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Review Paper

To the spectral theory of polynomial bundles in Hilbert space

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Spectral theory of operators is the essential direction of functional analysis and arose from the needs of ordinary differential equations and operator differential equations [1],[11]. Spectral theory of operator bundles took the important place in spectral theory of operators. In its turn the consideration of Cauchy's problem [1] with several initial conditionsled to the study of the general theory of equations of type

$$L(\lambda)=A_0+\lambda A_1B+\lambda^2A_2B^2+\ldots+\lambda^{n-1}A_{n-1}B^{n-1}+\lambda^nB^n, \qquad (1)$$

where $A_i\,(i=0,1,\ldots,n)$ and B are completely continuous operators acting in Hilbert space H The bundle $L(\lambda)$ led to the study of questions of multiple completeness and multiple decompositions on eigen and associated vectors of the polynomial bundle $L(\lambda)$ in Hilbert space. The bundle $L(\lambda)$ is known as Keldysh's bundle and it was studied in[1].In connection with the M.V.Keldysh's considerations it is known that n multiple completeness of eigen and associated vectors of bundle $L(\lambda)$ is true when theoperators $A_i\,(i=0,1,\ldots,n)$ are completely continuous, B is self-adjoint completely continuous operator with the restrictions on the location of its spectrum, besides operator B has he fiinite order an $KerB=\{0\}$.The fundamental result of Keldysh [1] was generalized by many authors in many different directions. Here we should note the works of J.E.Allakhverdiev[2],M.Q.Qasymov[3], A.Q.Kosstyuchenko and Q.V.Radzievskii[4], Q.Radzievskii[5] and many others. Theorems about multiple decompositions on eigen and associated vectors with brackets of the Keldysh bundle $L(\lambda)$ are proven in works of R.M.Dzhabarzadeh[6], V.N.Vizitey , A.S.Markus[8] when the operators A_iB^{-i} are bounded and $\varliminf_{k\to\infty} k\mu_k^{-p}=\infty$, also when the operators A_iB^{-i} are completely continuous and $\varliminf_{k\to\infty} k\mu_k^{-p}<\infty$. The sequence μ_k is the different modules of characteristic numbers of operator B arranged in descending order taking into account their multiplicity.

We should also note the result about summation by root subspaces of completely continuous operator using the generalized Abel method [9].

Let $A(\lambda)=E-A_n-\lambda A_1-\ldots-\lambda^{n-1}A_{n-1}-\lambda^nA_n$ be a polynomial bundle where A_i are bounded operators acting in Hilbert space H.

We introduce some definitions [1].]10],[11].

1.If for some nonzerovector y_0 we have $A(c)y_{0=}=y_0$ then y_0 is called an eigenvector of operator $A(\lambda)$ correspondingto eigenvalue c.

2. Vector y_k is called a $k-th$ -associated vector to the

eigenvector y_0 if the following equalities

$$
\begin{aligned}
&y = A(c)y \\
&y_1 = A(c)y_1 + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y \\
&\ldots\ldots\ldots \\
&y_k = A(c)y_k + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y_{k-1} + \ldots + \frac{1}{k!}\frac{\partial^k A(c)}{\partial c^k}y
\end{aligned}
\qquad (2)
$$

arefulfilled.

The system of linear independent eigen and associated vectors is called a chain of eigen and associated (e.a)vectors of operator $A(\lambda)$ corresponding to eigenvalue c . The number of e.a. vectors in the chain of e.a. vectors is called a length of eigenvector y_0 .Totality all independent eigensnd associated vectors corresponding to all eigen vectors with eigenvalue c is called the nultiplicity of eigenvalue c .

