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**Review Paper**

# Ricci Yamabe Soliton on Genearlized Sasakian Space Form

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**Abstract:** In this paper we studied Ricci Yamabe soliton in Generalized Sasakian Space Form(GSSF) satisfying the conditions: $R.S = L_S Q$, $W_3 S = fQ(g, S)$ and $R.R = L_R Q(S, R)$.

**Keywords:** Sasakian space form , Ricci-Yamabe soliton, Ricci pseudo-symmetric manifold, Ricci generalized pseudo-symmetric manifold.

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## 1. Introduction

Guler and Crasmareanu [2]introduced a new geometric flow known as Ricci-Yamabe flow which is the generalization of Ricci flow and Yamabe flow.

$$\frac{\partial}{\partial t} g(X,Y) = -2pS(X,Y) + qrg(X,Y), \qquad g(0) = g_0 \tag{1.1}$$

A solution to the Ricci Yamabe flow is called Ricci-Yamabe soliton, denoted as $(RYS)$ and its $(g, V, p, \lambda)$ on a Riemannaian manifolds $(M, g)$ such that

$$L_V \; g(X,Y) + 2pS(X,Y) + (2\lambda - qr)g(X,Y) = 0. \tag{1.2}$$

Following two cases axis from Ricci -Yamabe soliton
Case 1: Yamabe soliton , if p=0 then

$$L_V \; g(X,Y) + (2\lambda - qr)g(X,Y) = 0 \tag{1.3}$$

Case 2: Ricci soliton, if q=0 then

$$L_V \; g(X,Y) + 2pS(X,Y) + 2\lambda g(X,Y) = 0 \tag{1.4}$$

## II. Preliminaries

An n-dimensional smooth manifold $(M, g)$ is almost contact metric structure $(\phi, \xi, \eta, g)$ if it satisfies the following relations:

$$\phi^2(X) = -X + \eta(X)\xi, \phi(\xi) = 0, \eta(\xi) = 1, g(X, \xi) = \eta(X), \tag{2.1}$$

$$g(\phi X, Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for all vector X,Y and M. In the view of relation , we have

$$g(\phi X, Y) = -g(X, \phi Y), g(\phi X, X) = 0 \tag{2.3}$$

An n-dimensional generalized Sasakian Space Form (GSSF) is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.4}$$

for all vector $X, Y, Z$ on $M$, where $f_1$, $f_2$, $f_3$ are function on $M, R$ denotes curvature tensor, and $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$.

In a generalized Sasakian space form the following relation hold:

$$S(X,Y) = [(n-1)f_1 + 3f_2 - f_3]g(X,Y) - [3f_2 + (n-2)f_3]\eta(X)\eta(Y), \tag{2.5}$$

$$QX = [(n-1)f_1 + 3f_2 - f_3]X - [3f_2 + (n-2f_3)]\eta(X)\xi \tag{2.6}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] \tag{2.7}$$

$$R(\xi, X)Y = -R(X,\xi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X] \tag{2.8}$$

We define tensor $R.T$ and $Q(g,T)$ by

$$\begin{aligned} (R(X,Y).T)(X_1, X_2, X_3 \dots X_K) &= -T(R(X,Y)X_1, X_2, X_3 \dots X_K) \\ &-T(X_1, R(X,Y)X_2, X_3 \dots X_K) \\ &- \dots T(X_1, X_2, X_3 \dots R(X,Y)X_K) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} Q(g,T)(X_1, X_2, X_3 \dots X_K; X, Y) &= -T((X \wedge_g Y)X_1, X_2, X_3 \dots X_K) \\ &-T(X_1, (X \wedge_g Y)X_2, X_3 \dots X_K) \\ &- \dots - T(X_1, X_2, X_3 \dots (X \wedge_g Y)X_K) \end{aligned} \tag{2.10}$$

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y \tag{2.11}$$

We define the tensor $R.R$ and $R.S$ on $(M^n, g)$ by

$$(R(X,Y).R)(U,V)W = R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \tag{2.12}$$

and

$$(R(X,Y).S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V) \tag{2.13}$$

Where S is Ricci tensor and R is curvature tensor.

