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Review Paper

Comparing Solutions to the Linear Space-Fractional Telegraph Equation

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Abstract
The applications of the fractional differentiation for the mathematical modelling of real-world physical problems such as the earthquake modelling, and allometric scaling laws in biology, etc., have been widespread in this modern era. Fractional differential theory has gained much more attention as the fractional order system response ultimately converges to the integer order equations. Differential equations of fractional order have been successfully employed for modelling the so-called anomalous phenomena during last two decades. One of these nonlinear partial differential equations, the linear space-fractional telegraph equation. That applied into signal analysis for transmission, propagation of electrical signals, and so on. The aim of this article is to compare were the fractional Sumudu decomposition method (SDM), a double Sumudu matching transformation method, a finite difference scheme, a finite difference scheme based on a combination of the extended cubic B-splines (ExCuBs) method, and a quadratic spline functions with the solution of the linear space-fractional telegraph equation. We will conduct a comparison of the stability of the methods and convergence. In addition, numerical examples will be presented to illustrate the accuracy of these methods.

Keywords: *The linear space- Fractional Telegraph Equation, Sumudu Decomposition Method, AFinite Difference Scheme, AQuadratic Spline Functions,The Stability, The Convergence.*

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I. Introduction

The fractional derivative which simultaneously possesses memory and nonlocal property can describe different nonlinear phenomena more accurately and efficiently in comparison with the integer-order derivative. This makes fractional calculus a powerful tool for modelling the complex dynamical systems. Nowadays, Fractional calculus (FC) is an appropriate tool to describe the physical properties of materials [1-5]. Fractional order differential equations (FDEs) have been established for modelling of real phenomena in various fields such as physics, engineering, mechanics, control theory, economics, medical science, finance and etc. Many methods have been developed to solve the linear space-fractional telegraph equation including the fractional Sumudu decomposition method (SDM), a double Sumudu matching transformation method, a finite difference scheme, a finite difference scheme based on a combination of the extended cubic B-splines (ExCuBs) method, and a quadratic spline functions. Researchers who have used the Sumudu decomposition method (SDM) include Khan et al (2018), established a new and efficient analytical scheme for space fractional telegraph equation. The technique they have produced can be used for many classes of fractional order models including linear as well nonlinear [1]. The next year, A finite difference scheme based on a combination of the extended cubic B-splines (ExCuBs) method and Caputo's fractional derivative for the numerical solution of time fractional telegraph equation (TFTE) has been presented by Akram et al. The proposed method is investigated and good compatibility was found with the exact solution. Also, the proposed scheme is convergent of order O(τ 2–α + h2) and unconditionally stable [2]. In the same year, finite difference scheme is presented for the Generalized Time-Fractional Telegraph Equation (GTFTE) defined using Generalized Fractional Derivative (GFD) by Kumara et al. The generalization of fractional derivatives is done by introducing scale and weight functions, and for their particular choices, GFD reduces to Caputo and Riemann–Liouville derivatives. The proposed scheme is stable and convergent. The simulation results showed that numerical scheme is of Δt^(2-2α) order of convergence [3].Intends to obtain accurate and convergent numerical solutions of linear space-time matching telegraph fractional equations by means of a double Sumudu matching transformation methodby Hamza et al. in

2021. Them suggested method is applied successfully for obtaining the general solutions of several linear and on homogeneousconformable fractional telegraph equations. The effects show that them resolved method is efficient and can be applied for finding the general solutions of all cases related to the conformable fractional differential equations [4]. Recently, in 2022 Zaki et al. proposed a general framework that can be used to guide quadratic spline functions in order to create a numerical method for obtaining an approximation solution using the linear space-fractional telegraph equation. The numerical scheme is effective and reliable for the time–space fractional order telegraph equation. The obtained approximate numerical solutions are in good agreement with the approximate solutions [5].As the previous five methods have been used many times in recent years, we wanted to present a comparison between them to assist future researchers.

Section one outlines some previous studies on the linear space-fractional telegraph equation. Section two offers basic methods of solutions for the linear space-fractional telegraph equation. The third section showed the local truncation errors. In the fourth section will describe the stability analysis and the convergence. Section five addresses numerical illustration. In this section, we offer an example from each author as well as their results. In the final section, we offer some conclusions and highlight some areas for further development.

The generalised of the time–space fractional order telegraph equation of the form [5]:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u \quad x > 0, \quad 1 < \alpha \le 2, \qquad (1)$$

subject to boundary conditions

$$u(a,t) = \beta_1(t), \quad u(b,t) = \beta_2(t), \quad t > 0 \ .(2)$$

and the initial conditions

$$u(x,0) = f_1(x), \ \frac{\partial u(x,0)}{\partial t} = f_2(x), \quad a \le x \le \ b \qquad (3)$$

II. The Methods of solution

In this section, we will illustrate the linear space-fractional telegraph equation including the fractional Sumudu decomposition method (SDM), a double Sumudu matching transformation method, a finite difference scheme, a finite difference scheme based on a combination of the extended cubic B-splines (ExCuBs) method, and a quadratic spline function.

2.1 The fractional Sumudu decomposition method [1]

Khan et al.consider the following fractional telegraph equationand hence illustrate the basic idea for the mention method,

$${}_0^C D_x^{n\alpha} U(x,t) = A(x,t)\,\partial_t^2 U(x,t) + B(x,t)\,\partial_t U(x,t) + C(x,t)U(x,t) + g(x,t)(4)$$

where $0 < x < a, 0 < \alpha \le 1, \quad a \in \mathbb{R}$, and $A(x,t), B(x,t)$ and $C(x,t)$ are continuous functions.

Applying the Sumudu transform (ST) to both sides of Equation (4), they get the components of the approximation solution as the following, respectively:

$$\begin{aligned} &U_0(x,t) = \varrho(x,t) \\ &U_1(x,t) = \mathbb{S}^{-1}\big[u^{n\alpha}\mathbb{S}[A(x,t)\,\partial_t^2 U_0(x,t) + B(x,t)\,\partial_t U_0(x,t) + C(x,t)U_0(x,t)]\big], \\ &U_2(x,t) = \mathbb{S}^{-1}\big[u^{n\alpha}\mathbb{S}[A(x,t)\,\partial_t^2 U_1(x,t) + B(x,t)\,\partial_t U_1(x,t) + C(x,t)U_1(x,t)]\big], \\ &U_3(x,t) = \mathbb{S}^{-1}\big[u^{n\alpha}\mathbb{S}[A(x,t)\,\partial_t^2 U_2(x,t) + B(x,t)\,\partial_t U_2(x,t) + C(x,t)U_2(x,t)]\big], \\ &\vdots \\ &U_{n+1}(x,t) = \mathbb{S}^{-1}\big[u^{n\alpha}\mathbb{S}[A(x,t)\,\partial_t^2 U_n(x,t) + B(x,t)\,\partial_t U_n(x,t) + C(x,t)U_n(x,t)]\big]. \end{aligned} \qquad (5)$$

Thus, in view of relations given by (5), the components $U_0(x,t), U_1(x,t), U_2(x,t), U_3(x,t), \ldots$, are completely determined. As a result, the solution $U(x,t)$ of FTE (4) in a series form can be easily obtained

2.2 A finite difference scheme based on a combination of the extended cubic B-splines method [2]

Akram et al. considerd the following one-dimensional time fractional telegraph equation with reaction term:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + 2\lambda \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \mu u(x,t) = v\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (x,t) \in [a,b] \times [0,T](6)$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = f_1(x) \\ u_t(x,0) = f_2(x) \\ u(a,t) = g_1(t) \\ u(b,t) = g_2(t) \end{cases} \qquad (7)$$

where $0 < \alpha < 1$, and λ, μ, v are arbitrary positive constants. $f(x)$ is the forcing term and $f_1(x), f_2(x), g_1(x), g_2(x)$ are sufficiently smooth prescribed functions. If $\alpha = 1$ Eq. (6) becomes the one-dimensional hyperbolic telegraph equation. Time fractional derivatives $\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}}$ and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denote the Caputo fractional derivative of order 2α and α, respectively.

The approximated solution $U(x,t)$ of given model using ExCuBs to the exact solution $u(x,t)$ is described in the following form:

$U(x,t)=\sum_{i=-1}^{N+1} d_i(t)E_i(x,\lambda)$ (8)

where $d_i(t)$ are the time dependent unknown coefficients which are to be required by some particular restrictions. Each subinterval $[x_i,x_{i+1}]$ of basis function covers only three nonzero elements E_{i-1},E_i,E_{i+1}. The approximated solution u_j^n at the grid point (x_j,t_n) at the nth time level to the exact solution is defined as

$u_j^n=\sum_{j=i-1}^{i+1} d_j^n(t)E_j(x,\lambda)$, (9)

where $i=0,1,\dots,N$. Using the above approximation and basis functions, the values u_j^n and their necessary derivatives up to second order as given below:

$$\begin{aligned} u_i^n &= c_1 d_{i-1}^n + c_2 d_i^n + c_1 d_{i+1}^n,\\ (u_x)_i^n &= c_3 d_{i+1}^n - c_3 d_{i-1}^n,\\ (u_{xx})_i^n &= c_4 d_{i-1}^n - c_5 d_i^n + c_4 d_{i+1}^n, \end{aligned} \qquad (10)$$

where $c_1=\frac{4-\lambda}{24}, c_2=\frac{8+\lambda}{12}, c_3=\frac{1}{2h}, c_4=\frac{2+\lambda}{2h^2}, c_5=\frac{2+\lambda}{h^2}$.