3.M.V.Keldysh built the derivative systems with the help of the formulas

$$\left[\frac{d^k}{dt^k}e^{\lambda t}\left(x_k + \frac{1}{1!}x_{k-1} + \ldots + \frac{1}{k!}x_0\right)\right](t=0) \quad k = 1, 2, \ldots, s \qquad (3)$$

4. System of eigen and associated vectors of operator bundle $A(\lambda)$ in space $\boldsymbol{H}$ forms the $n-$ multiple complete system if any n elements $f_0, f_1, \ldots, f_{n-1}$ of the space $\boldsymbol{H}$ can be approximated with the help of linear combinations of elements $\{x_i^{(j)}\}_{i=1}^{\infty}, j = 0,1,2,\ldots,n-1$ in accordance with predetermined accuracy and the same coefficients, not depending on indices of elements $f_0, f_1, \ldots, f_{n-1}$.

5.Thesystemofsubspaces $\{M_k\}_{k=1}$ forms n -multiple basis if any element x may be presented in the form $x = \sum_{j=1}^{\infty}{}_1 x_j$ where x_j from M_j . The study of spectral properties of equation $L(\lambda)x = x$ in Hilbert space led to the study of the spectral properties of the equation

$$\overline{A}\overline{x} + \lambda\overline{B}\overline{x} = \overline{x} \quad (4)$$

where the operators $\overline{A} = \begin{pmatrix} A_0 & A_1 & \ldots & A_{n-1} \\ 0 & 0 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & 0 \end{pmatrix}$,

$$\overline{B}=\begin{pmatrix} 0 & 0 & 0 & ... & B \\ B & 0 & 0 & ... & 0 \\ 0 & B & 0 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ...B & 0 \end{pmatrix} \text{ and } \overline{E}=\begin{pmatrix} E & 0 & ... & 0 \\ 0 & E & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & E \end{pmatrix}$$

act in direct sum of n copies of Hilbert space H [6],[7],[8].
Let the following conditions
a) operators $A_i (i=0,1,...,n-1)$ and operator B are completely continuous
b) $Ker(E-A_0)=\{\vartheta\}$, $KerB=\{\vartheta\}$ are fulfilled.

From the conditions $Ker(E-A_0) =\{\vartheta\}$ and completely continuity of operator $\overline{A}$ follow that the operator $\overline{E}-\overline{A}$ has a bounded inverse. Besides completely continuity of operators $\overline{A}$ and $\overline{B}$ mean $(\overline{E}-\overline{A})^{-1}\overline{B}$ is a completely continuous operator, and it has the form

$$\begin{pmatrix} (E-A_0)^{-1} A_1 B & ... & (E-A_0)^{-1} A_{n-1} B & (E-A_0)^{-1} B \\ B & ... & 0 & 0 \\ . & ... & . & ... \\ 0 & ... & B & 0 \end{pmatrix}$$

where

$$\left(\overline{E}-\overline{A}\right)^{-1}=\begin{pmatrix} (E-A_0)^{-1} & (E-A_0)^{-1} A & ... & (E-A_0)^{-1} A_{n-1} \\ 0 & E & ... & 0 \\ . & . & ... & 0 \\ 0 & 0 & ... & E \end{pmatrix} \text{ and}$$

$$\overline{B}=\begin{pmatrix} 0 & 0 & 0 & ... & B \\ B & 0 & 0 & ... & 0 \\ 0 & B & 0 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ...B & 0 \end{pmatrix}$$

We introduce the new notations $\overline{T}=\dfrac{(\overline{E}-\overline{A})^{-1}\overline{B}+((\overline{E}-\overline{A})^{-1}\overline{B})^*}{2}$

and $\overline{S}=\dfrac{(\overline{E}-\overline{A})^{-1}\overline{B}-((\overline{E}-\overline{A})^{-1}\overline{B})^*}{2i}$

Let $\tilde{E}_t$ be the expansion of unity of operator $\overline{T}$ and $\tilde{F}_s$ is the expansion of unity of operator $\overline{S}$ [10].

