Quest Journals
Journal of Research in Applied Mathematics
Volume 10 ~ Issue 8 (2024) pp: 20-39
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

On General Wide Graph Algebras and Orbit Equivalence

AbdEllteaf Yahia(1) and Shawgy Hussein(2)
(1) Sudan University of Science and Technology, Sudan.
(2) Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.

Abstract
N.Brownlowe, T. M.Carlsen, and M. F. Whittaker [27] introduce the notion of orbit equivalence of directed graphs, following Matsumoto's notion of continuous orbit equivalence for topological Markov shifts. Theyshow that two graphs in which every cycle has an exit are orbit equivalent if and only if there is a diagonal-preserving isomorphism between their C^*-algebras. They show that it is necessary to assume that every cycle has an exit for the forward implication, but that the reverse implication holds for arbitrary graphs. As part of their analysis we follow their way to study of the arbitrary graphs $E_t$so we construct a groupoid$\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ from the graph algebra $C^*(E_t)$ and its diagonal subalgebra$\mathcal{D}(E_t)$ which generalises Renault's Weyl groupoid construction applied to $(C^*(E_t),\mathcal{D}(E_t))$. We show that $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$recovers the graph groupoid$\mathcal{G}_{E_t}$ without the assumption that every cycle in E_t has an exit, which is required to apply Renault's results to $(C^*(E_t),\mathcal{D}(E_t))$. We finish with applications of their results to out-splittings of graphs and to amplified graphs.
Keywords: *C^*-algebra, directed graph, orbit equivalence, groupoid.*

***Received** 15 Aug., 2024; **Revised** 28 Aug., 2024; **Accepted** 31 Aug., 2024*
Published** with open access at **www.questjournas.org

I. Introduction

The relationship between orbit equivalence and isomorphism of C^*-algebras has been studied extensively in the last 20 years. The first result of this type was the celebrated theorem of [5, Theorem 2.4], in which they showed that orbit equivalence for minimal dynamical systems on the Cantor set is equivalent to isomorphism of their corresponding crossed product C^*-algebras. The importance of Giordano, Putnam and Skau's result cannot be overstated. In general there is no direct method of checking whether two Cantor minimal systems are orbit equivalent. However, because the crossed product C^*-algebras are classifiable, Giordano, Putnam and Skau's result means that orbit equivalence can be determined using K-theory. The work in [5] has been generalised in many directions, including Tomiyama's results on topologically free dynamical systems on compact Hausdorff spaces [24], and their extension of [5, Theorem 2.4] to minimal $\mathbb{Z}^d$-actions on the Cantor set [6].

More recently, in [15]the authors have shown that two irreducible onesided topological Markov shifts (X_{A_t},σ_{A_t}) and (X_{B_t},σ_{B_t}) are continuously orbit equivalent if and only if the corresponding Cuntz-Krieger algebras $\mathcal{O}_{A_t}$ and $\mathcal{O}_{B_t}$ are isomorphic and $\det(I-A_t)=\det(I-B_t)$. The proof of Matsumoto and Matui's theorem relies on two key results. The first of these is [12, Theorem 1.1], in which Matsumoto proves that the following statements are equivalent:

(1) (X_{A_t},σ_{A_t}) and (X_{B_t},σ_{B_t}) are continuously orbit equivalent,
(2) there exists a $*$-isomorphism $\phi_t\colon\mathcal{O}_{A_t}\to\mathcal{O}_{B_t}$ which maps the maximal abelian subalgebra $\mathcal{D}_{A_t}$ onto $\mathcal{D}_{B_t}$, and
(3) the topological full group of (X_{A_t},σ_{A_t}) and the topological full group of (X_{B_t},σ_{B_t}) are spatially isomorphic.
(In [14, Theorem 1.1], Matsumoto showed that this is equivalent to the topological full groups being abstractly isomorphic.)

The second key result is [22, Proposition 4.13], which, as noticed by ([16, Theorem 5.1]), implies that there exists a $*$-isomorphism $\phi_t\colon\mathcal{O}_{A_t}\to\mathcal{O}_{B_t}$ that maps the maximal abelian subalgebra, or diagonal, $\mathcal{D}_{A_t}$ onto $\mathcal{D}_{B_t}$ if and only if the corresponding groupoids $\mathcal{G}_{A_t}$ and $\mathcal{G}_{B_t}$ are isomorphic.

In [27] the authors initiate the study of orbit equivalence of directed graphs, and they prove the analogous result to [22, Proposition 4.13] for graph algebras. In particular, as part of their main result we prove that if E_t and F_t

are two graphs in which every cycle has an exit, then the following are equivalent:
(1) There is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps the diagonal subalgebra $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$.
(2) The graph groupoids$\mathcal{G}_{E_t}$ and $\mathcal{G}_{E_t}$ are isomorphic as topological groupoids.
(3) The pseudogroups of E_t and F_t are isomorphic.
(4) The graphs E_t and F_t are orbit equivalent.

It is natural to ask whether every cycle having an exit is necessary for the results. In our main result we in fact prove that allimplicationhold for arbitrary directed graphs. It is only the mentioned implication that requires that every cycle has an exit (and we provide examples that show that this implication does not hold in general without the assumption that every cycle has an exit). Their analysis of these implications for arbitrary graphs provides their most technical innovation, which is the introduction of a groupoid$\mathcal{G}(C^*(E_t), \mathcal{D}(E_t))$ associated to $(C^*(E_t), \mathcal{D}(E_t))$ that we call the extended Weyl groupoid. The construction generalises Renault's Weyl groupoid construction from [22, Definition 4.11] applied to $(C^*(E_t), \mathcal{D}(E_t))$. We show that $\mathcal{G}_{(C^*(E_t), \mathcal{D}(E_t))}$ and $\mathcal{G}_{E_t}$ are isomorphic as topological groupoids for an arbitrary graph E_t, which can be deduced from Renault's results in [22] only when every cycle in E_t has an exit.

Two applications of the main theorem are considered. The first application shows that if two general graphs E_t and F_t are conjugate then there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$. As a corollary, we strengthen a result of [3, Theorem 3.2] on out-splitting of graphs. The second application adds three additional equivalences to Eilers, Ruiz, and Sørensen's complete invariant for amplified graphs [4, Theorem 1.1] (see [27]).

We provide background on graphs, their groupoids and their C^*-algebras. We define orbit equivalence of graphs and associate with each graph a pseudogroup which is the analogue of the topological full group Matsumoto has associated with each irreducible one-sided topological Markov shift, and we show that two graphs are orbit equivalent if and only if their pseudogroups are isomorphic. We also construct the extended Weyl groupoid$\mathcal{G}_{(C^*(E_t), \mathcal{D}(E_t))}$ from $(C^*(E_t), \mathcal{D}(E_t))$, and we show that $\mathcal{G}_{(C^*(E_t), \mathcal{D}(E_t))}$ and $\mathcal{G}_{E_t}$ are isomorphic as topological groupoids. We use this result to show that if there is a diagonal-preserving isomorphism from $C^*(E_t)$ to $C^*(F_t)$, then $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids. We finish the proof of the main theorem and provide examples. Finally, we give the two applications of the main theorem (see [27]).

Remark 1.1. We have learned that Xin Li has also considered orbit equivalence for directed graphs, and has independently proved that two graphs in which every cycle has an exit are orbit equivalent if and only if there is a diagonal-preserving isomorphism between their C^*-algebras.

II. Background on the Groupoidsand C^*-Algebras of Directed Graphs

We begin with graphs and their C^*-algebras. We recall the definitions of the boundary path space of a directed graph, graph C^*-algebras and graph groupoids.

2.1. Graphs and their C^*-algebras.For a more detailed treatment on graphs and their C^*-algebras (see [19]). However, we note that the directions of arrows defining a graph are reversed in this paper. We used this convention so that the results can easily be compared with the work of Matsumoto and Matui's work on shift spaces (see [27]).

A directed graph (also called a quiver) $E_t = (E_t^0, E_t^1, r_t, s_t)$ consists of countable sets E_t^0 and E_t^1, and range and source maps $r_t, s_t: E_t^1 \to E_t^0$. The elements of E_t^0 are called vertices, and the elements of E_t^1 are called edges.

A path μ of length n in E_t is a sequence of edges $\mu = \mu_1 ... \mu_n$ such that $r_t(\mu_i) = s_t(\mu_{i+1})$ for all $1 \leq i \leq n-1$. The set of paths of length n is denoted E_t^n. We denote by $|\mu|$ the length of μ. The range and source maps extend naturally to paths: $s_t(\mu) := s_t(\mu_1)$ and $r_t(\mu) := r_t(\mu_n)$. We regard the elements of E_t^0 as path of length 0, and for $v_t \in E_t^0$ we set $s_t(v_t) := r_t(v_t) := v_t$. For $v_t \in E_t^0$ and $n \in \mathbb{N}$ we denote by $v_t E_t^n$ the set of paths of length n with source v_t, and by $E_t^n v_t$ the paths of length n with range v_t. We define $E_t^* := \bigcup_{n \in \mathbb{N}} E_t^n$ to be the collection of all paths with finite length. For $v_t, w_t \in E_t^0$ let $v_t E_t^* w_t := \{\mu \in E_t^* : s_t(\mu) = v_t \text{ and } r_t(\mu) = w_t\}$. We define $(E_t^0)_{\text{reg}} := \{v_t \in E_t^0 : v_t E_t^1 \text{ is finite and nonempty }\}$ and $(E_t^0)_{\text{sing}} := E_t^0 \setminus (E_t^0)_{\text{reg}}$. If $\mu = \mu_1 \mu_2 \cdots \mu_m, \nu = \nu_1 \nu_2 \cdots \nu_n \in E_t^*$ and $r_t(\mu) = s_t(\nu)$, then we let $\mu\nu$ denote the path $\mu_1 \mu_2 \cdots \mu_m \nu_1 \nu_2 \cdots \nu_n$.

A loop (also called a cycle) in E_t is a path $\mu \in E_t^*$ such that $|\mu| \geq 1$ and $s_t(\mu) = r_t(\mu)$.

If μ is a loop and k is a positive integer, then μ^k denotes the loop $\mu\mu \cdots \mu$ where μ is repeated k-times. We say that the loop μ is simple if μ is not equal to ν^k for any loop ν and any integer $k \geq 2$. Notice than any loop μ is equal to ν^k for some simple loop ν and some positive integer k. An edge e is an exit to the loop μ if there exists i such that $s_t(e) = s_t(\mu_i)$ and $e \neq \mu_i$.A graph is said to satisfy condition (L) if every loop has an exit.

ACuntz-Krieger E_t-family $\{P, S\}$ consists of a set of mutually orthogonal projections $\{P_{v_t} : v_t \in E_t^0\}$ and partial isometries $\{S_e : e \in E_t^1\}$ satisfying

(CK1) $S_e^* S_e = P_{r_t(e)}$ for all $e \in E_t^1$;
(CK2) $S_e S_e^* \leq P_{s_t(e)}$ for all $e \in E_t^1$;

(CK3) $P_{v_t} = \sum_{e\in v_t E_t^1} S_e S_e^*$ for all $v_t \in (E_t^0)_{\text{reg}}$.

The graph C^*-algebra $C^*(E_t)$ is the universal C^*-algebra generated by a Cuntz-Krieger E_t-family. We denote by $\{p, s_t\}$ the Cuntz-Krieger E_t-family generating $C^*(E_t)$. There is a strongly continuous action $\gamma: C^*(E_t) \to \mathbb{T}$, called the gauge action, satisfying $\gamma_{z_t}(p_{v_t}) = p_{v_t}$ and $\gamma_{z_t}(s_e) = z_t s_e$, for all $z_t \in \mathbb{T}, v_t \in E_t^0, e \in E_t^1$. If $\{Q, T\}$ is a Cuntz-Krieger

E_t-family in a C^*-algebra B, then we denote by $\pi_{Q,T}$ the homomorphism $C^*(E_t) \to B$ such that $\pi_{Q,T}(p_{v_t}) = Q_{v_t}$ for all $v_t \in E_t^0$, and $\pi_{Q,T}(s_e) = T_e$ for all $e \in E_t^1$. an Huef and Raeburn's gauge invariant uniqueness theorem [7] says that $\pi_{Q,T}$ is injective if and only if there is an action β of $\mathbb{T}$ on the C^*-algebra generated by $\{Q, T\}$ satisfying $\beta_{z_t}(Q_{v_t}) = Q_{v_t}$ and $\beta_{z_t}(T_e) = z_t T_e$, for all $z_t \in \mathbb{T}, v_t \in E_t^0, e \in E_t^1$, and $Q_{v_t} \neq 0$ for all $v_t \in E_t^0$.

If $\mu = \mu_1 \cdots \mu_n \in E_t^n$ and $n \geq 2$, then we let $s_\mu := s_{\mu_1} \cdots s_{\mu_n}$. Likewise, we let $s_{v_t} := p_{v_t}$ if $v_t \in E_t^0$. Then $C^*(E_t) = \text{span}\{s_\mu s_\nu^* : \mu, \nu \in E_t^*, r_t(\mu) = r_t(\nu)\}$. The C^*-subalgebra $\mathcal{D}(E_t) := \text{span}\{s_\mu s_\mu^* : \mu \in E_t^*\}$ of $C^*(E_t)$ is a maximal abelian subalgebra if and only if every loop in E_t has an exit (see [17, Example 3.3]).

2.2. The boundary path space of a graph. An infinite path in E_t is an infinite sequence $(x_t)_1(x_t)_2 \ldots$ of edges in E_t such that $r_t(e_i) = s_t(e_{i+1})$ for all i. We let E_t^∞ be the set of all infinite paths in E_t. The source map extends to E_t^∞ in the obvious way. We let $|x_t| = \infty$ for $x_t \in E_t^\infty$. The boundary path space of E_t is the space

$$\partial E_t := E_t^\infty \cup \{\mu \in E_t^* : r_t(\mu) \in (E_t^0)_{\text{sing}}\}.$$

If $\mu = \mu_1\mu_2 \cdots \mu_m \in E_t^*, x_t = (x_t)_1(x_t)_2 \cdots \in E_t^\infty$ and $r_t(\mu) = s_t(x_t)$, then we let μx_t denote the infinite path $\mu_1\mu_2 \cdots \mu_m (x_t)_1(x_t)_2 \cdots \in E_t^\infty$.

For $\mu \in E_t^*$, the cylinder set of μ is the set

$$Z(\mu) := \{\mu x_t \in \partial E_t : x_t \in r_t(\mu)\, \partial E_t\},$$

where $r_t(\mu)\, \partial E_t := \{x_t \in \partial E_t : r_t(\mu) = s_t(x_t)\}$. Given $\mu \in E_t^*$ and a finite subset $F_t \subseteq r_t(\mu) E_t^1$ we define

$$Z(\mu \setminus F_t) := Z(\mu) \setminus \left(\bigcup_{e\in F_t} Z(\mu e) \right).$$

The boundary path space ∂E_t is a locally compact Hausdorff space with the topology given by the basis $\{Z(\mu \setminus F_t) : \mu \in E_t^*, F_t \text{ is a finite subset of } r_t(\mu)E_t^1\}$, and each such $Z(\mu \setminus F_t)$ is compact and open (see [25, Theorem 2.1 and Theorem 2.2]). Moreover, [25, Theorem 3.7] shows that there is a unique homeomorphism $(h_t)_{E_t}$ from ∂E_t to the spectrum of $\mathcal{D}(E_t)$ given by

$$(h_t)_{E_t}(x_t)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x_t \in Z(\mu), \\ 0 & \text{if } x_t \notin Z(\mu). \end{cases} \tag{2.1}$$

Our next lemma gives a description of the topology on the boundary path space, which we will need in the proof of Proposition 3.3 (see [27]).

Lemma 2.1. Every nonempty open subset of ∂E_t is the disjoint union of sets that are both compact and open.

Proof. Let U be a nonempty open subset of ∂E_t. For each $x_t \in U$ let

$B_{x_t} := \{(\mu, F_t) : \mu \in E_t^*, F_t \text{ is a finite subset of } r_t(\mu)E_t^1, x_t \in Z(\mu \setminus F_t) \subseteq U\}$.

If $(\mu, F_t) \in B_{x_t}$, then $x_t \in Z(\mu)$ and $x_t \notin Z(\mu e)$ for each $e \in F_t$. Let μ_{x_t} be the shortest $\mu \in E_t^*$ such that $(\mu, F_t) \in B_{x_t}$ for some finite subset F_t of $r_t(\mu)E_t^1$, and let $(F_t)_{x_t} := \cap \{F_t : (\mu_{x_t}, F_t) \in B_{x_t}\}$. Then $(\mu_{x_t}, (F_t)_{x_t}) \in B_{x_t}$ and $Z(\mu \setminus F_t) \subseteq Z(\mu_{x_t} \setminus (F_t)_{x_t})$ for all $(\mu, F_t) \in B_{x_t}$. It follows that if $x_t, y_t \in U$, then either $Z(\mu_{x_t} \setminus (F_t)_{x_t}) = Z(\mu_{y_t} \setminus (F_t)_{y_t})$ or $Z(\mu_{x_t} \setminus (F_t)_{x_t}) \cap Z(\mu_{y_t} \setminus (F_t)_{y_t}) = \emptyset$.

Since $U = \cup_{x_t\in U} Z(\mu_{x_t} \setminus (F_t)_{x_t})$ and each $Z(\mu_{x_t} \setminus (F_t)_{x_t})$ is open and compact, this shows that U is the disjoint union of sets that are both compact and open.

For $n \in \mathbb{N}$, let $\partial {E_t}^{\geq n} := \{x_t \in \partial E_t : |x_t| \geq n\}$. Then $\partial {E_t}^{\geq n} = \cup_{\mu\in E_t^n} Z(\mu)$ is an open subset of ∂E_t. We define the shift map on E_t to be the map $\sigma_{E_t} : \partial {E_t}^{\geq 1} \to \partial E_t$ given by $\sigma_{E_t}((x_t)_1(x_t)_2(x_t)_3 \cdots) = (x_t)_2(x_t)_3 \cdots$ for $(x_t)_1(x_t)_2(x_t)_3 \cdots \in \partial E_t^{\geq 2}$ and $\sigma_{E_t}(e) = r_t(e)$ for $e \in \partial E_t \cap E_t^1$.