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X) \tag{2.14}$$

$$Q(\xi) = (n-1)(f_1 - f_2)\xi \tag{2.15}$$

$$g(QX,Y) = S(X,Y) \tag{2.16}$$

Where S is Ricci tensor and Q is Ricci opeartor. By the above results, we prove the following sections.

**2.1 Definition:** A Generalized Sasakian Space Form (GSSF) $(M^n, g)$ is said to be Ricci -pseudo-symmetric if the tensor $R.S$ and $Q(g,S)$ are linearly dependent. This is equivalent to

$$R.S = L_S Q(g,S) \tag{2.17}$$

. **2.2 Definition:** A Generalized Sasakain Space Form (GSSF) $(M^n, g)$ is said to be Ricci generalized pseudo-symmetric if the tensor $R.R$ and $Q(S,R)$ are linearly dependent. This is equivalent to

$$R.R = L_R Q(S,R) \tag{2.18}$$

## 3. Ricci Yamabe soliton in Generalized Sasakian Space Form (GSSF)

Let $(g, \xi, \lambda)$ be a Ricci soliton in an n-dimensional Generalized Sasakian Space Form M . from (1.2) we have

$$L_V \; g(X,Y) + 2PS(X,Y) + (2\lambda - qr)g(X,Y) = 0 \tag{3.1}$$

putting $V = \xi$ in (3.1) , we get

$$L_\xi \; g(X,Y) + 2pS(X,Y) + (2\lambda - qr)g(X,Y) = 0 \tag{3.2}$$

For any X,Y $\in TM^n$ , where $L_\xi$ is the Lie derivative operator along the vector field $\xi$, S is the Ricci tensor field of the metric g and $\lambda$ is real constant.

$$(L_\xi \; g) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0 \tag{3.3}$$

Using (3.3) in (3.2), we get

$$S(X,Y) = -(\frac{\lambda}{p} - \frac{qr}{2p})g(X,Y) \tag{3.4}$$

Ricci Yamabe soliton is called shrinking, steady or expanding according as $\lambda$ is negative, zero or positive respectively.

## 4. Ricci Yamabe soliton in Ricci pseudo-symmetric Generalized Sasakian Space Form

We consider a Ricci pseudo- symmetric Generalized Sasakian Space Form(GSSF). Then from the from definition (2.17), we have

$$(R(X,Y).S)(U,V) = L_S Q(g,S)(X,Y;U,V) \tag{4.1}$$

using (2.10) in (4.1) ,we get

$$(R(X,Y).S)(U,V) = L_S((X \wedge_g Y).S)(U,V) \tag{4.2}$$

With the help of(2.10) and (2.13) we get from (4.2)

$$-S(R(X,Y)U,V) - S(U,R(X,Y)V) = L_S[-S((X \wedge_g Y)U,V) - S(U,(X \wedge_g Y)V)] \tag{4.3}$$

using (2.11)in (4.3) , we get,

$$-S(R(X,Y)U,V)-S(U,R(X,Y)V) = L_S[-g(Y,U)S(X,V)+g(X,U)S(Y,V)-g(Y,V)S(U,X)+g(X,V)S(U,Y)] \tag{4.4}$$

Putting $X = U = \xi$ in (4.4) , we get

$$-S(R(\xi,Y)\xi,V)-S(\xi,R(\xi,Y)V) = L_S[-g(Y,\xi)S(\xi,V)+g(\xi,\xi)S(Y,V)-g(Y,V)S(\xi,\xi)+g(\xi,V)S(\xi,Y)] \tag{4.5}$$

$$\begin{aligned}&- S((f_1 - f_3)[\eta(Y)\xi - Y],V) - S(\xi,(f_1 - f_3)[g(Y,V)\xi - \eta(V)Y + S(Y,V)] = L_S[-\eta(Y)(n-1)(f_1 - f_3)\eta(V)] \\ &- g(Y,V)(n-1)(f_1 - f_3) + \eta(V)(n-1)(f_1 - f_3)\eta(Y)]\end{aligned} \tag{4.6}$$