Since the operators $\overline{T}$ and $\overline{S}$ are bounded there is some numbers a,b,c,d that the following equalities take place

1. $\tilde{E}_a = 0$, $\tilde{E}_b = 1$

$\tilde{F}_c = 0$ $\tilde{F}_d = 1$

2. $\tilde{E}_{tm}\tilde{E}n = \tilde{E}_{\min(m,n)}$,

$\tilde{F}_l \tilde{F}_s = \tilde{F}_{\min(l,k)}$

3. $\tilde{E}_t - \tilde{E}_{t-0} = P_t$

$\tilde{F}_s - \tilde{F}_{s-0} = R_s$

where P_t is a projective operator that projects onto the eigen subspace of operator $\overline{T}$ corresponding to its eigenvalue t ,and R_s is projective operator that projects onto the eigen subspace of operator $\overline{S}$ corresponding to its eigenvalue s .

Theorem. Let $L(\lambda)$ be the operator bundle where $A_i\,(i = 0,1,...,n)$ and B are completely continuous operators acting in Hilbert space H , $Ker(E - A_0) = \{\vartheta\}$, $KerB = \{\vartheta\}$

If for some real numbers t and s two parameter projective operator $P_t R_s \neq 0$ then $(t - is)/(t^2 + s^2)$ is the eigenvalue of the bundle $L(\lambda)$.

Proof. The condition $P_t R_s \neq 0$ means that the range of operator $P_t R_s$ contains thou one nonzero vector $\overline{x}$,besides $P_t R_s \subset P_t$ and $P_t R_s \subset R_s$. The last means that $\overline{x}$ enters the range of both operators P_t and R_s . From[12] ,[13],[14]we have that t is the eigenvalue of operator $\overline{T}$,and s is the eigenvalue of operator $\overline{S}$.

$t + is$ is the eigenvalue of the operator $(\overline{E} - \overline{A})^{-1}\overline{B}$ that is $(\overline{E} - \overline{A})^{-1}\overline{B}\overline{x} = (t + is)\overline{x}$

Acting on both side on last equation by the operator $(t + is)^{-1}(\overline{E} - \overline{A})$ we have that $1/(t + is)$ is an eigenvalue of equation $\overline{A}\overline{x} + \lambda\overline{B}\overline{x} = \overline{x}$

It is known that eigenvalues of bundle $L(\lambda)$ and equation (4) coincide.For proof of this Theorem ii is enough to prove if $t + is$ is the eigenvalue c of operator $(\overline{E} - \overline{A})^{-1}\overline{B}$ then $(t + is)^{-1}$ is an eigenvalue of bundle $L(\lambda)$. From the conditions of Theorem follows the eigenvalues of operator $(\overline{E} - \overline{A})^{-1}\overline{B}$ could not be zero.

Really let $\overline{x} = (x_0, x_1, ..., x_{n-1})$ be the eigenvector of operator $(\overline{E} - \overline{A})^{-1}\overline{B}$ corresponding to eigenvalue λ then the equation $\overline{A}\overline{x} + \lambda^{-1}\overline{B}\overline{x} = \overline{x}$ may be presented in the form of system of equalities:

$$x_0 = A_0 x_0 + A_1 x_1 + ... + A_{n-1} x_{n-1} + \lambda^{-1} B x_{n-1}$$
$$x_1 = \lambda^{-1} B x_0$$
$$x_2 = \lambda^{-1} B x_1 \qquad (5)$$
$$.........$$
$$x_{n-1} = \lambda^{-1} B x_{n-2}$$

Sequentially expressing x_j through $\lambda^{-1} Bx_{j-1}$, x_{j-1} through $\lambda^{-1} Bx_{j-2}$ from(5) and continued this process, in the end we have

$$x_k = \lambda^{-k} B^k x_0 \quad (6)$$
$$k = 1, 2, ..., n-1$$

Substituting the expressions from (6) into the first equation of (5) we obtain that λ^{-1} is the eigenvalue of bundle $L(\lambda)$,the first component of eigenvector ofoperator $(\overline{E} - \overline{A})^{-1} \overline{B}$ is the eigenvector of $L(\lambda)$, the second, third and other components are the elements of first, second and other elements of corresponding derivative systems built by the formulas(3).Theorem is proven.

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