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Review Paper

About common eigenvalues and eigenvectors of two and more completely continuous operators.

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In mathematics there are several result about common eigenvalues of two bounded polynomial bundles acting each in their own Hilbert space. We can indicateat articles [1],[2].
In the special case of the bounded polynomial operator bundles depending on the identicalparameters with the same highest degrees of parameter the necessary and sufficient conditions for existence of common eigenvalue is given in the article of Khayniq [2].Balinskii [1] generalized this result to the case of polynomial bundles with the different highest degrees of the parameter.
Letbe two polynomial bundles depending on the same parameter and acting generallyspeaking each in their own Hilbertspace·.

Resultant of polynomial bundles $A(\lambda)$ and $B(\lambda)$ is the operatorpresented as the determinant

$$\operatorname{Re}s(A(\lambda),B(\lambda))=\begin{pmatrix} A_0\otimes E_2 & A_1\otimes E_2 & \dots & A_n\otimes E_2 & \dots & 0 \\ . & . & \dots & . & \dots & . \\ 0 & 0 & \dots A_0\otimes E_2 & A_1\otimes E_2 & \dots & A_n\otimes E_2 \\ E_1\otimes B_0 & E_1\otimes B_1 & \dots & E_1\otimes B_m & .. & 0 \\ . & . & \dots & . & \dots & . \\ . & . & \dots E_1\otimes B_0 & E_1\otimes B_1 & \dots. & E_1\otimes B_m \end{pmatrix}$$ *acting in*

the $(H_1\otimes H_2)^{n+m}$-direct sumof $n+m$ copies of tensor product space $H_1\otimes H_2$ and the nunber of rows $A_i\otimes E_2$ with the operators in the $\operatorname{Re}s(A_1(\lambda),A_2(\lambda))$ is equal to the highest degree of the parameter λ in the bundle $B(\lambda)$, and the number of rows with the operators $E_1\otimes B_j$ is equal to the highest degree of parameter λ of the bundle $A(\lambda)$

From]1],]2] we have: if all operators $A_i\,(i=0,1,\dots,n)$ and $B_j\,(j=0,1,\dots,m)$ are bounded

$$KerA_n=0\,,\ KerB_m=0$$

$$Ker\operatorname{Re}s(A(\lambda),B(\lambda))\neq 0$$

then the bundles $A(\lambda)$ and $B(\lambda)$ have a common point of their spectra.

This result for the purpose of application has been generalized on the case of several bounded polynomial bundles [3].

Now we give the sufficient, (in the special cases the necessary and sufficient) conditions for the existence of common eigenvalues and eigenvectors of two and more completely continuous operators.

Definitions[4],[5].[6]

1. λ is an eigenvalue of operator A ifthere is non-null element x such that $Ax-\lambda x=0$

2.Element x is called a root vector of height k if the following equalities

$$(A-\lambda E)^k x=0$$

$(A-\lambda E)^{k-1} x\neq 0$ are satisfied.

Naturally the eigenvector x of operator A is the root vector of operator A of height 1 .

Let A be bounded operator , λ is an eigenvalue, $x_1,...x_s$ are the corresponding eigenvectors.

Each eigenvector $x_{i,}$ of operator A corresponds the elements $x_{i,1}, x_{i,2,},\dots, x_{i,m_i}$ satisfying the conditions

,

$$(A-\lambda E)^k x_{i,k,k}=0 \text{ ,, } (A-\lambda E)^{k-1} x_{i,k}\neq 0$$

The all linear independent root vectors $x_{i,1}, x_{i,2,},\dots, x_{i,r}$ with the heights $1,2,...,r$ ($x_{i,s}-$ the root vector of height s of operator A corresponding to the its eigenvalue λ and to eigenvector $,x_{i,1}$)is called a Jordan chain or a Jordan cell. Operator A may be presented in the form

$A=\dfrac{A+A^*}{2}+i\dfrac{A-A^*}{2i}$ where $T=\dfrac{A+A^*}{2}$ and $S=\dfrac{A-A^*}{2i}$ are completely continuous self-adjoint operators in Hilbert space. Similar we have $\dfrac{B-B^*}{2i}$

