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Review Paper

A Minor Oversight on the Diagonal-Preserving Gauge-Invariant Isomorphisms of the Order Graph C^*-Algebras

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Abstract
We follow the approach of [16], methodology with a bit change of symbols realizing the mentioned ordered of graph C^*-algebras equipped with generalised gauge actions, and characterise in terms of ordered groupoids and groupoid cocycles when two ordered graph C^*-algebras are isomorphic by a diagonal-preserving isomorphism that intertwines the generalised gauge actions. Similarly, we apply this characterisation to show that two Cuntz–Krieger algebras are isomorphic by a diagonal-preserving isomorphism that intertwines the gauge actions if and only if the corresponding one-sided subshifts are eventually conjugate, and that the stabilisation of two Cuntz–Krieger algebras are isomorphic by a diagonal-preserving isomorphism that intertwines the gauge actions if and only if the corresponding two-sided subshifts are conjugate.
Keywords: *Ordered graph C^*-algebras, Ordered graph groupoids, Cuntz–Krieger algebras, Shift spaces.*

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I. Introduction

It is well-known that the properties of the Cuntz-Krieger algebra $\mathcal{O}_{A_t}$ of a finite square $\{0,1\}$-matrix A_t are closely connected to the properties of the one- and two-sided subshifts X_{A_t} and $\bar{X}_{A_t}$ of A_t. Cuntz and Krieger proved for example in [6, Proposition 2.17] that if A_t and B_t are finite square $\{0,1\}$-matrices both satisfying condition (I), and X_{A_t} and X_{B_t} are conjugate, then there exists a diagonal-preserving $*$-isomorphism between $\mathcal{O}_{A_t}$ and $\mathcal{O}_{B_t}$ which intertwines the gauge actions of $\mathcal{O}_{A_t}$ and $\mathcal{O}_{B_t}$ (the result actually says a bit more than that); and Cuntz proved in [5, Theorem 2.3] that if A_t and B_t are finite square $\{0,1\}$-matrices such that A_t and B_t and their transposes satisfy condition (I), and $\bar{X}_{A_t}$ and $\bar{X}_{B_t}$ are conjugate, then there exists a diagonal-preserving $*$-isomorphism between the stabilised Cuntz-Krieger algebras $\mathcal{O}_{A_t} \otimes \mathcal{K}$ and $\mathcal{O}_{B_t} \otimes \mathcal{K}$ which intertwines the gauge, actions of $\mathcal{O}_{A_t} \otimes \mathcal{K}$ and $\mathcal{O}_{B_t} \otimes \mathcal{K}$ (this result was first proved under the additional assumption that A_t and B_t are irreducible and aperiodic by [6, Theorem 3.8]). Cuntz also proved in [5, Theorem 2.4] that if A_t and B_t are finite square $\{0,1\}$-matrices such that A_t and B_t and their transposes satisfy condition (I), and $\bar{X}_{A_t}$ and $\bar{X}_{B_t}$ are flow equivalent, then there exists a diagonal-preserving $*$-isomorphism between the stabilised Cuntz-Krieger algebras $\mathcal{O}_{A_t} \otimes \mathcal{K}$ and $\mathcal{O}_{B_t} \otimes \mathcal{K}$ (this result was first proved under the additional assumption that A_t and B_t are irreducible and aperiodic by [6, Theorem 4.1]).

The connection between the properties of $\mathcal{O}_{A_t}$ and the properties of X_{A_t} and $\bar{X}_{A_t}$ was further highlighted in [11] when the authors presented, among several other interesting results, a converse to [6, Theorem 4.1] by proving that if A_t and B_t are finite square $\{0,1\}$-matrices that are irreducible and not permutation matrices, and there is a diagonal-preserving $*$-isomorphism between the stabilised Cuntz-Krieger algebras $\mathcal{O}_{A_t} \otimes \mathcal{K}$ and $\mathcal{O}_{B_t} \otimes \mathcal{K}$, then $\bar{X}_{A_t}$ and $\bar{X}_{B_t}$ are flow equivalent [11, Corollary 3.8]. To prove this result, Matsumoto and Matui used that the C^*-algebra $\mathcal{O}_{A_t}$ can be constructed as the (reduced) C^*-algebra of an étale groupoid and proved that if A_t and B_t are finite square $\{0,1\}$-matrices that are irreducible and not permutation matrices then there is a diagonal-preserving $*$-isomorphism between $\mathcal{O}_{A_t}$ and $\mathcal{O}_{B_t}$ if and only if the corresponding étale groupoids are isomorphic, and if and only if the one-sided subshifts X_{A_t} and X_{B_t} are continuously orbit equivalent [11, Theorem 2.3].

Matsumoto has since then used this aproach to study gauge-invariant isomorphisms of Cuntz-Krieger algebras [9,10], and has, among other results, proved a converse to [6, Proposition 2.17] in the irreducible case when he proved in [9, Theorem 1.2] that if A_t and B_t are finite square {0,1}-matrices that are irreducible and not permutation matrices, then there is a diagonal-preserving $*$-isomorphism between $\mathcal{O}_{A_t}$ and $\mathcal{O}_{B_t}$ that intertwines the gauge actions of $\mathcal{O}_{A_t}$ and $\mathcal{O}_{B_t}$ if and only if X_{A_t} and X_{B_t} are eventually conjugate.

In [8] the authors extended the definition of Cuntz-Krieger algebras when they used groupoids to construct C^*-algebras from directed ordered graphs that are not assumed to be finite. These ordered graph C^*-algebras have since attracted a lot of interest, and by using that ordered graph C^*-algebras can be constructed from groupoids, [11, Theorem 2.3] has recently been transfered to the setting of the ordered graph C^*-algebras [1,2].

Toke Meier Carlesn and James Rout in their paper [16] use groupoids to study diagonal-preserving isomorphisms of order graph C^*-algebras that intertwine generalised gauge actions. They characterises when there is a diagonal-preserving $*$-isomorphism between two ordered graph C^*-algebras that intertwines two generalised gauge actions. The characterisation is given in terms of cocycle-preserving isomorphisms of the corresponding ordered graph groupoids.

They also present a "stabilised version" of this result.

By specialising to ordinary gauge actions they show in Theorem 4.1 that there is a diagonal-preserving gauge-invariant $*$-isomorphism between two ordered graph C^*-algebras if and only if the two ordered graphs are eventually conjugate, and in Theorem 5.1 that if we consider two finite ordered graphs with no sinks or sources, then there is a diagonal-preserving gauge-invariant $*$-isomorphism between the stabilisation of the two ordered graph C^*-algebras if and only if the two-sided edge shifts of the two ordered graphs are conjugate. As corollaries, they prove in Corollary 4.2 a converse to [6, Proposition 2.17] by generalising [9, Theorem 1.2] to the non-irreducible case, and they prove in Corollary 5.2 a converse to [5, Theorem 2.3].

As with [11, Theorem 2.3], the results in this paper can be transfered to the setting of Leavitt path algebras, and they have done this in [3].

II. Definitions and Notation

We recall the definitions of directed ordered graphs and their boundary path spaces, ordered graph groupoids, and ordered graph C^*-algebras. This is standard and can be found (see for example [1] and [2]). Usually let $\mathbb{N}$ denote the set of nonnegative integers $\{0,1,2,...\}$.

2.1. Directed Ordered Graphs and their Boundary Path Spaces

A directed ordered graph E_t is a quadruple $E_t = (E_t^0, E_t^1, r_t, s_t)$ consisting of countable sets E_t^0 and E_t^1, and range and source maps $r_t, s_t: E_t^1 \to E_t^0$. An element of $v_t \in E_t^0$ is called a vertex and an element of $e_t \in E_t^1$ is called an edge.