For $n \geq 1$, we let $\sigma_{E_t}^n$ be the n-fold composition of σ_{E_t} with itself. We let $\sigma_{E_t}^0$ denote the identity map on ∂E_t. Then $\sigma_{E_t}^n$ is a local homeomorphism for all $n \in \mathbb{N}$. When we write $\sigma_{E_t}^n(x_t)$, we implicitly assume that $x_t \in \partial E_t^{\geq n}$. We say that $x_t \in \partial E_t$ is eventually periodic if there are $m, n \in \mathbb{N}, m \neq n$ such that $\sigma_{E_t}^m(x_t) = \sigma_{E_t}^n(x_t)$. Notice that $x_t \in \partial E_t$ is eventually periodic if and only if $x_t = \mu\nu\nu\nu \cdots$ for some path $\mu \in E_t^*$ and some loop $\nu \in E_t^*$ with $s_t(\nu) = r_t(\mu)$. By replacing ν by a subloop if necessary, we can assume that ν is a simple loop.

2.3. Graph groupoids. In [11],the authors defined groupoidC^*-algebras associated to a locally-finite directed graph with no sources. Their construction has been generalized to compactly aligned topological k-graphs in [26]. We will now explain this construction in the case that E_t is an arbitrary graph. The resulting groupoid is isomorphic to the one constructed in [18]. Let

$$\mathcal{G}_{E_t} := \{(x_t, m-n, y_t): x_t, y_t \in \partial E_t, m, n \in \mathbb{N}, \text{ and } \sigma^m(x_t) = \sigma^n(y_t)\},$$

with product $(x_t, k, y_t)(w_t, l, z_t) := (x_t, k+l, z_t)$ if $y_t = w_t$ and undefined otherwise, and inverse given by $(x_t, k, y_t)^{-1} := (y_t, -k, x_t)$. With these operations $\mathcal{G}_{E_t}$ is a groupoid (cf. [11, Lemma 2.4]). The unit space $\mathcal{G}_{E_t}^0$ of $\mathcal{G}_{E_t}$ is $\{(x_t, 0, x_t): x_t \in \partial E_t\}$ which we will freely identify with ∂E_t via the map $(x_t, 0, x_t) \mapsto x_t$ throughout the paper. We then have that the range and source maps $r_t, s_t: \mathcal{G}_{E_t} \to \partial E_t$ are given by $r_t(x_t, k, y_t) = x_t$ and $s_t(x_t, k, y_t) = y_t$.

We now define a topology on $\mathcal{G}_{E_t}$. Suppose $m, n \in \mathbb{N}$ and U is an open subset of $\partial E_t^{\geq m}$ such that the restriction of $\sigma_{E_t}^m$ to U is injective, V is an open subset of $\partial E_t^{\geq n}$ such that the restriction of $\sigma_{E_t}^n$ to V is injective, and that $\sigma_{E_t}^m(U) = \sigma_{E_t}^n(V)$, then we define

$$Z(U, m, n, V) := \{(x_t, k, y_t) \in \mathcal{G}_{E_t}: x_t \in U, k = m-n, y_t \in V, \sigma_{E_t}^m(x_t) = \sigma_{E_t}^n(y_t)\}. \quad (2.2)$$

Then $\mathcal{G}_{E_t}$ is a locally compact, Hausdorff, étale topological groupoid with the topology generated by the basis consisting of sets $Z(U, m, n, V)$ described in (2.2), see [11, Proposition 2.6] for an analogous situation. For $\mu, \nu \in E_t^*$ with $r_t(\mu) = r_t(\nu)$, let $Z(\mu, \nu) := Z(Z(\mu), |\mu|, |\nu|, Z(\nu))$. It follows that each $Z(\mu, \nu)$ is compact and open, and that the topology ∂E_t inherits when we consider it as a subset of $\mathcal{G}_{E_t}$ by identifying it with $\{(x_t, 0, x_t): x_t \in \partial E_t\}$ agrees with the topology described in the previous section.

Notice that for all $\mu, \nu \in E_t^*, U$ a compact open subset of $Z(\mu)$, and V a compact open subset of $Z(\nu)$, the collection $\{Z(U, |\mu|, |\nu|, V): \sigma_{E_t}^{|\mu|}(U) = \sigma_{E_t}^{|\nu|}(V)\}$ is a basis for the topology of $\mathcal{G}_{E_t}$. According to [26, Proposition 6.2], $\mathcal{G}_{E_t}$ is topologically amenable in the sense of [1, Definition 2.2.8]. It follows from [1, Proposition 3.3.5] and [1, Proposition 6.1.8] that the reduced and universal C^*-algebras of $\mathcal{G}_{E_t}$ are equal, and we denote this C^*-algebra by $C^*(\mathcal{G}_{E_t})$. We have (see [27]).

Proposition 2.2 (Cf. [11, Proposition 4.1]). Suppose E_t is a graph. Then there is a unique isomorphism $\pi_t: C^*(E_t) \to C^*(\mathcal{G}_{E_t})$ such that $\pi_t(p_{v_t}) = 1_{Z(v_t, v_t)}$ for all $v_t \in E_t^0$ and $\pi_t(s_e) = 1_{Z(e, r_t(e))}$ for all $e \in E_t^1$, and such that $\pi_t(\mathcal{D}(E_t)) = C_0(\mathcal{G}_{E_t}^0)$.

Proof. Using calculations along the lines of those used in the proof of [11, Proposition 4.1], it is straight forward to check that

$$\{Q, T\} := \{Q_{v_t} := 1_{Z(v_t, v_t)} \text{ and } T_e := 1_{Z(e, r_t(e))}: v_t \in E_t^0, e \in E_t^1\}$$

is a Cuntz-Krieger E_t-family. The universal property of $\{p, s\}$ implies that there is a $*$-homomorphism $\pi_t := (\pi_t)_{Q,T}: C^*(E_t) \to C^*(\mathcal{G}_{E_t})$ satisfying $\pi_t(p_{v_t}) = Q_{v_t}$ and $\pi_t(s_e) = T_e$.

An argument similar to the one used in the proof of [11, Proposition 4.1] shows that $C^*(\mathcal{G}_{E_t})$ is generated by $\{Q, T\}$, so π_t is surjective. The cocycle $(x_t, k, y_t) \mapsto k$ induces an action β of $\mathbb{T}$ on $C^*(\mathcal{G}_E)$ satisfying $\beta_{z_t}(Q_{v_t}) = Q_{v_t}$ and $\beta_{z_t}(T_e) = z_t T_e$, for all $z_t \in \mathbb{T}, v_t \in E_t^0, e \in E_t^1$ (see [21, Proposition $II.5.1$]), and since $Q_{v_t} = 1_{Z(v_t, v_t)} \neq 0$ for all $v_t \in E_t^0$, the gauge invariant uniqueness theorem of $C^*(\mathcal{G}_{E_t})$ ([2, Theorem 2.1]) implies that π_t is injective.

Since $\mathcal{D}(E_t)$ is generated by $\{s_\mu s_\mu^*: \mu \in E_t^*\}$ and $\pi_t(s_\mu s_\mu^*) = 1_{Z(\mu,\mu)}$, we have that π_t maps $\mathcal{D}(E_t)$ into $C_0(\mathcal{G}_{E_t}^0)$. An application of the Stone-Weierstrass theorem implies that $C_0(\mathcal{G}_{E_t}^0)$ is generated by $\{1_{Z(\mu,\mu)}: \mu \in E_t^*\}$. Hence $\pi_t(\mathcal{D}(E_t)) = C_0(\mathcal{G}_{E_t}^0)$.

Suppose $\mathcal{G}$ is a groupoid, the isotropy group of $x_t \in \mathcal{G}^0$ is the group $\mathrm{Iso}(x_t) := \{\gamma \in \mathcal{G}: s_t(\gamma) = r_t(\gamma) = x_t\}$. In [22], an étale groupoid is said to be topologically principal if the set of points of $\mathcal{G}^0$ with trivial isotropy group is dense. We will now characterize when $\mathcal{G}_{E_t}$ is topologically principal (see [27]).

Proposition 2.3. Let E_t be a graph. Then the graph groupoid $\mathcal{G}_{E_t}$ is topologically principal if and only if every loop in E_t has an exit.

Proof. Let $x_t \in \partial E_t$. We claim that $(x_t, 0, x_t)$ has nontrivial isotropy group if and only if x_t is eventually periodic. Indeed, suppose $(x_t, m-n, x_t) \in \mathrm{Iso}(x_t)$ with $m \neq n$, then $\sigma^m(x_t) = \sigma^n(x_t)$ and x_t is eventually periodic. On the other hand, suppose $x_t = \mu\lambda^\infty$, then $(x_t, (|\mu| + |\lambda|) - |\mu|, x_t) \in \mathrm{Iso}(x_t)$, proving the claim. Now observe that if v_t is a vertex such that there are two different simple loops α and β with $s_t(\alpha) = s_t(\beta) = v_t$, then any cylinder set $Z(\delta)$ for which $r_t(\delta) E_t^* v_t \neq \emptyset$ contains a y_t such that $(y_t, 0, y_t)$ has trivial isotropy. To see this, pick $\lambda \in r_t(\delta) E_t^* v_t$, then $y_t = \delta\lambda\alpha\beta\alpha^2\beta\alpha^3\beta \cdots$ has trivial isotropy since it is not eventually periodic.

Assume that every loop in E_t has an exit and suppose for contradiction that U is an nonempty open subset of ∂E_t such that $(x_t, 0, x_t)$ has nontrivial isotropy group for every $x_t \in U$. Note that $U \subseteq E_t^\infty$ since $y_t \in \partial E_t$ with $|y_t| < \infty$ implies that the isotropy group of $(y_t, 0, y_t)$ is trivial. Let $x_t \in U$. Since x_t has nontrivial isotropy group, there exist $\zeta_1 \in E_t^*$ and a loop η such that $\zeta_1\eta^\infty \in Z(\zeta_1\eta^k) \subseteq U$ for some $k \in \mathbb{N}$. Since η has an exit and $(x_t, 0, x_t)$ has nontrivial isotropy group for every $x_t \in U$, it follows that there is a $\zeta_2 \in r_t(\zeta_1) E_t^*$ such that $Z(\zeta_1\zeta_2) \subseteq U$ and such that $r_t(\zeta_2) E_t^* r_t(\zeta_1) = \emptyset$, for otherwise there would be two distinct simple loops based at $r_t(\zeta_1)$. By repeating this argument we get a sequences of paths $\zeta_1, \zeta_2, \zeta_3, \ldots$ such that $s_t(\zeta_{n+1}) =$

$r_t(\zeta_n), r_t(\zeta_{n+1})E_t^* r_t(\zeta_n) = \emptyset$ and $Z(\zeta_1\zeta_2 \dots \zeta_n) \subseteq U$ for all n. The element $y_t = \zeta_1\zeta_2\zeta_3 \dots$ then belongs to U, but since it only visits each vertex a finite number of times, $(y_t, 0, y_t)$ must have trivial isotropy, which contradicts the assumption that $(x_t, 0, x_t)$ has nontrivial isotropy group for every $x_t \in U$. Thus, $\mathcal{G}_{E_t}$ is topologically principal if every loop in E_t has an exit.

Conversely, if μ is a loop without exit and $x_t = \mu\mu\mu \dots$ then $(x_t, 0, x_t)$ is an isolated point in $\mathcal{G}_{E_t}^0$ with nontrivial isotropy group. Thus, $\mathcal{G}_{E_t}$ is not topologically principal if there is a loop in E_t without an exit.

Since the reduced and universal C^*-algebras of $\mathcal{G}_{E_t}$ are equal, it follows from [21, Proposition $II.4.2(i)$] that we can regard $C^*(\mathcal{G}_{E_t})$ as a subset of $C_0(\mathcal{G}_{E_t})$. For $f_t \in C^*(\mathcal{G}_{E_t})$ and $j \in \mathbb{Z}$, we let $\Phi_j(f_t)$ denote the restriction of f_t to $\{(x_t, k, y_t) \in \mathcal{G}_{E_t} : k = j\}$, and for $m \in \mathbb{N}$ we let $\Sigma_m(f_t) := \sum_{j=-m}^{m} \sum_t \left(1 - \frac{|j|}{m+1}\right) \Phi_j(f_t)$.

Proposition 2.4 (see [27]). Let E_t be a graph and let $f_t \in C^*(\mathcal{G}_{E_t})$. Then each $\Phi_k(f_t)$ and each $\Sigma_m(f_t)$ belong to $C^*(\mathcal{G}_{E_t})$, and $(\Sigma_m(f_t))_{m\in\mathbb{N}}$ converges to f_t in $C^*(\mathcal{G}_{E_t})$.

Proof. Let $j \in \mathbb{Z}$. The map $(x_t, k, y_t) \mapsto k$ is a continuous cocycle from $\mathcal{G}_{E_t}$ to $\mathbb{Z}$. For each $z_t \in \mathbb{T}$ there is a unique automorphism γ_{z_t} on $C^*(\mathcal{G}_{E_t})$ such that $\gamma_{z_t}(g_t)(x_t, k, y_t) = z_t^k g_t(x_t, k, y_t)$ for $g_t \in C^*(\mathcal{G}_{E_t})$ and $(x_t, k, y_t) \in \mathcal{G}_{E_t}$, and that the *map* $z_t \mapsto \gamma_{z_t}$ is a strongly continuous action of $\mathbb{T}$on $C^*(\mathcal{G}_{E_t})$ (see [21, Proposition $II.5.1$]). It follows that the integral $\int_{\mathbb{T}} \sum_t \gamma_{z_t}(f_t) z_t^{-j} dz_t$, where dz_t denotes the normalized Haar measure on $\mathbb{T}$, is welldefined and belongs to $C^*(\mathcal{G}_{E_t})$ (see for example [20, Section $C.2$]). Let $(x_t, k, y_t) \in \mathcal{G}_{E_t}$.

If $k \neq j$, then

$$\int_{\mathbb{T}} \sum_t \gamma_{z_t}(f_t) z_t^{-j} dz_t (x_t, k, y_t) = \int_{\mathbb{T}} \sum_t z_t^{k-j} dz_t f_t(x_t, k, y_t) = 0,$$

and if $k = j$, then

$$\int_{\mathbb{T}} \sum_t \gamma_{z_t}(f_t) z_t^{-j} dz_t (x_t, k, y_t) = \int_{\mathbb{T}} \sum_t z_t^{k-j} dz_t f_t(x_t, k, y_t) = f_t(x_t, k, y_t).$$

Thus, $\Phi_j(f_t) = \int_{\mathbb{T}} \sum_t \gamma_{z_t}(f_t) z_t^{-j} dz_t$ from which it follows that $\Phi_j(f_t) \in C^*(\mathcal{G}_{E_t})$.

Since each $\Sigma_m(f_t)$ is a linear combination of functions of the form $\Phi_j(f_t)$, each $\Sigma_m(f_t)$ belongs to $C^*(\mathcal{G}_{E_t})$.

For $m \in \mathbb{N}$, let $\sigma_m : \mathbb{T} \to \mathbb{R}$ be the Fejér's kernel defined by

$$\sigma_m(z_t) = \sum_{j=-m}^{m} \sum_t \left(1 - \frac{|j|}{m+1}\right) z_t^{-j}.$$

Then $\sigma_m(z_t) \geq 0$ for all $z_t \in \mathbb{T}$, $\int_T \sigma_m(z_t) dz_t = 1$, and

$$\begin{aligned} \Sigma_m(f_t) &= \sum_{j=-m}^{m} \sum_t \left(1 - \frac{|j|}{m+1}\right) \Phi_j(f_t) \\ &= \sum_{j=-m}^{m} \left(1 - \frac{|j|}{m+1}\right) \int_{\mathbb{T}} \sum_t \gamma_{z_t}(f_t) z_t^{-j} dz_t = \int_{\mathbb{T}} \sum_t \gamma_{z_t}(f_t) \sigma_m(z_t) dz_t. \end{aligned}$$

Thus

$$\|\Sigma_m(f_t)\| \leq \int_{\mathbb{T}} \sum_t \|\gamma_{z_t}(f_t)\| \sigma_m(z_t) dz_t = \sum_t \| f_t \|.$$

If $g_t \in C_c(\mathcal{G}_{E_t})$, then there is an $m_0 \in \mathbb{N}$ such that $\Phi_j(g_t) = 0$ for $|j| > m_0$. It follows that

$$\begin{aligned} \sum_t \|g_t - \Sigma_m(g_t)\| &= \sum_t \left\| g_t - \sum_{j=-m}^{m} \left(1 - \frac{|j|}{m+1}\right) \Phi_j(g_t) \right\| \\ &\leq \sum_t \left\| g_t - \sum_{j=-m}^{m} \Phi_j(g_t) \right\| + \sum_t \left\| \sum_{j=-m}^{m} \left(\frac{|j|}{m+1}\right) \Phi_j(g_t) \right\| \\ &\leq \sum_t \left\| \sum_{j=-m}^{m} \left(\frac{|j|}{m+1}\right) \Phi_j(g_t) \right\| \\ &\leq \sum_t \left\| \sum_{j=-m_0}^{m_0} \left(\frac{|j|}{m+1}\right) \Phi_j(g_t) \right\| \quad \text{for } m \geq m_0 \\ &\leq \sum_{j=-m_0}^{m_0} \sum_t \frac{|j|}{m+1} \|\Phi_j(g_t)\| \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

Thus, for any $\epsilon > 0$ there exists $g_t \in C_c(\mathcal{G}_{E_t})$ and an $M \in \mathbb{N}$ such that $\| f_t - g_t \| < \epsilon/3$ and $\|g_t - \Sigma_m(g_t)\| < \epsilon/3$ for any $m \geq M$, and then

$$\|f_t - \Sigma_m(f_t)\| \leq \| f_t - g_t \| + \|g_t - \Sigma_m(g_t)\| + \|\Sigma_m(g_t - f_t)\| < \epsilon$$

for any $m \geq M$. This shows that $(\Sigma_m(f_t))_{m\in\mathbb{N}}$ converges to f_t in $C^*(\mathcal{G}_{E_t})$.

