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Review Paper

Local Discontinuous Adaptive Finite Element Method for Steklov Eigenvalue Problems

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***ABSTRACT**: Using the flexibility of the finite element method to solve the solution problems on different shaped and natured elements, the local discontinuous Galerkin method can handle very complex boundary problems. Using the local discontinuous Galerkin method to perform a priori error estimation for the Steklov eigenvalue problem, we obtain a reasonable error estimation subspace, which can effectively solve the validity and reliability of the eigenfunction indicator subspace and the reliability of the eigenvalue error estimation indicator. We use precise numerical data obtained from MATLAB experiments as the basis for judging whether the conclusion is reasonable. Finally, combining theoretical analysis, we show that the method achieves optimal convergence order.*

KEYWORDS: *Steklov eigenvalue, hp Local discontinuous Galerkin metho, Self-adaptation, Error analysis*

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I. INTRODUCTION

Steklov eigenvalue problems have a wide range of applications in physics and engineering. The dynamics of isotropic elastic media is combined with some general conclusions to solve the free motion of particles or the constrained motion of particles [1]. The problem of determining the lower bound of the lowest frequency vibration of a rigid metal pendulum composed of a pendulum suspended on a steel wire is studied by integral equation method and composition method [2]. The lateral motion of the elastic string with mass at one end and the model of the transmission line tilting towards the circuit are investigated [3]. An approximate finite element analysis of structural vibration modes of coplanar incompressible fluids and a finite element analysis of numerical solutions of spectral problems in fluid-solid interactions are analyzed [4]. Fast Fourier-Galerkin method was adopted to solve Steklov's eigenvalue problem [5]. A finite element method for an effective 4-order Steklov eigenvalue problem over a spherical region is obtained by dimensionality reduction [6]. The local discontinuity Galerkin method was used to perform hp analysis on the convection diffusion equation to obtain the conclusion that the diameter of the partition element is optimal and the polynomial degree is suboptimal [7]. The discontinuous Galerkin method is used to perform prior and posterior estimates of Steklov eigenvalue problems to obtain the optimal convergence order [8]. The non-self-adjoint Steklov eigenvalue problem in the inverse scattering of the posterior error estimation and adaptive algorithm is discussed for the first time [9]. The improved prior error estimation and posterior error estimation of the inverse scattering eigenvalue problem are proved, and the reliability and efficiency of the posterior error estimation of the eigenfunction can reach higher order terms are proved, and the reliability of the eigenvalue estimator is analyzed [10].

II. BASIC THEORY PREOARATION

Set be a bounded domain with Lipshitz boundary $\partial\Omega$. and let n be the outward normal to $\partial\Omega$, consider the Steklov eigenvalue problem: Find $\lambda \in R$ and a nontrivial function $u \in H^1(\Omega)$, such that

$$\begin{cases} -\Delta u + u = 0, & in\ \Omega \\ \frac{\partial u}{\partial n} = \lambda u, & on\ \partial\Omega, \end{cases} \tag{2.1}$$

If $\mathbf{q} = \nabla u$, take the Green integral transformation of (2.1) to obtain the corresponding weak form, and define a continuous bilinear form $a(u, v)$, such that.

$$a(u,v) = (\nabla u, \nabla v) + (u,v), \forall u, v \in H^1(\Omega)\,,$$

where $(u,v) = \int_\Omega uvdx$, under these assumptions, there exist two positive constants unrelated to u , v two positive constants independent of γ and ρ, such that the bilinear form $a(\cdot,\cdot)$ satisfies

$$\begin{gathered} |a(u,v)| \leqslant \gamma \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \quad \forall u, v \in H^1(\Omega) \\ |a(v,v)| \geqslant \rho \|v\|_{1,\Omega}^2, \quad \forall v \in H^1(\Omega) \end{gathered} \tag{2.2}$$

The weak form of (2.1) is: Find $(\lambda, u) \in R \times H^1(\Omega)$, $u \neq 0$, such that the following equation is true,

$$a(u,v) = \lambda < u, v >, \forall v \in H^1(\Omega). \tag{2.3}$$

where $< u, v >= \int_{\partial\Omega} uvds$.

