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Review Paper

A fixed point theorem for weakly multiplicative contractive mappings in multiplicative metric space

Harsh Kumar, Manoj Kumar
Department of Mathematics
Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Abstract
In the present manuscript, a new notion of weakly multiplicative contractive mapping is introduced and a fixed point theorem in multiplicative metric space is proved by using this new notion.
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I. Introduction

In 2008, Bashirov *et al.* [2] introduced a new notion called multiplicative metric space (MMS for short). The main idea was that the usual triangular inequality was replaced by a 'multiplicative triangle inequality' as follows:

Definition 1.1. Let $\mathcal{B}$ be a nonempty set. A mapping $m^* : \mathcal{B} \times \mathcal{B} \to \Re_+$ satisfying the followings:
(1) $m^*(c, d) > 1$, for all $c, d \in \mathcal{B}$ and $m^*(c, d) = 1$ if and only if $c = d$;
(2) $m^*(c, d) = m^*(d, c)$ for all $c, d \in \mathcal{B}$;
(3) $m^*(c, d) \leq m^*(c, e).m^*(e, d)$ for all $c, d, e \in \mathcal{B}$; (multiplicative triangle inequality)
is called a multiplicative metric and the pair $(\mathcal{B}, m^*)$ is called a multiplicative metric space (MMS in short).

Example 1.2.[5] Let $m^* : \Re \times \Re \to [1, \infty)$ be defined as $m^*(c, d) = a^{|c-d|}$, where $c, d \in \Re$ and $a > 1$. Then m^* is a multiplicative metric and $(\Re, m^*)$ is a multiplicative metric space(usual).
One can refer to ([1, 3, 4, 5]) for detailed multiplicative metric topology.

Definition 1.3.[4] Let $(\mathcal{B}, m^*)$ be a multiplicative metric space. Then a sequence $\{c_n\}$ in $\mathcal{B}$ is said to be
(1) a multiplicative convergent to c if $m^*(c_n, c) \to 1$ as $n \to \infty$.
(2) a multiplicative Cauchy sequence if $m^*(c_n, c_p) \to 1$ as $n, p \to \infty$.

Remark: If every multiplicative Cauchy sequence in $(\mathcal{B}, m^*)$ is convergent to $c \in \mathcal{B}$, then $(\mathcal{B}, m^*)$ is called a complete multiplicative metric space.

II. Main Result

In this section, a new notion of weakly multiplicative contractive mapping is introduced and a fixed point theorem is proved for such kind of mappings.

Definition 2.1. Let $(\mathcal{B}, m^*)$ be a multiplicative metric space. A self map g on $\mathcal{B}$ is said to be weakly contractive, if there exists a function $\alpha : (1, \infty) \to [0, 1)$ with

$$\sup\{\alpha(c) : 1 < a \leq c \leq b\} < 1$$

and such that

$$m^*(gx, gy) \leq m^*(x, y)^{\alpha[m^*(x, y)]}. \tag{2.1}$$

Theorem 2.2. Let $(\mathcal{B}, m^*)$ be a multiplicative metric space and let $g : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying (2.1), then g has a unique fixed point.
Proof. Let $x \in \mathcal{B}$ be arbitrary. Consider the sequence $\{g^n x\}$.
If $m^*(g^n x, g^{n+1} x) = 1$, for some n, then $g g^n x = g^n x$, that is, $g^n x$ is a fixed point of g and so conclusion of Theorem follows.
Suppose now that $m^*(g^n x, g^{n+1} x) > 1$, for all $n \in \mathbb{N}$.

Then as $\alpha(c) < c$, for $c > 1$, from (2.1), we have that g is multiplicative contractive.
So, we get

$$m^*(g^n x,\ g^{n+1}x) = m^*(g g^{n-1}x,\ g g^n x) \leq m^*(g^{n-1}x,\ g^n x)^{\alpha[m^*(g^{n-1}x, g^n x)]} < m^*(g^{n-1}x,\ g^n x).$$

Thus $\{m^*(g^n x,\ g^{n+1}x)\}$ is a monotone decreasing sequence of reals and so it converges.
Let $\lim_{n\to\infty} m^*(g^n x,\ g^{n+1}x) = r$.
Now, we show that $r = 1$.
Suppose on the contrary that $r > 1$ and set
$\alpha = sup\{\alpha(c): 1 < r \leq c \leq m^*(x, gx)\}$.
Then $\alpha\big(m^*(g^n x,\ g^{n+1}x)\big) \leq \alpha$, for all $n \geq 0$, and so we have
$1 < r < m^*(g^n x,\ g^{n+1}x) \leq m^*(g^{n-1}x,\ g^n x)^{\alpha} \leq \ldots \leq m^*(x,\ gx)^{\alpha^n} \to 1$ as $n \to \infty$, a contradiction.
Therefore $r = 1$.
Now, we show that $\{g^n x\}$ is a multiplicative Cauchy sequence.
Let $\varepsilon > 1$ and set $1 < \alpha(\varepsilon) = \sup\{\alpha(c): \frac{\varepsilon}{2} \leq c \leq \varepsilon\}$.
Since $\lim_{n\to\infty} m^*(g^n x,\ g^{n+1}x) = 1$ and $\alpha(\varepsilon) - 1 > 0$, there exists $p \in \mathbb{N}$ such that

$$m^*(g^n x,\ g^{n+1}x) < \mathcal{E}^{\frac{\alpha(\varepsilon)-1}{2}}, \qquad (2.2)$$

for all $n \geq p$.
Let $n \geq p$ be any fixed positive integer.
We shall show by induction that

$$m^*(g^n x,\ g^l x) < \varepsilon, \qquad (2.3)$$

for all $l > n > p$.
For $l = n + 1$, (2.3) follows from (2.2).
Assume now that (2.3) holds for some $l \geq n + 1$.
If $m^*(g^n x,\ g^l x) \geq \frac{\varepsilon}{2}$, then from (2.1), we have
$m^*(g(g^n x),\ g(g^l x)) \leq m^*(g^n x,\ g^l x)^{\alpha(\varepsilon)} < \varepsilon^{\alpha(\varepsilon)}$.
Thus, by the multiplicative triangle inequality and (2.2), we get

$$m^*(g^n x,\ g^{l+1}x) \leq m^*(g^n x,\ g(g^n x)).m^*(g(g^n x),\ g(g^l x)) < \mathcal{E}^{\frac{\alpha(\varepsilon)-1}{2}}\ \varepsilon^{\alpha(\varepsilon)} < \mathcal{E}.$$

If $m^*(g^n x,\ g^l x) < \frac{\varepsilon}{2}$, then by the multiplicative triangle inequality and (2.2), we have

$$m^*(g^n x,\ g^{l+1}x) \leq m^*(g^n x,\ g^l x).m^*(g^l x,\ g^{l+1}x) < \frac{\varepsilon}{2}.\mathcal{E}^{\frac{\alpha(\varepsilon)-1}{2}} < \mathcal{E}.$$

Therefore, $m^*(g^n x,\ g^{l+1}x) < \mathcal{E}$, this completes the induction.
From (2.3), we conclude that $\{g^n x\}$ is a multiplicative Cauchy sequence. The multiplicative completeness of g guarantees the existence of some point $u \in \mathcal{B}$ such that $\lim_{n\to\infty} g^n x = u$.
By continuity of g, it follows that
$gu = g \lim_{n\to\infty} g^n x = \lim_{n\to\infty} g g^n x = u$.
Hence u is a fixed point of g. The uniqueness of fixed point follows from the multiplicative contractivity of g.

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