After simplification they obtain the recurrence relation in the following form:

$$\big((\beta_1+\beta_2+\mu)c_1-\nu c_4\big)d_{j-1}^{n+1}+\big((\beta_1+\beta_2+\mu)c_2-\nu c_5\big)d_j^{n+1}+\big((\beta_1+\beta_2+\mu)c_1-\nu c_4\big)d_{j+1}^{n+1}$$
$$=\begin{cases}(2\beta_1+\beta_2)\big(c_1d_{j-1}^n+c_2d_j^n+c_1d_{j+1}^n\big)-\beta_1\big(c_1d_{j-1}^{n-1}+c_2d_j^{n-1}+c_1d_{j+1}^{n-1}\big)\\ -\beta_1\sum_{k=1}^{n} b_k^{2\alpha}\big[c_1\big(d_{j-1}^{n+1-k}-2d_{j-1}^{n-k}+d_{j-1}^{n-1-k}\big)+c_2\big(d_j^{n+1-k}-2d_j^{n-k}\\ +d_j^{n-1-k}\big)+c_1\big(d_{j+1}^{n+1-k}-2d_{j+1}^{n-k}+d_{j+1}^{n-1-k}\big)\big]-\beta_2\sum_{k=1}^{n} b_k^{\alpha}\big[c_1\big(d_{j-1}^{n+1-k}\\ -d_{j-1}^{n-k}\big)+c_2\big(d_j^{n+1-k}-d_j^{n-k}\big)+c_1\big(d_{j+1}^{n+1-k}-d_{j+1}^{n-k}\big)\big]+f_j^{n+1}.\end{cases} \qquad (11)$$

The above system carries $(N+1)\times(N+3)$ dimensions. To solve the above system for unique solution we need two additional equations which will come from boundary conditions. Thus, the system has $(N+3)\times(N+3)$ dimensions.

2.3 A finite difference scheme method [3]

They present the FDS of the following partial differential equation called fractional telegraph equation (FTE),

$\frac{*\partial^{2\alpha}u(x,t)}{*\partial t^{2\alpha}}+2\lambda\frac{*\partial^{\alpha}u(x,t)}{*\partial t^{\alpha}}=c^2\frac{\partial^2u(x,t)}{\partial^2 x}+f(x,t)$,(12)

where $\alpha\in(0,1/2]$, and $c,\lambda>0$ are real numbers. We confine the above Eq. (12) in a bounded domain Ω such that $(x,t)\in\Omega=[a,b]\times[0,T]$ and under the initial condition $u(x,0)=\varphi(x)$ and boundary conditions $u(a,t)=g_1(t)$ and $u(b,t)=g_2(t)$ for all $>>0$. The notion $(*\,\partial^\alpha/*\,\partial t^\alpha)u(x,t)$ represents the generalized fractional partial derivative of order α with respect to t and the definition will be given in Eq. (14).

The definition of left/ forward GFD of order $\alpha\in\mathbb{R}^+$of type 2 of a function $u(t)$ is as follows

$\big(D_{0+;[z;w;2]}^{\alpha}u\big)(t)=\big(I_{0+;[z;w]}^{m-\alpha}D_{[z;w;L]}^{m}u\big)(t)$ (13)

provided the right-side of the Eq. (13) is finite, where $m-1<\alpha<m$, and $m\in\mathbb{N}^+$. In the above definition, GFD of type 2 is also the generalized form of the Caputo fractional derivative, so we will now call it generalized Caputo type fractional derivative. The above definitions are only given for the left/forward sense of generalized fractional integral and GFD. Those can also be defined in the right/backward sense.

They considered left Caputo type GFD in upcoming work because Caputo derivative is always depicting the real-world models. For the special case, $0<\alpha<1$, GFD in Eq. (2) will be given as

$\big(D_{0+;[z;w;2]}^{\alpha}u\big)(t)=\frac{[w(t)]^{-1}}{\Gamma(1-\alpha)}\int_0^t\frac{(w(\tau)u(\tau))'}{(z(t)-z(\tau))^{\alpha}}d\tau$ (14)

To get the numerical scheme for solving FTE, we divide space and time interval into N and M equal sub-intervals, respectively, in a way such that $R_{\Delta x}=\{x_i: 0\le i\le N\}$ is a uniform mesh of the interval $[a,b]$, where $x_i=a+i(\Delta x), 0\le i\le N$ with $\Delta x=\frac{b-a}{N}$, and $R_{\Delta t}=\{t_j: 0\le j\le M\}$ is a uniform mesh in time direction where $t_j=j(\Delta t), j=0,1,\dots,M$, with $t_0=0$ is the initial time, and x_0 and x_N denote the boundary points. For simplification, we write $u(x_i,t_j)=u_j^i$, $w(t_j)=w_j$, and $z(t_j)=z_j$. For the numerical solution, we firstly approximate the first term of Eq. (1) as follows

$\left[\frac{*\partial^{2\alpha}u(x,t)}{*\partial t^{2\alpha}}\right]_{(x_it_{j+1})}=\frac{[w(t_{j+1})]^{-1}}{\Gamma(1-2\alpha)}\int_0^{t_{j+1}}\frac{\frac{\partial}{\partial\tau}[w(\tau)u(x_i,\tau)]}{[z(t_{j+1})-z(\tau)]^{2\alpha}}d\tau$(15)

we can be written (15) as

$$\mu\left(u_{j+1}^{i+1}-2u_{j+1}^{i}+u_{j+1}^{i-1}\right)=\sum_{k=0}^{j}\left(a_k^j u_{k+1}^i-b_k^j u_k^i\right)+2\lambda\sum_{k=0}^{j}\left(s_k^j u_{k+1}^i-v_k^j u_k^i\right)-f_{j+1}^i,$$

for $0\le j\le M-1$ and $1\le i\le N-1$,

(16)

where, the coefficients are given as

$$\begin{aligned}
a_k^j &=\frac{w_{j+1}^{-1}w_{k+1}}{\Gamma(2-2\alpha)(z_{k+1}-z_k)}\left[\left(z_{j+1}-z_k\right)^{1-2\alpha}-\left(z_{j+1}-z_{k+1}\right)^{1-2\alpha}\right],\\
b_k^j &=\frac{w_{j+1}^{-1}w_{k}}{\Gamma(2-2\alpha)(z_{k+1}-z_k)}\left[\left(z_{j+1}-z_k\right)^{1-2\alpha}-\left(z_{j+1}-z_{k+1}\right)^{1-2\alpha}\right],\\
s_k^j &=\frac{w_{j+1}^{-1}w_{k+1}}{\Gamma(2-\alpha)(z_{k+1}-z_k)}\left[\left(z_{j+1}-z_k\right)^{1-\alpha}-\left(z_{j+1}-z_{k+1}\right)^{1-\alpha}\right],\\
v_k^j &=\frac{w_{j+1}^{-1}w_{k}}{\Gamma(2-\alpha)(z_{k+1}-z_k)}\left[\left(z_{j+1}-z_k\right)^{1-\alpha}-\left(z_{j+1}-z_{k+1}\right)^{1-\alpha}\right],\\
\mu &=\frac{c^2}{(\Delta x)^2}.
\end{aligned}\qquad(17)$$

2.4 Conformable Double Sumudu Transform method [4]

The methodology of team of Hamza is as follows. they present a double definition of Sumudu transformation, its properties, a description of conformable fractional, and fractional space-time telegraph equations. After that, they obtained the exact solutions of the conformable fractional space-time telegraph equations.

The following general time-space conformable fractional telegraph equation:

$$\frac{\partial^\beta}{\partial x^\beta}f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right)=\ a\frac{\partial^\alpha}{\partial t^\alpha}f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right)+b\frac{\partial^\gamma}{\partial t^\gamma}f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right)+cf\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right)+h\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right),\qquad(18)$$

having initial condition:

$$f\left(\frac{x^\beta}{\beta},0\right)=k_1\left(\frac{x^\beta}{\beta}\right),D_\alpha f\left(\frac{x^\beta}{\beta},0\right)=k_2\left(\frac{x^\beta}{\beta}\right),\qquad(19)$$

and boundary:

$$f\left(0,\frac{t^\alpha}{\alpha}\right)=r_1\left(\frac{t^\alpha}{\alpha}\right)\quad {}_xD_\beta f\left(0,\frac{t^\alpha}{\alpha}\right)=r_2\left(\frac{t^\alpha}{\alpha}\right),\qquad(20)$$

where $h\left(x^\beta/\beta,t^\alpha/\alpha\right)$ is the given function and a,b,c are constants.

By taking conformable double Sumudu transform for bout aids of Equation (18),

$$\begin{aligned}
&\frac{1}{u^2}F(u,v)-\frac{F(0,v)}{u^2}-\frac{1}{u}S_t^\alpha\left[\ {}_xD_\beta f\left(0,\frac{t^\alpha}{\alpha}\right)\right]\\
&=a\left[\frac{1}{v^2}F(u,v)-\frac{F(u,0)}{v^2}-\frac{1}{v}S_x^\beta\left[\ {}_tD_\alpha f\left(\frac{x^\beta}{\beta},0\right)\right]\right]\\
&\quad+b\left[\frac{F(u,v)}{v}-\frac{F(u,0)}{v}\right]+cF(u,v)+h(u,v).
\end{aligned}\qquad(21)$$

2.5 The quadratic spline functions method [5]

The space fractional partial derivative of order α in Eq. (1) is considered in the Caputo sense

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x,t_j)=\frac{1}{\Gamma(n-\alpha)}\int_g^x\frac{\partial^n u(s,t_j)}{\partial x^n}(x-s)^{n-\alpha-1}ds,\quad n-1<\alpha\le n.\qquad(22)$$

To set up the quadratic polynomial spline method, select an integer $N>0$ and time-step size $k>0$. With $h=\frac{b-a}{N}$, then mesh points $\left(x_i,t_j\right)$ are $x_i=a+ih$, for each $i=0,1,\dots,N$, and $t_j=jk$, $k=\Delta t$ for each $j=0,1,\dots$

Let ${Z_i}^j$ be an approximation to $u(x_i,t_j)$ obtained by the segment $P_i(x,t_j)$ of the spline function passing through the points (x_i,Z_i^j) and (x_{i+1},Z_{i+1}^j). Each segment has the form

$$P_i(x,t_j)=a_i(t_j)\ (x-x_i)^2+b_i(t_j)\ (x-x_i)+c_i(t_j).\qquad(23)$$

for each $i=0,1,\dots,N-1$. To obtain expressions for the coefficients of (23) in terms of ${Z_{i+1/2}}^j$, ${D_i}^j$, and $S_{i+1/2}^j$, we first define

$$P_i(x_{i+1/2},t_j)=Z_{i+1/2}^j\qquad(24)$$

$P_i^{(1)}(x_i,t_j)=D_i^j$ (25) $P_i^{(\alpha)}(x_{i+1/2},t_j)=\frac{\partial^\alpha}{\partial x^\alpha}P_i(x_{i+1/2},t_j)=S_{i+1/2}^j,\quad 1<\alpha\le 2,\quad x_i<x_{i+1/2}\le x_{i+1}.$
(26)

where $a_i\equiv a_i(t_j),\ b_i\equiv b_i(t_j),\ c_i\equiv c_i(t_j),\ d_i\equiv d_i(t_j)$ and $\theta=\omega h$. Eqs. (23), (24) and (25), give

$\frac{h^2}{4}a_i+\frac{h}{2}b_i+c_i=Z_{i+1/2}^j.$ (27)

$b_i=D_i^j.$ (28)

Using Eqs. (22), (23), and (26), we obtain

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x_{i+1/2},t_j)=\frac{1}{\Gamma(2-\alpha)}\int_{x_i}^{x_{i+1/2}}\frac{\partial^2P_i(s,t_j)}{\partial x^2}\left(x_{i+1/2}-s\right)^{1-\alpha}ds=S_{i+1/2}^j\ . \qquad (29)$$

This equation can be simplified as:

$\mu\ a_i=S_{i+1/2}^j$ (30)

where $\mu=\frac{2}{\Gamma(3-\alpha)}\left(\frac{h}{2}\right)^{2-\alpha}$.