Thus, we have

$$\begin{aligned}-(f_1 - f_3)\eta(Y)S(\xi,V) + (f_1 - f_3)S(Y,V) - g(Y,V)(f_1 - f_3)S(\xi,\xi) + \eta(V)(f_1 - f_3)S(Y,\xi) = \\ L_S[S(Y,V) - g(Y,V)(n-1)(f_1 - f_3)]\end{aligned} \tag{4.7}$$

$$\begin{aligned}-(f_1 - f_3)^2\eta(Y)(n-1)\eta(V) + (f_1 - f_3)S(Y,V) - g(Y,V)(f_1 - f_3)^2(n-1) + \eta(V)(f_1 - f_3)^2(n-1)\eta(Y) = \\ L_S[S(Y,V) - g(Y,V)(n-1)(f_1 - f_3)]\end{aligned} \tag{4.8}$$

from (4.8), we get

$$(f_1 - f_3)S(Y,V) - g(Y,V)(f_1 - f_3)^2(n-1) = L_S[S(Y,V) - g(Y,V)(n-1)(f_1 - f_3)] \tag{4.9}$$

Thus,we have

$$L_S[S(Y,V) - g(Y,V)(n-1)(f_1 - f_3)] - S(Y,V)(f_1 - f_3) + g(Y,V)(f_1 - f_3)^2(n-1) = 0 \tag{4.10}$$

from (4.10), we get

$$(L_S + (f_1 - f_3))[S(Y,V) - g(Y,V)(f_1 - f_3)(n-1)] = 0 \tag{4.11}$$

$$(L_S + (f_1 - f_3))[-(\frac{\lambda}{p} - \frac{qr}{2p})g(Y,V) - g(Y,V)(f_1 - f_3)(n-1)] = 0 \tag{4.12}$$

$$(L_S + (f_1 - f_3))g(Y,V)[-(\frac{\lambda}{p} - \frac{qr}{2p}) - (f_1 - f_3)(n-1)] = 0 \tag{4.13}$$

from (4.13), we get

$$(\frac{\lambda}{p} - \frac{qr}{2p}) = -(f_1 - f_3)(n-1) \tag{4.14}$$

$$\lambda = -(f_1 - f_3)(n-1)p + \frac{qr}{2} \tag{4.15}$$

now substituting $f_1$ and $f_3$ in (4.15) ,we get

$$\lambda = -(\frac{c+3}{4} - \frac{c-1}{4})(n-1)p + \frac{qr}{2} \tag{4.16}$$

Thus, we have

$$\lambda = -(n-1)p + \frac{qr}{2} \tag{4.17}$$

**Theorem:** A Ricci Yamabe soliton in Ricci pseudo-symmetric Generalized Sasakian Space Form (GSSF) is shrinking or steady or expanding accordingly as:

$$-(n-1)p + \frac{qr}{2} > 0, -(n-1)p + \frac{qr}{2} = 0, -(n-1)p + \frac{qr}{2} < 0 \tag{4.18}$$

If p=0 then we get $\lambda = \frac{qr}{2}$ . thus, we can state

**Corollary:** A q-Yamabe soliton in Ricci pseudo-symmetric Generalized Sasakian Space Form (GSSF) is shrinking or expanding or steady accordingly as:

$$\frac{qr}{2} > 0, \frac{qr}{2} < 0, \frac{qr}{2} = 0 \tag{4.19}$$

If q=0 then we get $\lambda = -(n-1)p$. thus, we have

**Corollary:** A p-Ricci soliton in Ricci pseudo-symmetric Generalized Sasakian Space Form (GSSF) is shrinking or expanding or steady as:

$$-(n-1)p > 0, -(n-1)p < 0, -(n-1)p = 0 \tag{4.20}$$

## 5. Ricci Yamabe Solitons in Ricci generalized pseudo-symmetric Generalized Sasakian Space Form (GSSF)

Consider a Ricci generalized pseudo-symmetric Generalized Sasakian Space Form (GSSF). Then from the definition (2.18), we have

$$R.R = L_R Q(S, R) \tag{5.1}$$

using(2.10) in(5.1) , we get,

$$(R(X,Y).R)(U,V)W = L_R((X \wedge_S Y).R)(U,V)W \tag{5.2}$$

With the help of (2.10) and (2.13), we get from

$$\begin{aligned} R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W = \\ L_R[(X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V)W - R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W] \end{aligned} \tag{5.3}$$

using (2.11) in(5.3) , we get

$$\begin{aligned} R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W = \\ L_R[S(Y,R(U,V)W)X - S(X,R(U,V)W)Y - S(Y,U)R(X,V)W + \\ S(X,U)R(Y,V)W - S(Y,V)R(U,X)W + S(X,V)R(U,Y)W - S(Y,W)R(U,V)X + S(X,W)R(U,V)Y] \end{aligned} \tag{5.4}$$

Putting $X = U = \xi$ in (5.4) , we get

$$R(\xi,Y)R(\xi,V)W - R(R(\xi,Y)\xi,V)W - R(\xi,R(\xi,Y)V)W - R(\xi,V)R(\xi,Y)W = \\ L_R[S(Y,R(\xi,V)W)\xi - S(\xi,R(\xi,V)W)Y - S(Y,\xi)R(\xi,V)W + S(\xi,\xi)R(Y,V)W - \\ S(Y,V)R(\xi,\xi)W + S(\xi,V)R(\xi,Y)W - S(Y,W)R(\xi,V)\xi + S(\xi,W)R(\xi,V)Y] \tag{5.5}$$

$$R(\xi,Y)(f_1-f_3)[g(V,W)\xi - \eta(W)V] - R((f_1-f_3)[\eta(Y)\xi - \eta(\xi)Y],V)W - R(\xi,(f_1-f_3)[g(Y,V)\xi - \eta(V)Y])W \\ -R(\xi,V)((f_1-f_3)[g(Y,W)\xi - \eta(W)Y] = L_R[S(Y,(f_1-f_3)[g(V,W)\xi - \eta(W)V]) \\ -S(\xi,(f_1-f_3)[g(V,W)\xi - \eta(W)V])Y - (n-1)(f_1-f_3)\eta(Y)(f_1-f_3)[g(V,W\xi - \eta(W)V)] + \\ (n-1)(f_1-f_3)R(Y,V)W - S(Y,V)(f_1-f_3)[g(\xi,W)\xi - \eta(W)\xi] \\ +(n-1)(f_1-f_3)\eta(V)(f_1-f_3)[g(Y,W)\xi - \eta(W)Y] - S(Y,W)(f_1-f_3)[g(V,\xi)\xi - \eta(\xi)V] \\ +(n-1)\eta(W)(f_1-f_3)(f_1-f_3)[g(V,Y)\xi - \eta(Y)V]] \tag{5.6}$$