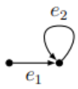
III. Orbit Equivalence and Pseudogroups

Here we introduce the notion of orbit equivalence of two graphs, which is a natural generalisation of Matsumoto's continuous orbit equivalence for topological Markov shifts from [12]. We also define the pseudogroup of a graph using Renault's pseudogroups associated to groupoids[22], and then show that two graphs are orbit equivalent if and only if their pseudogroups are isomorphic (see [27]).

Definition 3.1. Two graphs E_t and F_t are orbit equivalent if there exist a homeomorphism $h_t\colon \partial E_t \to \partial F_t$and continuous functions $k_1, l_1\colon \partial E_t^{\geq 1} \to \mathbb{N}$ and $k_1', l_1'\colon \partial F_t \geq^1\to \mathbb{N}$ such that

$$\sigma_{F_t}^{k_1(x_t)}\left(h_t\left(\sigma_{E_t}(x_t)\right)\right) = \sigma_{F_t}^{l_1(x_t)}(h_t(x_t)) \text{ and } \sigma_{E_t}^{k_1'(y_t)}\left(h_t^{-1}\left(\sigma_{F_t}(y_t)\right)\right) \sigma_{E_t}^{l_1'(y_t)}(h_t^{-1}(y_t)), \tag{3.1}$$

for all $x_t \in \partial E_t^{\geq 1}, y_t \in \partial F_t \geq 1$.



Example 3.2 (see [27]). Consider the graphs

$$(f_t)_1$$
$$E_t F_t$$
$$(f_t)_2$$

Then $\partial E_t = \{e_1e_2e_2 \dots, e_2e_2 \dots\}$ and $\partial F_t = \{(f_t)_1(f_t)_2(f_t)_1(f_t)_2 \dots, (f_t)_2(f_t)_1(f_t)_2(f_t)_1 \dots\}$. The $map\ h_t$: $\partial E_t \to \partial F_t$ given by

$$h_t(e_1e_2e_2 \dots) = (f_t)_1(f_t)_2(f_t)_1(f_t)_2 \dots \text{ and } h_t(e_2e_2 \dots) = (f_t)_2(f_t)_1(f_t)_2(f_t)_1 \dots$$

is a homeomorphism. Consider $k_1, l_1\colon \partial E_t^{\geq 1} \to \mathbb{N}$ given by $k_1(e_1e_2e_2 \dots) = 1$ and $k_1(e_2e_2 \dots) = 0$, and $l_1(e_1e_2e_2 \dots) = 0 = l_1(e_2e_2 \dots)$. Then k_1 and l_1 are continuous, and we have $\sigma_{F_t}^{k_1(x_t)}\left(h_t\left(\sigma_{E_t}(x_t)\right)\right) = \sigma_{F_t}^{l_1(x_t)}(h_t(x_t))$ for all $x_t \in \partial E_t^{\geq 1}$. Similarly the functions $k_1', l_1'\colon \partial F_t^{\geq 1} \to \mathbb{N}$ given by
$k_1'((f_t)_1(f_t)_2(f_t)_1(f_t)_2 \dots) = 0$ and $k_1'((f_t)_2(f_t)_1(f_t)_2(f_t)_1 \dots) = 1$, and
$l_1'((f_t)_1(f_t)_2(f_t)_1(f_t)_2 \dots) = 1$ and $l_1'((f_t)_2(f_t)_1(f_t)_2(f_t)_1 \dots) = 0$, are continuous and satisfy

$$\sigma_{E_t}^{k_1'(y_t)}\left(h_t^{-1}\left(\sigma_{F_t}(y_t)\right)\right) = \sigma_{E_t}^{l_1'(y_t)}(h_t^{-1}(y_t)) \quad \text{for all } y_t \in \partial F_t^{\geq 1}.$$

Hence E_t and F_t are orbit equivalent.

Sections 5 and 6 contain further examples of orbit equivalent graphs.

In Section 3 of [22], Renault constructs for each étale groupoid $\mathcal{G}$ a pseudogroup in the following way: Define a bisection to be a subset A_t of $\mathcal{G}$ such that the restriction of the source map of $\mathcal{G}$ to A_t and the restriction of the range map of $\mathcal{G}$ to A_t both are injective.

The set of all open bisections of $\mathcal{G}$ forms an inverse semigroup $\mathcal{S}$ with product defined by $A_tB_t = \{\gamma\gamma'\colon (\gamma, \gamma') \in (A_t \times B_t) \cap \mathcal{G}^{(2)}\}$ (where $\mathcal{G}^{(2)}$ denote the set of composable pairs of), and the inverse of A_t is defined to be the image of A_t under the inverse map of $\mathcal{G}$. Each $A_t \in \mathcal{S}$ defines a unique homeomorphism $\alpha_{A_t}\colon s_t(A_t) \to r_t(A_t)$ such that $\alpha(s_t(\gamma)) = r_t(\gamma)$ for $\gamma \in A_t$. The set $\{\alpha_{A_t}\colon A_t \in \mathcal{S}\}$ of partial homeomorphisms on $\mathcal{G}^0$ is the pseudogroup of $\mathcal{G}$.

When E_t is a graph, then we call the pseudogroup of the étale groupoid $\mathcal{G}_{E_t}$ the pseudogroup of E_t and denote it by $\mathcal{P}_{E_t}$.

We will now give two alternative characterizations of the partial homeomorphisms of ∂E_t that belong to $\mathcal{P}_{E_t}$. (see [27]).

Proposition 3.3. Let E_t be a graph, let U and V be open subsets of ∂E_t, and let $\alpha\colon V \to U$ be a homeomorphism. Then the following are equivalent:

(1) $\alpha \in \mathcal{P}_{E_t}$.

(2) For all $x_t \in V$, there exist $m, n \in \mathbb{N}$ and an open subset V' such that $x_t \in V' \subseteq V$, and such that $\sigma_{E_t}^m(x_t') = \sigma_{E_t}^n\left(\alpha(x_t')\right)$ for all $x' \in V'$.

(3) There exist continuous functions $m, n\colon V \to \mathbb{N}$ such that $\sigma_{E_t}^{m(x_t)}(x_t) = \sigma_{E_t}^{n(x_t)}(\alpha(x_t))$ for all $x_t \in V$.

Proof.(1) $\Rightarrow$ (2): Suppose $\alpha \in \mathcal{P}_{E_t}$. Let $A_t \in \mathcal{S}$ be such that $\alpha = \alpha_{A_t}$. Let $x_t \in V$. Then there is a unique $\gamma \in A_t$ such that $s_t(\gamma) = x_t$, and then $r_t(\gamma) = \alpha(x_t)$. Since A_t is an open subset of $\mathcal{G}_{E_t}$, there are $m, n \in \mathbb{N}$, an open subset U' of $\partial E_t^{\geq m}$ such that the restriction of $\sigma_{E_t}^m$ to U' is injective, and an open subset V' of $\partial E_t^{\geq n}$ such that the restriction of $\sigma_{E_t}^n$ to V' is injective and $\sigma_{E_t}^m(U') = \sigma_{E_t}^n(V')$, and such that $\gamma \in Z(U', m, n, V') \subseteq A_t$. Then $x_t \in V' \subseteq V$ and $\sigma_{E_t}^m(x_t') = \sigma_{E_t}^n(\alpha(x_t'))$ for all $x_t' \in V'$.
(2) $\Longrightarrow$ (3): Assume that for all $x_t \in V$, there exist $m, n \in \mathbb{N}$ and an open subset V' such that $x_t \in V' \subseteq V$, and such that $\sigma_{E_t}^m(x_t') = \sigma_{E_t}^n(\alpha(x_t'))$ for all $x_t' \in V'$. According to Lemma 2.1, V is the disjoint union of sets that are both compact and open. Since ∂E_t is locally compact, it follows that there exists a family $\{V_i : i \in I\}$ of mutually disjoint compact and open sets and a family $\{(m_i, n_i) : i \in I\}$ of pairs of nonnegative integers such that $V = \bigcup_{i \in I} V_i$ and $\sigma_{E_t}^{m_i}(x_t) = \sigma_{E_t}^{n_i}(\alpha(x_t))$ for $x_t \in V_i$. Define $m, n : V \to \mathbb{N}$ by setting $m(x_t) = m_i$ and $n(x_t) = n_i$ for $x_t \in V_i$. Then m and n are continuous and $\sigma_{E_t}^{m(x_t)}(x_t) = \sigma_{E_t}^{n(x_t)}(\alpha(x_t))$ for all $x_t \in V$.
(3) $\Longrightarrow$ (1): Assume that $m, n : V \to \mathbb{N}$ are continuous functions such that $\sigma_{E_t}^{m(x_t)}(x_t) = \sigma_{E_t}^{n(x_t)}(\alpha(x_t))$ for all $x_t \in V$. Then there exist for each $x_t \in V$ a compact and open subset V_{x_t} such that $x_t \in V_{x_t} \subseteq V, m(x_t') = m(x_t)$ and $n(x_t') = n(x_t)$ for all $x_t' \in V_{x_t}$, the restriction of $\sigma_{E_t}^{n(x_t)}$ to V_{x_t} is injective, and the restriction of $\sigma_{E_t}^{m(x_t)}$ of $\alpha(V_{x_t})$ is injective. According to Lemma 2.1, V is the disjoint union of sets that are both compact and open. It follows that there exists a family $\{V_i : i \in I\}$ of mutually disjoint compact and open sets and a family $\{(m_i, n_i) : i \in I\}$ of pairs of nonnegative integers such that $V = \bigcup_{i \in I} V_i, m(x_t) = m_i$ and $n(x_t) = n_i$ for all $x_t \in V_i$, the restriction of $\sigma_{E_t}^{n_i}$ to V_i is injective, and the restriction of $\sigma_{E_t}^{m_i}$ of $\alpha(V_i)$ is injective. Let $A_t := \bigcup_{i \in I} Z(\alpha(V_i), m_i, n_i V_i)$.Then $A_t \in \mathcal{S}$ and $\alpha = \alpha_{A_t}$, so $\alpha \in \mathcal{P}_{E_t}$.
Suppose that E_t and F_t are two graphs and that there exists a homeomorphism $h_t : \partial E_t \to \partial F_t$. Let U and V be open subsets of ∂E_t and let $\alpha : V \to U$ be a homeomorphism.
We denote by $h_t \circ \mathcal{P}_{E_t} \circ h_t^{-1} := \{h_t \circ \alpha \circ h_t^{-1} : \alpha \in \mathcal{P}_{E_t}\}$. We say that the pseudogroups of E_t and F_t are isomorphic if there is a homeomorphism $h_t : \partial E_t \to \partial F_t$ such that $h_t \circ \mathcal{P}_{E_t} \circ h_t^{-1} = \mathcal{P}_{F_t}$. We can now state the main result of this section (see [27]).
Proposition 3.4. Let E_t and F_t be two graphs. Then E_t and F_t are orbit equivalent if and only if the pseudogroups of E_t and F_t are isomorphic.
To prove this Proposition we will use the following result.
Lemma 3.5 [27]. Suppose two graphs E_t and F_t are orbit equivalent, $h_t : \partial E_t \to \partial F_t$ is a homeomorphism and $k_1, l_1 : \partial E_t^{\geq 1} \to \mathbb{N}$ and $k_1', l_1' : \partial F_t^{\geq 1} \to \mathbb{N}$ are continuous functions satisfying (3.1). Let $n \in \mathbb{N}$. Then there exist continuous functions $k_n, l_n : \partial E_t^{\geq n} \to \mathbb{N}$ and $k_n', l_n' : \partial F_t^{\geq n} \to \mathbb{N}$ such that

$$\sigma_{F_t}^{k_n(x_t)}\left(h_t\left(\sigma_{E_t}^n(x_t)\right)\right) = \sigma_{F_t}^{l_n(x_t)}\left(h_t(x_t)\right) \quad \textit{and} \quad \sigma_{E_t}^{k_n'(y_t)}\left(h_t^{-1}\left(\sigma_{F_t}^n(y_t)\right)\right) = \sigma_{E_t}^{l_n'(y_t)}(h_t^{-1}(y_t)), \tag{3.2}$$

for all $x_t \in \partial E_t^{\geq n}, y_t \in \partial F_t \geq n$.
Proof. There is nothing to prove for $n = 0$ and $n = 1$. We will prove the general case by induction. Let $m \geq 1$ and suppose that the lemma holds for $n = m$. Let $x_t \in \partial E_t^{\geq m+1}$. Then

$$\sigma_{F_t}^{k_1\left(\sigma_{E_t}^m(x_t)\right)}\left(h_t\left(\sigma_{E_t}^{m+1}(x_t)\right)\right) = \sigma_{F_t}^{l_1\left(\sigma_{E_t}^m(x_t)\right)}\left(h_t\left(\sigma_{E_t}^m(x_t)\right)\right)$$

and

$$\sigma_{E_t}^{k_m(x_t)}\left(h_t\left(\sigma_{E_t}^m(x_t)\right)\right) = \sigma^{l_m(x_t)}\left(h_t(x_t)\right).$$

Let

$$k_{m+1}(x_t) \quad := k_1\left(\sigma_{E_t}^m(x_t)\right) + \max\{l_1\left(\sigma_{E_t}^m(x_t)\right), k_m(x_t)\} - l_1\left(\sigma_{E_t}^m(x_t)\right) \tag{3.3}$$

$$l_{m+1}(x_t) \quad := l_m(x_t) + \max\left\{l_1\left(\sigma_{E_t}^m(x_t)\right), k_m(x_t)\right\} - k_m(x_t). \tag{3.4}$$

Then

$$\sigma_{F_t}^{k_{m+1}(x_t)}\left(h_t\left(\sigma_{E_t}^{m+1}(x_t)\right)\right) = \sigma_{F_t}^{l_{m+1}(x_t)}\left(h_t(x_t)\right).$$

Since k_1, l_1, k_m, and l_m are continuous, it follows that $k_{m+1}, l_{m+1} : \partial E_t^{\geq m+1} \to \mathbb{N}$ defined by (3.3) and (3.4) are also continuous.
Similarly, if we define $k_{m+1}', l_{m+1}' : \partial F_t \geq m+1 \to \mathbb{N}$ by letting

$$k_{m+1}'(y_t) \quad := k_1'\left(\sigma_{F_t}^m(y_t)\right) + \max\{l_1'\left(\sigma_{F_t}^m(y_t)\right), k_m'(y_t)\} - l_1'\left(\sigma_{F_t}^m(y_t)\right)$$

$$l_{m+1}'(y_t) \quad := l_m(y_t) + max\{l_1'\left(\sigma_{F_t}^m(y_t)\right), k_m'(y_t)\} - k_m'(y_t)$$

for $y_t \in \partial F_t \geq m+1$, then k_{m+1}' and l_{m+1}' are continuous, and

$$\sigma_{E_t}^{k_{m+1}'(y_t)}\left(h_t^{-1}\left(\sigma_{F_t}^{m+1}(y_t)\right)\right) = \sigma_{E_t}^{l_{m+1}'(y_t)}(h_t^{-1}(y_t))$$

for all $y_t \in \partial E_t^{\geq m+1}$. Thus, the lemma also holds for $n = m + 1$, and the general result holds by induction.

Proof of Proposition 3.4. Suppose E_t and F_t are orbit equivalent. Then there exists a homeomorphism $h_t\colon \partial E_t \to \partial F_t$ and, for each $n \in \mathbb{N}$, there exists continuous functions $k_n, l_n\colon \partial E_t^{\geq n} \to \mathbb{N}$ satisfying the first equation of (3.2). Let $(\alpha\colon V \to U) \in \mathcal{P}_{E_t}$, and let $m, n\colon V \to \mathbb{N}$ be continuous functions such that $\sigma_{E_t}^{m(x_t)}(x_t) = \sigma_{E_t}^{n(x_t)}(\alpha(x_t))$ for all $x_t \in V$.
Let $y_t \in h_t(V)$. Then

$$\sigma_{F_t}^{l_{n(h_t^{-1}(y_t))}(\alpha(h_t^{-1}(y_t)))}\left(h_t\left(\alpha(h_t^{-1}(y_t))\right)\right) = \sigma_{F_t}^{k_{n(h_t^{-1}(y_t))}(\alpha(h_t^{-1}(y_t)))}\left(h_t\left(\sigma_{E_t}^{n(h_t^{-1}(y_t))}\left(\alpha(h_t^{-1}(y_t))\right)\right)\right)$$
$$= \sigma_{F_t}^{k_{n(h_t^{-1})(y_t)})}\left(\alpha(h_t^{-1}(y_t))\right)\left(h_t\left(\sigma_{E_t}^{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t))\right)\right),$$

and

$$\sigma_{F_t}^{k_{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t))}\left(h_t\left(\sigma_{E_t}^{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t))\right)\right) = \sigma_{F_t}^{l_{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t))}(y_t).$$

So if we let

$$m'(y_t) := l_{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t)) + \max\left\{k_{n(h_t^{-1}(y_t))}\left(\alpha(h_t^{-1}(y_t))\right), k_{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t))\right\} - k_{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t)) \tag{3.5}$$

and

$$n'(y_t) := l_{n(h_t^{-1}(y_t))}\left(\alpha(h_t^{-1}(y_t))\right) + max\left\{k_{n(h_t^{-1}(y_t))}\left(\alpha(h_t^{-1}(y_t))\right), k_{m(h_t^{-1}(y_t))}(h_t^{-1}(y_t))\right\} - k_{n(h_t^{-1}(y_t))}\left(\alpha\left(h_t^{-1}(y_t)\right)\right), \tag{3.6}$$

then $\sigma_{F_t}^{m'(y_t)}(y_t) = \sigma^{n'(y_t)}\left(h_t\left(\alpha(h_t^{-1}(y_t))\right)\right)$. Since $h_t^{-1}, m, n,$ and α are continuous, it follows that $m', n'\colon h_t(V) \to \mathbb{N}$ defined by (3.5) and (3.6) are also continuous. Thus, it follows from Proposition 3.3 that $h_t \circ \alpha \circ h_t^{-1} \in \mathcal{P}_{F_t}$.A similar argument proves that if $\alpha' \in \mathcal{P}_{F_t}$,
then $h_t^{-1} \circ \alpha' \circ h_t \in \mathcal{P}_{E_t}$. Thus $h_t \circ \mathcal{P}_{E_t} \circ h_t^{-1} = \mathcal{P}_{F_t}$ and the pseudogroups of E_t and F_t are isomorphic.
Now suppose that $h_t\colon \partial E_t \to \partial F_t$ is a homeomorphism such that $h_t \circ \mathcal{P}_{E_t} \circ h_t^{-1} = \mathcal{P}_{F_t}$.
Fix $e \in E_t^1$ and let $\alpha_e := \sigma_{E_t}\big|_{Z(e)}$. Then α_e is a homeomorphism from $Z(e)$ to $\alpha_e(Z(e))$ and since $\alpha_e(x_t) = \sigma_{E_t}(x_t)$ for all $x_t \in Z(e)$, it follows from Proposition 3.3 that $\alpha_e \in \mathcal{P}_{E_t}$.
Thus $h_t \circ \alpha_e \circ h_t^{-1} \in \mathcal{P}_{F_t}$ by assumption. It follows from Proposition 3.3 that there are continuous functions $m'_e, n'_e\colon h_t(Z(e)) \to \mathbb{N}$ such that

$$\sigma_{F_t}^{n'_e(y_t)}\left(h_t\left(\alpha_e(h_t^{-1}(y_t))\right)\right) = \sigma_{F_t}^{m'_e(y_t)}(y_t) \quad \text{for } y_t \in h_t\big(Z(e)\big).$$

Define functions $k_1, l_1\colon \partial E_t^{\geq 1} \to \mathbb{N}$ by $k_1(x_t) = n'_{(x_t)_1}(h_t(x_t))$ and $l_1(x_t) = m'_{(x_t)_1}(h_t(x_t))$, which are continuous because the $Z(e)$ are pairwise-disjoint compact open sets covering $\partial E_t \geq 1$.
Then for each $x_t = (x_t)_1(x_t)_2 \cdots \in \partial E_t$ we have

$$\sigma_{F_t}^{l_1(x_t)}\big(h_t(x_t)\big) = \sigma_{F_t}^{m'_{(x_t)_1}(h_t(x_t))}\big(h_t(x_t)\big) = \sigma_{F_t}^{n'_{(x_t)_1}(h_t(x_t))}\left(h_t\left(\alpha_{(x_t)_1}(x_t)\right)\right) = \sigma_{F_t}^{k_1(x_t)}\left(h_t\left(\sigma_{E_t}(x_t)\right)\right).$$

Hence k_1 and l_1 satisfy the first equation from (3.1).A similar argument gets the second equation from (3.1). Thus E_t and F_t are orbit equivalent.