A path of length n in E_t is a sequence of edges $\mu = \mu_1 ... \mu_n$ such that $r_t(\mu_i) = s_t(\mu_{i+1})$ for all $1 \le i \le n-1$. We denote by E_t^n the collection of all paths of length n, and we define, $E_t^* := \bigcup_{n\in\mathbb{N}} E_t^n$ to be the collection of all paths of finite length. We write $|\mu|$ for the length of $\mu \in E_t^*$. The range and source maps extend to paths: $r_t(\mu) := r_t(\mu_n)$ and $s_t(\mu) := s_t(\mu_1)$.

We regard the vertices $v_t \in E_t^0$ as paths of length 0, and set $r_t(v_t) := s_t(v_t) := v_t$. For $v_t \in E_t^0$ and $n \in \mathbb{N}$, we define $v_t E_t^n := \{\mu \in E_t^n : s_t(\mu) = v_t\}$. The set of regular vertices is given by $(E_t^0)_{\text{reg}} := \{v_t \in E_t^0 : v_t E_t^1$ is finite and nonempty $\}$ and the set of singular vertices by $(E_t^0)_{\text{sing}} := E_t^0 \setminus (E_t^0)_{\text{reg}}$. If $\mu = \mu_1 ... \mu_m, \nu = \nu_1 ... \nu_n \in E_t^*$ with $r_t(\mu) = s_t(\nu)$, then we let $\mu\nu := \mu_1 ... \mu_m \nu_1 ... \nu_n \in E_t^*$. A cycle (sometimes called a loop in the literature) in E_t is a path $\mu \in E_t^* \setminus E_t^0$ such that $s_t(\mu) = r_t(\mu)$. An edge e_t is an exit to the cycle μ if there exists i such that $s_t(e_t) = s_t(\mu_i)$ and $e_t \neq \mu_i$. A ordered graph is said to satisfy condition (L) if every cycle has an exit.

An infinite path in E_t is an infinite sequence $x_1 x_2 ...$ of edges in E_t such that $r_t(x_i) = s_t(x_{i+1})$ for all i. We let E_t^∞ be the set of all infinite paths in E_t. The source map extends to E_t^∞ by setting $s_t(x) := s_t(x_1)$. We let $|x| = \infty$ for $x \in E_t^\infty$. The boundary path space of E_t is the space $\partial E_t := E_t^\infty \cup \left\{\mu \in E_t^* : r_t(\mu) \in (E_t^0)_{\text{sing}}\right\}$. If $\mu = \mu_1 ... \mu_m \in E_t^*$ and $x = x_1 x_2 \cdots \in E_t^\infty$ with $r_t(\mu) = s_t(x)$, then we let $\mu x = \mu_1 ... \mu_m x_1 x_2 \cdots \in E_t^\infty$. For $v_t \in E_t^0$, we define $v_t\, \partial E_t := \{x \in \partial E_t : s_t(x) = v_t\}$.

For $\mu \in E_t^*$, the cylinder set of μ is the set $Z(\mu) := \{\mu x \in \partial E_t : x \in r_t(\mu)\, \partial E_t\} \subseteq \partial E_t$.

For $\mu \in E_t^*$ and a finite subset $F_t \subseteq r_t(\mu) E_t^1$, we define $Z(\mu \setminus F_t) := Z(\mu) \setminus \left(\bigcup_{e_t \in F_t} Z(\mu e_t)\right)$. The boundary path space ∂E_t is a locally compact Hausdorff space with the topology given by the basis $\{Z(\mu \setminus F_t) : \mu \in E_t^*, F_t$ is a finite subset of $r_t(\mu)\, \partial E_t\}$, and each such $Z(\mu \setminus F_t)$ is compact and open (see [15, Theorem 2.1 and Theorem 2.2]).

For $n \in \mathbb{N}$, let $\partial E_t^{\ge n} := \{x \in \partial E_t : |x| \ge n\} \subseteq \partial E_t$. Then $\partial E_t^{\ge n} = \bigcup_{\mu \in E_t^n} Z(\mu)$ is an open subset of ∂E_t. Define the edge shift map $\sigma_{E_t} : \partial E_t^{\ge 1} \to \partial E_t$ by $\sigma_{E_t}(x_1 x_2 x_3 ...) = x_2 x_3 ...$ for $x_1 x_2 x_3 \cdots \in \partial E_t^{\ge 2}$ and $\sigma_{E_t}(e_t) = r_t(e_t)$ for $e_t \in$

$\partial E_t \cap E_t^1$. Let $\sigma_{E_t}^0$ be the identity map on ∂E_t, and for $n \geq 1$, let $\sigma_{E_t}^n$ be the n-fold composition of σ_{E_t} with itself. Then $\sigma_{E_t}^n$ is a local homeomorphism for all $n \in \mathbb{N}$. When we write $\sigma_{E_t}^n(x)$, we implicitly assume that $x \in \partial E_t^{\geq n}$. (see [16]).

2.2. Ordered Graph Groupoids

The ordered graph groupoid $\mathcal{G}_{E_t}$ of a directed ordered graph E_t is given by

$$\mathcal{G}_{E_t} := \{(x, m-n, y) \in \partial E_t \times \mathbb{Z} \times \partial E_t : m, n \in \mathbb{N} \text{ and } \sigma_{E_t}^m(x) = \sigma_{E_t}^n(y)\},$$

with partially-defined product $(x, m-n, y)(w, m'-n', z) := (x, m+m'-(n+n'), z)$ if $y = w$ and undefined otherwise, inverse operation $(x, m-n, y)^{-1} := (y, n-m, x)$, and range and source maps $r_t(x, m-n, y) := x$ and $s_t(x, m-n, y) := y$.

The groupoid $\mathcal{G}_{E_t}$ is a locally compact Hausdorff étale topological groupoid when equipped with the topology generated by subsets of the form

$$Z(U, m, n, V) := \{(x, m-n, y) \in \mathcal{G}_{E_t} : x \in U, y \in V, \sigma_{E_t}^m(x) = \sigma_{E_t}^n(y)\},$$

where $m, n \in \mathbb{N}, U$ is an open subset of $\partial E_t^{\geq m}$ such that $\sigma_{E_t}^m$ is injective on U, V is an open subset of $\partial E_t^{\geq n}$ such that $\sigma_{E_t}^n$ is injective on V, and $\sigma_{E_t}^m(U) = \sigma_{E_t}^n(V)$. For $\mu, \nu \in E_t^*$ with $r_t(\mu) = r_t(\nu)$, let $Z(\mu, \nu) := Z(Z(\mu), |\mu|, |\nu|, Z(\nu))$. The map $x \mapsto (x, 0, x)$ is a homeomorphism from ∂E_t to the unit space $\mathcal{G}_{E_t}^0$ of $\mathcal{G}_{E_t}$.

In this paper, a cocycle of a groupoid $\mathcal{G}$ is a groupoid homomorphism $c: \mathcal{G} \to \mathbb{Z}$, where we consider $\mathbb{Z}$ to be a groupoid with product and inverse given by the usual group operations. The function $c_{E_t}: \mathcal{G}_{E_t} \to \mathbb{Z}$ given by $c_{E_t}((x, m-n, y)) = m-n$ is a continuous cocycle.

All isomorphisms between groupoids considered in this paper are, in addition to preserving the groupoid structure, homeomorphisms (see [16]).

2.3. Ordered Graph C^*-Algebras

The ordered graph C^*-algebra of a directed ordered graph E_t is the universal C^*-algebra $C^*(E_t)$ generated by mutually orthogonal projections $\{p_{v_t} : v_t \in E_t^0\}$ and partial isometries $\{(s_t)_{e_t} : e_t \in E_t^1\}$ satisfying

(CK1) $(s_t)_{e_t}^*(s_t)_{e_t} = p_{r_t(e_t)}$ for all $e_t \in E_t^1$;

(CK2) $(s_t)_{e_t}(s_t)_{e_t}^* \leq p_{s_t(e_t)}$ for all $e_t \in E_t^1$;

(CK3) $p_{v_t} = \sum_{e_t \in v_t E_t^1} (s_t)_{e_t}(s_t)_{e_t}^*$ for all $v_t \in (E_t^0)_{\mathrm{reg}}$.