Thus, we have

$$(f_1-f_3)g(V,W)R(\xi,Y)\xi - (f_1-f_3)g(V,W)\eta(W)R(\xi,Y)V - (f_1-f_3)\eta(Y)R(\xi,V)W + (f_1-f_3)R(Y,V)W - \\ g(Y,V)(f_1-f_3)R(\xi,\xi)W + \eta(V)R(\xi,Y)W(f_1-f_3)(-f_1-f_3)g(Y,W)R(\xi,V)\xi + (f_1-f_3)\eta(W)R(\xi,V)Y \\ = L_R[(f_1-f_3)g(V,W)\xi S(Y,\xi) - S(Y,V)\eta(W)\xi(f_1-f_3) - (f_1-f_3)S(\xi,\xi)g(V,W)Y + S(\xi,V)\eta(W)Y(f_1-f_3) \\ -(n-1)\eta(Y)(f_1-f_3)^2 g(V,W)\xi + (n-1)(f_1-f_3)^2\eta(Y)\eta(W)V + (n-1)(f_1-f_3)R(Y,V)W \\ -S(Y,V)(f_1-f_3)[\eta(W)\xi - \eta(W)\xi] - (n-1)(f_1-f_3)^2\eta(V)g(Y,W)\xi \\ -(n-1)(f_1-f_3)^2\eta(V)\eta(W)Y - S(Y,W)(f_1-f_3)\eta(V)\xi + S(Y,W)V(f_1-f_3) \\ +(n-1)\eta(W)(f_1-f_3)^2 g(V,Y)\ \xi - (n-1)\eta(W)(f_1-f_3)^2\eta(Y)V] \tag{5.7}$$

From (5.7), we get

$$\eta(Y)(f_1-f_3)^2 g(V,W)\xi - (f_1-f_3)^2 g(V,W)Y - (f_1-f_3)^2\eta(W)g(Y,V)\xi + (f_1-f_3)^2\eta(W)\eta(V)Y \\ -(f_1-f_3)^2\eta(Y)g(V,W)\xi + (f_1-f_3)^2\eta(W)\eta(Y)V \\ +(f_1-f_3)R(Y,V)W + \eta(V)(f_1-f_3)^2 g(Y,W)\xi - \eta(V)\eta(W)Y(f_1-f_3)^2 \\ -(f_1-f_3)^2\eta(V)g(Y,W)\xi + (f_1-f_3)^2 g(Y,W)V + (f_1-f_3)^2\eta(W)g(Y,V)\xi \\ -(f_1-f_3)^2\eta(W)\eta(Y)V = L_R[(n-1)(f_1-f_3)^2 g(V,W)\xi\eta(Y) \\ -S(Y,V)\eta(W)\xi(f_1-f_3) - (n-1)(f_1-f_3)^2\eta(Y)g(V,W)\xi + (f_1-f_3)^2\eta(Y)\eta(W)V \\ +(n-1)(f_1-f_3)R(Y,V)W + (n-1)(f_1-f_3)^2\eta(V)g(Y,W)\xi \\ -(n-1)(f_1-f_3)^2\eta(V)\eta(W)Y - S(Y,W)(f_1-f_3)\eta(V)\xi \\ -S(Y,W)V(f_1-f_3) + (n-1)\eta(W)(f_1-f_3)^2 g(V,Y)\xi - (n-1)\eta(W)(f_1-f_3)^2\eta(Y)V] \tag{5.8}$$

Thus, we have

$$-(f_1-f_3)^2g(V,W)Y+(f_1-f_3)R(Y,V)W+(f_1-f_3)^2g(Y,W)V=L_R[-S(Y,W)(f_1-f_3)\eta(V)\xi-(f_1-f_3)^2+ g(V,W)Y+(n-1)(f_1-f_3)R(Y,V)W+(n-1)(f_1-f_3)^2\eta(V)g(Y,W)\xi- S(Y,W)(f_1-f_3)\eta(V)\xi+S(Y,W)V(f_1-f_3)+(n-1)g(Y,V)\xi\eta(W)] \tag{5.9}$$

Taking the inner product with respect to Z in (5.9), we get

$$-(f_1-f_3)^2g(V,W)g(Y,Z)-g(R(Y,V)W,Z)(f_1-f_3)+g(Y,W)g(V,Z)(f_1-f_3)^2= l_R[-S(Y,V)\eta(W)g(\xi,Z)-(f_1-f_3)^2(n-1)g(V,W)g(Y,Z)+(n-1)(f_1-f_3)g(Y,Z)+ (n-1)(f_1-f_3)g(R(Y,V)W,Z)+(n-1)(f_1-f_3)^2\eta(V)g(Y,W)\eta(Z)-S(Y,W)(f_1-f_3)\eta(V)\eta(Z)- S(Y,W)(f_1-f_3)\eta(V)\eta(Z)+S(Y,W)(f_1-f_3)g(V,Z)+(f_1-f_3)g(Y,V)\eta(Z)\eta(W)] \tag{5.10}$$