Then Eq. (23) becomes

$$Z_{i+3/2}^j-2Z_{i+1/2}^j+Z_{i-1/2}^j=\delta(S_{i+3/2}^j+6S_{i+1/2}^j+S_{i-1/2}^j),\quad i=1,2,\dots,N-2. \qquad (31)$$

where $\delta=\frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}$. As $\alpha\to 2$, system (31) reduces to

$$Z_{i+3/2}^j-2Z_{i+1/2}^j+Z_{i-1/2}^j=\frac{h^2}{8}(S_{i+3/2}^j+6S_{i+1/2}^j+S_{i-1/2}^j),\quad i=1,2,\dots,N-2. \qquad (32)$$

III. Truncation error and a class of methods [5]

Expanding (31) in a Taylor series in terms of $y(x_j,t_m)$ and its derivatives and using (1) respectively, they obtain the truncation error as follows:

$$T^*{}_i^j=\left(u_{i-1/2}^j+u_{i+3/2}^j\right)-2u_{i+1/2}^j-\delta\left(D_x^2u_{i-1/2}^j+D_x^2u_{i+3/2}^j\right)-6\delta D_x^2u_{i+1/2}^j \qquad (33)$$

can be obtained by expanding this equation in Taylor series in terms of $u(x_{i+1/2},t_j)$ and its derivatives as follows

$$T^*{}_i^j=(h^2-8\delta)D_x^2u_{i+1/2}^j+\left(\frac{h^4}{12}-\delta h^2\right)D_x^4u_{i+1/2}^j+\left(\frac{h^6}{360}-\frac{\delta h^4}{12}\right)D_x^6u_{i+1/2}^j+\cdots. \qquad (34)$$

Since $\delta=\frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}$ then the last expression can be simplified as

$$T^*{}_i^j=h^\alpha(h^{2-\alpha}-8\theta)D_x^2u_{i+1/2}^j+h^{2+\alpha}\left(\frac{h^{2-\alpha}}{12}-\theta\right)D_x^4u_{i+1/2}^j+$$
$$h^{4+\alpha}\left(\frac{h^{2-\alpha}}{360}-\frac{\theta}{12}\right)D_x^6u_{i+1/2}^j+\cdots. \qquad (35)$$

where $\theta=\frac{\Gamma(3-\alpha)}{2^{\alpha+1}}$. From this expression of the local truncation error, our scheme is $O(h^\alpha),\quad 1<\alpha\le 2$.

$$S_i^j=\frac{\partial^\alpha Z_i^j}{\partial x^\alpha}=\frac{\partial^2Z_i^j}{\partial t^2}+\frac{\partial Z_i^j}{\partial t}+Z_i^j \qquad (36)$$

$$S_i^j=\frac{\partial^\alpha Z_i^j}{\partial x^\alpha}\approx\frac{Z_i^{j+1}-2Z_i^j+Z_i^{j-1}}{k^2}+\frac{Z_i^{j+1}-Z_i^{j-1}}{2k}+Z_i^j \qquad (37)$$

which can be discretized as follows:

$$S_{i-1/2}^j=\frac{\partial^\alpha Z_{i-1/2}^j}{\partial x^\alpha}\approx\frac{Z_{i-1/2}^{j+1}-2Z_{i-1/2}^j+Z_{i-1/2}^{j-1}}{k^2}+\frac{Z_{i-1/2}^{j+1}-Z_{i-1/2}^{j-1}}{2k}+Z_{i-1/2}^j$$
$$S_{i+1/2}^j=\frac{\partial^\alpha Z_{i+1/2}^j}{\partial x^\alpha}\approx\frac{Z_{i+1/2}^{j+1}-2Z_{i+1/2}^j+Z_{i+1/2}^{j-1}}{k^2}+\frac{Z_{i+1/2}^{j+1}-Z_{i+1/2}^{j-1}}{2k}+Z_{i+1/2}^j \qquad (38)$$
$$S_{i+3/2}^j=\frac{\partial^\alpha Z_{i+3/2}^j}{\partial x^\alpha}\approx\frac{Z_{i+3/2}^{j+1}-2Z_{i+3/2}^j+Z_{i+3/2}^{j-1}}{k^2}+\frac{Z_{i+3/2}^{j+1}-Z_{i+3/2}^{j-1}}{2k}+Z_{i+3/2}^j$$

Using formulas (38) in (31) gives the following useful systems

$$A\ Z_{i-1/2}^{j+1}+B\ Z_{i+1/2}^{j+1}+A\ Z_{i+3/2}^{j+1}=A^*Z_{i-1/2}^j+$$
$$B^*Z_{i+1/2}^j+A^*Z_{i+3/2}^j+\hat{A}\ Z_{i-1/2}^{j-1}+\hat{B}\ Z_{i+1/2}^{j-1}+\hat{C}\ Z_{i+3/2}^{j-1} \qquad (39)$$

where

$$A=\frac{\delta}{k^2}+\frac{\delta}{2k},A^*=1+\frac{2\delta}{k^2}-\delta,\hat{A}=\frac{-\delta}{k^2}+\frac{\delta}{2k},$$
$$B=\frac{6\delta}{k^2}+\frac{3\delta}{k},B^*=-2+\frac{12\delta}{k^2}-6\delta,\ and\ \hat{B}=\frac{-6\delta}{k^2}+\frac{3\delta}{k}. \qquad (40)$$

System (39) consists of *N-2*equations in *N* unknowns. To get a solution to this system, we need 2-additional equations. Using the boundary conditions (2), that are $Z_0^j=\beta_1(t),\quad Z_{N+1}^j=\beta_2(t)$, we can obtain the following equations: Suppose that $Z_{1/2}^j$ is linearly interpolated between Z_0^j and $Z_{3/2}^j$

$-3Z_{1/2}^{j} + Z_{3/2}^{j} = -2Z_{0}^{j} = -2\beta_1, \quad j \geq 0$ (41)

In a similar manner,

$Z_{N-3/2}^{j} - 3Z_{N-1/2}^{j} = -2Z_{N}^{j} = -2\beta_2, \quad j \geq 0$ (42)

Eq. (39) implies that the *(j+1)*st time step requires values from the *(j)*st and *(j-1)*st time steps. This produces a minor starting problem, since values for *j=0*are given by the first part in Eq. (3)

$Z_i^0 = u(x_i, 0) = f_1(x_i), \quad i = 1, \ldots, N.$ (43)

but values for *j=0*, which are needed in Eq. (40) to computeZ_i^1, must be obtained from the first part in (3)

$$\frac{\partial Z_i^0}{\partial t} = u_t(x_i, 0) = f_2(x_i), \quad i = 1, \ldots, N.$$

One approach is to replace $\frac{\partial Z_i^0}{\partial t}$ by a forward-difference approximation

$f_2(x_i) = \frac{\partial Z_i^0}{\partial t} = \frac{Z_i^1 - Z_i^0}{k} + o(k)$ (44)

which gives us

$Z_i^1 = Z_i^0 + k f_2(x_i), \quad i = 1, \ldots, N.$ (45)

IV. Stability analysis and convergence

In this section, we will illustrate the stability by different methods and the convergence:

4.1 The Method of [1]

They produced the stability of our numerical scheme based on the SDM. For this, they considered a Banach space $(\mathcal{B}, \|.\|)$ and $\mathcal{F}: \mathcal{B} \to \mathcal{B}$. Let $\psi_{n+1} = f(\mathcal{F}, \psi_n)$ be a recursive technique and H($\mathcal{F}$) be the set of fixed points of $\mathcal{F}$ at least containing one point, say $p \in H(\mathcal{F})$. They assume that $\psi_n \subset \mathcal{B}$ and define $\text{err}_n = \|\psi_{n+1} - f(\mathcal{F}, \psi_n)\|$. If $\lim_{n\to\infty} \psi^n = p$, then $\psi_{n+1} = f(\mathcal{F}, \psi_n)$ is said to be H-stable.

Theorem 4.1.

Let $(\mathcal{B}, \|.\|)$ be a Banach space and $\mathcal{F}: \mathcal{B} \to \mathcal{B}$ satisfy the following condition

$\|\mathcal{B}_x - \mathcal{B}_y\| \leq C\|x - \mathcal{B}_x\| + c \| x - y \|$ (46)

for all $x, y \in \mathcal{B}, C \geq 0, 0 \leq c \leq 1$. Then $\mathcal{B}$ is picard $\mathcal{B} -$ stable.

Our Picard $\mathcal{B}$-stability result is now given by the following result.

Theorem 4.2.