IV. The Extended Weyl Groupoidof $(C^*(E_t), \mathcal{D}(E_t))$

Proposition 2.2 says that the pair $(C^*(E_t), \mathcal{D}(E_t))$ is an invariant of $\mathcal{G}_{E_t}$, in the sense that if E_t and F_t are two graphs such that $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids, then there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$. In this section we show that $\mathcal{G}_{E_t}$ is an invariant of $(C^*(E_t), \mathcal{D}(E_t))$, in the sense that if there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$,
then $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids.

To prove this result (see [27]) we build a groupoid from $(C^*(E_t), \mathcal{D}(E_t))$ that we call the extended Weyl groupoid, which generalises Renault's Weyl groupoid construction from [22] applied to $(C^*(E_t), \mathcal{D}(E_t))$. Recall from [22] that Weyl groupoids are associated to pairs (A_t, B_t) consisting of a C^*-algebra A_t and an abelian C^*-subalgebra B_t which contains an approximate unit of A_t. The Weyl groupoid construction has the property that if $\mathcal{G}$ is a topologically principal étale Hausdorff locally compact second countable groupoid and $A_t = C^*_{\text{red}}(\mathcal{G})$ and $B_t = C_0(\mathcal{G}^0)$, then the associated Weyl groupoid is isomorphic to $\mathcal{G}$ as a topological groupoid.

We will modify Renault's construction for pairs $(C^*(E_t),\mathcal{D}(E_t))$ to obtain a groupoid $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ such that $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ and $\mathcal{G}_{E_t}$ are isomorphic as topological groupoids, even when $\mathcal{G}_{E_t}$ is not topologically principal. We will then show that if E_t and F_t are two graphs such that there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$, then $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ and $\mathcal{G}_{(C^*(F_t),\mathcal{D}(F_t))}$, and thus $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids.

As in [22] (and originally defined in [9]), we define the normaliser of $\mathcal{D}(E_t)$ to be the set
$N(\mathcal{D}(E_t)) := \{n \in C^*(E_t): ndn^*, n^*dn \in \mathcal{D}(E_t) \text{ for all } d \in \mathcal{D}(E_t)\}$.

According to [22, Lemma 4.6], $nn^*, n^*n \in \mathcal{D}(E_t)$ for $n \in N(\mathcal{D}(E_t))$. Recalling the definition of $(h_t)_{E_t}$ given in (2.1), for $n \in N(\mathcal{D}(E_t))$, we let $\mathrm{dom}(n) := \{x_t \in \partial E_t: (h_t)_{E_t}(x_t)(n^*n) > 0\}$ and $\mathrm{ran}(n) := \{x_t \in \partial E_t: (h_t)_{E_t}(x_t)(nn^*) > 0\}$. It follows from [22, Proposition 4.7] that, for $n \in N(\mathcal{D}(E_t))$, there is a unique homeomorphism $\alpha_n: \mathrm{dom}(n) \to \mathrm{ran}(n)$ such that, for all $d \in \mathcal{D}(E_t)$,

$$(h_t)_{E_t}(x_t)(n^*dn) = (h_t)_{E_t}(\alpha_n(x_t))(d)(h_t)_{E_t}(x_t)(n^*n). \tag{4.1}$$

From [22, Lemma 4.10] we also know that $\alpha_{n^*} = \alpha_n^{-1}$ and $\alpha_{mn} = \alpha_m \circ \alpha_n$ for each $m, n \in N(\mathcal{D}(E_t))$.

The following lemma gives an insight into how the homeomorphisms α_n work. We collect further properties of these homeomorphisms in Lemma 4.2.

Lemma 4.1 (see [27]). Let E_t be a graph. For each $\mu, \nu \in E_t^*$ with $r_t(\mu) = r_t(\nu)$ we have $s_\mu s_\nu^* \in N(\mathcal{D}(E_t))$ with
$\mathrm{dom}(s_\mu s_\nu^*) = Z(\nu), \mathrm{ran}(s_\mu s_\nu^*) = Z(\mu)$ and $\alpha_{s_\mu s_\nu^*}(\nu z_t) = \mu z_t$ for all $z_t \in r_t(\nu)\,\partial E_t$.

Proof. Let $\mu, \nu \in E_t^*$ with $r_t(\mu) = r_t(\nu)$. For each $\lambda \in E_t^*$ we have

$$(s_\mu s_\nu^*)^* s_\lambda s_\lambda^* (s_\mu s_\nu^*) = \begin{cases} s_\nu s_\nu^* & \text{if } \mu = \lambda\mu' \\ s_{\nu\lambda'} s_{\nu\lambda'}^* & \text{if } \lambda = \mu\lambda' \\ 0 & \text{otherwise .} \end{cases}$$

So $(s_\mu s_\nu^*)^* s_\lambda s_\lambda^* (s_\mu s_\nu^*) \in \mathcal{D}(E_t)$, and it follows that $(s_\mu s_\nu^*)^* d (s_\mu s_\nu^*) \in \mathcal{D}(E_t)$ for all $d \in \mathcal{D}(E_t)$.A similar argument shows that $(s_\mu s_n u^*) d (s_\mu s_\nu^*)^* \in \mathcal{D}(E_t)$ for all $d \in \mathcal{D}(E_t)$, and
hence $s_\mu s_\nu^* \in N(D(E_t))$. We have

$$(h_t)_{E_t}(x_t)\left((s_\mu s_\nu^*)^* s_\mu s_\nu^*\right) = (h_t)_{E_t}(x_t)(s_\nu s_\nu^*) = \begin{cases} 1 & \text{if } x_t \in Z(\nu) \\ 0 & \text{if } x_t \notin Z(\nu), \end{cases}$$

and hence $\mathrm{dom}(s_\mu s_\nu^*) = Z(\nu)$.A similar calculation gives $\mathrm{ran}(s_\mu s_\nu^*) = Z(\mu)$.

Now suppose $z_t \in r_t(\nu)\,\partial E_t$. We use (4.1) and (4.2) to get

$$\begin{aligned} (h_t)_{E_t}\left(\alpha_{s_\mu s_\nu^*}(\nu z_t)\right)(s_\lambda s_\lambda^*) &= (h_t)_{E_t}(\nu z_t)\left((s_\mu s_\nu^*)^* s_\lambda s_\lambda^* (s_\mu s_\nu^*)\right)(h_t)_{E_t}(\nu z_t)\left((s_\mu s_\nu^*)^* s_\mu s_\nu^*\right) \\ &= \begin{cases} (h_t)_{E_t}(\nu z_t)(s_\nu s_\nu^*) & \text{if } \mu = \lambda\mu' \\ (h_t)_{E_t}(\nu z_t)(s_{\nu\lambda'} s_{\nu\lambda'}^*) & \text{if } \lambda = \mu\lambda' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \mu z_t \in Z(\lambda) \\ 0 & \text{otherwise} \end{cases} \\ &= (h_t)_{E_t}(\mu z_t)(s_\lambda s_\lambda^*). \end{aligned} \tag{4.2}$$

It follows that $(h_t)_{E_t}\left(\alpha_{s_\mu s_\nu^+}(\nu z_t)\right) = (h_t)_{E_t}(\mu z_t)$, and hence $\alpha_{s_\mu s_\nu^*}(\nu z_t) = \mu z_t$.

Denote by $\partial(E_t)_{\mathrm{iso}}$ the set of isolated points in ∂E_t. Notice that $x_t \in \partial E_t$ belongs to $\partial(E_t)_{\mathrm{iso}}$ if and only if the characteristic function $1_{\{x_t\}}$ belongs to $C_0(\partial E_t)$. For $x_t \in \partial(E_t)_{\mathrm{iso}}$, we let p_{x_t} denote the unique element of $\mathcal{D}(E_t)$ satisfying that $(h_t)_{E_t}(y_t)(p_{x_t})$ is 1 if $y_t = x_t$ and zero otherwise.

Lemma 4.2 (see [27]). Let E_t be a graph, $n \in N(\mathcal{D}(E_t))$ and $x_t \in \partial(E_t)_{\mathrm{iso}} \cap \mathrm{dom}(n)$. Then

(a) $np_{x_t}n^* = (h_t)_{E_t}(x_t)(n^*n)p_{\alpha_n(x_t)}$,

(b) $n^*p_{\alpha_n(x_t)}n = (h_t)_{E_t}(x_t)(n^*n)p_{x_t}$, and

(c) $np_{x_t} = p_{\alpha_n(x_t)}n$.

Proof. We use Equation (4.1) with $d = nn^*$ to get

$$(h_t)_{E_t}(x_t)(n^*n)^2 = (h_t)_{E_t}(x_t)(n^*nn^*n) = (h_t)_{E_t}(\alpha_n(x_t))(nn^*)(h_t)_{E_t}(x_t)(n^*n),$$

which implies that

$$(h_t)_{E_t}(\alpha_n(x_t))(nn^*) = (h_t)_{E_t}(x_t)(n^*n). \tag{4.3}$$

Note that this is a positive number because $x_t \in \mathrm{dom}(n)$. For (a) we again use (4.1) to get

$$(h_t)_{E_t}(y_t)\left(\left((h_t)_{E_t}(\alpha_n(x_t))(nn^*)\right)^{-1} np_{x_t}n^*\right) = \left((h_t)_{E_t}(\alpha_n(x_t))(nn^*)\right)^{-1}(h_t)_{E_t}(y_t)\left(np_{x_t}n^*\right)$$
$$= \left((h_t)_{E_t}(\alpha_n(x_t))(nn^*)\right)^{-1}(h_t)_{E_t}(\alpha_n+(y_t))\left(p_{x_t}\right)(h_t)_{E_t}(y_t)(nn^*)$$
$$= \begin{cases} 1 & \text{if } y_t = \alpha_n(x_t) \\ 0 & \text{otherwise .}\end{cases}$$

By the defining property of $p_{\alpha_n(x_t)}$ we now have $p_{\alpha_n(x_t)} = \left((h_t)_{E_t}(\alpha_n(x_t))(nn^*)\right)^{-1} np_{x_t}n^*$.
Using (4.3) gives $np_{x_t}n^* = (h_t)_{E_t}(x_t)(n^*n)p_{\alpha_n(x_t)}$, which is (a). Identity (b) follows from (a) by replacing n with n^* and x_t with $\alpha_n(x_t)$ and then use (4.3).
To prove (c) we first notice that

$$(h_t)_{E_t}(y_t)\left(\left((h_t)_{E_t}(x_t)(n^*n)\right)^{-1} n^*np_{x_t}\right) = \left((h_t)_{E_t}(x_t)(n^*n)\right)^{-1}(h_t)_{E_t}(y_t)(n^*n)(h_t)_{E_t}(y_t)\left(p_{x_t}\right)$$
$$= \begin{cases} 1 & \text{if } x_t = y_t \\ 0 & \text{if } x_t \neq y_t.\end{cases}$$

Hence by the defining property of p_{x_t} we have

$$n^*np_{x_t} = (h_t)_{E_t}(x_t)(n^*n)p_{x_t}. \tag{4.4}$$

We now use (4.4) and (a) to get (c):

$$np_{x_t} = n\left(\left((h_t)_{E_t}(x_t)(n^*n)\right)^{-1} n^*np_{x_t}\right) = \left((h_t)_{E_t}(x_t)(n^*n)\right)^{-1} np_{x_t}n^*n = p_{\alpha_n(x_t)}n.$$

Lemma 4.3 (see [27]). Suppose $x_t \in \partial(E_t)_{\text{iso}}$. If x_t is not eventually periodic, then $p_{x_t}C^*(E_t)p_{x_t} = p_{x_t}\mathcal{D}(E_t)p_{x_t} = \mathbb{C}p_{x_t}$. If $x_t = \mu\eta\eta\eta\cdots$ for some $\mu \in E_t^*$ and some simple loop $\eta \in E_t^*$ with $s_t(\eta) = r_t(\mu)$, then $p_{x_t}C^*(E_t)p_{x_t}$ is isomorphic to $C(\mathbb{T})$ by the isomorphism mapping $p_{x_t}s_\mu s_\eta s_\mu^* p_{x_t}$ to the identity function on $\mathbb{T}$, and $p_{x_t}\mathcal{D}(E_t)p_{x_t} = \mathbb{C}p_{x_t}$.
Proof. Let $\left(\mathcal{G}_{E_t}\right)_{x_t}^{x_t}$ denote the isotropy group $\{\gamma \in \mathcal{G}: s_t(\gamma) = r_t(\gamma) = x_t\}$ of $(x_t, 0, x_t)$.
Assume that x_t is not eventually periodic. Then $\left(\mathcal{G}_{E_t}\right)_{x_t}^{x_t} = \{(x_t, 0, x_t)\}$. Proposition 2.2 implies that there is an isomorphism from $p_{x_t}C^*(E_t)p_{x_t}$ to $C^*\left(\left(\mathcal{G}_{E_t}\right)_{x_t}^{x_t}\right)$, and consequently $p_{x_t}C^*(E_t)p_{x_t} = p_{x_t}\mathcal{D}(E_t)p_{x_t} = \mathbb{C}p_{x_t}$, completing the first assertion in the lemma.
Assume then that $x_t = \mu\eta\eta\eta\cdots$ for some $\mu \in E_t^*$ and some simple loop $\eta \in E_t^*$ with $s_t(\eta) = r_t(\mu)$. We then have that $\left(\mathcal{G}_{E_t}\right)_{x_t}^{x_t} = \{(x_t, k|\eta|, x_t): k \in \mathbb{Z}\}$. Now Proposition 2.2 implies that there is an isomorphism from $p_{x_t}C^*(E_t)p_{x_t}$ to $C(\mathbb{T})$ which maps $p_{x_t}s_\mu s_\eta s_\mu^* p_{x_t}$ to the identity function on $\mathbb{T}$, and that $p_{x_t}\mathcal{D}(E_t)p_{x_t} = \mathbb{C}p_{x_t}$.
The extended Weyl groupoid associated to $(C^*(E_t), \mathcal{D}(E_t))$ is built using an equivalence relation defined on pairs of normalisers and boundary paths. For isolated boundary paths x_t the equivalence relation is defined using a unitary in the corner of $C^*(E_t)$ determined by p_{x_t}.
Lemma 4.4 (see [27]). Let E_t be a graph. For $x_t \in \partial(E_t)_{\text{iso}}, n_1, n_2 \in N(\mathcal{D}(E_t)), x_t \in \text{dom}(n_1) \cap \text{dom}(n_2)$, and $\alpha_{n_1}(x_t) = \alpha_{n_2}(x_t)$, we denote

$$U_{(x_t,n_1,n_2)} := \left((h_t)_{E_t}(x_t)(n_1^*n_1n_2^*n_2)\right)^{-1/2} p_{x_t}n_1^*n_2p_{x_t}.$$

Then
(1) $U_{(x_t,n_1,n_2)}U_{(x_t,n_1,n_2)}^* = U_{(x_t,n_1,n_2)}^*U_{(x_t,n_1,n_2)} = p_{x_t}$, and
(2) $U_{(x_t,n_1,n_2)}^* = U_{(x_t,n_2,n_1)}$.
Moreover, if $n_3 \in N(\mathcal{D}(E_t)), x_t \in \text{dom}(n_3)$, and $\alpha_{n_3}(x_t) = \alpha_{n_1}(x_t) = \alpha_{n_2}(x_t)$, then
(3) $U_{(x_t,n_1,n_2)}U_{(x_t,n_2,n_3)} = U_{(x_t,n_1,n_3)}$.
Proof. Suppose that $x_t \in \partial(E_t)_{\text{iso}}, n_1, n_2 \in N(\mathcal{D}(E_t)), x_t \in \text{dom}(n_1) \cap \text{dom}(n_2)$, and $\alpha_{n_1}(x_t) = \alpha_{n_2}(x_t)$. First note that since $x_t \in \text{dom}(n_1) \cap \text{dom}(n_2)$, we have $(h_t)_{E_t}(n_1^*n_1), (h_t)_{E_t}(n_2^*n_2) > 0$,
and the formula for $U_{(x_t,n_1,n_2)}$ makes sense. We now claim that

$$p_{x_t}n_1^*n_2p_{x_t}n_2^*n_1p_{x_t} = (h_t)_{E_t}(x_t)(n_1^*n_1n_2^*n_2)p_{x_t}. \tag{4.5}$$