Putting $V=W=e_i$ in (5.10), we get

$$-n(f_1-f_3)^2g(Y,Z)+(f_1-f_3)S(Y,Z)+(f_1-f_3)^2g(Y,Z)= L_R[-(f_1-f_3)^2(n-1)ng(Y,Z)+(n-1)(f_1-f_3)S(Y,Z)+S(Y,Z)(f_1-f_3)] \tag{5.11}$$

$$S(Y,Z)(f_1-f_3)(nL_R-1)-(nL_R-1)g(Y,Z)(n-1)(f_1-f_3)^2=0 \tag{5.12}$$

$$(nL_R-1)[S(Y,Z)(f_1-f_3)-(nL_R-1)g(Y,Z)(n-1)(f_1-f_3)^2]=0 \tag{5.13}$$

$$(nL_R-1)[-(\frac{\lambda}{p}-\frac{qr}{2p})g(Y,Z)(f_1-f_3)-(nL_R-1)g(Y,Z)(n-1)(f_1-f_3)]=0 \tag{5.14}$$

From(5.14), we get

$$(\frac{\lambda}{p}-\frac{qr}{2p})=-(n-1)(f_1-f_3) \tag{5.15}$$

Thus,we have

$$\lambda=-(n-1)(f_1-f_3)p+\frac{qr}{2} \tag{5.16}$$

now substituting $f_1$ and $f_3$ in (5.16) ,we get

$$\lambda=-(\frac{c+3}{4}-\frac{c-1}{4})(n-1)p+\frac{qr}{2} \tag{5.17}$$

From(5.17), we get

$$\lambda=-(n-1)p+\frac{qr}{2} \tag{5.18}$$

**Theorem:** A Ricci Yamabe solitons in Ricci generalized pseudo-symmetric Generalized Sasakian Space Form (GSSF) is shrinking or steady or expanding accordingly as:

$$-(n-1)p+\frac{qr}{2}>0, -(n-1)p+\frac{qr}{2}=0, -(n-1)p+\frac{qr}{2}<0 \tag{5.19}$$

If p=0 then we get $\lambda=\frac{qr}{2}$ . thus, we can state

**Corollary:** A q-Yamabe soliton in Ricci generalized pseudo-symmetric Generalized Sasakian Space Form (GSSF) is shrinking or expanding or steady accordingly as:

$$\frac{qr}{2} > 0, \frac{qr}{2} < 0, \frac{qr}{2} = 0 \tag{5.20}$$

If q=0 then we get $\lambda = -(n-1)p$. thus, we have

**Corollary:** A p-Ricci soliton in Ricci generalized pseudo-symmetric Generalized Sasakian Space Form (GSSF) is shrinking or expanding or steady as:

$$-(n-1)p > 0, -(n-1)p < 0, -(n-1)p = 0 \tag{5.21}$$

## 6. Ricci Yamabe soliton in Generalized Sasakian Space Form (GSSF) satisfying the curvature condition Q·R=0

Consider a Generalized Sasakian Space Form satisfying the curvature condition

$$Q.R = 0 \tag{6.1}$$

i.e

$$(Q.R)(X,Y)Z = 0 \tag{6.2}$$

using(2.12) in (6.2), we get

$$(Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ = 0 \tag{6.3}$$

Putting $X = Z = \xi$ in(6.3), we get

$$Q(R(\xi,Y)\xi) - R(Q\xi,Y)\xi - R(\xi,QY)\xi - R(\xi,Y)Q\xi = 0 \tag{6.4}$$

$$Q((f_1 - f_3)[\eta(Y)\xi - Y]) - R(Q\xi,Y)\xi - (f_1 - f_3)\eta(QY)\xi + (f_1 - f_3)\eta(\xi)(QY) - R(\xi,Y)Q\xi = 0 \tag{6.5}$$