Let $\mathcal{F}: \mathcal{B} \to \mathcal{B}$ be an operator defined as bellow

$\mathcal{F}(\psi_n(x, y)) = \mathbb{S}^{-1}[u^{n\alpha}\mathbb{S}[A(x,t)\,\partial_t^2 + B(x,t)\,\partial_t + C(x,t)\mathrm{I}]U_n(x,t)],$ (47)

where

$\| A(x,t) \| \leq k_1, \quad \| B(x,t) \| \leq k_2,$ and $\| C(x,t) \| \leq k_3.$ (48)

Then, $\mathcal{B}$ is Picard $\mathcal{B}$-stable provided that $\lambda_\mathrm{i} > 0, (i = 1,2)$, and the following relations hold:

(i) $\|\partial_t^2\psi_n(x,y) - \partial_t^2\psi_m(x,y)\| \leq \lambda_1\|\psi_n(x,y) - \psi_m(x,y)\|$,

(ii) $\|\partial_t\psi_n(x,y) - \partial_t\psi_m(x,y)\| \leq \lambda_2\|\psi_n(x,y) - \psi_m(x,y)\|$,

(iii) $k_1\lambda_1 + k_2\lambda_2 + k_3 < 1$.

Proof:

In order to prove the existence of a fixed point of the operator $\mathcal{F}$, we consider $n, m \in \mathbb{N}$, and

$$\begin{aligned}
&\| \mathcal{F}\psi_n(x,y) - \mathcal{F}\psi_m(x,y) \| \\
&= \| \mathbb{S}^{-1}[u^{n\alpha}\mathbb{S}[A(x,t)\,\partial_t^2 + B(x,t)\,\partial_t + C(x,t)I]U_n(x,t)]\psi_n \\
&\quad -\mathbb{S}^{-1}[u^{n\alpha}\mathbb{S}[A(x,t)\,\partial_t^2 + B(x,t)\,\partial_t + C(x,t)I]U_n(x,t)]\psi_m \| \\
&\leq \| A(x,t) \| \|\partial_t^2\psi_n(x,y) - \partial_t^2\psi_m(x,y)\| + \| B(x,t) \| \|\partial_t\psi_n(x,y) - \partial_t\psi_m(x,y)\| \\
&\quad + \| C(x,t) \| \|\psi_n - \psi_m\| \leq (k_1\lambda_1 + k_2\lambda_2 + k_3)\|\psi_n - \psi_m\| \leq \|\psi_n - \psi_m\|
\end{aligned}$$

Consequently, with the help of Theorem 4.1, we can conclude that the operator $\mathcal{F}$ is Picard $\mathcal{F} -$ Stable.

4.2 The Method of [2, 5]

Akram et al. used the Von Neumann stability analysis to investigate the stability of proposed scheme. Consider the growth factor in the form of a one Fourier mode as

$U_j^n = \xi^n e^{iwhj}$(49)

with no forcing term. Here $i = \sqrt{-1}, w$ and h are the mode number and the element size, respectively. They have

$$[(\beta_1+\beta_2+\mu)c_1-vc_4]U_{j-1}^{n+1}+[(\beta_1+\beta_2+\mu)c_2-vc_5]U_j^{n+1} + [(\beta_1+\beta_2+\mu)c_1-vc_4]U_{j+1}^{n+1} = (2\beta_1+\beta_2)c_1U_{j-1}^{n}+(2\beta_1+\beta_2)c_2U_j^{n}+(2\beta_1+\beta_2)c_1U_{j+1}^{n}-\beta_1\big[c_1U_{j-1}^{n-1} + c_2U_j^{n-1}+c_1U_j^{n-1}\big]-\beta_1\sum_{k=1}^{n} b_k^{2\alpha}\big[c_1\big(U_{j-1}^{n-k+1}-2U_{j-1}^{n-k}+U_{j-1}^{n-k-1}\big)+c_2\big(U_j^{n-k+1} - 2U_j^{n-k}+U_j^{n-k-1}\big)+c_1\big(U_{j+1}^{n-k+1}-2U_{j+1}^{n-k}+U_{j+1}^{n-k-1}\big)\big]-\beta_2\sum_{k=1}^{n} b_k^{\alpha}\big[c_1\big(U_{j-1}^{n-k+1} - U_{j-1}^{n-k}\big)+c_2\big(U_j^{n-k+1}-U_j^{n-k}\big)+c_1\big(U_{j+1}^{n-k+1}-U_{j+1}^{n-k}\big)\big]. \quad (50)$$

The above equation shows a round off error equation. Consider Eq. (49) to be the solution, then the above equation becomes

$$\xi^{n+1}=\frac{1}{\omega}\big[(1+\alpha_1)\xi^n-\alpha_1\xi^{n-1}-\alpha_1\sum_{k=1}^{n} b_k^{2\alpha}[\xi^{n-k+1}-2\xi^{n-k}+\xi^{n-k-1}] -\alpha_2\sum_{k=1}^{n} b_k^{\alpha}[\xi^{n-k+1}-\xi^{n-k}]\big] \quad (51)$$

where $\alpha_1=\frac{\beta_1}{\beta_1+\beta_2}, \alpha_2=\frac{\beta_2}{\beta_1+\beta_2}$ and

$$\omega=1+\frac{\mu}{\beta_1+\beta_2}+\frac{12v(2+\lambda)\sin^2 wh/2}{h^2(\beta_1+\beta_2+\mu)(6+(\lambda-4)\sin^2 wh/2)}$$

Obviously $\omega\geq 1$, for all $\lambda>-2$.

Proposition 4.1. Let $\xi^n, n=0,1,\dots,T\times M$, be the solution of Eq. (49), we have

$|\xi^n|\leq(1+\alpha_1)|\xi^0|,\quad n=0,1,\dots,T\times M$ (52)

where α_1 is a positive constant.

Thus $|\xi^{n+1}|=|U_j^{n+1}|\leq(1+\alpha_1)|\xi^0|=(1+\alpha_1)|U_j^0|$, so that $\|U_j^{n+1}\|_2\leq(1+\alpha_1)\|\xi^0\|_2$. (53)

Thus, one concludes that the proposed numerical scheme is unconditionally stable.

The convergence of proposed technique using the LopezMarcos method, which plays a significant role in the theory of convergence analysis of fractional type equation.Assume that $\Omega_x=\{x_j; 0\leq j\leq N\}$ and $\Omega_t=\{t_n; 0\leq n\leq M\}$ be the equidistant partitioning of intervals $[a,b]$ and $[0,T]$ with the step size h and τ, respectively. Consider u_j^n be the approximated solution at the grid point (x_j,t_n) and $V=\{v_j; 0\leq j\leq N\}, W=\{w_j; 0\leq j\leq N\}$ be the two functions defined on Ω_x. We define difference notation as follows:

$\delta^2V=v_{j+1}-2v_j+v_{j-1},\quad \delta V=v_{j+1}-v_j$

$\|V\|^2=(V,V),\quad (V,W)=\sum_{j=1}^{N} hv_jw_j,$ (54)

$(V_{xx},V)=-(V_x,V_x),\quad (V,W_x)=-(V_x,W).$

We also suppose that u_{tt}, u_{xxxx} are continuous over the intervals $[a,b]$ and $[0,T]$, and that there is a positive constant F, such that

$|u_{tt}|\leq F,\quad |u_{xxxx}|\leq F$ (55)

The above values are different at different locations and independent of j,n,h,τ and for $(x,t)\in\Omega_x\times\Omega_t$

Proposition 5.1. Let $\{z_0,z_1,\dots,z_n,\dots\}$ be a monotonically decreasing sequence with the properties $z_n\geq 0$ and $z_{n+1}+z_{n-1}\geq 2z_n$. Then, for any positive integer S and for each vector $(v_1,v_2,\dots,v_S)$ with S real entries, we have

$\sum_{n=0}^{S-1}\left(\sum_{m=0}^{n} z_mV_{n+1-m}\right)V_{n+1}\geq 0$ (56)

So for the proposed scheme, we have

$$\beta_1\sum_{j=0}^{n} b_j^{2\alpha}\big(u(x,t_{n-j+1})-2u(x,t_{n-j})+u(x,t_{n-j-1})\big)+\beta_2\sum_{j=0}^{n} b_j^{\alpha}\big(u(x,t_{n-j+1}) - u(x,t_{n-j})\big)+\mu u(x,t^{n+1})=v\frac{\partial^2u(x,t^{n+1})}{\partial x^2}+f(x,t^{n+1})+O(\tau^{2-\alpha}+h^2) \quad (57)$$

and

$$\beta_1\sum_{j=0}^{n} b_j^{2\alpha}(u^{n-j+1}-2u^{n-j}+u^{n-j-1})+\beta_2\sum_{j=0}^{n} b_j^{\alpha}(u^{n-j+1}-u^{n-j})+\mu u^{n+1} = v\frac{\partial^2u^{n+1}}{\partial x^2}+f^{n+1} \quad (58)$$

where u_j^n and $u(x_j,t_n)$ are an approximated and the exact solution at point (x_j,t_n), respectively.

Theorem 5.1. Suppose that $u(x,t)$ and u_j^n be the solutions of given model and Eq. (57), respectively, and $u(x,t)$ satisfies the smoothness condition (55), then we have

$\|E^{n+1}\|\leq O(\tau^{2-\alpha}+h^2)$ (59)

where $E_j^{n+1}=u(x_j,t_{n+1})-u_j^{n+1}$, for every $t\geq 0$ and suitably small h and τ.

