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Review Paper

About common eigenvalues and eigenvectors of two and more polynomial operator bundles in Hilbert space

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In this article we give sufficient conditions for the existence of common eigenvalues and eigenvectors of two and more polynomial operator bundles. There are several results about the common eigenvalues of two bounded polynomial bundles acting each in their own Hilbert space. [1],[2]. In the case of the bounded polynomial operator bundles depending on the identicalparameter with the same highest degrees of parameter necessary and sufficient sconditions for existence of common eigenvalue is given in the article of Khayniq [2]. Balinskii [1] generalized this result to the case of polynomial bundles with the different highest degrees of the parameter.

Let
$$A(\lambda)=A_0+\lambda A_1+\lambda^2A_2+...+\lambda^nA_n,$$
$$B(\lambda)=B_0+\lambda B_1+\lambda^2B_2+...+\lambda^mB_m$$
be two polynomial bundles acting in their own Hilbert space and depending on the same parameter.

Definition

1. c is an eigenvalue of operator bundle $A(\lambda)$ if there is nonzero vector x such that $A(c)x=x$.Vector x is called an eigenvector of bundle $A(\lambda)$ corresponding to the eigenvalue c

Resultant [2] of polynomial operator bundles $A(\lambda)$ and $B(\lambda)$ is the operator presented as the determinant

$$\operatorname{Re}s(A(\lambda),B(\lambda))=\begin{pmatrix} A_0\otimes E_2 & A_1\otimes E_2 & ... & A_n\otimes E_2 & ... & 0 \\ . & . & ... & . & ... & . \\ 0 & 0 & ...A_0\otimes E_2 & A_1\otimes E_2 & ... & A_n\otimes E_2 \\ E_1\otimes B_0 & E_1\otimes B_1 & ... & E_1\otimes B_m & .. & 0 \\ . & . & ... & . & ... & . \\ . & . & ...E_1\otimes B_0 & E_1\otimes B_1 & .... & E_1\otimes B_m \end{pmatrix}$$

which acts in the space $(H_1\otimes H_2)^{n+m}$ -direct sum of $n+m$ copies of tensor product space $H_1\otimes H_2$ and the number of rows with the operators $A_j\otimes E_2\ (j=1.2,...,n)$ in the $\operatorname{Re}s(A_1(\lambda),A_2(\lambda))$ is equal to the highest degree of the parameter λ in the bundle $B(\lambda)$, and the number of rows with the operators $E_1\ \otimes B_j\ (j=1,2,...,m)$ is equal to the highest degree of parameter λ in the bundle $A(\lambda)$

From [1],]2] we have: if all operators $A_i(i=0,1,...,n)$ and $B_j(j=0,1,...,m)$ are bounded $KerA_n=0$, $KerB_m=0$ $Ker\operatorname{Re}s(A(\lambda),B(\lambda))\neq 0$

then the bundles $A(\lambda)$ and $B(\lambda)$ have a common point of their spectra.

This result for the purpose of application has been generalized on the case of several bounded polynomial bundles [3].

Some results about existence of common eigenvalues and eigenvectors of two completely continuous operators are given in [4].

Now we provide sufficient condition for existence of common eigenvalues of two and more completely continuous operator polynomial bundles.

Let be $A(\lambda)=A_0+\lambda A_1B+\lambda^2A_2B^2+...+\lambda^{n-1}A_{n-1}B^{n-1}+\lambda^nB^n,$ (1)

where $A_i(i=0,1,...,n)$ are bounded, and B is completely continuous operators acting in Hilbert space H ,

and

$$C(\lambda)=C_0+\lambda C_1D+...+C^{\,m-1}C_{m-1}D^m+\lambda^mD^m \quad (2)$$

where $C_i(i=0,1,...,m-1)$ are bounded, and D iscompletely continuous operators acting in Hilbert space H . These operator bundles are famous as Keldysh's bundle[7].

.The study of these operator bundles arose with the considerations of operator differential equations with the several initial conditions.

Without loss of generality all operations we carry out with the bundle $LA\lambda)$ The study of the spectral properties of the bundle $A(\lambda)$ leads to the study of the spectral properties of the equation (3)

where the operators $\overline{A}=\begin{pmatrix}A_0 & A_1 & ... & A_{n-1}\\ 0 & 0 & ... & 0\\ . & . & ... & .\\ 0 & 0 & ... & 0\end{pmatrix}$

, $\overline{B}=\begin{pmatrix}0 & 0 & 0 & ... & B\\ B & 0 & 0 & ... & 0\\ 0 & B & 0 & ... & 0\\ . & . & . & ... & .\\ 0 & 0 & 0 & ...B & 0\end{pmatrix}$ and $\overline{E}=\begin{pmatrix}E & 0 & ... & 0\\ 0 & E & ... & 0\\ . & . & ... & .\\ 0 & 0 & ... & E\end{pmatrix}$

act in direct sum $H^{\,n}$ of n copies of Hilbert space H [7],[8],[9].