To see this, we apply identities (a) and (b) of Lemma 4.2 to get

$$\begin{aligned} p_{x_t} n_1^* n_2 p_{x_t} n_2^* n_1 p_{x_t} &= p_{x_t} n_1^* (n_2 p_{x_t} n_2^*) n_1 p_{x_t} \\ &= p_{x_t} n_1^* \left((h_t)_{E_t}(x_t)(n_2^* n_2) p_{\alpha_{n_2}(x_t)} \right) n_1 p_{x_t} \\ &= (h_t)_{E_t}(x_t)(n_2^* n_2) p_{x_t} n_1^* p_{\alpha_{n_1}(x_t)} n_1 p_{x_t} \\ &= (h_t)_{E_t}(x_t)(n_1^* n_1)(h_t)_{E_t}(x_t)(n_2^* n_2) p_{x_t} \\ &= (h_t)_{E_t}(x_t)(n_1^* n_1 n_2^* n_2) p_{x_t}. \end{aligned}$$

We now use (4.5) to get

$$U_{(x_t,n_1,n_2)} U^*_{(x_t,n_1,n_2)} = \left((h_t)_{E_t}(n_1^* n_1 n_2^* n_2) \right)^{-1} p_{x_t} n_1^* n_2 p_{x_t} n_2^* n_1 p_{x_t} = p_{x_t}$$

Similarly, and using that $p_{x_t} C^*(E_t) p_{x_t}$ is commutative, we have

$$\begin{aligned} U^*_{(x_t,n_1,n_2)} U_{(x_t,n_1,n_2)} &= \left((h_t)_{E_t}(n_1^* n_1 n_2^* n_2) \right)^{-1} p_{x_t} n_2^* n_1 p_{x_t} n_1^* n_2 p_{x_t} \\ &= \left((h_t)_{E_t}(n_1^* n_1 n_2^* n_2) \right)^{-1} p_{x_t} n_1^* n_2 p_{x_t} n_2^* n_1 p_{x_t} \\ &= p_{x_t}. \end{aligned}$$

So (1) holds.
Identity (2) holds because $n_1^* n_1, n_2^* n_2 \in \mathcal{D}(E_t)$, and hence

$$\begin{aligned} U^*_{(x_t,n_1,n_2)} := \left((h_t)_{E_t}(x_t)(n_1^* n_1 n_2^* n_2) \right)^{-1/2} p_{x_t} n_2^* n_1 p_{x_t} &= \left((h_t)_{E_t}(x_t)(n_2^* n_2 n_1^* n_1) \right)^{-1/2} p_{x_t} n_2^* n_1 p_{x_t} \\ &= U_{(x_t,n_2,n_1)}. \end{aligned}$$

We use identities (a) and (c) of Lemma 4.2 to get

$$\begin{aligned} U_{(x_t,n_1,n_2)} U_{(x_t,n_2,n_3)} &= \left((h_t)_{E_t}(x_t)(n_1^* n_1 n_2^* n_2)(h_t)_{E_t}(x_t)(n_2^* n_2 n_3^* n_3) \right)^{-1/2} p_{x_t} n_1^* n_2 p_{x_t} n_2^* n_3 p_{x_t} \\ &= \left((h_t)_{E_t}(x_t)(n_2^* n_2) \right)^{-1} \left((h_t)_{E_t}(x_t)(n_1^* n_1 n_3^* n_3) \right)^{-1/2} p_{x_t} n_1^* \left((h_t)_{E_t}(x_t)(n_2^* n_2) p_{\alpha_{n_2}(x_t)} \right) n_3 p_{x_t} \\ &= \left((h_t)_{E_t}(n_1^* n_1 n_3^* n_3) \right)^{-1/2} p_{x_t} n_1^* p_{\alpha_{n_3}(x_t)} n_3 p_{x_t} \\ &= \left((h_t)_{E_t}(n_1^* n_1 n_3^* n_3) \right)^{-1/2} p_{x_t} n_1^* n_3 p_{x_t} \\ &= U_{(x_t,n_1,n_3)}. \end{aligned}$$

So (3) holds.
Notation 4.5. For x_t, n_1 and n_2 as in Lemma 4.4 we denote

$$\lambda_{(x_t,n_1,n_2)} := (h_t)_{E_t}(x_t)(n_1^* n_1 n_2^* n_2).$$

So $U_{(x_t,n_1,n_2)} = \lambda^{-1/2}_{(x_t,n_1,n_2)} p_{x_t} n_1^* n_2 p_{x_t}$. It follows from identity (1) of Lemma 4.4 that $U_{(x_t,n_1,n_2)}$ is a unitary element of $p_{x_t} C^*(E_t) p_{x_t}$. We denote by $\left[U_{(x_t,n_1,n_2)}\right]_1$ the class of $U_{(x_t,n_1,n_2)}$ in $K_1\left(p_{x_t} C^*(E_t) p_{x_t}\right)$.
Proposition 4.6 (see [27]). Let E_t be a graph. For each $(x_t)_1, (x_t)_2 \in \partial E_t$ and $n_1, n_2 \in N(\mathcal{D}(E_t))$ such that $(x_t)_1 \in \operatorname{dom}(n_1)$ and $(x_t)_2 \in \operatorname{dom}(n_2)$ we write $(n_1, (x_t)_1) \sim (n_2, (x_t)_2)$ if either
(a) $(x_t)_1 = (x_t)_2 \in \partial(E_t)_{\text{iso}}, \alpha_{n_1}((x_t)_1) = \alpha_{n_2}((x_t)_2)$, and $\left[U_{((x_t)_1,n_1,n_2)}\right]_1 = 0$; or
(b) $(x_t)_1 = (x_t)_2 \notin \partial(E_t)_{\text{iso}}$ and there is an open set V such that $(x_t)_1 \in V \subseteq \operatorname{dom}(n_1) \cap \operatorname{dom}(n_2)$ and $\alpha_{n_1}(y_t) = \alpha_{n_2}(y_t)$ for all $y_t \in V$.
Then $\sim$ is an equivalence relation on $\{(n, x_t) : n \in N(\mathcal{D}(E_t)), x_t \in \operatorname{dom}(n)\}$.
Proof. The only nontrivial parts to prove are that $\sim$ is symmetric and transitive when the boundary paths are isolated points. Suppose $(n_1, (x_t)_1) \sim (n_2, (x_t)_2)$ with $x_t := (x_t)_1 = (x_t)_2 \in \partial(E_t)_{\text{iso}}$. We know from Lemma 4.4(2) that $U_{(x_t,n_2,n_1)} = U^*_{(x_t,n_1,n_2)}$. So

$$\left[U_{(x_t,n_1,n_2)}\right]_1 = 0 \Longrightarrow \left[U_{(x_t,n_2,n_1)}\right]_1 = \left[U^*_{(x_t,n_1,n_2)}\right]_1 = 0,$$

and hence $(n_2, (x_t)_2) \sim (n_1, (x_t)_1)$.
For transitivity, suppose $(n_1, (x_t)_1) \sim (n_2, (x_t)_2)$ and $(n_2, (x_t)_2) \sim (n_3, (x_t)_3)$ with $x_t := (x_t)_1 = (x_t)_2 = (x_t)_3 \in \partial(E_t)_{\text{iso}}$. We know from Lemma 4.4(2) that $U_{(x_t,n_1,n_2)} U_{(x_t,n_2,n_3)} = U_{(x_t,n_1,n_3)}$. So

$$\left[U_{((x_t,n_1,n_2)}\right]_1 = 0 = \left[U_{(x_t,n_2,n_3)}\right]_1 \Longrightarrow \left[U_{(x_t,n_1,n_3)}\right]_1 = \left[U_{(x_t,n_1,n_2)}\right]_1 \left[U_{(x_t,n_2,n_3)}\right]_1 = 0,$$

and hence $(n_1, (x_t)_1) \sim (n_3, (x_t)_3)$.
Proposition 4.7 (see [27]). Let E_t be a graph, and $\sim$ the equivalence relation on $\{(n, x_t) : n \in N(\mathcal{D}(E_t)), \quad x_t \in \operatorname{dom}(n)\}$ from Proposition 4.6. Denote the collection of equivalence classes by $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$. Define a partially-defined product on $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ by

$$[(n_1, (x_t)_1)][(n_2, (x_t)_2)] := [(n_1 n_2, (x_t)_2)] \quad \text{if } \alpha_{n_2}((x_t)_2) = (x_t)_1,$$

and undefined otherwise. Define an inverse map by $[(n, x_t)]^{-1} := [(n^*, \alpha_n(x_t))]$. Then these operations make $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ into a groupoid.

Proof. We only check that composition and inversion are well-defined. That composition is associative and every element is composable with its inverse (in either direction) is left to the reader. To see that composition is well-defined, suppose $[(n_1,(x_t)_1)] = [(n_1',(x_t)_1')]$, and $[(n_2,(x_t)_2)] = [(n_2',(x_t)_2')]$ with $[(n_1,(x_t)_1)]$ and $[(n_2,(x_t)_2)]$ composable. We need to show that $[(n_1',(x_t)_1')]$ and $[(n_2',(x_t)_2')]$ are also composable with

$$[(n_1n_2,(x_t)_2)] = [(n_1'n_2',(x_t)_2')]. \tag{4.6}$$

We immediately know that $(x_t)_1 = (x_t)_1', (x_t)_2 = (x_t)_2', (x_t)_2 = \alpha_{n_2}^{-1}((x_t)_1), \alpha_{n_1}((x_t)_1) = \alpha_{n_1'}((x_t)_1')$, and $\alpha_{n_2}((x_t)_2) = \alpha_{n_2'}((x_t)_2')$. This gives

$$\alpha_{n_2'}^{-1}((x_t)_1') = \alpha_{n_2'}^{-1}((x_t)_1) = \alpha_{n_2'}^{-1}\left(\alpha_{n_2}((x_t)_2)\right) = \alpha_{n_2'}^{-1}\left(\alpha_{n_2'}((x_t)_2')\right) = (x_t)_2'.$$

So $\alpha_{n_2'}((x_t)_2') = (x_t)_1'$, and hence $[(n_1',(x_t)_1')]$ and $[(n_2',(x_t)_2')]$ are composable.

To see that (4.6) holds we have two cases:

Case 1: Suppose $(x_t)_1 \notin \partial(E_t)_{\text{iso}}$. Then $(x_t)_1' = (x_t)_1 \notin \partial(E_t)_{\text{iso}}, (x_t)_2 = \alpha_{n_2}^{-1}((x_t)_1) \notin \partial(E_t)_{\text{iso}}$, and $(x_t)_2' = \alpha_{n_2'}^{-1}((x_t)_1') \notin \partial(E_t)_{\text{iso}}$ We also know there exists an open set V_1 such that $(x_t)_1 \in V_1 \subseteq \operatorname{dom}(n_1) \cap \operatorname{dom}(n_1')$ with $\alpha_{n_1}\big|_{V_1} = \alpha_{n_1'}\big|_{V_1}$, and an open set V_2 such that $(x_t)_2 \in V_2 \subseteq \operatorname{dom}(n_2) \cap \operatorname{dom}(n_2')$ with $\alpha_{n_2}\big|_{V_2} = \alpha_{n_2'}\big|_{V_2}$. Let $V := V_2 \cap \alpha_{n_2}^{-1}(V_1)$, which is an open set containing $(x_t)_2$. We claim that

$$V \subseteq \operatorname{dom}(n_1n_2) \cap \operatorname{dom}(n_1'n_2').$$

To see this, let $x_t \in V$. Then using (4.1) we have

$$(h_t)_{E_t}(x_t)((n_1n_2)^*n_1n_2) = (h_t)_{E_t}\left(\alpha_{n_2}(x_t)\right)(n_1^*n_1)(h_t)_{E_t}(x_t)(n_2^*n_2),$$

which is positive because $\alpha_{n_2}(x_t) \in \operatorname{dom}(n_1)$ and $x_t \in \operatorname{dom}(n_2)$. So $V \subseteq \operatorname{dom}(n_1n_2)$.A similar argument gives $V \subseteq \operatorname{dom}(n_1'n_2')$, and so the claim holds. For each $x_t \in V$ we have $\alpha_{n_2}(x_t) = \alpha_{n_2'}(x_t) \in V_1$, which means

$$\alpha_{n_1n_2}(x_t) = \alpha_{n_1}\left(\alpha_{n_2}(x_t)\right) = \alpha_{n_1'}\left(\alpha_{n_2'}(x_t)\right) = \alpha_{n_1'n_2'}(x_t).$$

So $\alpha_{n_1n_2}\big|_V = \alpha_{n_1'n_2'}\big|_V$. Hence $(n_1n_2,(x_t)_2) \sim (n_1'n_2',(x_t)_2')$, and (4.6) holds in this case.

Case 2: Suppose $(x_t)_1 \in \partial(E_t)_{\text{iso}}$. Then $(x_t)_1' = (x_t)_1 \in \partial(E_t)_{\text{iso}}, (x_t)_2 = \alpha_{n_2}^{-1}((x_t)_1) \in \partial(E_t)_{\text{iso}}$, and $(x_t)_2' = \alpha_{n_2'}^{-1}((x_t)_1') \in \partial(E_t)_{\text{iso}}$. We also have $\alpha_{n_1}((x_t)_1) = \alpha_{n_1'}((x_t)_1'), \alpha_{n_2}((x_t)_2) = \alpha_{n_2'}((x_t)_2')$, and hence

$$\alpha_{n_1n_2}((x_t)_2) = \alpha_{n_1}\left(\alpha_{n_2'}((x_t)_2')\right) = \alpha_{n_1}((x_t)_1') = \alpha_{n_1'}((x_t)_1') = \alpha_{n_1'}\left(\alpha_{n_2'}((x_t)_2')\right) = \alpha_{n_1'n_2'}((x_t)_2').$$

To get $(n_1n_2,(x_t)_2) \sim (n_1'n_2',(x_t)_2')$ in this case it now suffices to show that $\left[U_{((x_t)_2,n_1n_2,n_1'n_2')}\right]_1 = 0$. We use that $\alpha_{n_2}((x_t)_2) = (x_t)_1$ and $\alpha_{n_2'}((x_t)_2') = (x_t)_1' = (x_t)_1$ and apply Lemma 4.2(c) twice to get

$$\begin{aligned} p_{(x_t)_2}n_2^*n_1^*n_1'n_2'p_{(x_t)_2} &= \left(n_2p_{(x_t)_2}\right)^*n_1^*n_1'\left(n_2'p_{(x_t)_2}\right) = \left(p_{\alpha_{n_2}}((x_t)_2)n_2\right)^*n_1^*n_1'p_{\alpha_{n_2'}}((x_t)_2)n_2' \\ &= n_2^*p_{(x_t)_1}n_1^*n_1'p_{(x_t)_1}n_2'. \end{aligned}$$

Now we can write

$$\begin{aligned} U_{((x_t)_2,n_1n_2,n_1'n_2')} &= \lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'n_2')}p_{(x_t)_2}n_2^*n_1^*n_1'n_2'p_{(x_t)_2} \\ &= \lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'n_2')}n_2^*p_{(x_t)_1}n_1^*n_1'p_{(x_t)_1}n_2' \\ &= \lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'r_2')}\lambda^{1/2}_{((x_t)_1,n_1,n_1')}n_2^*\left(\lambda^{-1/2}_{((x_t)_1,n_1,n_1')}p_{(x_t)_1}n_1^*n_1'p_{(x_t)_1}\right)n_2' \\ &= \lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{((x_t)_1,n_1,n_1')}n_2^*U_{((x_t)_1,n_1,n_1')}n_2' \end{aligned}$$

Since $(n_1,(x_t)_1) \sim (n_1',(x_t)_1')$ implies that $U_{((x_t)_1,n_1,n_1')}$ is homotopic to $p_{(x_t)_1}$, we see that $U_{((x_t)_2,n_1n_2,n_1'n_2')}$ is homotopic to

$$\lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{((x_t)_1,n_1,n_1')}n_2^*p_{(x_t)_1}n_2'.$$

We use Lemma 4.2(c) to get

$$n_2^*p_{(x_t)_1}n_2'n_2^*p_{\alpha_{n_2}((x_t)_2)}p_{\alpha_{n_2'}((x_t)_2')}n_2' = p_{(x_t)_2}n_2^*n_2'p_{(x_t)_2}.$$

Hence $U_{((x_t)_2,n_1n_2,n_1'n_2')}$ is homotopic to

$$\begin{aligned} \lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{((x_t)_1,n_1,n_1')}n_2^*p_{(x_t)_1}n_2' &= \lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{((x_t)_1,n_1,n_1')}p_{(x_t)_2}n_2^*n_2'p_{(x_t)_2} \\ &= \lambda^{-1/2}_{((x_t)_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{((x_t)_1,n_1,n_1')}\lambda^{1/2}_{((x_t)_2,n_2,n_2')}U_{((x_t)_2,n_2,n_2').} \end{aligned}$$

But $(n_2,(x_t)_2) \sim (n_2',(x_t)_2')$ implies that $U_{((x_t)_2,n_2,n_2')}$ is homotopic to $p_{(x_t)_2}$, and hence $U_{((x_t)_2,n_1n_2,n_1'n_2')}$ is homotopic to $p_{(x_t)_2}$. This says that $\left[U_{((x_t)_2,n_1n_2,n_1'n_2')}\right]_1 = 0$, as desired.

This complete the proof that composition is well-defined. To see that inversion is well-defined, suppose $[(n_1,(x_t)_1)] = [(n_1',(x_t)_1')]$. We need to show that $\left[\left(n_1^*,\alpha_{n_1}((x_t)_1)\right)\right] = \left[\left((n_1')^*,\alpha_{n_1'}((x_t)_1')\right]$. We again have two cases.