There is a strongly continuous action $\lambda^{E_t}: \mathbb{T} \to \mathrm{Aut}(C^*(E_t))$, called the gauge action, satisfying $\lambda_z^{E_t}(p_{v_t}) = p_{v_t}$ and $\lambda_z^{E_t}((s_t)_{e_t}) = z(s_t)_{e_t}$ for $z \in \mathbb{T}, v_t \in E_t^0$ and $e_t \in E_t^1$.

We let $(s_t)_{v_t} := p_{v_t}$ for $v_t \in E_t^0$, and for $n \geq 2$ and $\mu = \mu_1 \dots \mu_n \in E_t^n$, we let $(s_t)_\mu := (s_t)_{\mu_1} \dots (s_t)_{\mu_n}$. Then $\mathrm{span}\{(s_t)_\mu (s_t)_\nu^* : \mu, \nu \in E_t^*, r_t(\mu) = r_t(\nu)\}$ is dense in $C^*(E_t)$. We define $\mathcal{D}(E_t)$ to be the closure of $\mathrm{span}\{(s_t)_\mu (s_t)_\mu^* : \mu \in E_t^*\}$ in $C^*(E_t)$. Then $\mathcal{D}(E_t)$, called the diagonal of $C^*(E_t)$, is an abelian subalgebra of $C^*(E_t)$, and is isomorphic to the C^*-algebra $C_0(\partial E_t)$.

Moreover, $\mathcal{D}(E_t)$ is a maximal abelian subalgebra of $C^*(E_t)$ if and only if E_t satisfies condition (L) (see [12, Example 3.3]).

Theorem 3.7 of [15] shows that there is a unique homeomorphism $(h_t)_{E_t}$ from ∂E_t to the spectrum of $\mathcal{D}(E_t)$ given by

$$(h_t)_{E_t}(x)\big((s_t)_\mu (s_t)_\mu^*\big) = \begin{cases} 1 & \text{if } x \in Z(\mu), \\ 0 & \text{if } x \notin Z(\mu). \end{cases}$$

There is a $*$-isomorphism from the C^*-algebra of $\mathcal{G}_{E_t}$ to $C^*(E_t)$ that maps $C_0(\mathcal{G}_{E_t}^0)$ onto $\mathcal{D}(E_t)$ (see [2, Proposition 2.2] and [8, Proposition 4.1]). (see also [16]).

III. Gauge-Invariant Isomorphisms of Ordered Graph C^*-Algebras and Cocycle-Preserving Isomorphisms of Ordered Graph Groupoids

In this section, we show the two main results of [16] Theorem 3.1 and Theorem 3.3.

Let E_t be a directed ordered graph, and let $k: E_t^1 \to \mathbb{R}$ be a function. Then k extends to a function $k: E_t^* \to \mathbb{R}$ given by $k(v_t) = 0$ for $v_t \in E_t^0$ and $k((e_t)_1 \dots (e_t)_n) = k((e_t)_1) + \cdots + k((e_t)_n)$ for $(e_t)_1 \dots (e_t)_n \in E_t^n, n \geq 1$. We then get a continuous cocycle $c_k: \mathcal{G}_{E_t} \to \mathbb{R}$ given by $c_k((\mu x, |\mu| - |\nu|, \nu x)) = k(\mu) - k(\nu)$ and a generalised gauge action $\gamma^{E_t,k}: \mathbb{R} \to \mathrm{Aut}(C^*(E_t))$ given by $\gamma_{t_0}^{E_t,k}(p_{v_t}) = p_{v_t}$ for $v_t \in E_t^0$ and $\gamma_{t_0}^{E_t,k}((s_t)_{e_t}) = e^{ik(e_t)t_0}(s_t)_{e_t}$ for $e_t \in E_t^1$.

If $k(e_t) = 1$ for all $e_t \in E_t^1$, then $\gamma_{t_0}^{E_t,k} = \lambda_{e^{it_0}}^{E_t}$ for all $t_0 \in \mathbb{R}$, where λ^{E_t} is the usual gauge action on $C^*(E_t)$.

Theorem 3.1 [16]. Let E_t and F_t be directed graphs and $k: E_t^1 \to \mathbb{R}$ and $l: F_t^1 \to \mathbb{R}$ functions. The following are equivalent.

(1) There is an isomorphism $\Phi: \mathcal{G}_{E_t} \to \mathcal{G}_{F_t}$ satisfying $c_l(\Phi(\eta)) = c_k(\eta)$ for $\eta \in \mathcal{G}_{E_t}$.
(2) There is a $*$-isomorphism $\Psi: C^*(E_t) \to C^*(F_t)$ satisfying $\Psi(\mathcal{D}(E_t)) = \mathcal{D}(F_t)$ and $\gamma_t^{F_t,l} \circ \Psi = \Psi \circ \gamma_t^{E_t,k}$ for $t \in \mathbb{R}$.

To prove the implication (2) $\Longrightarrow$ (1) of Theorem 3.1, we need a lemma. We recall the extended Weyl groupoid $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ constructed in [2] from a graph C^*-algebra and its diagonal subalgebra. As defined in [13], the normaliser of $\mathcal{D}(E_t)$ is the set

$$N(\mathcal{D}(E_t)) := \{n \in C^*(E_t): ndn^*, n^*dn \in \mathcal{D}(E_t) \text{ for all } d \in \mathcal{D}(E_t)\}.$$

By [2, Lemma 4.1], $(s_t)_\mu (s_t)_\nu^* \in N(\mathcal{D}(E_t))$ for all $\mu, \nu \in E_t^*$ with $r_t(\mu) = r_t(\nu)$. For $n \in N(\mathcal{D}(E_t))$, let $\mathrm{dom}(n) := \{x \in \partial E_t: (h_t)_{E_t}(x)(n^*n) > 0\}$ and $\mathrm{ran}(n) := \{x \in \partial E_t: (h_t)_{E_t}(x)(nn^*) > 0\}$. It follows from [13, Proposition 4.7] that, for $n \in N(\mathcal{D}(E_t))$, there is a unique homeomorphism $\alpha_n: \mathrm{dom}(n) \to \mathrm{ran}(n)$ such that $(h_t)_{E_t}(x)(n^*dn) = (h_t)_{E_t}(\alpha_n(x))(d)(h_t)_{E_t}(x)(n^*n)$ for all $d \in \mathcal{D}(E_t)$.

The extended Weyl groupoid $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ is the collection of equivalence classes for an equivalence relation on $\{(n,x): n \in N(\mathcal{D}(E_t)), x \in \mathrm{dom}(n)\}$ with the partiallydefined product
$[(n_1,x_1)][(n_2,x_2)] := [(n_1n_2,x_2)]$ if $\alpha_{n_2}(x_2) = x_1$ and the inverse operation $[(n,x)]^{-1} := [(n^*,\alpha_n(x))]$ (see [2, Proposition 4.7]). By [2, Proposition 4.8], the map $\phi_{E_t}: (\mu x, |\mu| - |\nu|, \nu x) \to [((s_t)_\mu (s_t)_\nu^*, \nu x)]$ is a groupoid isomorphism between $\mathcal{G}_{E_t}$ and $\mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$.

Lemma 3.2 [16]. Let E_t be a directed graph, $k: E_t^1 \to \mathbb{R}$ a function, and $\eta \in \mathcal{G}_{E_t}$.
(a) There exist $n \in N(\mathcal{D}(E_t))$ and $x \in \mathrm{dom}(n)$ such that $\phi_{E_t}(\eta) = [(n,x)]$ and $\gamma_{t_0}^{E_t,k}(n) = e^{ic_k(\eta)t_0}n$ for all $t_0 \in \mathbb{R}$.
(b) Suppose $r_t \in \mathbb{R}$ and that there are $n' \in N(\mathcal{D}(E_t))$ and $x' \in \mathrm{dom}(n')$ such that $\phi_{E_t}(\eta) = [(n',x')]$ and $\gamma_{t_0}^{E_t,k}(n') = e^{ir_tt_0}n'$ for all $t_0 \in \mathbb{R}$. Then $r_t = c_k(\eta)$.

