}'\sigma_\hbar, \mathcal{G}'\rho_\hbar), \beta(\mathcal{F}'\sigma_\hbar, \mathcal{G}'\rho_\hbar), s^4 d_\gamma(\mathcal{F}'\sigma_\hbar, \mathcal{G}'\rho_\hbar), \mathcal{M}(\sigma_\hbar, \rho_\hbar))$

$\mathcal{M}(\sigma_\hbar, \rho_\hbar) - \alpha(\sigma_\hbar, \rho_\hbar), \beta(\sigma_\hbar, \rho_\hbar), s^4 d_\gamma(\sigma_\hbar, \rho_\hbar)$

since the pairs $(\mathcal{F}', \mathcal{Q})$ and $(\mathcal{G}', \mathcal{G}')$ are (α, β)-admissible mappings,

$d_\gamma(\sigma_\hbar, \rho_\hbar) \leq \alpha(\sigma_\hbar, \rho_\hbar), \beta(\sigma_\hbar, \rho_\hbar), s^4 d_\gamma(\sigma_\hbar, \rho_\hbar) < \mathcal{M}(\sigma_\hbar, \rho_\hbar)$. ...(23)

Contrary to that

$$\mathcal{M}(\vartheta_{2n_k}, \rho_\hbar) = max\{d_\gamma(\mathcal{Q}\sigma_\hbar, \mathcal{G}'\rho_\hbar), d_\gamma(\mathcal{F}'\sigma_\hbar, \mathcal{Q}\sigma_\hbar), d_\gamma(\mathcal{G}'\rho_\hbar, \mathscr{R}\rho_\hbar), \frac{d_\gamma(\mathcal{F}'\sigma_\hbar, \mathcal{G}'\rho_\hbar) + d_\gamma(\mathcal{G}'\rho_\hbar, \mathcal{Q}\sigma_\hbar)}{2s}\}$$

$$= max\{d_\gamma(\sigma_\hbar, \rho_\hbar), d_\gamma(\sigma_\hbar, \sigma_\hbar), d_\gamma(\rho_\hbar, \rho_\hbar), \frac{d_\gamma(\sigma_\hbar, \rho_\hbar) + d_\gamma(\rho_\hbar, \sigma_\hbar)}{2s}\}$$

$$d_\gamma(\sigma_\hbar, \rho_\hbar) > 0.$$

Hence from equation (22) and (23)

$d_\gamma(\sigma_\hbar, \rho_\hbar) \leq \alpha(\sigma_\hbar, \rho_\hbar), \beta(\sigma_\hbar, \rho_\hbar), s^4 d_\gamma(\sigma_\hbar, \rho_\hbar) < \mathcal{M}(\sigma_\hbar, \rho_\hbar) = d_\gamma(\sigma_\hbar, \rho_\hbar)$ a contradiction.

Hence $\sigma_\hbar = \rho_\hbar$.

Corollary 3.3: Let $(\mathscr{X}_\jmath, d_\gamma)$ be a TVS-valued CbMS with $s > 1$ and $\mathcal{F}', \mathscr{R} : \mathscr{X}_\jmath \times \mathscr{X}_\jmath \to \mathscr{X}_\jmath$ and $\alpha, \beta: \mathscr{X}_\jmath \times \mathscr{X}_\jmath \to [0, \infty)$ be mappings such that the given conditions hold:

$\zeta(\alpha(\mathcal{F}'\vartheta, \mathcal{F}'\eta), \beta(\mathcal{F}'\vartheta, \mathcal{F}'\eta), s^4 d_\gamma(\mathcal{F}'\vartheta, \mathcal{F}'\eta), \mathcal{M}(\vartheta, \eta)) \geq 0$ for all $\vartheta, \eta \in \mathscr{X}_\jmath$ and $\zeta \in \mathcal{Z}_\varphi$ ...(24)

where $\mathcal{M}(\vartheta, \eta) = max\{d_\gamma(\mathscr{R}\vartheta, \mathscr{R}\eta), d_\gamma(\mathcal{F}'\vartheta, \mathscr{R}\vartheta), d_\gamma(\mathcal{F}'\eta, \mathscr{R}\eta), \frac{d_\gamma(\mathcal{F}'\vartheta, \mathscr{R}\eta) + d_\gamma(\mathscr{R}\vartheta, \mathcal{F}'\eta)}{2s}\}$

Let one of the pair $(\mathcal{F}', \mathscr{R})$ satisfies the b-$(E.A)$-property and $\mathscr{R}(\vartheta)$ is closed. Therefore the pair $(\mathcal{F}', \mathscr{R})$ has a common fixed-point. Further, if this pair is weakly compatible, then $\mathcal{F}'$ and $\mathscr{R}$ have a unique coincidence fixed-point.