Using the Cauchy-Schwarz inequality, we get

$\|E^{n+1}\|^2 \leq \frac{1}{\mu}(p^{k+1}, E^{k+1}) \leq \frac{1}{\mu}\|p^{k+1}\|\|E^{k+1}\|$. (60)
From the above inequality, we can get the desired result,
$\|E^{n+1}\| \leq O(\tau^{2-\alpha} + h^2)$. (61)
But Zaki et al. used of Eqs. (15) and (39) gives us the characteristic equation in the form

$$\zeta^{j+1}\left\{ A\, e^{((i-1)q\phi h)} + B\ \ e^{(iq\phi h)} + A\, e^{((i+1)q\emptyset h)}\right\} =$$

$$\zeta^{j}\left\{A^*\, e^{((i-1)q\phi h)} + B^*\, e^{(iq\phi h)} + A^* e^{((i+1)q\emptyset h)}\right\} +$$

$$\zeta^{j-1}\left\{\hat{A}\, e^{((i-1)q\phi h)} + \hat{B}\ \ e^{(iq\phi h)} + \hat{A}\ \ e^{((i+1)q\emptyset h)}\right\}. \qquad (62)$$

This equation can be rewritten in the simple form

$a\zeta^2 + b\zeta + c = 0$ (63)
where

$$a = \left(A\, e^{(-q\phi h)} + B\ + A\ e^{(q\phi h)}\right),\ b = -\left(A^*\, e^{(-q\phi h)} + B^* + A^* e^{(q\phi h)}\right),$$
$$and\ c = -\left(\hat{A}\, e^{(-q\phi h)} + \hat{B}\ + \hat{A}\ e^{(q\phi h)}\right)$$

For the stability, they must have $|\zeta_{\pm}| \leq 1$. So they have three cases
Case 1: The discriminant of the Quadratic equation (63) is zero, that is
$\psi^2 - 1 = 0$, in that case $\varsigma_{\pm} = \pm\sqrt{\frac{c}{a}} = \pm\sqrt{\frac{2-k}{2+k}}$, $0 < k < 1$ and the stability condition, $|\zeta_{\pm}| \leq 1$, is satisfied.
Case 2: Discriminant is less than zero, that is $\psi^2 - 1 < 0$, in this case

$$\varsigma_{\pm} = \sqrt{\frac{c}{a}}\left(-\psi \pm q\sqrt{1-\psi^2}\right) = \sqrt{\frac{2-k}{2+k}}\left(-\psi \pm q\sqrt{1-\psi^2}\right) \Rightarrow$$

the stability condition, $|\zeta_{\pm}| \leq 1$, is satisfied.
Case3: The discriminant is greater than zero. This means that one of ζ_+ and
ζ_- does not satisfy the stability condition.
Thus, for stability we must have $\psi^2 - 1 \leq 0$

$$-1 \leq \psi \leq 1$$
$$-1 \leq \frac{b}{2\sqrt{ac}} \leq 1$$

Since $\sqrt{ac} > 0 \Rightarrow -2\sqrt{ac} \leq b \leq 2\sqrt{ac}$

$$-\frac{2\delta}{k^2}(3 + \cos\varphi)\sqrt{4-k^2} \leq 2(1-\cos\varphi) - 2\delta(3+\cos\varphi)\left(\frac{2}{k^2} - 1\right) \leq \frac{2\delta}{k^2}(3+\cos\varphi)\sqrt{4-k^2}$$

The right above inequality takes the form:

$$2(1-\cos\varphi) \leq \frac{2\delta}{k^2}(3+\cos\varphi)\left(\sqrt{4-k^2} + 2 - k^2\right)$$

Which is satisfied for k<< $\delta$where$h$issmall enough.
But the left above inequality takes the form:

$$-2(1-\cos\varphi) \leq \frac{2\delta}{k^2}(3+\cos\varphi)\left(\sqrt{4-k^2} - 2 + k^2\right)$$

Which is satisfied for k<< $\delta$where$h$issmall enough, then the method is conditionally stable.

4.3 The Method of [3]

They can easily check the properties of coefficients a_k^j, b_k^j, s_k^j and v_k^j in the following lemma, which is useful to prove the stability of the finite difference scheme:
Lemma 4.1. If $0 < \alpha \leq 1/2$, the weight function $w(t)$ is positive and increasing, the scale function $z(t)$ is non-negative and strictly increasing. Under these conditions, the following results hold:
(i) $a_k^j > b_k^j > 0$, and $w_k a_k^j = w_{k+1} b_k^j$,
(ii) $s_k^j > v_k^j > 0$, and $w_k s_k^j = w_{k+1} v_{kj}^{j'}$, and
(iii) if $w(t)$ is a nonzero constant, then $a_k^j = b_k^j, s_k^j = v_k^j$ for all $k = 0,1,2,\ldots,j$.

Theorem 4.3 (Lax-Richertmyer Theorem).
For consistent numerical methods to approximate the solution of a well-posed linear differential equation, stability and convergence are equivalent.

Theorem 4.4. If the scale function $z(t)$ is positive and increasing and, weight function $w(t)$ is positive and non-decreasing, then the numerical scheme (16) is stable.
Proof. Here, we need to prove only the stability of the homogeneous part of the iteration scheme (16). So, let the numerical solution of Eq. (12) is of the form $u_{j+1}^{m} = \delta_{j+1}e^{i\theta my}$, where, $i = \sqrt{-1}$ (unit of complex numbers), $\theta \in \mathbb{R}$ and, $1 \le m \le N-1$. So, we can write the homogeneous part of Eq. (16) as,

$$\mu\delta_{j+1}e^{i\theta(m-1)y} + \left(-2\mu - a_j^j - 2\lambda s_j^j\right)\delta_{j+1}e^{i\theta my} + \mu\delta_{j+1}e^{i\theta(m+1)y} = -b_j^j\delta_j e^{i\theta my} - 2\lambda q_j^j\delta_j e^{i\theta my} + \sum_{k=0}^{j-1}\left(a_k^j\delta_{j+1}e^{i\theta my} - b_k^j\delta_j e^{i\theta my}\right) + 2\lambda\sum_{k=0}^{j-1}\left(s_k^j\delta_{j+1}e^{i\theta my} - v_k^j\delta_j e^{i\theta my}\right).$$
(64)

This implies

$$\mu\delta_{j+1}e^{-i\theta y} + \left(-2\mu - a_j^j - 2\lambda s_j^j\right)\delta_{j+1} + \mu\delta_{j+1}e^{i\theta y} = -b_j^j\delta_j - 2\lambda q_j^j\delta_j + \sum_{k=0}^{j-1}\left(a_k^j\delta_{j+1} - b_k^j\delta_j\right) + 2\lambda\sum_{k=0}^{j-1}\left(s_k^j\delta_{j+1} - v_k^j\delta_j\right),$$

$$-\mu\delta_{j+1}e^{-i\theta y} + \left(2\mu + a_j^j + 2\lambda s_j^j\right)\delta_{j+1} - \mu\delta_{j+1}e^{i\theta y} = b_j^j\delta_j + 2\lambda q_j^j\delta_j - \sum_{k=0}^{j-1}\left(a_k^j\delta_{j+1} - b_k^j\delta_j\right) - 2\lambda\sum_{k=0}^{j-1}\left(s_k^j\delta_{j+1} - v_k^j\delta_j\right),$$

$$\mu\delta_{j+1}\left(-e^{-i\theta y} + 2 - e^{i\theta y}\right) + \left(a_j^j + 2\lambda s_j^j\right)\delta_{j+1} = \left(b_j^j + 2\lambda v_j^j\right)\delta_{j+1} - \sum_{k=0}^{j-1}\left(a_k^j\delta_{j+1} - b_k^j\delta_j\right) - 2\lambda\sum_{k=0}^{j-1}\left(s_k^j\delta_{j+1} - v_k^j\delta_j\right)$$(65)

From Lemma 4.1, we have $a_k^j \ge b_k^j, s_k^j \ge v_k^j$ for all $k = 0,1,2,\dots,j$ and $j = 0,1,2,\dots,M-1$. Hence, from Eq. (65), we get

$$\delta_{j+1} = \frac{\left(b_j^j + 2\lambda v_j^j\right)\delta_j - \sum_{k=0}^{j-1}\left(a_k^j\delta_{j+1} - b_k^j\delta_j\right) - 2\lambda\sum_{k=0}^{j-1}\left(s_k^j\delta_{j+1} - v_k^j\delta_j\right)}{(2\mu - 2\mu\cos\theta y) + \left(a_j^j + 2\lambda s_j^j\right)},$$

$$\delta_{j+1} = \frac{\left(b_j^j + 2\lambda v_j^j\right)\delta_j}{[2\mu - 2\mu\cos(\theta y)] + \left(a_j^j + 2\lambda s_j^j\right)} - \frac{\sum_{k=0}^{j-1}\left(a_k^j\delta_{j+1} - b_k^j\delta_j\right)}{[2\mu - 2\mu\cos(\theta y)] + \left(a_j^j + 2\lambda s_j^j\right)} - \frac{2\lambda\sum_{k=0}^{j-1}\left(s_k^j\delta_{j+1} - v_k^j\delta_j\right)}{[2\mu - 2\mu\cos(\theta y)] + \left(a_j^j + 2\lambda s_j^j\right)}.$$
(66)

Since $2\mu \ge 0$ and $1 - \cos(\theta y) \ge 0$ this implies $2\mu - 2\mu\cos(\theta y) \ge 0$ and from Eq. (66) we get

$$\delta_{j+1} \le \frac{\left(b_j^j + 2\lambda v_j^j\right)\delta_j}{\left(a_j^j + 2\lambda s_j^j\right)} - \frac{\sum_{k=0}^{j-1}\left(a_k^j\delta_{j+1} - b_k^j\delta_j\right)}{[2\mu - 2\mu\cos(\theta y)] + \left(a_j^j + 2\lambda s_j^j\right)} - \frac{2\lambda\sum_{k=0}^{j-1}\left(s_k^j\delta_{j+1} - v_k^j\delta_j\right)}{[2\mu - 2\mu\cos(\theta y)] + \left(a_j^j + 2\lambda s_j^j\right)} \le \delta_j.$$
(67)

Since, each term in summation is nonnegative which implies that $\delta_{j+1} \le \delta_j \le \cdots \le \delta_2 \le \delta_1 \le \delta_0$, therefore, $\delta_{j+1} = \left|u_{j+1}^m\right| \le \delta_0 = |u_0^m| = |u_0|$. Hence, $\|u_{j+1}\|_{1^2} \le \|u_0\|_{1^2}$ and the stability of numerical scheme (16)is proved.
The consistency of the numerical scheme (16) is easy to prove, and so the numerical scheme (16) is convergent (from Theorem 4.3).

V. Numerical Examples

In this section, we obtain numerical solutions of equations(1,……) for a numerical example.