Proof. (a): Choose $\mu, \nu \in E_t^*$ with $r_t(\mu) = r_t(\nu)$ such that $\eta \in Z(\mu,\nu)$, and let $n := (s_t)_\mu (s_t)_\nu^*$ and $x := s_t(\eta)$. Then $x \in \mathrm{dom}(n)$, $\phi_{E_t}(\eta) = [(n,x)]$, and $\gamma_{t_0}^{E_t,k}(n) = e^{i(k(\mu)-k(\nu))t_0}n = e^{ic_k(\eta)t_0}n$ for all $t_0 \in \mathbb{R}$.
(b): Let π denote the isomorphism from $C^*(E_t)$ to $C^*(\mathcal{G}_{E_t})$ given in [2, Proposition 2.2]. As in [2], we think of $C^*(\mathcal{G}_{E_t})$ as a subset of $C_0(\mathcal{G}_{E_t})$ and define $supp\ '(f_t) := \{\zeta \in \mathcal{G}_{E_t}: f_t(\zeta) \neq 0\}$ for $f_t \in C_0(\mathcal{G}_{E_t})$. Since $\pi\left(\gamma_{t_0}^{E_t,k}((s_t)_\mu (s_t)_\nu^*)\right)(\zeta) = e^{ic_k(\zeta)t_0}\pi((s_t)_\mu (s_t)_\nu^*)(\zeta)$ for $\mu, \nu \in E_t^*, t \in \mathbb{R}$ and $\zeta \in \mathrm{supp}'\left(\pi((s_t)_\mu (s_t)_\nu^*)\right)$, and $\mathrm{span}\{(s_t)_\mu (s_t)_\nu^*: \mu, \nu \in E_t^*\}$ is dense in $C^*(E_t)$, we have $\pi\left(\gamma_{t_0}^{E_t,k}(n')\right)(\zeta) = e^{ic_k(\zeta)t_0}\pi(n')(\zeta)$ for $t_0 \in \mathbb{R}$ and $\zeta \in \mathrm{supp}'(\pi(n'))$. Since $\gamma_{t_0}^{E_t,k}(n') = e^{ir_tt_0}n'$ for all $t_0 \in \mathbb{R}$, it follows that $\mathrm{supp}'(\pi(n')) \subseteq c_k^{-1}(r_t)$. It follows that $r_t = c_k(\eta)$ because if $r_t \neq c_k(\eta) = k(\mu) - k(\nu)$, then we would have that $\{\eta' \in \mathrm{supp}'\left(\pi((s_t)_\nu (s_t)_\mu^* n')\right): s_t(\eta') = r_t(\eta') = s_t(\eta)\} = \{(s_t(\eta), m, s_t(\eta))\}$ for some $m \in \mathbb{Z} \setminus \{0\}$, from which, together with the definition of the equivalence class $[(n',x')]$(see [2, Proposition 4.6]), it would follow that $\phi_{E_t}(\eta) = \left[\left((s_t)_\mu (s_t)_\nu^*, s_t(\eta)\right)\right] \neq [(n',x')]$.

Proof of Theorem 3.1. (1) $\Longrightarrow$ (2) : Suppose that $\Phi: \mathcal{G}_{E_t} \to \mathcal{G}_{F_t}$ is an isomorphism such that $c_l(\Phi(\eta)) = c_k(\eta)$ for $\eta \in \mathcal{G}_{E_t}$. It then follows from [2, Proposition 2.2] that there is a $*$-isomorphism $\Psi: C^*(E_t) \to C^*(F_t)$ satisfying $\Psi(\mathcal{D}(E_t)) = \mathcal{D}(F_t)$ and $\gamma_{t_0}^{F_t,l} \circ \Psi = \Psi \circ \gamma_{t_0}^{E,k}$ for $t_0 \in \mathbb{R}$.
(2) $\Longrightarrow$ (1): Since $\Psi(\mathcal{D}(E_t)) = \mathcal{D}(F_t)$, it follows from [2, Proposition 4.11] that Ψ induces an isomorphism $\psi: \mathcal{S}_{(C^*(E_t),\mathcal{D}(E_t))} \to \mathcal{G}_{(C^*(F_t),\mathcal{D}(F_t))}$. Define $\Phi := \phi_{F_t}^{-1} \circ \psi \circ \phi_{E_t}$. Then $\Phi: \mathcal{G}_{E_t} \to \mathcal{G}_{F_t}$ is a groupoid isomorphism. It remains to check that Φ is cocyle-preserving.
Fix $\eta \in \mathcal{G}_{E_t}$. By Lemma 3.2$(a)$ there exists $[(n,x)] \in \mathcal{G}_{(C^*(E_t),\mathcal{D}(E_t))}$ such that $\phi_{E_t}(\eta) = [(n,x)]$ and $\gamma_{t_0}^{E_t,k}(n) = e^{ic_k(\eta)t_0}n$ for all $t_0 \in \mathbb{R}$. We then have $\phi_{F_t}(\Phi(\eta)) = [(\Psi(n),\kappa(x))] \in \mathcal{G}_{(C^*(F_t),\mathcal{P}(F_t))}$, where κ is a homeomorphism from ∂E_t onto ∂F_t (see [2, Proposition 4.11]), and $\gamma_{t_0}^{F_t,l}(\Psi(n)) = e^{ic_k(\eta)t_0}\Psi(n)$ for all $t_0 \in \mathbb{R}$, so $c_k(\eta) = c_l(\Phi(\eta))$ by Lemma 3.2(b).
Next, we present and prove a "stabilised version" of Theorem 3.1. We denote by $\mathcal{K}$ the compact operators on $\ell^2(\mathbb{N})$, and by $\mathcal{C}$ the maximal abelian subalgebra of $\mathcal{K}$ consisting of diagonal operators.
As in [4], for a directed graph E_t, we denote by SE_t the graph obtained by attaching a head $\ldots (e_t)_{3,v_t}(e_t)_{2,v_t}(e_t)_{1,v_t}$ to every vertex $v_t \in E_t^0$ (see [14, Definition 4.1]). For a function $k: E_t^1 \to \mathbb{R}$, we let $\bar{k}: SE_t^1 \to \mathbb{R}$ be the function given by $\bar{k}(e_t) = k(e_t)$ for $e_t \in E_t^1$, and $\bar{k}\left((e_t)_{i,v_t}\right) = 0$ for $v_t \in E_t^0$ and $i = 1,2, \ldots.$

Theorem 3.3 (see [16]). Let E_t and F_t be directed ordered graphs and $k: E_t^1 \to \mathbb{R}$ and $l: F_t^1 \to \mathbb{R}$ functions. The following are equivalent.
(A) There is an isomorphism $\Phi: \mathcal{G}_{SE_t} \to \mathcal{G}_{SF_t}$ satisfying $c_l(\Phi(\eta)) = c_{\bar{k}}(\eta)$ for $\eta \in \mathcal{G}_{SE_t}$.

(B) There is a $*$-isomorphism $\Psi: C^*(E_t) \otimes \mathcal{K} \to C^*(F_t) \otimes \mathcal{K}$ satisfying $\Psi(\mathcal{D}(E_t) \otimes \mathcal{C}) = \mathcal{D}(F_t) \otimes \mathcal{C}$ and $\left(\gamma_{t_0}^{F_t,l} \otimes \mathrm{Id}_{\mathcal{K}}\right) \circ \Psi = \Psi \circ \left(\gamma_{t_0}^{E_t,k} \otimes \mathrm{Id}_{\mathcal{K}}\right)$ for $t_0 \in \mathbb{R}$.