Let the operators $A_i(i=0,1,...,n-1)$ are bounded ,andoperator B is completely continuous

a) operator $E-A_0$ has a bounded inverse.

From condition b)and from operator $\overline{A}$ bounded it follows that the operator $\overline{E}-\overline{A}$ has a bounded inverse. In addition thereis operator $\overline{B}$ isa completely continuous then $(\overline{E}-\overline{A})^{-1}\overline{B}$ is a completely continuous operator, and it has the form $\begin{pmatrix} (E-A_0)^{-1}A_1B & \cdots & (E-A_0)^{-1}A_{n-1}B & (E-A_0)^{-1}B \\ B & \cdots & 0 & 0 \\ . & \cdots & . & \cdots \\ 0 & \cdots & B & 0 \end{pmatrix}$

when

$$\left(\overline{E}-\overline{A}\right)^{-1}=\begin{pmatrix} (E-A_0)^{-1} & (E-A_0)^{-1}A & \cdots & (E-A_0)^{-1}A_{n-1} \\ 0 & E & \cdots & 0 \\ . & . & \cdots & 0 \\ 0 & 0 & \cdots & E \end{pmatrix}$$ and

$$\overline{B}=\begin{pmatrix} 0 & 0 & 0 & \cdots & B \\ B & 0 & 0 & \cdots & 0 \\ 0 & B & 0 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots B & 0 \end{pmatrix}$$

We introduce the new operators[5] $\overline{T}=\dfrac{(\overline{E}-\overline{A})^{-1}\overline{B}+((\overline{E}-\overline{A})^{-1}\overline{B})^{*}}{2}$

and $\overline{S}=\dfrac{(\overline{E}-\overline{A})^{-1}\overline{B}-((\overline{E}-\overline{A})^{-1}\overline{B})^{*}}{2i}$

Operators $\overline{T}$ and $\overline{S}$ are self-adjoint and completely continuous andthey act in space H^{n}. Let $\tilde{E}_t$ be the expansion of unity of operator $\overline{T}$,and $\tilde{F}_S$ is the expansion of unity of operator $\overline{S}$ [6].

Since the operators $\overline{T}$ and $\overline{S}$ are bounded there is some four numbers a,b,c,d that the following equalities take place

1. $\tilde{E}_a=0$, $\tilde{E}_b=1$

$\tilde{F}_c=0$ $\tilde{F}_d=1$ (4)

2. $\tilde{E}_{tm}\tilde{E}n=\tilde{E}_{\min(m,n)}$,

$\tilde{F}_l\tilde{F}_s=\tilde{F}_{\min(l,k)}$

3. $\tilde{E}_t-\tilde{E}_{t-0}=P_t$

$$\tilde{F}_s - \tilde{F}_{s-0} = R_s$$

where P_t is a projective operator that projects onto the eigen subspace of operator $\overline{T}$ corresponding to its eigenvalue t ,and R_s is projective operator that projects onto the eigen subspace of operator $\overline{S}$ corresponding to its eigenvalue s .

Similar investigations we carry out with the operator bundle $C(\lambda)$ .The study of spectral properties of operator bundle $C(\lambda)$ led to to the study of equation

$$\overline{C}\overline{y} + \lambda \overline{D}\overline{y} = \overline{y} \quad (5)$$

that acts in H^m -direct sum of m copies of Hilbert space H .

b) Let theollowing conditions are true:operators $C_i (i = 0,1,...,m-1)$ are bounded and operator D is completely continuous

c) operator $E - C_0$ has a bounded inverse.

rom the condition b)and there isoperator $\overline{C}$ is bounded it follows that the operator $\overline{E} - \overline{C}$ has a bounded inverse and in additionoperator $(\overline{E} - \overline{C})^{-1}\overline{D} =$

$$\begin{pmatrix} (E-C_0)^{-1} C_1 D & \ldots & (E-C_0)^{-1} C_{m-1} D & (E-C_0)^{-1} D \\ D & \ldots & 0 & 0 \\ . & \ldots & . & \ldots \\ 0 & \ldots & D & 0 \end{pmatrix}$$

is a completely continuous

whereand $\overline{D} = \begin{pmatrix} 0 & 0 & 0 & \ldots & D \\ D & 0 & 0 & \ldots & 0 \\ 0 & D & 0 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots D & 0 \end{pmatrix} 0$

We consider the new operators[5]

$$\overline{M} = \frac{(\overline{E} - \overline{C})^{-1}\overline{D} - +(\overline{E} - \overline{C})^{-1}\overline{D})^*}{2}$$

and $\overline{N} = \dfrac{(\overline{E} - \overline{C})^{-1}\overline{D} - ((\overline{E} - \overline{C})^{-1}\overline{D})^*}{2i}$

Operators $\overline{M}$

and $\overline{N}$ are self-adjoint and completelycontinuous.