Case 1: Suppose that $(x_t)_1 = (x_t)_1' \notin \partial(E_t)_{\text{iso}}$. We know that there is open V such that $(x_t)_1 \in V \subseteq \mathrm{dom}(n_1) \cap \mathrm{dom}(n_1')$ and $\alpha_{n_1}\big|_V = \alpha_{n_1'}\big|_V$.A straightforward argument shows that the open set $V' := \alpha_{n_1}(V)$ satisfies $\alpha_{n_1}((x_t)_1) \in V' \subseteq \mathrm{dom}(n_1^*) \cap \mathrm{dom}((n_1')^*)$ and $\alpha_{n_1^*}\big|_{V'} = \alpha_{(n_1')^*}\big|_{V'}$. So $\left[\left(n_1^*,\alpha_{n_1}((x_t)_1)\right)\right] = \left[\left((n_1')^*,\alpha_{n_1'}((x_t)_1')\right]$ in this case.

Case 2: Suppose that $(x_t)_1 = (x_t)_1' \in \partial(E_t)_{\text{iso}}$. We have to show that $\left[U_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)}\right]_1 = 0$.

We use Lemma 4.2(c) to get

$$\begin{aligned} U_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)} &= \lambda^{-1/2}_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)} p_{\alpha_{n_1}((x_t)_1)} n_1 (n_1')^* p_{\alpha_{n_1}}((x_t)_1) \\ &= \lambda^{-1/2}_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)} n_1 p_{(x_t)_1} (n_1')^*. \end{aligned}$$

Since $(n_1,(x_t)_1) \sim (n_1',(x_t)_1')$, we have $U_{((x_t)_1,n_1,n_1')}$ homotopic to $p_{(x_t)_1}$. Hence $U_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)}$ is homotopic to

$$\lambda^{-1/2}_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)} n_1 U_{((x_t)_1,n_1,n_1')} (n_1')^* = \lambda^{-1/2}_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)} \lambda^{-1/2}_{((x_t)_1,n_1,n_1')} n_1 p_{(x_t)_1} n_1^* n_1' p_{(x_t)_1} (n_1')^*.$$

Now, using (4.3) we have

$$\lambda^{-1/2}_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)} \lambda^{-1/2}_{((x_t)_1,n_1,n_1')} = (h_t)_{E_t}((x_t)_1)(n_1^* n_1)^{-1} (h_t)_{E_t}((x_t)_1)((n_1')^* n_1')^{-1}.$$

So $U_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)}$ is homotopic to

$$\begin{aligned} &(h_t)_{E_t}((x_t)_1)(n_1^* n_1)^{-1} (h_t)_{E_t}((x_t)_1)((n_1')^* n_1')^{-1} n_1 p_{(x_t)_1} n_1^* n_1' p_{(x_t)_1} (n_1')^* \\ &= \left((h_t)_{E_t}((x_t)_1)(n_1^* n_1)^{-1} n_1 p_{(x_t)_1} n_1^*\right)\left((h_t)_{E_t}((x_t)_1)((n_1')^* n_1')^{-1} n_1' p_{(x_t)_1} (n_1')^*\right) \\ &= p_{\alpha_{n_1}((x_t)_1)}, \end{aligned}$$

where the last equality follows from Lemma 4.2(a). Hence $\left[U_{\left(\alpha_{n_1}((x_t)_1),n_1^*,(n_1')^*\right)}\right]_1 = 0$.

We equip $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ with the topology generated by $\{\{[(n,x_t)]: x_t \in \mathrm{dom}(n)\}: n \in N(\mathcal{D}(E_t))\}$. It can be proven directly that $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ is a topological groupoid with this topology, however, it also follows from our next result.

Proposition 4.8 [27]. Let E_t be a graph. Then $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ is a topological groupoid, and $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ and $\mathcal{G}_{E_t}$ are isomorphic as topological groupoids.

Remark 4.9. If $\mathcal{G}_{E_t}$ is topological principally, which we know from Proposition 2.3 is equivalent to E_t satisfying condition (L), then $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ is isomorphic to the Weyl groupoid $\mathcal{G}_{C_0\left(\mathcal{G}^0_{E_t}\right)}$ of $\left(C^*\left(\mathcal{G}_{E_t}\right), C_0\left(\mathcal{G}^0_{E_t}\right)\right)$ as in [22]. In this case the isomorphism of $\mathcal{G}_{E_t}$ and $\mathcal{G}_{(C^+(E_t),\mathcal{D}(E_t))}$ proved below follows from [22, Proposition 4.14].

To prove Proposition 4.8 we need the following result. The proof can be deduced from the proof of [22, Proposition 4.8], but we include a proof for completeness. As in [22], we let $\mathrm{supp}'(f_t) := \{y_t \in \mathcal{G}_{E_t}: f_t(\gamma) \neq 0\}$ for $f_t \in C^*\left(\mathcal{G}_{E_t}\right)$.

Lemma 4.10 (see [27]). Let E_t be a graph and $\pi_t: C^*(E_t) \to C^*\left(\mathcal{G}_{E_t}\right)$ the isomorphism from Proposition 2.2. Let $n \in N(\mathcal{D}(E_t))$, and $f_t := \pi_t(n)$. Then $\mathrm{supp}'(f_t)$ satisfies

(i) $s(\mathrm{supp}'(f_t)) = \mathrm{dom}(n)$;

(ii) $(x_t,k,y_t) \in \mathrm{supp}'(f_t) \Longrightarrow \alpha_n(y_t) = x_t$; and

(iii) $y_t \in \mathrm{dom}(n) \Longrightarrow (\alpha_n(y_t),k,y_t) \in \mathrm{supp}'(f_t)$ for some $k \in \mathbb{Z}$.

Proof. Identity (i) follows because

$$(h_t)_{E_t}(y_t)(n^*n) = \pi_t(n^*n)(y_t,0,y_t) = f_t^* f_t(y_t,0,y_t) = \sum_{\substack{\gamma \in \mathcal{G}_{E_t} \\ s(\gamma)=(y_t)}} |f_t(\gamma)|^2.$$

For (ii) we first consider the function $f_t^* g_t f_t$ where g_t is any element of $\pi_t(\mathcal{D}(E_t)) = C_0\left(\mathcal{G}^{(0)}_{E_t}\right)$. Using the convolution product we have

$$f_t^* g_t f_t(y_t,0,y_t) = \sum_{\substack{\gamma \in \mathcal{G}_{E_t} \\ s(\gamma)=(y_t)}} \sum_{t} |f_t(\gamma)|^2 g_t(r(\gamma)) \text{ for all } x_t \in \partial E_t. \tag{4.7}$$

Alternatively, we can also apply (4.1) to, say, $g_t = \pi_t(d)$ to get

$$f_t^* g_t f_t(y_t, 0, y_t) = \pi_t(n^* dn)(y_t, 0, y_t) = (h_t)_{E_t}(y_t)(n^* dn) \quad = (h_t)_{E_t}\big(\alpha_n(y_t)\big)(d)(h_t)_{E_t}(y_t)(n^* n) \\ = g_t\big(\alpha_n(y_t), 0, \alpha_n(y_t)\big)|f_t(y_t, 0, y_t)|^2. \tag{4.8}$$

Now suppose for contradiction that $(x_t, k, y_t) \in \mathrm{supp}'(f_t)$ but $\alpha_n(y_t) \neq x_t$. Choose $g_t \in C_0\big(\mathcal{G}_{E_t}^{(0)}\big)$ a positive function with $g_t(x_t, 0, x_t) = 1$ and $g_t(\alpha_n(y_t), 0, \alpha_n(y_t)) = 0$. Then (4.7) gives

$$f_t^* g_t f_t(y_t, 0, y_t) \geq |f_t(x_t, k, y_t)|^2 g_t(x_t, 0, x_t) > 0,$$

whereas (4.8) gives

$$f_t^* g_t f_t(y_t, 0, y_t) = g_t\big(\alpha_n(y_t), 0, \alpha_n(y_t)\big)|f_t(y_t, 0, y_t)|^2 = 0.$$

So (ii) holds.

Implication (iii) follows immediately from (i) and (ii).

Proof of Proposition 4.8. Let $(x_t, k, y_t) \in \mathcal{G}_{E_t}$. Then there are $\mu, \nu \in E_t^*$ and $z_t \in \partial E_t$ such that $x_t = \mu z_t, y_t = \nu z_t$, and $k = |\mu| - |\nu|$. We know from Lemma 4.1 that $s_\mu s_\nu^* \in N(\mathcal{D}(E_t)), y_t \in \mathrm{dom}\big(s_\mu s_\nu^*\big)$, and that $\alpha_{s_\mu s_\nu^*}(y_t) = x_t$. Define $\phi_t \colon \mathcal{G}_{E_t} \to \mathcal{G}_{(C^*(E_t), \mathcal{D}(E_t))}$ by

$$\phi_t\big((x_t, k, y_t)\big) = \big[\big(s_\mu s_\nu^*, y_t\big)\big].$$

It is routine to check that ϕ_t is well-defined, in the sense that if $\mu, \nu, \mu', \nu' \in E_t^*, z_t, z_t' \in \partial E_t, \mu z_t = \mu' z_t', \nu z_t = \nu' z_t'$, and $|\mu| - |\nu| = |\mu'| - |\nu'|$, then $\big[\big(s_\mu s_\nu^*, \nu z_t\big)\big] = \big[\big(s_{\mu'} s_{\nu'}^*, \nu' z_t'\big)\big]$. It is also routine to check that ϕ_t is a groupoid homomorphism. We now have to show that ϕ_t is a homeomorphism.

To show that ϕ_t is injective, assume that $\phi_t((x_t, k, y_t)) = \phi_t\big((x_t', k', y_t')\big)$. Then $x_t = x_t'$ and $y_t = y_t'$. Suppose for contradiction that $k \neq k'$. Then y_t must be eventually periodic, because otherwise we would have $\alpha_{s_n s_\lambda^\dagger}(y_t) \neq \alpha_{s_{k'} s_{\lambda'}^*}(y_t)$ for $|\kappa| - |\lambda| = k$ and $|\kappa'| - |\lambda'| = k'$.

Thus $x_t = \mu\eta\eta\eta \cdots$ and $y_t = \nu\eta\eta\eta \cdots$ for some $\mu, \nu \in E_t^*$ and a simple loop $\eta \in E_t^*$ such that $s_t(\eta) = r_t(\mu) = r_t(\nu)$. It follows that $\phi_t((x_t, k, y_t)) = \big[\big(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*, y_t\big)\big]$ and $\phi_t\big((x_t, k', y_t)\big) = \Big[\Big(s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^* y_t\Big)\Big]$ where m, n, m', n' are nonnegative integers such that $|\mu(\eta)^m| - |\nu(\eta)^n| = k$ and $\big|\mu(\eta)^{m'}\big| - \big|\nu(\eta)^{n'}\big| = k'$. Suppose that η has an exit. Then $y_t \notin \partial(E_t)_{\mathrm{iso}}$ and there is a $\zeta \in E_t^*$ such that $s_t(\zeta) = s_t(\eta), |\zeta| \leq |\eta|$, and $\zeta \neq \eta_1\eta_2 \cdots \eta_{|\zeta|}$ (where $\eta = \eta_1\eta_2 \cdots \eta_{|\eta|}$). Then for any open set U with $y_t \in U \subseteq \mathrm{dom}\big(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^* \cap \mathrm{dom}\Big(s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^*\Big)$, there is a positive integer l such that $\emptyset \neq Z(\nu(\eta)^l \zeta) \subseteq U$, and that $\alpha_{s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*}(z_t) \neq \alpha_{s_{\mu(\eta)^{m'}} s_{\nu(\eta)'}^*}(z_t)$ for any $z_t \in Z(\mu(\eta)^l \zeta)$. This contradicts the assumption that $\phi_t((x_t, k, y_t)) = \phi_t\big((x_t, k', y_t)\big)$. If η does not have an exit, then $y_t \in \partial(E_t)_{\mathrm{iso.}}$. Without loss of generality assume $k > k'$, then we can use Lemma 4.2(c) to compute

$$\Big[p_{y_t}\big(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*\big)^* s_{\mu(\eta)^{m'}} s_{\nu(\eta)^n}^* p_{y_t}\Big]_1 = \Big[p_{y_t} s_\nu s_{\eta^{k-k'}} s_\nu^* p_{y_t}\Big]_1 = \Big[\big(p_{y_t} s_\nu s_\eta s_\nu^* p_{y_t}\big)^{k-k'}\Big]_1,$$

and the second assertion in Lemma 4.3 implies that $\Big[\big(p_{y_t} s_{y_t} s_n s_\nu^* p_{y_t}\big)^{k-k'}\Big]_1 \neq 0$. Thus,

and hence $\big(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*, y_t\big) \nsim \Big(s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^*, y_t\Big)$. But this means $\phi_t((x_t, k, y_t)) \neq \phi_t\big((x_t, k', y_t)\big)$, which is a contradiction. So we must have $k = k'$, and hence ϕ_t is injective.

To show that ϕ_t is surjective, let $[(n, x_t)]$ be an arbitrary element of $\mathcal{G}_{(C^*(E_t), \mathcal{D}(E_t))}$. Let $f_t := \pi_t(n)$, where $\pi_t \colon C^*(E_t) \to C^*\big(\mathcal{G}_{E_t}\big)$ is the isomorphism from Proposition 2.2. We know from (iii) of Lemma 4.10 that $(\alpha_n(x_t), k, x_t) \in \mathrm{supp}'(f_t)$ for some $k \in \mathbb{Z}$. Suppose first that $x_t \notin \partial(E_t)_{\mathrm{iso}}$. Choose $\mu, \nu \in E_t^*$, a clopen neighborhood U of $\alpha_n(x_t)$, and a clopen neighborhood V of x_t such that $U \subseteq Z(\mu), V \subseteq Z(\nu), \sigma_{E_t}^{|\mu|}(U) = \sigma_{E_t}^{|\nu|}(V), k = |\mu| - |\nu|$, and $Z(U, |\mu|, |\nu|, V) \subseteq \mathrm{supp}'(f_t)$. Then $\alpha_{s_\mu s_\nu^*}(y_t) = \alpha_n(y_t)$ for all $y_t \in V$, and hence $\phi_t(\alpha_n(x_t), |\mu| - |\nu|, x_t) = \big[\big(s_\mu s_\nu^*, x_t\big)\big] = [(n, x_t)]$.

Now suppose that $x_t \in \partial(E_t)_{\mathrm{iso}}$ is not eventually periodic. Choose $\mu, \nu \in E_t^*$ and $z_t \in \partial E_t$ such that $x_t = \nu z_t, \alpha_n(x_t) = \mu z_t$, and $k = |\mu| - |\nu|$. It follows from Lemma 4.3 that $\Big[U_{(x_t, n, s_\mu s_\nu^*)}\Big]_1 = 0$ (because $K_1(\mathbb{C}) = 0$), and thus that $\phi_t\big((\alpha_n(x_t), k, x_t)\big) = \big[\big(s_\mu s_\nu^*, x_t\big)\big] = [(n, x_t)]$. Assume then that x_t is eventually periodic. Then there are $\mu, \nu \in E_t^*$ and a simple loop $\eta \in E_t^*$ such that $s_t(\eta) = r_t(\mu) = r_t(\nu), x_t = \nu\eta\eta\eta \cdots, \alpha_n(x_t) = \mu\eta\eta\eta \cdots$, and $k = |\mu| - |\nu|$. Choose positive integers l and m such that $\Big[U_{(x_t, n, s_\mu s_\nu^*)}\Big]_1 = l - m$. Then by Lemma 4.3 we have

$$\big[p_{x_t} s_\mu s_{\eta^1} s_{\eta^m}^* s_\nu^* p_{x_t}\big]_1 = \Big[\big(p_{x_t} s_\mu s_\eta s_\nu^* p_{x_t}\big)^l \big(p_{x_t} s_\mu s_\eta^* s_\nu^* p_{x_t}\big)^m\Big]_1 = l - m,$$

and hence $\Big[U_{(x_t, n, s_\mu s_\nu^*)}\Big]_1 = \big[p_{x_t} s_\nu s_{\eta\eta} s_{\eta^m}^* s_\nu^* p_{x_t}\big]_1$. Since

$$(h_t)_{E_t}\big(s_{\nu\eta^m}s^*_{\nu\eta^m}\big)^{-1/2}U_{(x_t,n,s_\mu s_\nu^*)}\big(p_{x_t}s_\nu s_{\eta'}s^*_{\eta^m}s^*_\nu p_{x_t}\big)^* = U_{(x_t,n,s^{ms}_{\mu\eta}s^*_{\nu\eta})},$$

We have $\Big[U_{(x_t,n,s^m_{\mu\eta}s^*\ {}'_{\nu\eta})}\Big]_1 = \Big[U_{(x_t,n,s_\mu s_\nu^*)}\Big]_1 - \Big[p_{x_t}s_\nu s_{\eta^l}s^*_{\eta^m}s^*_\nu p_{x_t}\Big]_1 = 0$. Hence

$$\phi_t\big((\alpha_n(x_t),|\mu(\eta)^m|-|\nu(\eta)^l|,x_t)\big) = \Big[\Big(s_{\mu(\eta)^m}s^*_{\nu(\eta)^l},x_t\Big)\Big] = [(n,x_t)],$$

which shows that ϕ_t is surjective.

To see that ϕ_t is open, let $\mu,\nu \in E_t^*$ and let U and V be clopen subsets of ∂E_t such that $U \subseteq Z(\mu), V \subseteq Z(\nu)$, and $\sigma^{|\mu|}_{E_t}(U) = \sigma^{|\nu|}_{E_t}(V)$. Then there is a $p_V \in \mathcal{D}(E_t)$ such that $(h_t)_{E_t}(x_t)(p_V) = 1$ if $x_t \in V$, and $(h_t)_{E_t}(x_t)(p_V) = 0$ if $x_t \in \partial E_t \setminus V$; and then $\phi_t(Z(U,|\mu|,|\nu|,V)) = \{[s_\mu s_\nu^* p_V, x_t] : x_t \in \mathrm{dom}(s_\mu s_\nu^* p_V)\}$. This shows that ϕ_t is open.