Proof. By [14, Theorem 4.2], there is an isomorphism $\rho: C^*(E_t) \otimes \mathcal{K} \to C^*(SE_t)$. Let $\{\theta_{i,j}\}_{i,j \in \mathbb{N}}$ be the canonical generators of $\mathcal{K}$. For $v_t \in E_t^0$ and $i \in \mathbb{N}$ denote by μ_{i,v_t} the path $(e_t)_{i,v_t}(e_t)_{i-1,v_t} \dots (e_t)_{1,v_t}$ in SE_t (if $i = 0$, then we let $\mu_{i,v_t} = v_t$). One can check that the isomorphism $\rho: C^*(E_t) \otimes \mathcal{K} \to C^*(SE_t)$ can be chosen such that $\rho\left(p_{v_t} \otimes \theta_{i,j}\right) = (s_t)_{\mu_{i,v_t}}(s_t)^*_{\mu_{j,v_t}}$ for $v_t \in E_t^0$ and $\rho\left((s_t)_{e_t} \otimes \theta_{i,j}\right) = (s_t)_{\mu_{i,s_t(c)}}(s_t)_{e_t}(s_t)^*_{\mu_{j,r_t(c)}}$ for $e_t \in E_t^1$. Routine calculations then show that $\rho(\mathcal{D}(E_t) \otimes \mathcal{C}) = \mathcal{D}(SE_t)$ and that $\rho \circ \left(\gamma_{t_0}^{E_t,k} \otimes \mathrm{Id}_{\mathcal{K}}\right) = \gamma_{t_0}^{SE_t,\bar{k}} \circ \rho$ for $t_0 \in \mathbb{R}$. The equivalence (A) $\Leftrightarrow$ (B) now follows from the equivalence (1) $\Leftrightarrow$ (2) of Theorem 3.1 applied to the ordered graphs SE_t and SF_t and the functions $\bar{k}$ and $\bar{l}$.

IV. Eventual Conjugacy of Ordered Graphs

Let E_t and F_t be directed ordered graphs. Following [9], we say that E_t and F_t are eventually conjugate if there exists a homeomorphism $h_t: \partial E_t \to \partial F_t$ and continuous maps $k: \partial E_t^{\geq 1} \to \mathbb{N}$ and $k': \partial F_t^{\geq 1} \to \mathbb{N}$ such that $\sigma_{F_t}^{k(x)}\left(h_t(\sigma_{E_t}(x))\right) = \sigma_{F_t}^{k(x)+1}(h_t(x))$ for all $x \in \partial E_t^{\geq 1}$ and $\sigma_{E_t}^{k'(y)}\left(h_t^{-1}(\sigma_{F_t}(y))\right) = \sigma_{E_t}^{k'(y)+1}(h_t^{-1}(y))$ for all $y \in \partial F_t^{\geq 1}$. We call such a homeomorphism $h_t: \partial E_t \to \partial F_t$ an eventual conjugacy.

Notice that if $h_t: \partial E_t \to \partial F_t$ is a conjugacy in the sense that $\sigma_{F_t}(h_t(x)) = h_t\left(\sigma_{E_t}(x)\right)$ for all $x \in \partial E_t^{\geq 1}$, then h_t is an eventual conjugacy (in this case we can take k and k' to be constantly equal to 0).

Theorem 4.1 (see [16]). Let E_t and F_t be directed ordered graphs. The following are equivalent.

(i) E_t and F_t are eventually conjugate.

(ii) There is an isomorphism $\Phi: \mathcal{G}_{E_t} \to \mathcal{G}_{F_t}$ satisfying $c_{F_t}(\Phi(\eta)) = c_{E_t}(\eta)$ for $\eta \in \mathcal{G}_{E_t}$.

(iii) There is a $*$-isomorphism $\Psi: C^*(E_t) \to C^*(F_t)$ satisfying $\Psi(\mathcal{D}(E_t)) = \mathcal{D}(F_t)$ and $\lambda_z^{F_t} \circ \Psi = \Psi \circ \lambda_z^{E_t}$ for $z \in \mathbb{T}$.

Proof. $(ii) \Leftrightarrow (iii)$: Let $k: E_t^1 \to \mathbb{R}$ and $l: F_t^1 \to \mathbb{R}$ both be constantly equal to 1.

Then $c_k = c_{E_t}, c_l = c_{F_t}$, and $\gamma_{t_0}^{E_t,k} = \lambda_{e^{it_0}}^{E_t}$ and $\gamma_{t_0}^{F_t,l} = \lambda_{e^{it_0}}^{F_t}$ for all $t_0 \in \mathbb{R}$. It therefore follows from Theorem 3.1 that (ii) and (iii) are equivalent.

$(i) \implies (ii)$: Suppose $h_t: \partial E_t \to \partial F_t$ is an eventual conjugacy. Then the map $(x, n, y) \mapsto (h_t(x), n, h_t(y))$ is an isomorphism $\Phi: \mathcal{G}_{E_t} \to \mathcal{G}_{F_t}$ satisfying $c_{F_t}(\Phi(\eta)) = c_{E_t}(\eta)$ for $\eta \in \mathcal{G}_{E_t}$.

$(ii) \implies (i)$: Suppose $\Phi: \mathcal{G}_{E_t} \to \mathcal{G}_{F_t}$ is an isomorphism such that $c_{F_t}(\Phi(\eta)) = c_{E_t}(\eta)$ for $\eta \in \mathcal{G}_{E_t}$. Then the restriction of Φ to $\mathcal{G}_{E_t}^0$ is a homeomorphism onto $\mathcal{G}_{F_t}^0$. Since the map $x \mapsto (x, 0, x)$ is a homeomorphism from ∂E_t onto $\mathcal{G}_{E_t}^0$, and $y \mapsto (y, 0, y)$ is a homeomorphism from ∂F_t onto $\mathcal{G}_{F_t}^0$, it follows that there is a homeomorphism $h_t: \partial E_t \to \partial F_t$ such that $\Phi((x, 0, x)) = (h_t(x), 0, h_t(x))$ for all $x \in \partial E_t$. Since $c_{F_t}(\Phi(\eta)) = c_{E_t}(\eta)$ for all $\eta \in \mathcal{G}_{E_t}$, it follows that $\Phi((x, n, y)) = (h_t(x), n, h_t(y))$ for all $(x, n, y) \in \mathcal{G}_{E_t}$. Let $e_t \in E_t^1$.

Then $\Phi(Z(e_t, r_t(e_t)))$ is an open and compact subset of $c_{F_t}^{-1}(1)$. It follows that there exist an n, mutually disjoint open subsets $U_1, \dots, U_n$ of ∂F_t, mutually disjoint open subsets $V_1, \dots, V_n$ of ∂F_t, and $k_1, \dots, k_n \in \mathbb{N}$ such that $\Phi\left(Z\left(e_t, r_t(e_t)\right)\right) = \bigcup_{i=1}^{n} Z(U_i, k_i + 1, k_i, V_i)$.

Define $k_{e_t}: Z(e_t) \to \mathbb{N}$ by $k_{e_t}(x) = k_i$ for $x \in h_t^{-1}(U_i)$. Then k_{e_t} is continuous and $\sigma_{F_t}^{k_c(x)}\left(h_t\left(\sigma_{E_t}(x)\right)\right) = \sigma_{F_t}^{k_c(x)+1}(h_t(x))$ for $x \in Z(e_t)$. By doing this for each $e_t \in E_t^1$, we get a continuous map $k: \partial E_t^{\geq 1} \to \mathbb{N}$ such that $\sigma_{F_t}^{k(x)}\left(h_t\left(\sigma_{E_t}(x)\right)\right) = \sigma_{F_t}^{k(x)+1}(h_t(x))$ for all $x \in \partial E_t^{\geq 1}$.