Let $\tilde{K}_t$, be the expansion of unity of operator $\overline{M}$,and $\tilde{L}_s$, is the expansion of unity of operator $\overline{N}$ correspoinding to its eigenvalue s [8],[9].Since the operators $\overline{M}$ and N are bounded thenfor some four numbers a,b,c,d similar to 1.and2. from (4)equalities are true.

Further we suppose

$\tilde{K}_t - \tilde{K}_{t-0}, = S_t$

$$\tilde{L}_s - \tilde{L}_{s-0} = H_s$$

where S_t is a projective operator that projects onto the eigen subspace of operator $\overline{M}$ corresponding to its eigenvalue t ,and H_s is projective operator that projects onto the eigen subspace of operator $\overline{N}$ corresponding to its eigenvalue s .

Theorem 1.
If the following conditions are true:

a) $A_i(i = 0,1,...,n-1)$,, $C_i(i = 0,1,...,m-1)$ are bounded, B and D are completely continuousoperators whichact in space H .

b)operators $E - A_0$ and

$E - C_0$ have bounded inverses.

c) $P_a R_b \neq 0$ (6),

$S_a H_b \neq 0$ (7)

then $1/a + ib$ isa common eigenvalue of bundles (1) and (2)

Proof. From[4,5]it follows that condition (6) means $\overline{a+ib}$ is an eigenvalue of operator $(\overline{E} - \overline{A})^{-1}\overline{B}$ (7)means that $a + ib$ isan eigenvalue of operator $(\overline{E} - \overline{C})^{-1}\overline{D}$

The last means that $1/a + ib$ is the common eigenvalue ofbundles (1) and (2), Theorem ` is prover.

Remark. The bundles (1) and (2) can act in different Hilbert spaces.

Theorem 2.
Let the all conditions of the Theorem1are fulfilled, $m = n$,for some four real numbers a,b,c,d $P_a R_b S_c H_d \neq 0$,

then bundles(1) and (2)have a common eigenvector

Proof. From $P_a R_b S_c H_d \neq 0 \ \lambda^{m-1}$ it follows that the range of projective operator $P_a R_b S_c H_d$ contains at least one element $\tilde{x}$. The range of projective operator $P_a R_b S_c H_d \neq 0$ is contained in the ranges of both projective operators $P_a R_b$ And $S_c H_d \neq 0$ and $P_a R_b \tilde{S}_c \tilde{H}_d \subset P_a R_b$

$$P_a R_b S_c \tilde{H}_d \subset \tilde{S}_c \tilde{H}_d \neq 0_d$$

Element $\tilde{x}$ enters the common eigen subspaceof both operators $(\overline{E}-A)^{-1}\overline{B}$ with the eigenvalue $a+ib^{-}$ and $(\overline{E}-\overline{C})^{-1}\overline{D}$ with the eigenvalue $c+id$ .It is known[5] thatthe first component of element $\tilde{x}$ is the common eigenvector of operator bundles (1) and(2).

Accord to the Theorem1 the condition $P_a R_b \neq 0$ means that $1/a+ib$ is the eigenvalue of (1) ,and $S_c H_d \neq 0$ means that the $1/c=id$ is the eigenvalue of bundle (2) .

Theorem 2 is proven.

Let

$$A_i(\lambda) = A_{0,i} + \lambda A_{1,i} B_i + \lambda^2 A_{2,i} B_i^2 + \ldots + \lambda^{n_i-1} A_{n-1,i} B_i^{n_i-1} + \lambda^{n_i} B_i^{n_i}$$

$$i = 1, 2, \ldots, s \quad (8)$$

be s polynomial operator bundles acting in Hilbert space.It is not difficult to distribute the results of Theorem1 and Theorem2 on the case of more than two polynomial operator bundles.

Theorem 3.

Let the following conditions:

a) $A_{k,i}(k = 0, 1, \ldots, n_k : i = 1, 2, \ldots, s)$ are bounded, B_i arecompletely continuous operatorsacting in space H .

b)operators

$E - A_{0,i}, i = 1, 2, \ldots, s$ have a bounded inverses.

c) $P_{a,i} R_{b,i} \neq 0$ are fulfilled then $a+ib$ is a common eigenvalue of all operator bundles(8).

Theorem4.

Let conditions a) and b) of Theorem3 are true and $\cap P_{a,i} R_{b,i} \neq 0$ then the bundles (8) have a common eigenvector.

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