Let $\mathcal{T}_h = \{\kappa\}$ be a shape-regular mesh of Ω. The diameter of a face e (an edge when d=2) is denoted by h_e, the diameter of a cell $\kappa \in \mathcal{T}_h$ is denoted by h_κ, The set of faces of cells $\Gamma_h = \Gamma_h^i \cup \Gamma_h^b$ where Γ_h^idenotes the interior faces set and Γ_h^bdenotes the set of faces lying on the boundary $\partial\Omega$.
$p_\kappa \geqslant 1$ indicates the highest degree of polynomial in unit $\kappa \in \mathcal{T}_h$, where $\underline{p} = \{p_\kappa\}_{\kappa\in\mathcal{T}_h}$, the hp finite element space is defined as
$S^{\underline{p}}(\mathcal{T}_h) = \{u \in L^2(\Omega) : u|_\kappa \in S^{p_\kappa}(\kappa), \forall \kappa \in \mathcal{T}_h\}$

when the element κ is a triangle, $S^{p_\kappa}(\kappa)$ is the $p_\kappa$polynomial space $p^{p_K}(K)$ over κ. Introduce the space of piecewise H^s functions space of degree s:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_\kappa \in H^s(\kappa), \forall \kappa \in \mathcal{T}_h\}.$$

The auxiliary variable q = ∇u is introduced, then (2.1) can be rewritten as follows.

$$\begin{cases} -\nabla \cdot \mathbf{q} + u = 0, & in\ \Omega \\ \dfrac{\partial u}{\partial n} = \lambda u, & on\ \partial\Omega, \end{cases} \tag{2.4}$$

$V_h = S^{\underline{p}}(\mathcal{T}_h)$ and $Q_h = S^{\underline{p}}(\mathcal{T}_h)^2$ represent the hp local discontinuous finite element space, then the hp-ldg format of the approximation problem of (2.5), find $(\lambda_h, u_h) \in C \times S^{\underline{p}}(\mathcal{T}_h)$, $u_h \neq 0$, for all $\kappa \in \mathcal{T}_h$, $\forall v \in V_h$, $\mathbf{t} \in \mathbf{Q}_h$,such that

$$\int_\kappa \mathbf{q}_h \cdot \nabla v dx - \int_{\partial\kappa} \widehat{\mathbf{q}}_h \cdot \mathbf{n}_\kappa v ds + \int_\kappa u_h v dx = 0, \quad \forall\, v \in \mathrm{V}_h \tag{2.5}$$

$$\int_\kappa \mathbf{q}_h \cdot \mathbf{t} dx - \int_{\partial\kappa} \hat{\mathrm{u}}_h \cdot \mathbf{n}_\kappa v ds + \int_\kappa u_h \nabla \cdot \mathbf{t} dx = 0, \quad \forall\, \mathbf{t} \in \mathrm{Q}_h \tag{2.6}$$

where $v \in V_h$, n_κ is the unit normal vector of $\partial\kappa$, and $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$ are numerical fluxes, which are the approximations of the traces of $\mathbf{u}$ and $\mathbf{q}$ on $\partial\kappa$. Define the mean and hop of v on e:

$$\{\{v\}\} = \frac{1}{2}(v^+ + v^-) \qquad \{\{\mathbf{r}\}\} = \frac{1}{2}(\mathbf{r}^+ + \mathbf{r}^-)$$

$$[\![v]\!] = \frac{1}{2}(v^+\mathbf{n}_\kappa^+ + v^-\mathbf{n}_\kappa^-) \qquad [\![\mathbf{r}]\!] = \mathbf{r}^+\mathbf{n}_\kappa^+ + \mathbf{r}^-\mathbf{n}_\kappa^-$$