A continuous map $k': \partial F_t^{\geq 1} \to \mathbb{N}$ such that $\sigma_{E_t}^{k'(y)}\left(h_t^{-1}\left(\sigma_{F_t}(y)\right)\right) = \sigma_{E_t}^{k'(y)+1}(h_t^{-1}(y))$ for all $y \in \partial F_t^{\geq 1}$ can be constructed in a similar way. Thus, h_t is an eventual conjugacy.

Let A_t be a finite square $\{0,1\}$-matrix, and assume that every row and every column of A_t is nonzero. As in [6], we denote by $\mathcal{O}_{A_t}$ the Cuntz-Krieger algebra of A_t with gauge action λ^{A_t} and canonical abelian subalgebra $\mathcal{D}_{A_t}$, and by $\left(X_{A_t}, \sigma_{A_t}\right)$ the one-sided subshift of A_t (if A_t does not satisfy condition (I), then we let $\mathcal{O}_{A_t}$ denote the universal Cuntz-Krieger algebra $\mathcal{AO}_{A_t}$ introduced in [7]). As in [9], we say that $\left(X_{A_t}, \sigma_{A_t}\right)$ and $\left(X_{B_t}, \sigma_{B_t}\right)$ are eventually one-sided conjugate if there is a homeomorphism $h_t: X_{A_t} \to X_{B_t}$ and continuous maps $k: X_{A_t} \to \mathbb{N}$ and $k': X_{B_t} \to \mathbb{N}$ such that $\sigma_{B_t}^{k(x)}\left(h_t\left(\sigma_{A_t}(x)\right)\right) = \sigma_{B_t}^{k(x)+1}(h_t(x))$ for all $x \in X_{A_t}$, and $\sigma_{A_t}^{k'(y)}\left(h_t^{-1}\left(\sigma_{B_t}(y)\right)\right) = \sigma_{A_t}^{k'(y)+1}(h_t^{-1}(y))$ for all $y \in X_{B_t}$.

We obtain from Theorem 4.1 the following corollary which was proved in the irreducible case by Kengo Matsumoto in [9, Theorem 1.2], and which can be seen as a kind of a converse to [6, Proposition 2.17].

Corollary 4.2 (see [16]). Let A_t and B_t be finite square {0,1}-matrices, and assume that every row and every column of A_t and B_t is nonzero. There is a $*$-isomorphism $\Psi: \mathcal{O}_{A_t} \to \mathcal{O}_{B_t}$ satisfying $\Psi(\mathcal{D}_{A_t}) = \mathcal{D}_{B_t}$ and $\lambda_z^{B_t} \circ \Psi = \Psi \circ \lambda_z^{A_t}$ for all $z \in \mathbb{T}$ if and only if (X_{A_t}, σ_{A_t}) and (X_{B_t}, σ_{B_t}) are eventually one-sided conjugate.

Proof. Let $(E_t)_{A_t}$ be the graph of A_t, i.e., $(E_t^0)_{A_t}$ is the index set of $A_t, (E_t)_{A_t} = \{(i,j) \in (E_t)_{A_t} \times (E_t)_{A_t} : A_t(i,j) = 1\}$, and $r_t((i,j)) = j$ and $s_t((i,j)) = i$ for $(i,j) \in (E_t)^1_{A_t}$. Then $\partial(E_t)_{A_t} = (E_t)^\infty_{A_t}$, and there is a homeomorphism from X_{A_t} to $(E_t)^\infty_{A_t}$ that intertwines σ_{A_t} and $\sigma_{(E_t)_{A_t}}$. It follows that (X_{A_t}, σ_{A_t}) and (X_{B_t}, σ_{B_t}) are eventually one-sided conjugate if and only if $(E_t)_{A_t}$ and $(E_t)_{B_t}$ are eventually one-sided conjugate.

It is well-known that there is a $*$-isomorphism $\Psi: \mathcal{O}_{A_t} \to C^*((E_t)_{A_t})$ satisfying $\Psi(\mathcal{D}_{A_t}) = \mathcal{D}((E_t)_{A_t})$ and $\lambda_z^{(E_t)_{A_t}} \circ \Psi = \Psi \circ \lambda_z^{A_t}$ for all $z \in \mathbb{T}$. The corollary therefore follows from the equivalence of (i) and (iii) in Theorem 4.1.

Remark 4.3. Kengo Matsumoto has strengthened [9, Theorem 1.2] in [10] and shown that if the matrices A_t and B_t in Corollary 4.2 are irreducible and not permutation matrices, then the two conditions in Corollary 4.2 are equivalent to several other interesting conditions, for example to the condition that there is a $*$-isomorphism $\Psi: \mathcal{O}_{A_t} \to \mathcal{O}_{B_t}$ satisfying $\Psi(\mathcal{D}_{A_t}) = \mathcal{D}_{B_t}$ and $\Psi(\mathcal{F}_{A_t}) = \mathcal{F}_{B_t}$, where $\mathcal{F}_{A_t}$ is the fixed point algebra of λ^{A_t} and $\mathcal{F}_{B_t}$ is the fixed point algebra of λ^{B_t}. (see [16]).

V. Conjugacy of Two-Sided Shifts of Finite Type

For a finite directed graph E_t with no sinks or sources, we define $\bar{X}_{E_t}$ to be the two-sided edge shift

$$\bar{X}_{E_t} := \{(x_n)_{n\in\mathbb{Z}} : x_n \in E_t^1 \text{ and } r_t(x_n) = s_t(x_{n+1}) \text{ for all } n \in \mathbb{Z}\}$$

equipped with the induced topology of the product topology of $(E_t^1)^Z$ (where each copy of E_t^1 is given the discrete topology), and let $\bar{\sigma}_{E_t}: \bar{X}_{E_t} \to \bar{X}_{E_t}$ be the homeomorphism given by $(\bar{\sigma}_{E_t}(x))_m = x_{m+1}$ for $x = (x_n)_{n\in\mathbb{Z}} \in \bar{X}_{E_t}$.

If E_t and F_t are finite directed ordered graphs with no sinks or sources, then a conjugacy from $\bar{X}_{E_t}$ to $\bar{X}_{F_t}$ is a homeomorphism $\phi: \bar{X}_{E_t} \to \bar{X}_{F_t}$ such that $\bar{\sigma}_{F_t} \circ \phi = \phi \circ \bar{\sigma}_{E_t}$. The shift spaces $\bar{X}_{E_t}$ and $\bar{X}_{F_t}$ are said to be conjugate if there is a conjugacy from $\bar{X}_{E_t}$ to $\bar{X}_{F_t}$.

Recall that for a directed graph E_t, we denote by SE_t the graph obtained by attaching a head $\dots (e_t)_{3,v_t}(e_t)_{2,v_t}(e_t)_{1,v_t}$ to every vertex $v_t \in E_t^0$ (see [14, Definition 4.1]). Define a function $\bar{k}_{E_t}: (SE_t)^1 \to \mathbb{R}$ by $\bar{k}_{E_t}(e_t) = 1$ for $e_t \in E_t^1$ and $\bar{k}_{E_t}((e_t)_{i,v_t}) = 0$ for $v_t \in E_t^0$ and $i = 1,2, \dots$.

Theorem 5.1 (see [16]). Let E_t and F_t be directed ordered graphs. The following two conditions are equivalent.

(I) There is an isomorphism $\Phi: \mathcal{G}_{SE_t} \to \mathcal{G}_{SF_t}$ satisfying $c_{\bar{k}_{F_t}}(\Phi(\eta)) = c_{\bar{k}_{E_t}}(\eta)$ for $\eta \in \mathcal{G}_{SE_t}$.