$B==\dfrac{B+B^*}{2}+i\dfrac{B-B^*}{2i}$ where $B\dfrac{B+B^*}{2}$ and $N=\dfrac{B-B^*}{2i}$ are $N=\dfrac{B+B^*}{2}$ also completely continuous self-adjoint operators in H

Let E_t and F_s be the family of projective operators with the following properties[4],[5[,[6]

1. $E_a=0$, $E_b=1$

$F_c=0$, $F_d=1$

2. $E_m E_n=E_k$, $k=\min(m,n)$

$F_pF_q = F_r$ $r = \min(p,q)$

3. $E_t - E_{t-0} = P_t$, $F_s - F_{s-0} = R_s$

where P_t is projective operator that projects onto eigen subspace of operator T , corresponding to its eigenvalue t and R_s is the projective operator that projects onto eigen subspace of operator S with the eigenvalue s .

Similar the expansions of unity K_t and L_k of operators M and N , correspondingly, satisfy the properties

1. $K_c = 0$, $K_d = 1$

$L_c = 0$, $L_d = 1$

2., $K_mK_n = K_s$ $s = \min(m,n)$

$L_pL_q = L_r$, $r = \min(p,q)$

3. $K_t - K_{t-0} = S_t$, $L_t - L_{t-0} = H_t$

where S_t is projective operator onto eigen subspace of operator M , corresponding to its eigenvalue t and H_t - is a projective operator onto eigen subspace of operator N , corresponding to its eigenvalues t .

Theorem 1. If $R(P_aR_b) \neq \{0\}$ and $R(S_aH_b) \neq \{0\}$ then the completely continuous operators A and B have a common eigenvalue $a+ib$.

Proof. The condition $R(P_aR_b) \neq \{0\}$ means that $P_a \neq \{0\}$ and $R_b \neq \{0\}$. Similar from $R(S_aH_b) \neq \{0\}$ follows that $S_a \neq \{0\}$ and $H_b \neq \{0\}$ According to definition of projective operators P_a , R_b and also S_a , H_b we have that a is eigenvalue of operator $T = \frac{A+A^*}{2}$ and $M = B\frac{B+B^*}{2}$, b b is the eigenvalue of operators $S = \frac{A-A^*}{2i}$ and $N = \frac{B-B^*}{2i}$.

[7] . Condition $R(P_aR_b) \neq \{0\}$ means that $a+ib$ is eigenvalue of operator $A.$ Similar the condition $R(S_aH_b) \neq \{0\}$ meams that $a+ib$ is eigenvalue of operator $B.$ Theorem1 is proven.

Theorem 2. If operators A and B are completely continuous and normal then the conditions $R(P_a R_b\ ) \neq \{0\}$ and $R(S_a H_b) \neq \{0\}$ are necessary

and sufficient for these operators to have a common eigenvalue $a+ib$.
Proof. Sufficient of conditions
$R(P_a R_b\ ) \neq \{0\}$ and $R(S_a H_b) \neq \{0\}$ is proven inTheoren1.

Necessaty. Let $a+ib$ is an ejgenvalue of both operators A and B . If $a+ib$ is eigenvalue of operator A and x corresponding eigenvector x then $A(a+ib)x$ and $A^* x = (a-ib)x.$ x From formulas for operators T and S we have $Tx = ax$ and $Sx = bx.$

Last means P_a, R_b are not equal to zero and $R(P_a R_b\ ) \neq \{0\}$ .Similar we prove $R(S_a H_b) \neq \{0\}$

Theorem 2 is proven.

If there are several completely continuous operators A_m . For each operator $A_m = T_m + iS_m$

where. $T_m = \frac{A_m + A_m^*}{2}$ and $S_m = \frac{A_m - A_m^*}{2_i}$

Operators T_m and S_m are completely continuous operators acting in Hilbert space H .Introducing the expansions of unity for the operators T_m and S_m and taking into account their properties we obtain thatprojective operator $P_{m,t}$ projects onto eigensubspace of operator T_m , corresponding to its eigenvalue t and the projective operator $R_{m,s}$ projects onto eigen subspace of operator $S_{m,}$ with the eigenvalue s

Teorem3.