To prove that ϕ_t is continuous we will show that $\phi_t^{-1}(\{[(n,y_t)] : y_t \in \mathrm{dom}(n)\})$ is open for each $n \in N(\mathcal{D}(E_t))$. Fix $n \in N(\mathcal{D}(E_t))$ and $z_t \in \mathrm{dom}(n)$. We claim that there is an open subset $V_{(n,z_t)}$ in $\mathcal{G}_{E_t}$ such that

$$\phi_t^{-1}([(n,z_t)]) \in V_{(n,z_t)} \subseteq \phi_t^{-1}(\{[(n,y_t)] : y_t \in \mathrm{dom}(n)\}).$$

Let $\pi_t : C^*(E_t) \to C^*(\mathcal{G}_{E_t})$ be the isomorphism from Proposition 2.2, and $f_t := \pi_t(n)$.

We know from (iii) of Lemma 4.10 that $(\alpha_n(z_t),k,z_t) \in \mathrm{supp}'(f_t)$ for some $k \in \mathbb{Z}$.

Suppose that there are two different integers k_1 and k_2 such that both $(\alpha_n(z_t),k_1,z_t)$ and $(\alpha_n(z_t),k_2,z_t)$ belong to $supp\ '(f_t)$. Then there are $\mu_1,\nu_1,\mu_2,\nu_2 \in E_t^*$ such that $(\alpha_n(z_t),k_1,z_t) \in Z(\mu_1,\nu_1), (\alpha_n(z_t),k_2,z_t) \in Z(\mu_2,\nu_2)$ and $Z(\mu_1,\nu_1), Z(\mu_2,\nu_2) \subseteq \mathrm{supp}'(f_t)$.

Without loss of generality we can assume that $|\mu_1| = |\mu_2|$, and then we have $\mu_1 = \mu_2$.

We also have $\nu_1 = \nu_2\xi$ or $\nu_2 = \nu_1\xi$ for some $\xi \in E_t^* \setminus E_t^0$; we assume that $\mu_2 = \mu_1\xi$ and denote $\mu := \mu_1 = \mu_2$.

We claim that z_t is an isolated point, and that

$$s_t\big(Z(\mu,\nu_1)\big) \cap s_t\big(Z(\mu,\nu_1\xi)\big) = \{(z_t)\}.$$

To see this, suppose $(x_t) \in s_t\big(Z(\mu,\nu_1)\big) \cap s_t\big(Z(\mu,\nu_1\xi)\big)$. Then $x_t = \nu_1 y_t$ for some y_t such that $\alpha_n(y_t) = \mu y_t$, and $x_t = \nu_1\xi y_t'$ for some y_t' such that $\alpha_n(y_t) = \mu y_t'$. It follows that $y_t' = y_t = \xi y_t'$, and hence $y_t = \nu_1\xi\xi\xi$ .... So

$$s_t\big(Z(\mu,\nu_1)\big) \cap s_t\big(Z(\mu,\nu_1\xi)\big) = \{(\nu_1\xi\ \dots)\} = \{(z_t)\},$$

and hence z_t is an isolated point. Now $\phi_t^{-1}([(n,z_t)]) = (\mu\xi\ \dots, |\mu| - |\nu_1|, \nu_1\xi\ \dots)$ is isolated because $\{\phi_t^{-1}([(n,z_t)])\} = Z(\{\mu\xi,\dots\},|\mu|,|\nu_1|,\{\nu_1\xi\ \dots\})$ is open. So in this case we take $V_{(n,z_t)} = \{\phi_t^{-1}([(n,z_t)])\}$.

Now assume that there is a unique k such that $(\alpha_n(z_t),k,z_t) \in \mathrm{supp}'(f_t)$. Choose $\mu,\nu \in E_t^*$ with $r_t(\mu) = r_t(\nu)$ and an open subset $V \subseteq r_t(\mu)\,\partial E_t$ such that $(\alpha_n(z_t),k,z_t) \in Z(\mu V,|\mu|,|\nu|,\nu V) \subseteq \mathrm{supp}'(f_t)$. Lemma 4.10 implies that $\alpha_n(x_t) = \alpha_{s_\mu s_\nu^*}(x_t)$ for all $x_t \in \nu V$.

We aim to find an open subset $W \subseteq \nu V$ such that $z_t \in W$ and

$$\Big[U_{(x_t,n,s_\mu s_\nu^*)}\Big]_1 = 0 \text{ for all } x_t \in W \cap \partial(E_t)_{\mathrm{iso}}\,; \tag{4.9}$$

for then we have $\phi_t\big((\alpha_n(x_t),|\mu|-|\nu|,x_t)\big) = \big[(s_\mu s_\nu^*,x_t)\big] = [(n,x_t)]$ for all $x_t \in W$, and the open subset $V_{(n,z_t)} := Z(\alpha_n(W),|\mu|,|\nu|,W)$ satisfies the desired $\phi_t^{-1}([(n,z_t)]) \in V_{(n,z_t)} \subseteq \phi_t^{-1}(\{[(n,y_t)] : y_t \in \mathrm{dom}(n)\})$.

Let $\delta := |f_t(\alpha_n(z_t),k,z_t)|$. Then

$$(h_t)_{E_t}(z_t)(n^*n) = f_t^*f_t(z_t,0,z_t) = \sum_{\substack{\gamma\in\mathcal{G}_{E_t}\\ s(\gamma)=(z_t)}}\sum_t |f_t(\gamma)|^2 = \delta^2.$$

Choose an open subset $V_0 \subseteq \nu V$ such that $z_t \in V_0$ and $(h_t)_{E_t}(x_t)(n^*n) > (\delta/2)^2$ for all $x_t \in V_0$. Define

$$g_t := f_t^* 1_{Z(\mu,\nu)} - \lambda(f_t^*f_t)^{\frac{1}{2}},$$

where $\lambda = \overline{f_t(\alpha_n(z_t),k,z_t)}/|f_t(\alpha_n(z_t),k,z_t)| \in \mathbb{T}$. We claim that $g_t(z_t,j,z_t) = 0$ for all $j \in \mathbb{Z}$.

When $j = 0$ we have

$$g_t(z_t,0,z_t) = \sum_{\gamma_1\gamma_2=(z_t,0,z_t)}\sum_t f_t^*(\gamma_1)1_{Z(\mu,\nu)}(\gamma_2) - \lambda\sum_{\eta_1\eta_2=(z_t,0,z_t)}\sum_t \big(f_t^*(\eta_1)f_t(\eta_2)\big)^{\frac{1}{2}}.$$

Implication (ii) of Lemma 4.10 ensures that the only terms in the sums which produce nonzero entries are $\gamma_1,\eta_1 = (z_t,-k,\alpha_n(z_t))$ and $\gamma_2,\eta_2 = (\alpha_n(z_t),k,z_t)$. Hence

$$g_t(z_t,0,z_t) = \overline{f_t(\alpha_n(z_t),k,z_t)} - \lambda|f_t(\alpha_n(z_t),k,z_t)| = 0.$$

When $j \neq 0$, both terms in the expression for g_t contain $f_t(\alpha_n(z_t),k-j,z_t)$, which is zero. Hence $g_t(z_t,j,z_t) = 0$.

Use Proposition 2.4 to choose $m \in \mathbb{N}$ such that $\|g_t - \Sigma_m(g_t)\| < \delta/2$. Since $g_t(z_t,j,z_t) = 0$ for all $j \in \mathbb{Z}$, there is an open set W such that $z_t \in W \subseteq V_0$ and

$$\left|\left(1-\frac{|j|}{m+1}\right)g_t(x_t,j,x_t)\right| < \frac{\delta}{2(m+1)} \tag{4.10}$$

DOI: 10.35629/0743-10082039

for all $-m \le j \le m$ and $x_t \in W$. Then $|\Sigma_m(g_t)(x_t, j, x_t)| < \delta/2$ for all $(x_t, j, x_t) \in \mathcal{G}_{E_t}$ with $x_t \in W$. It follows from the definition of the norm on $C^*(\mathcal{G}_{E_t})$ that for all $x_t \in W \cap \partial(E_t)_{\text{iso}}$ we have

$$\left\|\sum_t \pi_t(p_{x_t})\Sigma_m(g_t)\pi_t(p_{x_t})\right\| \le \sum_t \left|\sum_{\gamma\in\mathcal{G}_{E_t}} \left(\pi_t(p_{x_t})\Sigma_m(g_t)\pi_t(p_{x_t})\right)(\gamma)\right|$$
$$= \left|\sum_{\gamma\in\mathcal{G}_{E_t}} \sum_{\gamma_1\ {}_2\gamma_3=\gamma} \sum_t 1_{\{(x_t,0,x_t)\}}(\gamma_1)\Sigma_m(g_t)(\gamma_2)1_{\{(x_t,0,x_t)\}}(\gamma_3)\right|$$
$$= \left|\sum_{j\in\mathbb{Z}} \sum_t \Sigma_m(g_t)(x_t, j, x_t)\right| < \frac{\delta}{2}.$$

Hence

$$\left\|\sum_t \pi_t(p_{x_t})g_t\pi_t(p_{x_t})\right\| \le \sum_t \|g_t - \Sigma_m(g_t)\| + \sum_t \|\pi_t(p_{x_t})\Sigma_m(g_t)\pi_t(p_{x_t})\| < \delta. \tag{4.11}$$

We now claim that $\left\|U_{(x_t,n,s_\mu s_\nu^*)} - \lambda p_{x_t}\right\| < 2$ for all $x_t \in W \cap \partial(E_t)_{\text{iso}}$. To see this, first note that

$$\pi_t(p_{x_t}) = (f_t^* f_t)^{-\frac{1}{2}}(x_t, 0, x_t)(f_t^* f_t)^{\frac{1}{2}}(x_t, 0, x_t)\pi_t(p_{x_t}) = (f_t^* f_t)(x_t, 0, x_t)^{-\frac{1}{2}}(f_t^* f_t)^{\frac{1}{2}}\pi_t(p_{x_t}).$$

Thus

$$\begin{aligned}\pi_t\left(U_{(x_t,n,s_\mu s_\nu^*)} - \lambda p_{x_t}\right) &= \pi_t\left((h_t)_{E_t}(x_t)(n^*n)^{-1/2}p_{x_t}n^*s_\mu s_\nu^* p_{x_t} - \lambda p_{x_t}\right)\\ &= (f_t^* f_t)(x_t, 0, x_t)^{-\frac{1}{2}}\pi_t(p_{x_t})f_t^* 1_{Z(\mu,\nu)}\pi_t(p_{x_t}) - \lambda\pi_t(p_{x_t})\\ &= (f_t^* f_t)(x_t, 0, x_t)^{-1/2}\left(\pi_t(p_{x_t})f_t^* 1_{Z(\mu,\nu)}\pi_t(p_{x_t}) - \lambda(f_t^* f_t)^{1/2}\pi_t(p_{x_t})\right)\\ &= (f_t^* f_t)(x_t, 0, x_t)^{-\frac{1}{2}}\pi_t(p_{x_t})g_t\pi_t(p_{x_t}).\end{aligned}$$

Using (4.11) we now get

$$\begin{aligned}\left\|U_{(x_t,n,s_\mu s_\nu^*)} - \lambda p_{x_t}\right\| = \left\|\pi_t\left(U_{(x_t,n,s_\mu s_\nu^*)} - \lambda p_{x_t}\right)\right\| &= (f_t^* f_t)(x_t, 0, x_t)^{-1/2}\|\pi_t(p_{x_t})g_t\pi_t(p_{x_t})\|\\ &< (f_t^* f_t)(x_t, 0, x_t)^{-\frac{1}{2}}\delta.\end{aligned}$$

Recall that $x_t \in W \cap \partial(E_t)_{\text{iso}} \subseteq V_0$, and hence $(f_t^* f_t)(x_t, 0, x_t)^{-1/2} = (h_t)_{E_t}(x_t)(n^*n)^{-1/2} < 2/\delta$ So

$$\left\|U_{(x_t,n,s_\mu s_\nu^*)} - \lambda p_{x_t}\right\| < 2.$$

But this means $\left[U_{(x_t,n,s_\mu s_\nu^*)}\right]_1 = 0$, and so W satisfies the desired (4.9). As mentioned, this means $V_{(n,z_t)} := Z(\alpha_n(W), |\mu|, |\nu|, W)$ satisfies

$$\phi_t^{-1}([(n, z_t)]) \in V_{(n,z_t)} \subseteq \phi_t^{-1}(\{[(n, y_t)]: y_t \in \text{dom}(n)\}),$$

as required.

Proposition 4.11 (see [27]). Let E_t and F_t be two graphs. If there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ to $\mathcal{D}(F_t)$, then $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ and $\mathcal{G}_{(C^*(F_t),\mathcal{D}(F_t))}$ are isomorphic as topological groupoids, and consequently $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids.

Proof. Suppose ϕ_t is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ to $\mathcal{D}(F_t)$.

Then there is a homeomorphism $\kappa: \partial E_t \to \partial F_t$ such that $(h_t)_{E_t}(x_t)(f_t) = (h_t)_{F_t}(\kappa(x_t))\phi_t(f_t)$ for all $f_t \in \mathcal{D}(E_t)$ and all $x_t \in \partial E_t$. It is routine to check that the *map* $[(n, x_t)] \mapsto [(\phi_t(n), \kappa(x_t))]$ is an isomorphism between the topological groupoids $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ and $\mathcal{G}_{(C^*(F_t),\mathcal{D}(F_t))}$. Then Proposition 4.8 implies that $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids.

V. Main Result and Examples

Theorem 5.1 (see [27]). Let E_t and F_t be graphs. Consider the following four statements.

(1) There is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$.

(2) The graph groupoids $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids.

(3) The pseudogroups of E_t and F_t are isomorphic.

(4) E_t and F_t are orbit equivalent.

Then (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) and (2) $\Rightarrow$ (3). If E_t and F_t satisfy condition (L), then (3) $\Rightarrow$ (2) and the four statements are equivalent.

Proof. (1) $\Rightarrow$ (2) is proved in Proposition 4.11. (2) $\Rightarrow$ (1) follows from Proposition 2.2. (3) $\Leftrightarrow$ (4) is proved in Proposition 3.4. (2) $\Rightarrow$ (3) follows directly from the definition of the pseudogroups $\mathcal{P}_{E_t}$ and $\mathcal{P}_{F_t}$.

Assume that E_t and F_t satisfy condition (L). Then it follows from Proposition 2.3 and [22, Proposition 3.6(i)] that $\mathcal{G}_{E_t}$ is isomorphic to the groupoid of germs of the pseudogroup $\mathcal{P}_{E_t}$ constructed on page 8 of [22], and that $\mathcal{G}_{F_t}$ is isomorphic to the groupoid of germs of the pseudogroup $\mathcal{P}_{F_t}$. It follows that if $\mathcal{P}_{E_t}$ and $\mathcal{P}_{F_t}$ are isomorphic, then $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic. Thus (3) $\Longrightarrow$ (2), and all 4 statements are equivalent when E_t and F_t satisfy condition (L). We have the following (see [27]).

Example 5.2. We show that (3) does not imply (2) in general. Consider the single vertex and single loop graphs

$$E_t F_t$$

We have $\partial E_t = \{v_t\}$ and $\partial F_t = \{ee\ldots\}$. So E_t and F_t are orbit equivalent, but $C^*(E_t) \cong \mathbb{C}$ is not isomorphic to $C^*(F_t) \cong C(\mathbb{T})$. Obviously F_t does not satisfy condition (L), so E_t and F_t provide a simple counterexample to the equivalence of statements (1) and (4) of Theorem 5.1 without the presence of condition (L).

Example 5.3. The graphs

$$E_t F_t$$

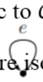
provide a similar counterexample to the equivalence of statements (1) and (4) of Theorem 5.1 without the presence of condition (L). In this case $\partial E_t = \mathbb{N} = \partial F_t$ (and, unlike Example 5.2, the shift map is defined on all of ∂E_t and ∂F_t), but $C^*(E_t) \cong \mathcal{K} \neq \mathcal{K} \otimes C(\mathbb{T}) \cong C^*(F_t)$.

Example 5.4. There exist graphs E_t and F_t such that $C^*(E_t)$ and $C^*(F_t)$ are isomorphic, and $\mathcal{D}(E_t)$ and $\mathcal{D}(F_t)$ isomorphic, but E_t and F_t are not orbit equivalent.

Consider for example the graphs $(E_t)_2$ and $(E_t)_2^-$below.

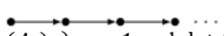
$$(E_t)_2 (E_t)_2^-$$

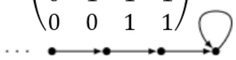
It follows from [19, Remark 2.8] that the C^*-algebra of $(E_t)_2$ is isomorphic to $\mathcal{O}_2$ (see for example [23]) and that the C^*-algebra of $(E_t)_2^-$is isomorphic to $\mathcal{O}_2^-$(see for example [23]).



It is proved in [23, Lemma 6.4] that $\mathcal{O}_2$ and $\mathcal{O}_2^-$ are isomorphic. We also have that $\mathcal{D}((E_t)_2)$ and $\mathcal{D}((E_t)_2^-)$because both $\partial(E_t)_2$ and $\partial(E_t)_2^-$are Cantor sets. However, $(E_t)_2$ and $(E_t)_2^-$ cannot be orbit equivalent because if they were, then it would follow from Theorem 5.1 and [15, Theorem 3.6] that $\det(I-(A_t)_2) = \det(I-(A_t)_2^-)$where

$$(A_t)_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } (A_t)_2^- = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$





However, $\det(I-(A_t)_2) = -1$ and $\det(I-(A_t)_2^-) = 1$.

VI. Applications

Here we provide two applications of Theorem 5.1.The first result shows that conjugacy of general graphs implies that their C^*-algebras are isomorphic and the isomorphism decends to their maximal abelian subalgebras. As a corollary we obtain a strengthening of [3, Theorem 3.2].The second application adds three additional equivalences to [4, Theorem 1.1], which provides a complete invariant for amplified graphs (see [27]).

6.1. Conjugacy and out-splitting. Two graphs E_t and F_t are said to be conjugate if there is a homeomorphism $h_t: \partial E_t \to \partial F_t$ such that $h_t(\partial E_t^{\geq 1}) = \partial F_t \geq^1$ and $h_t\big(\sigma_{E_t}(x_t)\big) = \sigma_{F_t}(h_t(x_t))$ for all $x_t \in \partial E_t^{\geq 1}$. It is routine to verify that if E_t and F_t are conjugate, then they are also orbit equivalent. Thus Theorem 5.1 implies that if E_t and F_t both satisfy condition (L) and they are conjugate, then there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$. In Theorem 6.1 we will prove that if E_t and F_t are conjugate, then $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic, and hence there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$, even if E_t and F_t do not satisfy condition (L). As a corollary, we strengthen [3, Theorem 3.2] for out-splittings of graphs.