(II) There is a $*$-isomorphism $\Psi: C^*(E_t) \otimes \mathcal{K} \to C^*(F_t) \otimes \mathcal{K}$ satisfying $\Psi(\mathcal{D}(E_t) \otimes \mathcal{C}) = \mathcal{D}(F_t) \otimes \mathcal{C}$ and $(\lambda_z^{F_t} \otimes \mathrm{Id}_{\mathcal{K}}) \circ \Psi = \Psi \circ (\lambda_z^{E_t} \otimes \mathrm{Id}_{\mathcal{K}})$ for $z \in \mathbb{T}$.

If E_t and F_t are finite ordered graphs with no sinks or sources, then (I) and (II) are equivalent to the following condition.

(III) The two-sided edge shifts $\bar{X}_{E_t}$ and $\bar{X}_{F_t}$ are conjugate.

Proof. An argument similar to the one used in the proof of Theorem 4.1 shows that the equivalence of (I) and (II) follows by applying Theorem 3.3 to the functions $k: E_t^1 \to \mathbb{R}$ and $l: F_t^1 \to \mathbb{R}$ that are constantly equal to 1.

It remains to establish $(I) \iff (III)$. Suppose that E_t and F_t are finite ordered graphs with no sinks or sources. Then $\partial E_t = E_t^\infty$ and $\partial F_t = F_t^\infty$. As in the proof of Theorem 3.3, for $v_t \in E_t^0$ and $i \in \mathbb{N}$, we denote by μ_{i,v_t} the path $(e_t)_{i,v_t}(e_t)_{i-1,v_t} \dots (e_t)_{1,v_t}$ in SE_t (if $i = 0$, then we let $\mu_{i,v_t} = v_t$). In the proof of [4, Lemma 4.1], it was shown that there is a homeomorphism $(SE_t)^\infty \to E_t^\infty \times \mathbb{N}$ satisfying $\mu_{i,s_t(x)}x \to (x,i)$ for $x \in E_t^\infty$ and $i \in \mathbb{N}$. We identify $(SE_t)^\infty$ with $E_t^\infty \times \mathbb{N}$.

$(III) \implies (I)$: Suppose $\bar{X}_{E_t}$ and $\bar{X}_{F_t}$ are conjugate. Then there is a conjugacy $\phi: \bar{X}_{E_t} \to \bar{X}_{F_t}$ and an $l \in \mathbb{N}$ such that if $x, x' \in \bar{X}_{E_t}$ with $x_n = x'_n$ for all $n \geq 0$, then $(\phi(x))_n = (\phi(x'))_n$ for all $n \geq 0$, and if $y, y' \in \bar{X}_{F_t}$ with $y_n = y'_n$ for all $n \geq 0$, then $(\phi^{-1}(y))_n = (\phi^{-1}(y'))_n$ for all $n \geq l$. It follows that there is a continuous map $\pi: E_t^\infty \to F_t^\infty$ such that $(\pi((x_k)_{k\in\mathbb{N}}))_n = (\phi(x))_n$ for $x = (x_k)_{k\in\mathbb{Z}} \in \bar{X}_{E_t}$ and $n \in \mathbb{N}$. Then π is surjective, $\pi \circ \sigma_{E_t} = \sigma_{F_t} \circ \pi$, and if $\pi(x) = \pi(x')$ for $x, x' \in E_t^\infty$, then $\sigma_{E_t}^l(x) = \sigma_{E_t}^l(x')$.

For an infinite path $x = (x_n)_{n\in\mathbb{N}}$ and $k \in \mathbb{N}$, we write $x_{[0,k)}$ for the finite path $x_0 x_1 \dots x_{k-1}$ of length k. It follows from the continuity of π and the compactness of E_t^∞ that we can choose $L \geq l$ such that $x_{[0,L)} = x'_{[0,L)} \implies \pi(x)_{[0,l)} = \pi(x')_{[0,l)}$. Define an equivalence relation $\sim$ on E_t^L by $\mu \sim \nu$ if there are $x \in Z(\mu)$ and $x' \in Z(\nu)$ such that $\pi(x) = \pi(x')$ (that $\sim$ is transitive follows from the fact that if $\mu, \nu, \eta \in E_t^L, x, x' \in E_t^\infty$, and $\pi(\mu x) = \pi(\nu x)$ and $\pi(\nu x') = \pi(\eta x')$, then $\pi(\mu x') = \pi(\eta x')$. Then $\pi(x) = \pi(x')$ if and only if $x_{[0,L)} \sim x'_{[0,L)}$ and $\sigma_{E_t}^l(x) = \sigma_{E_t}^l(x')$.

For each equivalence class $B_t \in E_t^L/\sim$ choose a partition $\{(A_t)_\mu : \mu \in B_t\}$ of $\mathbb{N}$ and bijections $(f_t)_\mu : (A_t)_\mu \to \mathbb{N}$. The map $\psi: (x,n) \mapsto \left(\pi(x), (f_t)^{-1}_{x_{[0,L}}(n)\right)$ is then a homeomorphism from $(SE_t)^\infty \to (SF_t)^\infty$. It is routine to check that

$$\Phi: \big((x,n),k,(x',n')\big) \mapsto \left(\psi(x,n), k + n' + (f_t)^{-1}_{x_{[0,L)}}(n) - n - (f_t)^{-1}_{x'_{[0,L)}}(n'), \psi(x',n')\right)$$

is a groupoid isomorphism from $\mathcal{G}_{SE_t}$ to $\mathcal{G}_{SF_t}$ satisfying $c_{\bar{k}_{F_t}}(\Phi(\eta)) = c_{\bar{k}_{E_t}}(\eta)$ for $\eta \in \mathcal{G}_{SE_t}$.

(I) $\Longrightarrow$ (III): Suppose $\Phi: \mathcal{G}_{SE_t} \to \mathcal{G}_{SF_t}$ is an isomorphism satisfying $c_{\bar{k}_{F_t}}(\Phi(\eta)) = c_{\bar{k}_{E_t}}(\eta)$ for $\eta \in \mathcal{G}_{SE_t}$. For $x \in E_t^\infty$, we have $(x,0) \in (SE_t)^\infty$ and $((x,0),0,(x,0)) \in \mathcal{G}_{SE_t}$.

Since Φ is an isomorphism, we have $\Phi((x,0),0,(x,0)) = ((y,m),0,(y,m))$ for some uniquely determined $y \in F_t^\infty$ and $m \in \mathbb{N}$. Define $\psi: E_t^\infty \to F_t^\infty$ by $\psi(x) := y$. Since Φ is continuous, the map $(x,0) \mapsto (y,m)$ is continuous, so ψ is also continuous.

We have $\left((x,0),1,\left(\sigma_{E_t}(x),0\right)\right) \in \mathcal{S}_{SE_t}$ for $x \in E_t^\infty$. By the cocyle condition there exist $m, m' \in \mathbb{N}$ such that $\Phi\left((x,0),1,\left(\sigma_{E_t}(x),0\right)\right) = \left((\psi(x),m),1+m-m',\left(\psi\left(\sigma_{E_t}(x)\right),m'\right)\right) \in \mathcal{G}_{SE_t}$. Hence there exists $l \in \mathbb{N}$ such that $\sigma_{F_t}^{l+1}(\psi(x)) = \sigma_{F_t}^{l}\left(\psi\left(\sigma_{E_t}(x)\right)\right)$; let $l(x)$ denote the smallest such number. We check that $l: E_t^\infty \to \mathbb{N}$ is continuous. Suppose $(x^n)_{n\in\mathbb{N}}$ in E_t^∞ converges to x. Then $\Phi\left((x^n,0),1,\left(\sigma_{E_t}(x^n),0\right)\right) \to \Phi\left((x,0),1,\left(\sigma_{E_t}(x),0\right)\right)$ since Φ is continuous and $\psi(x^n) \to \psi(x)$ and $\psi\left(\sigma_{E_t}(x^n)\right) \to \psi\left(\sigma_{E_t}(x)\right)$ since ψ is continuous. It follows that there is an $N \in \mathbb{N}$ such that for $n \geq N$, we have that $\sigma_{F_t}^{l(x)+1}\left(\psi(x^n)\right) = \sigma_{F_t}^{l(x)}\left(\psi\left(\sigma_{E_t}(x^n)\right)\right)$ and either $l(x) = 0$ or $\sigma_{F_t}^{l(x)}\left(\psi(x^n)\right) \neq \sigma_{F_t}^{l(x)-1}\left(\psi\left(\sigma_{E_t}(x^n)\right)\right)$. Hence $l(x^n) = l(x)$ for $n \geq N$.