Let $A_m (m=1,2,...,s)$ are completely continuous operators actin in Hilbert space and $R(S_{m,a} H_{m,b}) \neq \{0\}$

Then all operators A_m have a common sigenvalue Proof of Theorem 3 sinilar to proof of Theorem2.

Theorem4.

Let for some four real numbers a,b,c,d the projective operator $P_a R_b S_c H_d \neq 0$ projects onto nonzero subspace then completely continuous operators A and B in Hilbert spacehave a common eigenvector.

Really[7] [8].projectiveoperator $P_a R_b S_c H_d$ projects onto intersection of eigen subspaces of operators A with eigenvalue $a+ib$ and operator B with eigenvalue $c+id$.

We have $P_a R_b S_c H_d \subset P_a R_b$

$P_a R_b S_c H \ \subset S_c H_{dd}$

$P_a R_b S_c H_d \neq 0$

The condition $R(P_a R_b) \neq \{0\}$ means that $a+ib$ is the eigenvalue of operator A.

The condition $R(S_c H_d) \neq \{0\}$ means $c+id$ is the eigenvalue of operator B

The condition $R(P_a R_b S_c H_d) \neq \{0\}$ means that the operators A and B have common eigenvector so projection of projective operator $P_a R_b S_c H_d$ is contained in projections of operators $P_a R_b$ and $S_c H_d$. Each element from $R((P_a R_b S_c H_d))$ is common eigenvector of operators A and B

Theorem4 is proven.

Theorem5.

Let be $A_m (m=1,2,...,s)$ completely continuous operators acting in Hilbert space and for several complex numbers $a_m + ib_m$ $P_{1,a_1} R_{1,b_1} ..P_{s,a_s} R_{s,b_s} \neq 0$ (or $\bigcap_{m=1}^{S} P_{m,a_m} R_{m,b_m} \neq 0$)

$S_{m,a} H_{m,b}) \neq \{0\}$ $A_m (m=1,2,...,s)$.

The proof of Theorem5 similar to proof of Therem4

References

[1]. Balinskii A.I. The generalization of notion of Bezutiant and Resultant .DAN Ukr. SSR, ser.fiz.math-and tech. sciences, 1980, 2,p.3-6(in Ukrainian)

[2]. Khayniq Q. Abstract analoque of Resultant of two polynomial bundles. Functional analysis and its applications,1977,2,issue3,p.94

[3]. Dzhabsrzadeh R.M. About cexistence of common eigenvalue of some operator bundles that depend polynomial on parameter InternasionalTopoloji Confranc.2-9 okt.,1987,Baku.Tez.2 p.93.

[4]. Q.E. Shilov. Mathematical analysis. State Publishing of physical and mathematical literature. B-71 (in Russian)

[5]. N.I. Akhiezer andI.M.Glazman The theory of linear operators in Hilbert space. Science Publishng. Chief reduction of physical and mathematical literature. Moscow 1966 (in Russian)

[6]. I.Ts.Gokhberq, M.G. Krein.Introduction to the theory of linear not self-adjoint operators in Hilbert space ,Publisuing House "Nauka",Moskva,1964

[7]. R.M.Dzhabarzafeh.K.A.Alimardanova.Eigenvalues of a complete continuous operators in Hilbert space,Modern problems of Mathematics and Mechanics,Abstracts of International Confrance dedicated to the memory of genius and Azerbaijani scientist and thinker Nasireddin Tusi,Baku,2024,pp.244-245

[8]. R.Dzhaarzadeh.About common eigenvectors of two completely continuous operators in Hilbert space. Modern problems of Mathematics and Mechanics,Abstracts of International Confrance dedicatrd to the memory of genius and Azerbaijani scientist and thinker Nasireddin Tusi,Baku,2024,pp. 242-243