Theorem 6.1 (see [27]). Let E_t and F_t be graphs. If E_t and F_t are conjugate, then $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids, and hence there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$.

Proof. Let $h_t: \partial E_t \to \partial F_t$ be a homeomorphism such that $h_t(\partial E_t^{\geq 1}) = \partial F_t \geq 1$ and $h_t\big(\sigma_{E_t}(x_t)\big) = \sigma_{F_t}(h_t(x_t))$

for all $x_t \in \partial E_t \geq 1$. Define $\phi_t: \mathcal{G}_{E_t} \to \mathcal{G}_{F_t}$ by $\phi_t((x_t, k, y_t)) = (h_t(x_t), k, h_t(y_t))$. Then ϕ_t is a homeomorphism, and $\mathcal{G}_{E_t}$ and $\mathcal{G}_{F_t}$ are isomorphic as topological groupoids. Then Theorem 5.1 implies that there is an isomorphism from $C^*(E_t)$ to $C^*(F_t)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}(F_t)$.

As a corollary we are able to strengthen [3, Theorem 3.2]. Before we state the corollary we recall the terminology of [3].

Let E_t be a graph and let $\mathcal{P}$ be a partition of E_t^1 constructed in the following way. For each $v_t \in E_t^0$ with $v_t E_t^1 \neq \emptyset$, partition $v_t E_t^1$ into disjoint nonempty subsets $\mathcal{E}_{v_t}^1, \ldots, \mathcal{E}_{v_t}^{m(v_t)}$ where $m(v_t) \geq 1$, and let $m(v_t) = 0$ when $v_t E_t^1 = \emptyset$. The partion $\mathcal{P}$ is proper if for each $v_t \in E_t^0$ we have that $m(v_t) < \infty$ and that $\mathcal{E}_{v_t}^i$ is infinite for at most one i. The out-split of E_t with respect to $\mathcal{P}$ is the graph $(E_t)_{s_t}(\mathcal{P})$ where

$$(E_t)_{s_t}(\mathcal{P})^0 := \{v_t^i : v_t \in E_t^0, 1 \leq i \leq m(v_t)\} \cup \{v_t : v_t \in E_t^0, m(v_t) = 0\},$$

$$(E_t)_{s_t}(\mathcal{P})^1 := \{e^j : e \in E_t^1, 1 \leq j \leq m(r_t(e))\} \cup \{e : e \in E_t^1, m(r_t(e)) = 0\}$$

and $r_t, s_t : (E_t)_{s_t}(\mathcal{P})^1 \to (E_t)_{s_t}(\mathcal{P})^0$ are given by

$s_t(e^j) := s_t(e)^i$ and $r_t(e^j) := r_t(e)^j$ for $e \in \mathcal{E}_{s_t(e)}^i$ with $m(r_t(e)) \geq 1$, and

$s_t(e) := s_t(e)^i$ and $r_t(e) := r_t(e)$ for $e \in \mathcal{E}_{s_t(e)}^i$ with $m(r_t(e)) = 0$.

Corollary 6.2 (see [27]). Let $\mathcal{P}$ be a proper partition of E_t^1 as above. Then E_t and $(E_t)_{s_t}(\mathcal{P})$ are conjugate and there is an isomorphism from $C^*(E_t)$ to $C^*\big((E_t)_{s_t}(\mathcal{P})\big)$ which maps $\mathcal{D}(E_t)$, onto $\mathcal{D}\big((E_t)_{s_t}(\mathcal{P})\big)$.

Proof. Notice that since $\mathcal{P}$ is proper, we have that $v_t^i \in (E_t)_{s_t}(\mathcal{P})^0_{\text{reg}}$ if $v_t \in (E_t^0)_{\text{reg}}$, and that if $v_t E_t^1$ is infinite, then $v_t^i (E_t)_{s_t}(\mathcal{P})^1$ is infinite for exactly one i.

For $x_t = (x_t)_0 (x_t)_1 \cdots \in \partial E_t$, let $h_t(x_t) = y_t = (y_t)_0 (y_t)_1 \cdots \in \partial (E_t)_{s_t}(\mathcal{P})$ be defined by $h_t(x_t)$ having the same length as x_t and

$$(y_t)_n := \begin{cases} (x_t)_n & \text{if } m\big(r_t((x_t)_n)\big) = 0, \\ (x_t)_n^j & \text{if } (x_t)_{n+1} \in \mathcal{E}_{r_t(e)}^j, \\ (x_t)_n^j & \text{if } r_t((x_t)_n)^j (E_t)_{s_t}(\mathcal{P})^1 \text{ is infinite and the length of } x_t \text{ is } n. \end{cases}$$

Then the map $x_t \mapsto h_t(x_t)$ is a homeomorphism from ∂E_t to $\partial (E_t)_{s_t}(\mathcal{P}), h_t(\partial E_t^{\geq 1}) = \partial (E_t)_{s_t}(\mathcal{P})^{\geq 1}$ and $h_t\big(\sigma_{E_t}(x_t)\big) = \sigma_{(E_t)_{s_t}(\mathcal{P})}(h_t(x_t))$ for all $x_t \in \partial E_t^{\geq 1}$. Thus, E_t and $(E_t)_{s_t}(\mathcal{P})$ are conjugate and it follows from Theorem 6.1 that there is an isomorphism from $C^*(E_t)$ to $C^*\big((E_t)_{s_t}(\mathcal{P})\big)$ which maps $\mathcal{D}(E_t)$ onto $\mathcal{D}\big((E_t)_{s_t}(\mathcal{P})\big)$.

Remark 6.3. To see that orbit equivalence is weaker than conjugacy consider the graphs E_t and F_t from Example 3.2. We have already seen that E_t and F_t are orbit equivalent.

They are not, however, conjugate because $\sigma_{E_t}(e_2 e_2 \ldots) = e_2 e_2 \ldots$ and $\sigma_{F_t}(y_t) \neq y_t$ for all $y_t \in \partial F_t$, and fixed points are a conjugacy invariant.

6.2. Amplified graphs and orbit equivalence. In [4], a graph is called amplified if whenever there is an edge between two vertices in the graph, there are infinitely many. Theorem 1.1 in [4] characterises when the C^*-algebras of amplified graphs are isomorphic. Using our main result, we improve this result by adding three additional equivalences, see Theorem 6.4. Before we precisely state the result of [4] and our improvement, we will first recall the notation of [4].

If E_t is a graph, then the amplification of E_t is the graph $\bar{E}_t$ defined by $\bar{E}_t^0 := E_t^0, \bar{E}_t^1 := \{e(v_t, w_t)^n : e \in E_t^1, s_t(e) = v_t, r_t(e) = w_t, n \in \mathbb{N}\}, s_t(e(v_t, w_t)^n) := v_t$, and $r_t(e(v_t, w_t)^n) := w_t$.

It is routine to see that a graph E_t is amplified if and only if $E_t = \bar{E}_t$.

If E_t is a graph, then the transitive closure of E_t is the graph tE_t defined by $E_t^0 := E_t^0, tE_t^1 := E_t^1 \cup \{e(v_t, w_t) : \mu \in E_t^* \setminus (E_t^0 \cup E_t^1), s_t(\mu) = v_t, r_t(\mu) = w_t\}$, with source and range maps that extend those of E_t and satisfy $s_t(e(v_t, w_t)^n) := v_t$, and $r_t(e(v_t, w_t)^n) := w_t$.

Theorem 1.1 of [4] says that if E_t and F_t are graphs with E_t^0 and F_t^0 finite, then the following 6 statements are equivalent.

(1) The graphs $\overline{tE_t}$ and $\overline{tF_t}$ are isomorphic, in the sense that there are bijections $\phi_t^0 : \overline{tE_t^0} \to \overline{tF_t^0}$ and $\phi_t^1 : \overline{tE_t^1} \to \overline{tF}^1$ such that $s_t(\phi_t^1(e)) = \phi_t^0(s_t(e))$ and $r_t(\phi_t^1(e)) = \phi_t^0(r_t(e))$ for all $e \in \overline{tE_t^1}$.

(2) The C^*-algebras $C^*(\overline{tE})$ and $C^*(\overline{tF_t})$ are isomorphic.

(3) The C^*-algebras $C^*(\bar{E}_t)$ and $C^*(\bar{F}_t)$ are isomorphic.

(4) The C^*-algebras $C^*(\bar{E}_t)$ and $C^*(\bar{F}_t)$ are stably isomorphic.

(5) The tempered primitive ideal spaces $\text{Prim}^\tau(C^*(\bar{E}_t))$ and $\text{Prim}^\tau(C^*(\bar{F}_t))$ are isomorphic (see [4, Definition 4.8]).

(6) The ordered filtered K-theories $F_tK(C^*(\bar{E}_t))$ and $F_tK(C^*(\bar{F}_t))$ of $C^*(\bar{E}_t)$ and $C^*(\bar{F}_t)$ are isomorphic (see [4, Definition 4.4]).
The following result improves on [4, Theorem 1.1].
Theorem 6.4. Let E_t and F_t be graphs with E_t^0 and F_t^0 finite. Then each of the following 3 statements is equivalent to each of the statements $(1)-(6)$ above.
(7) The graphs $\bar{E}_t$ and $\bar{F}_t$ are orbit equivalent.
(8) The graph groupoids $\mathcal{G}_{\bar{E}_t}$ and $\mathcal{G}_{\bar{F}_t}$ are isomorphic as topological groupoids.
(9) There exists an isomorphism from $C^*(\bar{E}_t)$ to $C^*(\bar{F}_t)$ which maps $\mathcal{D}(\bar{E}_t)$ onto $\mathcal{D}(\bar{F}_t)$.
Hence the statements $(1)-(9)$ are all equivalent.
To prove Theorem 6.4 we need two results. We start with a modification of [4, Theorem 3.8].
Lemma 6.5 (see [27]). Let E_t be a graph and $\mu=\mu_1\mu_2\dots,\mu_m\in E_t^*$. Let F_t be the graph with $F_t^0:=E_t^0, F_t^1:=E_t^1\cup\{\mu^n:n\in\mathbb{N}\}$, and range and source maps that extend those of E_t and satisfy $s_t(\mu^n):=s_t(\mu)$ and $r_t(\mu^n):=r_t(\mu)$. If the set $\{e\in E_t^1: s_t(e)=s_t(\mu), r_t(e)=r_t(\mu)\}$ is infinite, then E_t and F_t are orbit equivalent.
Proof. Let $A_t:=\{e\in E_t^1: s_t(e)=s_t(\mu), r_t(e)=r_t(\mu)\}$ and assume A_t is infinite. Then there are injective functions $\eta_1:\mathbb{N}\to A_t$ and $\eta_2:A_t\to A_t$ such that $\eta_1(\mathbb{N})\cap\eta_2(A_t)=\emptyset$ and $\eta_1(\mathbb{N})\cup\eta_2(A_t)=A_t$. For each $x_t\in\partial F_t$, let $h_t(x_t)$ be the element of ∂E_t obtained by, for each $n\in\mathbb{N}$, replacing every occurence of μ^n by the path $\eta_1(n)\mu_2\mu_3\dots\mu_m$ and, for each $e\in A_t$, replacing every occurence of the path $a\mu_2\mu_3\dots\mu_m$ by the path $\eta_2(a)\mu_2\mu_3\dots\mu_m$.
Then $x_t\mapsto h_t(x_t)$ is a homeomorphism from ∂F_t to ∂E_t.
Define $k_1,l_1:\partial F_t\geq 1\to\mathbb{N}$ by

$$k_1(x_t):=0 \text{ for all } x_t\in\partial F_t^{\geq 1},\quad l_1(x_t):=\begin{cases} m & \text{if } x_t\in\cup_{n\in\mathbb{N}} Z(\mu^n),\\ 1 & \text{if } x_t\notin\cup_{n\in\mathbb{N}} Z(\mu^n).\end{cases}$$

Then k_1 and l_1 are both continuous, and $\sigma_{E_t}^{k_1(x_t)}\left(h_t\left(\sigma_{F_t}(x_t)\right)\right)=\sigma_{E_t}^{l_1(x_t)}\left(h_t(x_t)\right)$ for all $x_t\in\partial F_t^{\geq 1}$.
Similarly, define $k_1',l_1':\partial E_t^{\geq 1}\to\mathbb{N}$ by

$$k_1'(y_t):=\begin{cases} m-1 & \text{if } y_t\in\cup_{e\in\eta_1(\mathbb{N})} Z(e\mu_2\mu_3\dots\mu_m),\\ 0 & \text{if } y_t\in\cup_{e\in\eta_2(A_t)} Z(e\mu_2\mu_3\dots\mu_m),\\ 0 & \text{if } y_t\notin\cup_{e\in A_t} Z(e\mu_2\mu_3\dots\mu_m).\end{cases}$$

$$l_1'(y_t):=1 \text{ for all } x_t\in\partial F_t^{\geq 1}.$$

Then k_1' and l_1' are both continuous, and $\sigma_{F_t}^{k_1'(y_t)}\left(h_t^{-1}\left(\sigma_{E_t}(y_t)\right)\right)=\sigma_{F_t}^{l_1'(y_t)}(h_t^{-1}(y_t))$ for all $y_t\in\partial E_t^{\geq 1}$. This shows that E_t and F_t are orbit equivalent.
Proposition 6.6 [27]. Let E_t be a graph with E_t^0 finite. Then $\bar{E}_t$ and $\overline{tE_t}$ are orbit equivalent.
Proof. Notice that $\overline{tE_t}$ can be obtained from $\bar{E}_t$ by adding infinitely many edges from v_t to w_t whenever there is a path from v_t to w_t. Thus, that $\bar{E}_t$ and $\overline{tE_t}$ are orbit equivalent follows from finitely many applications of Lemma 6.5.
Proof of Theorem 6.4. Since both $\bar{E}_t$ and $\bar{F}_t$ satisfy condition (L), it follows from our main theorem that $(7)-(9)$ are equivalent, and it is obvious that (9) implies (3). Proposition 6.6 shows that (1) implies (7).

References

[1]. C. Anantharaman-Delaroche and J. Renault, Amenable Groupoids, Monographies de L'EnseignementMath´ematique, vol. 36, L'EnseignementMath´ematique, Geneva, 2000.
[2]. T. Bates, J. Hong, I. Raeburn, andW. Szymanski, The ideal structure of the C_-algebras of infinite graphs, Illinois J. Math. 46 (2002), 1159–1176.
[3]. T. Bates and D. Pask, Flow equivalence of graph algebras, Ergodic Th. and Dynam. Sys. 24 (2004), 367–382.
[4]. S. Eilers, E. Ruiz, and A. Sørensen, Amplified graph C^*-algebras, M¨unster J. Math. 5 (2012), 121–150.
[5]. T. Giordano, I. Putnam, and C. Skau, Topological orbit equivalence and C^*-crossed products, J. ReineAngew. Math. 469 (1995), 51–111.
[6]. T. Giordano, H. Matui, I. Putnam, and C. Skau, Orbit equivalence for Cantor minimal Zd-systems, Invent. Math. 179 (2010), 119–158.
[7]. A. an Huef and I. Raeburn, The ideal structure of Cuntz-Krieger algebras, Ergodic Th. and Dynam. Sys. 17 (1997), 611–624.
[8]. A. Kumjian, On localizations and simple C^*-algebras, Pacific J. Math. 112 (1984), 141–192.
[9]. A. Kumjian, On C_-diagonals, Can. J. Math. 4 (1986), 969–1008.
[10]. A. Kumjian, D. Pask, and I. Raeburn Cuntz-Krieger algebras of directed graphs, Pacific J. Math. 184 (1998), 161–174.
[11]. A. Kumjian, D. Pask, I. Raeburn, and J. Renault, Graphs, groupoids, and Cuntz-Krieger algebras, J. Func. Anal. 144 (1997), 505–541.
[12]. K. Matsumoto, Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras, Pacific J. Math. 246 (2010), 199–225.
[13]. K. Matsumoto, Orbit equivalence of one-sided subshifts and the associated C^*-algebras, Yokohama Math. J. 56 (2010), 59–85.
[14]. K. Matsumoto, Full groups of one-sided topological Markov shifts, preprint, arXiv:1205.1320.
[15]. K. Matsumoto and H. Matui, Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras, to appear in Kyoto J. Math., arXiv1307.1299.

[16]. H. Matui, Homology and topological full groups of ´etalegroupoids on totally disconnected spaces, Proc. Lond. Math. Soc. 104 (2012), 27–56.
[17]. G. Nagy and S. Reznikoff, Pseudo-diagonals and uniqueness theorems, Proc. Amer. Math. Soc. 142 (2014), 263–275.
[18]. A.L.T Paterson, Graph Inverse Semigroups, Groupoids and their C^*-Algebras, J. Operator Theory, 48 (2002), 645–662.
[19]. I. Raeburn, Graph Algebras, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.
[20]. I. Raeburn and D.Williams, Morita Equivalence and Continuous-Trace C^*-Algebras, Mathematical Surveys and Monographs Volume 60, the American Mathematical Society, Providence, RI, 1998, xiv+327.
[21]. J.N. Renault, A Groupoid Approach to C^*-Algebras, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin 1980.
[22]. J.N. Renault, Cartansubalgebras in C^*-algebras, Irish Math. Soc. Bull. 61 (2008), 29–63.
[23]. M. Rørdam, Classification of Cuntz-Krieger algebras, K-Theory 9 (1995), 31–58.
[24]. J. Tomiyama, Topological full groups and structure of normalizers in transformation group C^*-algebras, Pacific J. Math. 173 (1996), 571–583.
[25]. S. Webster, The path space of a directed graph, Proc. Amer. Math. Soc. 142 (2014), 213–225.
[26]. T. Yeend, Groupoid models for the C^*-algebras of topological higher-rank graphs, J. Operator Theory 57 (2007), 95–120.
[27]. Nathan Brownlowe, Toke Meier Carlsen, and Michael F. Whittaker, Graph Algebras and Orbit Equivalence, Ergod. Th. Dynam. Sys. 37 (2017) 389-417.