5.1 The Results of [1]:
Example 5.1.
Consider the one-dimensional space-time FTE

$$\begin{aligned} &{}_0^cD_x^{2\alpha}U(x,t) = D_t^2U(x,t) + D_tU(x,t) + U(x,t), \\ &0 < x < 1, 0 < \alpha \le 1, 0 < \beta \le 1, \quad t > 0, \end{aligned} \quad (68)$$

with the initial and boundary conditions

$$\begin{aligned} &U_0(0,t) = e^{-t}, \quad t \ge 0 \\ &U_x(0,t) = e^{-t}, \quad t \ge 0 \\ &U(x,0) = e^x, \quad 0 < x < 1, \\ &U_t(x,0) = 0, \quad 0 < x < 1. \end{aligned} \quad (69)$$

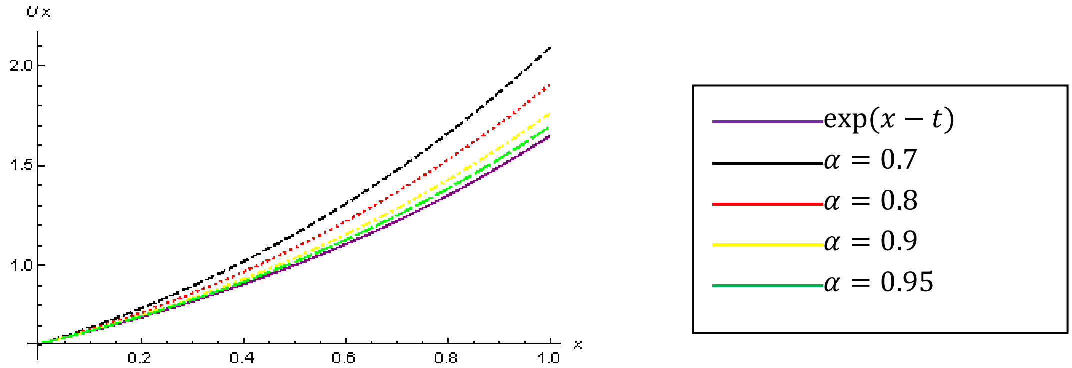


Figure 5.1. Plot of approximate solutions U (x, t) at different values of α at t = 0. 5 and compression with exact solution $e^{(x-t)}$

Example 5.2.
Consider the following nonlinear space-fractional telegraph equation

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial^2 t} + 2\frac{\partial u}{\partial t} + u^2 - e^{2x-4t} + e^{x-2t} \quad (70)$$

subjected to the following initial and boundary conditions

$$0 < \alpha \leq 1, t > 0, 0 < x < 1,$$

$$u(0,t) = 0, u_x(0,t) = e^x, \quad u(x,0) = 0, u_t(x,0) = -2e^x. \quad (71)$$

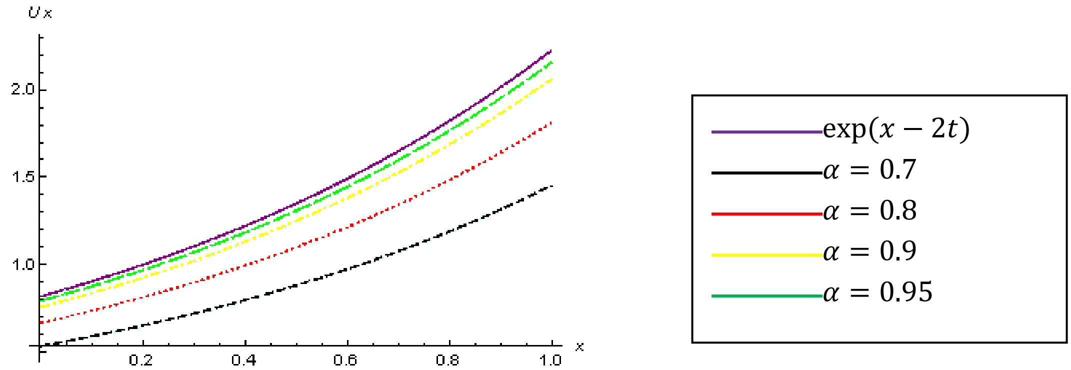


Figure 5.2 Plot of approximate solutions $u(x, t)$ at different value of α at t = 0.1

5.2 The results of [2]

some numerical experiments are discussed to demonstrate the feasibility of the proposed method. The calculated error norms are established by absolute L^∞ and Euclidean L^2 norms

Example5.3 Consider the TFTE of the form

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) (72)$$

where $f(x,t)$ is suitable with the exact solution $u(x,t) = t^{2+\alpha}\sin(2\pi x)$, for all $(x,t) \in [0,1] \times [0,1]$, where the initial and boundary conditions are

$$\begin{cases} f_1(x) = 0, \\ f_2(x) = 0, \\ g_1(t) = 0, \\ g_2(t) = 0. \end{cases} (73)$$

In Table 5.1, we calculate the L^2-norm for different spatial and temporal step size $h = 5$, $\tau = \frac{1}{M}$, $(M = 20,40,80)$. In Table 5.2, we determine the order of convergence from the computed data and present maximum absolute errors at different space-time step sizes. We give L^2-norm and maximum errors for $\alpha = 0.6,0.7,0.8,0.9$. Table 5.3 shows the maximum absolute error, the Euclidean norm, the order of convergence and the CPU time for $\alpha = 0.75$ and $\tau = \frac{1}{100}$. In additional, the compatibility of the exact and approximated solution for example5.3 can be viewed in Fig. 5.3

Table 5.1 comparison of L^2 norm with $h = 5\tau = \frac{1}{M}$ at $T = 1$ for example 5.3

α	L^2-norm[2]			L^2-normproposedmethod		
	M_1=20	M_2=40	M_3=80	M_1=20	M_2=40	M_3=80
0.6	4.7812E-03	1.0955E-03	2.2623E-04	1.8893E-03	9.1390E-04	1.6673E-04
0.7	4.5879E-03	1.0289E-03	2.0119E-04	8.0515E-04	2.1220E-04	5.2584E-05
0.8	4.3196E-03	9.3157E-04	1.6304E-04	6.0531E-04	5.1956E-05	3.4099E-06
0.9	3.9411E-03	7.8419E-04	2.1547E-04	2.0000E-05	3.2034E-06	2.4375E-07

Table 5.2 comparison of maximum error with $h = 5\tau = \frac{1}{M}$ at $T = 1$ for example 5.3

α	L^∞-normproposedmethod				
	M_1=20	M_2=40	M_3=80	Order=($\frac{M1}{M2}$)	Order=($\frac{M2}{M3}$)
0.6	2.6719E-03	1.2925E-03	2.3579E-04	1.04776	2.45455
0.7	1.1387E-03	3.0010E-04	7.4366E-05	1.92380	2.01273
0.8	8.5604E-04	7.3478E-05	4.8224E-06	3.54230	3.92949
0.9	2.8285E-05	4.5303E-06	3.4471E-07	2.64239	3.71615

Table5.3Maximum absolute errors and Euclidean norm L^2 for example 5.3

N	Proposedmethod			
	L^∞-norm	L^2-norm	Orderofconvergence	CPUtime
05	4.5584E-04	3.3892E-04	…	0.234002
10	9.1361E-05	6.7926E-05	2.31889	0.249602
20	1.5847E-05	1.1206E-05	2.52731	0.265202
40	9.2497E-07	6.5406E-07	4.09870	0.624004

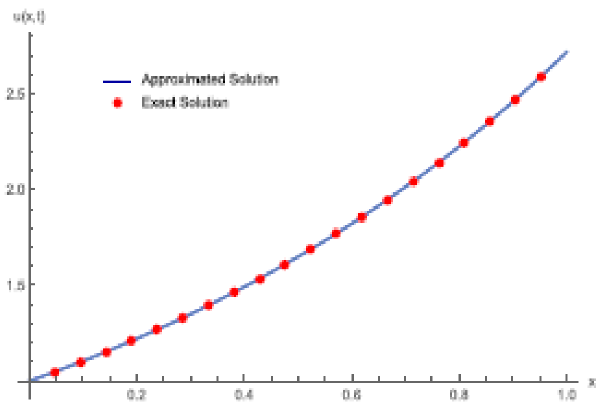


Figure 5.3 Comparison plots of exact solutions and approximated solutions for example5.3.

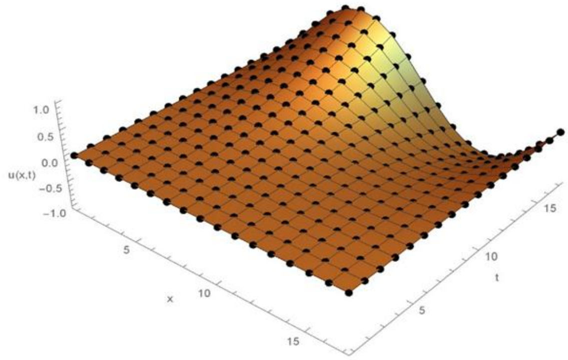


Figure 5.43D plot for the exact and approximated solution of example 5.3

5.3 The results of [3]

Example 5.4. Consider Eq. (12) with force term $f(x,t) = \frac{2x(x-1)t^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{4\lambda x(x-1)t^{2-\alpha}}{\Gamma(3-\alpha)} - 2c^2t^2$, and initial and boundary conditions as $u(x,0) = 0, x \in [0,1], u(0,t) = u(1,t) = 0, t > 0$. With $z(t) = t, w(t) = 1$, the exact solution of the Eq. (12) will be $u(x,t) = x(x-1)t^2$. We solve Eq. (12) by the numerical finite difference scheme is given in Eq. (16) with step size $\Delta x = 0.01, \Delta t = 0.001$. The analytical and the numerical solutions taking $\alpha = 0.2,0.3,0.4$ are shown in Figs. 5.5 and 5.6: F-1 to F-6. The maximum absolute errors (MAE) and the order of convergence (CO) are calculated for different step sizes and the obtained results for $\alpha = 0.2$ and $\alpha = 0.4$ are given in Table 5.4 and Table 5.5, respectively. From Table 5.4 and Table 5.5, we can assume that the numerical scheme is stable, and CO is $(\Delta t)^{2-2\alpha}$.

Table 5.4. MAE and CO for example5.4. with $\gamma = 0.5, c^2 = 1, \alpha = 0.2$.

Δt	Δx	MAE	Convergence order (CO)
$\frac{1}{10}$	$\frac{1}{10}$	0.00026	-
$\frac{1}{20}$	$\frac{1}{20}$	0.000085	1.6130
$\frac{1}{40}$	$\frac{1}{40}$	0.000028	1.6020
$\frac{1}{80}$	$\frac{1}{80}$	0.0000092	1.6057

Table 5.5. MAE and CO for example 5.4. with $\gamma = 0.5, c^2 = 1, \alpha = 0.4$.