Since E_t^∞ is compact, it follows that there is an $L \in \mathbb{N}$ such that $\sigma_{E_t}^{L+1}(\psi(x)) = \sigma_{F_t}^{L}\left(\psi\left(\sigma_{E_t}(x)\right)\right)$ for all $x \in E_t^\infty$. Define $\varphi := \sigma_{F_t}^L \circ \psi: E_t^\infty \to F_t^\infty$. Then φ is continuous and satisfies $\varphi \circ \sigma_{E_t} = \sigma_{F_t} \circ \varphi$.

For $x = (x_n)_{n\in\mathbb{Z}}$ in $\bar{X}_{E_t}$ or $\bar{X}_{F_t}$ and $k \in \mathbb{Z}$, let $x_{[k,\infty)}$ denote the infinite path $x_k x_{k+1} \ldots$ Define $\bar{\varphi}: \bar{X}_{E_t} \to \bar{X}_{F_t}$ by $(\bar{\varphi}(x))_{[k,\infty)} = \varphi\left(x_{[k,\infty)}\right)$ for $x \in \bar{X}_{E_t}$ and $k \in \mathbb{Z}$.

Since $\varphi \circ \sigma_{E_t} = \sigma_{F_t} \circ \varphi$, it follows that $\bar{\varphi}$ is well-defined. It is routine to check that $\bar{\varphi}$ is continuous and that $\bar{\varphi} \circ \bar{\sigma}_{E_t} = \bar{\sigma}_{F_t} \circ \bar{\varphi}$. We will next show that $\bar{\varphi}$ is also bijective. It will then follow that $\bar{\varphi}$ is a conjugacy and thus that $\bar{X}_{E_t}$ and $\bar{X}_{F_t}$ are conjugate.

We first show that $\bar{\varphi}$ is injective. Suppose $x = (x_n)_{n\in\mathbb{N}}, x' = (x'_n)_{n\in\mathbb{N}} \in E_t^\infty$ and $\varphi(x) = \varphi(x')$. Choose $m, m' \in \mathbb{N}$ such that $\Phi((x,0),0,(x,0)) = ((\psi(x),m),0,(\psi(x),m))$ and $\Phi\big((x',0),0,(x',0)\big) = \big((\psi(x'),m'),0,(\psi(x'),m')\big)$. Since $\sigma_{F_t}^L(\psi(x)) = \sigma_{F_t}^L\big(\psi(x')\big)$, it follows that $\big((\psi(x),m),m-m',(\psi(x'),m')\big) \in \mathcal{G}_{SF_t}$ and thus that

$$\big((x,0),0,(x',0)\big) = \Phi^{-1}\left(\big((\psi(x),m),m-m',(\psi(x'),m')\big)\right) \in \mathcal{G}_{SE_t}.$$

It follows that there is a $k \in \mathbb{N}$ such that $\sigma_{E_t}^k(x) = \sigma_{E_t}^k(x')$. Let $k\big((x,x')\big)$ be the smallest such k. An argument similar to the one used to prove that $l: E_t^\infty \to \mathbb{N}$ is continuous, shows that $k: \{(x,x') \in E_t^\infty \times E_t^\infty : \varphi(x) = \varphi(x')\} \to \mathbb{N}$ is continuous. Since $\{(x,x') \in E_t^\infty \times E_t^\infty : \varphi(x) = \varphi(x')\}$ is closed in $E_t^\infty \times E_t^\infty$ and thus compact, it follows that there exists $K \in \mathbb{N}$ such that $\sigma_{E_t}^K(x) = \sigma_{E_t}^K(x')$ for all $x, x' \in E^\infty$ satisfying $\varphi(x) = \varphi(x')$. The injectivity of $\bar{\varphi}$ easily follows.

Next, we show that $\bar{\varphi}$ is surjective. Suppose $y \in F_t^\infty$. Then

$$\Phi^{-1}((y,0),0,(y,0)) = ((x,n),0,(x,n))$$

for some $x \in E_t^\infty$ and some $n \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that

$$\Phi\big((x,0),0,(x,0)\big) = \big((\psi(x),m),0,(\psi(x),m)\big).$$

Since $((x,0),-n,(x,n)) \in \mathcal{G}_{SE_t}$ and $\Phi((x,0),-n,(x,n)) = ((\psi(x),m),m,(y,0))$, it follows that there is an $h_t \in \mathbb{N}$ such that $\sigma_{F_t}^{h_t}(\psi(x)) = \sigma_{F_t}^{h_t}(y)$. An argument similar to the one used in the previous paragraph, then shows that there is an $H \in \mathbb{N}$ such that for each $y \in F_t^\infty$ there is an $x \in E_t^\infty$ such that $\sigma_{F_t}^H(\psi(x)) = \sigma_{F_t}^H(y)$. The surjectivity of $\bar{\varphi}$ easily follows.

Let A_t be a finite square $\{0,1\}$-matrix, and assume that every row and every column of A_t is nonzero. As in [6], we denote by $\left(\bar{X}_{A_t}, \bar{\sigma}_{A_t}\right)$ the two-sided subshift of A_t. It follows from [5, Theorem 2.3] that if A_t and B_t are finite square $\{0,1\}$-matrices such that A_t and B_t and their transpose satisfy condition (I), and $\bar{X}_{A_t}$ and $\bar{X}_{B_t}$ are conjugate, then there exists a $*$-isomorphism $\Psi: \mathcal{O}_{A_t} \otimes \mathcal{K} \to \mathcal{O}_{B_t} \otimes \mathcal{K}$ such that $\Psi\left(\mathcal{D}_{A_t} \otimes \mathcal{C}\right) = \mathcal{D}_{B_t} \otimes \mathcal{C}$ and $\left(\lambda_z^{B_t} \otimes \mathrm{Id}_{\mathcal{K}}\right) \circ \Psi = \Psi \circ \left(\lambda_z^{A_t} \otimes \mathrm{Id}_{\mathcal{K}}\right)$ for all $z \in \mathbb{T}$. As a corollary to Theorem 5.1, we now prove the converse.

Corollary 5.2 (see [16]). Let A_t and B_t be finite square $\{0,1\}$-matrices, and assume that every row and every column of A_t and B_t is nonzero. There is a $*$-isomorphism $\Psi: \mathcal{O}_{A_t} \otimes \mathcal{K} \to \mathcal{O}_{B_t} \otimes \mathcal{K}$ such that $\Psi(\mathcal{D}_{A_t} \otimes \mathcal{C}) = \mathcal{D}_{B_t} \otimes \mathcal{C}$ and $(\lambda_z^{B_t} \otimes \mathrm{Id}_{\mathcal{K}}) \circ \Psi = \Psi \circ (\lambda_z^{A_t} \otimes \mathrm{Id}_{\mathcal{K}})$ for all $z \in \mathbb{T}$ if and only if $(\bar{X}_{A_t}, \bar{\sigma}_{A_t})$ and $(\bar{X}_{B_t}, \bar{\sigma}_{B_t})$ are conjugate.

Proof. As in the proof of Corollary 4.2, let $(E_t)_{A_t}$ be the graph of A_t, and $(E_t)_{B_t}$ the graph of B_t. The result then follows from the equivalence of $(II) \iff (VI)$ of Theorem 5.1 applied to the ordered graphs $(E_t)_{A_t}$ and $(E_t)_{B_t}$.

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