Proof: It is implied by the first theorem by selecting $\mathcal{F}' = \mathcal{G}'$ and $\mathcal{Q} = \mathscr{R}$.

Example 3.4: Let $\zeta(\vartheta, \eta) = \frac{99}{100}s - t$ for all μ, η in $[0, 1), \mathscr{X}_\jmath = [0, 1]$ and let $d_\gamma : \mathscr{X}_\jmath \times \mathscr{X}_\jmath \to [0, 1)$ as given below:

$$d_\gamma(\vartheta, \eta) = \begin{cases} 0, & \vartheta = \eta \\ \vartheta + \eta, & \vartheta \neq \eta \end{cases}$$

Let us define $\mathcal{F}', \mathcal{G}', \mathcal{Q}$ and $\mathscr{R}: \mathscr{X}_\jmath \to \mathscr{X}_\jmath$ and $\alpha, \beta : \mathscr{X}_\jmath \times \mathscr{X}_\jmath \to [0, 1]$ such that $\mathcal{F}'(\vartheta) = \vartheta^3, \mathcal{G}'(\vartheta) = \begin{cases} 0, & 0 \leq \vartheta < \frac{1}{3} \\ \frac{1}{27}, & \frac{1}{3} \leq \vartheta < 1 \end{cases}$

$\mathscr{R}(\vartheta) = 3\vartheta, \mathcal{Q}(\vartheta) = \begin{cases} \frac{\vartheta}{3} & 0 \leq \vartheta < \frac{1}{3} \\ \frac{1}{27} & \frac{1}{3} \leq \vartheta < 1 \end{cases}$ and $\alpha(\vartheta,\eta) = \begin{cases} 1, & \vartheta,\eta \in [0,1] \\ 0 & otherwise \end{cases}$,

$\beta(\vartheta,\rho) = \begin{cases} 1, & \vartheta,\eta \in [0,1] \\ 0 & otherwise \end{cases}$.

Let $\{\vartheta_n\}$ be the sequence in $\mathscr{X}_\jmath$ in such a way that $\vartheta_n = 1/27 + 1/n$ and then $lim_{n\to\infty}\mathcal{F}'(\vartheta_n) = lim_{n\to\infty}\mathcal{Q}(\vartheta_n) = \frac{1}{27}$.

So the pairs satisfies the $b-(E.A)$-property.

To verify the condition for every $\vartheta,\eta \in \mathscr{X}_\jmath\ if\ \vartheta = \eta = 0$ or $\vartheta = \eta = \frac{1}{3}$ then equation (24) is satisfied.

If $\vartheta,\eta \in [0,\frac{1}{3}]$ then $d_\gamma(\mathcal{F}'\vartheta,\mathcal{G}'\eta) = (\vartheta^3+0) = \vartheta^3$,$d_\gamma(\mathcal{F}'\vartheta,\mathscr{R}\eta) = (\vartheta^3+\vartheta) = \vartheta^3+\vartheta$,

$d_\gamma(\mathcal{G}'\eta,\mathcal{Q}\vartheta) = (0+\frac{\vartheta}{3}) = \frac{\vartheta}{3}$, $d_\gamma(\mathcal{Q}\vartheta,\mathscr{R}\eta) = (\frac{\vartheta}{3}+\vartheta) = \frac{\vartheta}{3}+\vartheta$, $d_\gamma(\mathcal{G}'\eta,\mathscr{R}\eta) = (0+\vartheta) = \vartheta$ and

$d_\gamma(\mathcal{F}'\vartheta,\mathcal{Q}\vartheta) = (\vartheta^3+\frac{\vartheta}{3}) = \vartheta^3+\frac{\vartheta}{3}$.

Now we consider $\frac{1}{2}min\{\vartheta^4,\frac{\vartheta}{3}\} \leq \vartheta^3$;

So we have $\frac{99}{100}\mathcal{M}(\vartheta,\eta) - \alpha(\mathcal{F}'\vartheta,\mathcal{G}'\eta),\beta(\mathcal{F}'\vartheta,\mathcal{G}'\eta),s^4 d_\gamma(\mathcal{F}'\vartheta,\mathcal{G}'\eta) \geq 0$.

Thus $\frac{99}{100} \times \frac{4}{3}\vartheta - s^4 \times \vartheta^3 \geq 0$.

Hence the conditions of equation (24) are satisfied and $\vartheta = 0$ is the unique coincidence point.

4. Conclusion

In this attempt, we studied Sujuki type-($\mathcal{Z}_\varphi(\alpha,\beta)$)-type rational contraction with respect to simulation function using (α,β)-admissible mappings and proved FPts for two couples of mappings in TVS-valued CbMS. The results obtained are generalization of many existing results in the literature.

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