Δt	Δx	MAE	Convergence order (CO)
$\frac{1}{10}$	$\frac{1}{10}$	0.0013	-
$\frac{1}{20}$	$\frac{1}{20}$	0.00056	1.2150
$\frac{1}{40}$	$\frac{1}{40}$	0.00024	1.2224
$\frac{1}{80}$	$\frac{1}{80}$	0.0001	1.2630

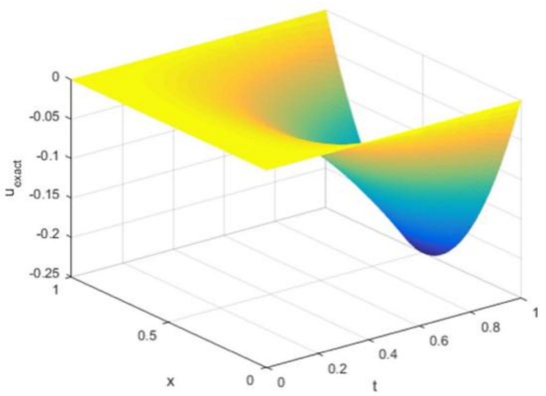
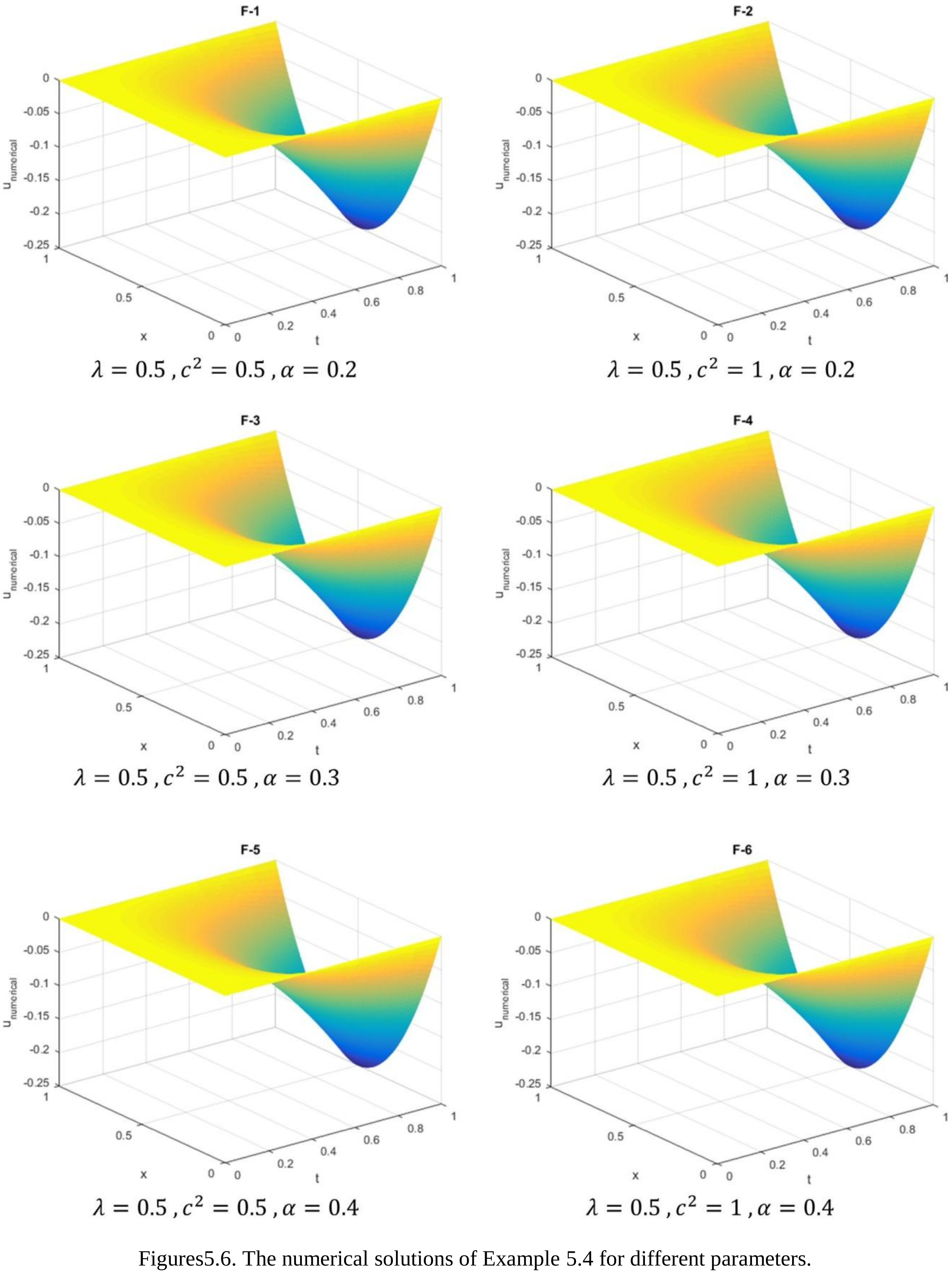


Figure5.5. The analytical solution of example 5.4.



Figures5.6. The numerical solutions of Example 5.4 for different parameters.

5.4 The results of [4]

Example 5.5. Consider the homogenous fractional telegraph equation:

$$ {}_xD_\beta^{(2)} f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) - \; {}_tD_\alpha^{(2)} f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) - \; {}_tD_\alpha f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) - f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) = 0, \quad (74)$$

with the conditions

$$\left.\begin{array}{l} f\left(0,\frac{t^\alpha}{\alpha}\right) = e^{-t^\alpha/\alpha}, f\left(\frac{x^\beta}{\beta},0\right) = e^{x^\beta/\beta} \\ {}_xD_\beta f\left(0,\frac{t^\alpha}{\alpha}\right) = e^{-t^\alpha/\alpha}, \quad {}_tD_\alpha f\left(\frac{x^\beta}{\beta},0\right) = -e^{x^\beta/\beta} \end{array}\right],(75)$$

where ${}_xD_\beta^{(2)}$ and ${}_tD_\alpha^{(2)}$ mean two times conformable fractional derivative of function $f(x^\beta/\beta, t^\alpha/\alpha)$.

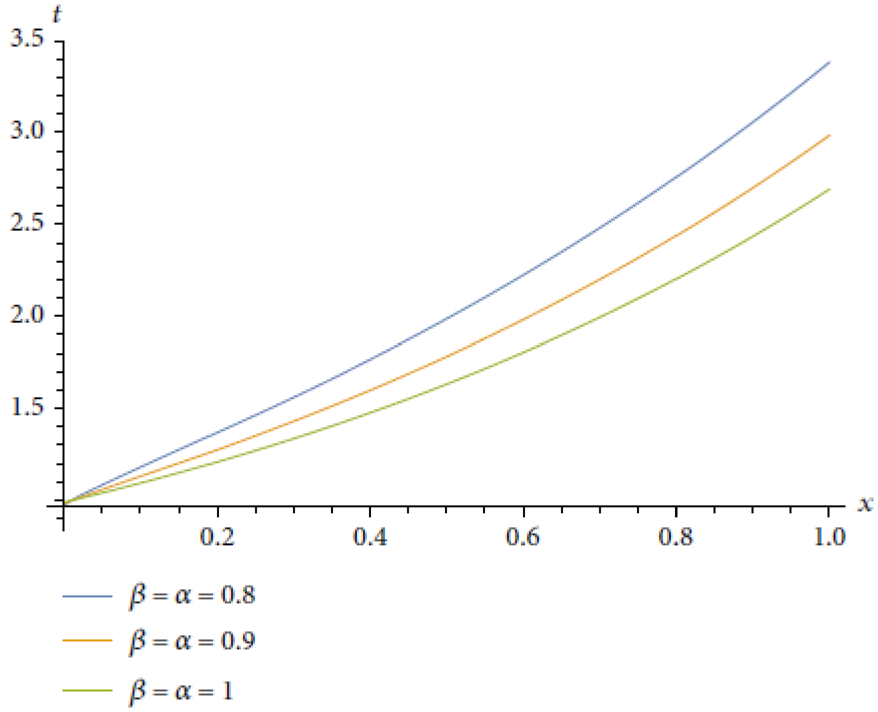


Figure 5.7: The figure shows the solution of (74), when $t = 0{:}01$, $0 \le x \le 1$.

Example 5.6. Regard nonhomogenous space-time fractional telegraph equation:

$${}_xD_\beta^{(2)} f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) - {}_tD_\alpha^{(2)} f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) - {}_tD_\alpha f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) - f\left(\frac{x^\beta}{\beta},\frac{t^\alpha}{\alpha}\right) = -2e^{x^\beta/\beta + t^\alpha/\alpha} \quad (76)$$

with the constraints

$$\left.\begin{array}{l} f\left(0,\frac{t^\alpha}{\alpha}\right) = e^{t^\alpha/\alpha}, f\left(\frac{x^\beta}{\beta},0\right) = e^{x^\beta/\beta} \\ {}_xD_\beta f\left(0,\frac{t^\alpha}{\alpha}\right) = e^{t^\alpha/\alpha}, \quad {}_tD_\alpha f\left(\frac{x^\beta}{\beta},0\right) = e^{x^\beta/\beta} \end{array}\right], \quad (77)$$

where ${}_xD_\beta^{(2)}$ and ${}_tD_\alpha^{(2)}$ mean two times conformable fractional derivative of function $f(x^\beta/\beta, t^\alpha/\alpha)$.