From (6.5), we get

$$(f_1 - f_3)\eta(Y)Q\xi - (f_1 - f_3)QY - R(Q\xi,Y)\xi - (f_1 - f_3)[\eta(QY)\xi - \eta(\xi)QY] - (f_1 - f_3)[g(Y,Q\xi)\xi - \eta(Q\xi)Y] \tag{6.6}$$

Thus, we have

$$(f_1 - f_3)\eta(Y)Q\xi - R(Q\xi,Y)\xi - (f_1 - f_3)\eta(QY)\xi - (f_1 - f_3)S(Y,\xi)\xi + (f_1 - f_3)S(\xi,\xi)Y \tag{6.7}$$

Taking inner product with $\xi$ in (6.7), we get

$$(f_1 - f_3)\eta(Y)g(Q\xi,\xi) - g(R(Q\xi,Y)\xi,\xi) - 2(f_1 - f_3)S(Y,\xi)\xi + (f_1 - f_3)S(\xi,\xi)\eta(Y) = 0 \tag{6.8}$$

From (6.8), we get

$$(f_1 - f_3)\eta(Y)S(\xi,\xi) - g((f_1 - f_3)[\eta(Y)Q\xi - \eta(Q\xi)Y],\xi) = 0 \tag{6.9}$$

Thus, we have

$$(f_1 - f_3)S(\xi,\xi)\eta(Y) - (f_1 - f_3)S(Y,\xi) = 0 \tag{6.10}$$

From (6.10), we get

$$(f_1 - f_3)^2(n-1)\eta(Y) + (f_1 - f_3)(\frac{\lambda}{p} - \frac{qr}{2p})\eta(Y) = 0 \tag{6.11}$$

Thus, we have

$$(f_1 - f_3)\eta(Y)[(f_1 - f_3)(n-1) + (\frac{\lambda}{p} - \frac{qr}{2p})] = 0 \tag{6.12}$$

From (6.12), we get

$$(\frac{\lambda}{p} - \frac{qr}{2p}) = -(n-1)(f_1 - f_3) = 0 \tag{6.13}$$

Thus, we have

$$\lambda = -(n-1)(f_1 - f_3)p + \frac{qr}{2} \tag{6.14}$$

now substituting $f_1$ and $f_3$ in (6.14) ,we get

$$\lambda = -(\frac{c+3}{4} - \frac{c-1}{4})(n-1)p + \frac{qr}{2} \tag{6.15}$$

From (6.15), we get

$$\lambda = -(n-1)p + \frac{qr}{2} \tag{6.16}$$

**Theorem:** A Ricci Yamabe solitons in Generalized Sasakian Space Form (GSSF) satisfying the curvature condition $Q.R = 0$ is shrinking or steady or expanding accordingly as:

$$-(n-1)p + \frac{qr}{2} > 0, -(n-1)p + \frac{qr}{2} = 0, -(n-1)p + \frac{qr}{2} < 0 \tag{6.17}$$

If p=0 then we get $\lambda = \frac{qr}{2}$ . thus, we can state

**Corollary:** A q-Yamabe soliton in Generalized-Sasakian Space Form (GSSF) satisfying the curvature condition $Q.R = 0$ is shrinking or expanding or steady accordingly as:

$$\frac{qr}{2} > 0, \frac{qr}{2} < 0, \frac{qr}{2} = 0 \tag{6.18}$$

If q=0 then we get $\lambda = -(n-1)p$. thus, we have

**Corollary:** A p-Ricci soliton in Generalized Sasakian Space Form (GSSF) satisfying the curvature condition $Q.R = 0$ is shrinking or expanding or steady as:

$$-(n-1)p > 0, -(n-1)p < 0, -(n-1)p = 0 \tag{6.19}$$

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