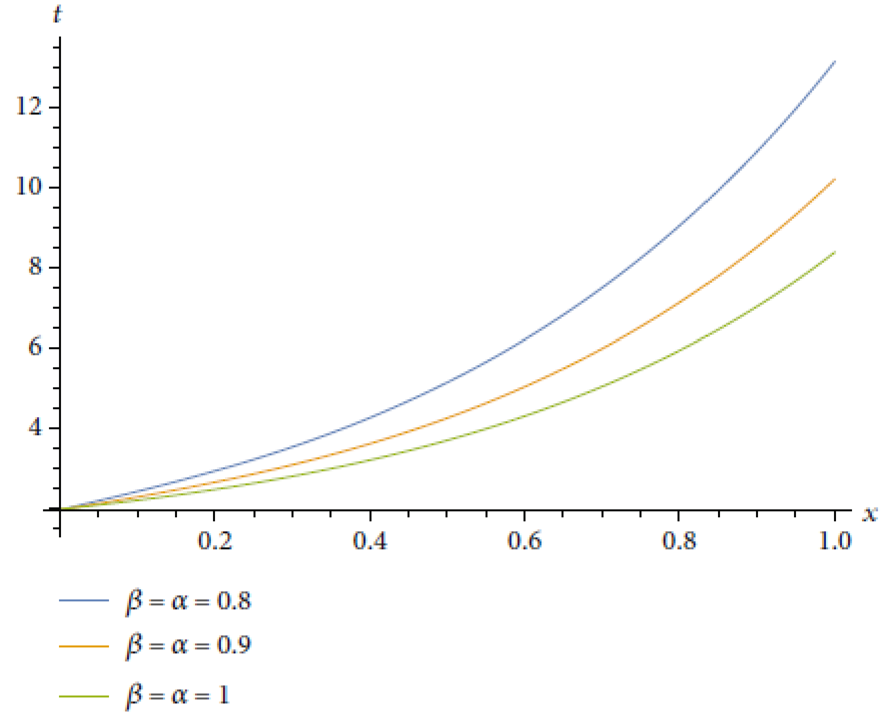


Figure 5.8: The figure shows the solution of (76), when $t = 0{:}01$, $0 \leq x \leq 1$.

5.5 The results of [5]

Example 5.7.

Consider the following linear space-fractional telegraph equation [22]

$\frac{\partial^{1.5}u}{\partial x^{1.5}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u \quad x > 0,$ (78)

Subject to the initial condition

$u(x, 0) = 0.$ (79)

and boundary conditions

$u(0.0125, t) \approx exp(-t)\,(1 + 0.0125) + \frac{0.0125^{1.5}}{\Gamma(5/2)} + \frac{0.0125^{2.5}}{\Gamma(7/2)} + \frac{0.0125^{3}}{\Gamma(4)} + \frac{0.0125^{4}}{\Gamma(5)} + \ldots\,,$ (80)

and

$u(1.0125, t) \approx exp(-t)\,(1 + 1.0125) + \frac{1.0125^{1.5}}{\Gamma(5/2)} + \frac{1.0125^{2.5}}{\Gamma(7/2)} + \frac{1.0125^{3}}{\Gamma(4)} + \frac{1.0125^{4}}{\Gamma(5)} + \ldots$.(81)

Then the exact solution is

$$u(x, t) \approx exp(-t)\left(1 + x + \frac{x^{1.5}}{\Gamma(5/2)} + \frac{x^{2.5}}{\Gamma(7/2)} + \frac{x^{3}}{\Gamma(4)} + \frac{x^{4}}{\Gamma(5)} + \ldots\right).$$

Table 5.6: Our numerical method for the time–space fractional order telegraph equation when k=0.000005, and h=0.025 and $\alpha = 1.5$

x	t=0.1	t=0.15
0.1	1.0139125288963764	0.9596371905024942
0.2	1.1480559941970572	1.0862170007301811
0.3	1.3004476445553682	1.2302168726205736

0.4	1.4710342588062175	1.3914776194134673
0.5	1.6607354930479643	1.5708357143835627
0.6	1.8709521529921287	1.7696035239597443
0.7	2.1034072173406435	1.9894042815310826
0.8	2.3600811054248516	2.2321069494266474
0.9	2.6431880681647740	2.4998194340024652
1.0	2.9649893224621964	2.8167740246016137

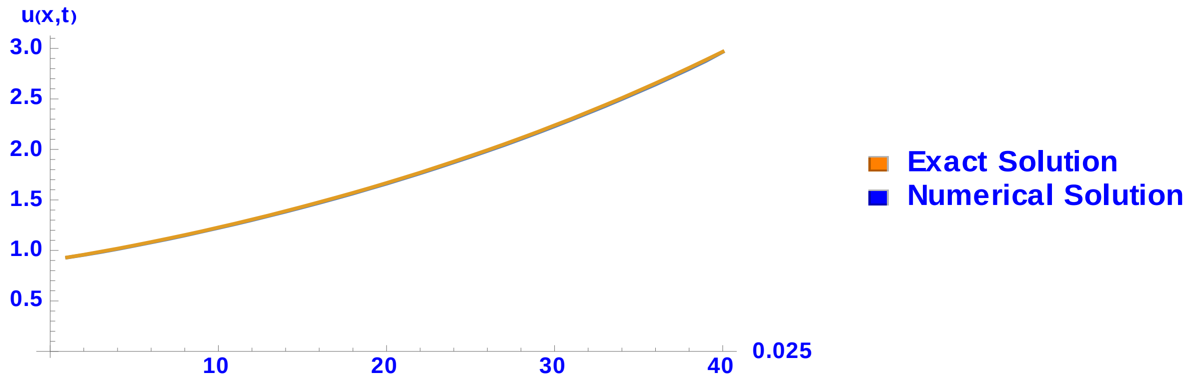


Figure. 5.9. Our numerical method for the time–space fractional order telegraph equation when t=0.1, k=0.000005, and h=0.025 and $\alpha = 1.5$

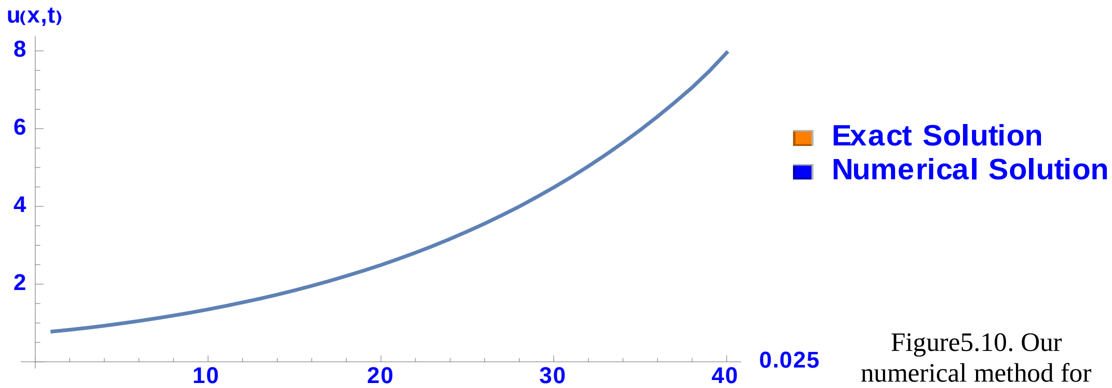


Figure5.10. Our numerical method for the time–space fractional order telegraph equationwhen t=0.15, k=0.000005, and h=0.025 and $\alpha = 1.5$

It is clear in Figure 5.9 the blue curve does not appear here and in the figure 5.10 the yellow curve does not appear also because the numerical results are very close to the exact results.

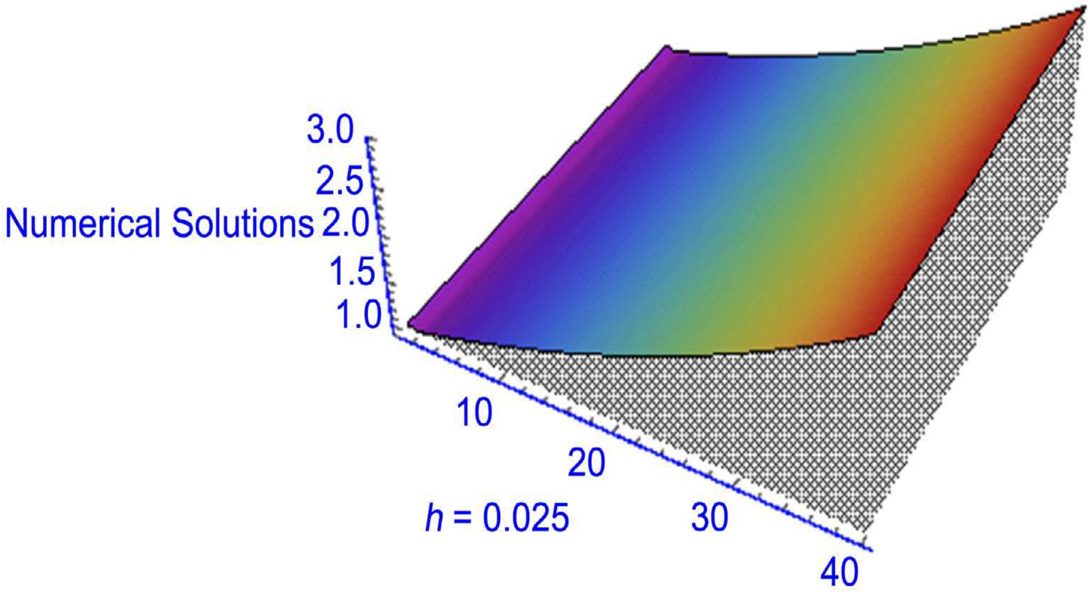


Figure 5.11. The 3-D behavior of the numerical solutions for the time–space fractional order telegraph equation from t=0.0005 to t=0.05, k=0.0005, and h=0.025, $\alpha = 1.75$

VI. Conclusion

Nowadays, the fractional derivative and fractional differential equation have many applications in various fields in physics, chemistry and other science. The fractional telegraph equation especially is applied into physics and engineering, and so on. In this article, we compare were the fractional Sumudu decomposition method (SDM), a double Sumudu matching transformation method, a finite difference scheme, a finite difference scheme based on a combination of the extended cubic B-splines (ExCuBs) method, and a quadratic spline functions with the solution of the linear space-fractional telegraph equation. We discussed the solutions and truncation error methods. In addition, the stability analysis of these methods was shown to be conditionally stable. Also, the numerical solutions are convergent. Furthermore, the obtained approximate numerical solutions maintain good accuracy compared with the exact solutions. Finally, we presented a large set of examples for these methods.

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