Where e is a surface consisting of two neighboring faces of κ^+ and κ^- common interior faces.v and $\mathbf{r}$ are smooth functions on $\kappa^\pm$ and $v^\pm$ and $\mathbf{r}^\pm$ are traces on the boundaries of $\partial\kappa^\pm$, defines the mean and jump of v and $\mathbf{r}$ on e, $v_+ = v|_{\kappa+}$, $v^- = v|_{\kappa^-}$, where $e = \partial\kappa^+ \cap \partial\kappa^-$, the n_κ is the outward normal vector from κ^+ to κ^-, then we have

$$\sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} v\mathbf{q}\cdot\mathbf{n}ds = \int_{\mathcal{E}} \{\{\mathbf{q}\}\}[\![v]\!]\,ds + \int_{\mathcal{E}_I} [\![\mathbf{q}]\!]\{\{v\}\}ds.$$

the definition of numerical flux of $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$ are as follows:

$$\hat{\mathbf{u}}|_e = \begin{cases} \{\{u\}\} + \eta[\![u]\!] & e\subset\Gamma_h^i \\ u & e\subset\Gamma_h^b \end{cases} \qquad \hat{\mathbf{q}}|_e = \begin{cases} \{\{\mathbf{q}\}\} - \alpha[\![u]\!] - \eta[\![\mathbf{q}]\!] & e\subset\Gamma_h^i \\ \mathbf{q} & e\subset\Gamma_h^b \end{cases}$$

where the parameters α and η are chosen appropriately, and to define the parameters, a function of h and p are introduced into the relative local unit size and approximation degree in $L^\infty(\mathcal{E})$, where $\eta = h^{-1}p^2$, such that $\|\ \eta\ \|_{\infty,\mathcal{E}_J} \leqslant \beta$, where $\alpha > 0$ and $\beta > 0$ are constants independent of the mesh size, the

$$h = h(x) = \begin{cases} min\{h_{\kappa^+}, h_{\kappa^-}\}, & x\in e_{\kappa^+\cap\kappa^-}, \\ h_\kappa & x\in e_{\kappa\cap\Omega}, \end{cases} \quad p = p(x) = \begin{cases} max\{p_{\kappa^+}, p_{\kappa^-}\}, & x\in e_{\kappa^+\cap\kappa^-}, \\ p_\kappa, & x\in e_{\kappa\cap\Omega}, \end{cases}$$

where $e_{\kappa^+\cap\kappa^-} = \mathrm{int}(\partial\kappa^+\cap\partial\kappa^-)$, $e_{\kappa\cap\Omega} = \mathrm{int}\,(\partial\kappa\cap\partial\Omega)$.
Define the lifting operator $\Phi(u)\in\mathbf{Q}_h$, $\mathbf{t}\in\mathbf{Q}_h$, $u\in V(h)+H^1(\Omega)$, such that

$$\int_\Omega \Phi(u)\mathbf{t}dx = \int_{\Gamma_h^i} [\![u]\!]\{\{\mathbf{t}\}\} - \eta[\![u]\!][\![\mathbf{t}]\!]ds, \tag{2.7}$$

Since $\mathbf{q} = \nabla u$, then

$$\int_\Omega \mathbf{q}\cdot\mathbf{t}dx = \int_\Omega \nabla_h u\mathbf{t}dx - \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}(u-\hat{u})\mathbf{t}\cdot n_\kappa ds$$

$$= \int_\Omega \nabla_h u\mathbf{t}dx - \int_{\Gamma_h^i} [\![u]\!]\{\{\mathbf{t}\}\} - \eta[\![u]\!][\![\mathbf{t}]\!]ds, \tag{2.8}$$

For the source problem, using Green's formula, (2.5) and the definition of numerical flux, $\mathbf{q}\in\mathbf{Q}_h$, which yields

$$\int_\Omega \mathbf{q}\cdot\nabla v dx + \int_\Omega uvdx - \int_{\Gamma_h^i} (\{\{\mathbf{q}\}\} - \alpha[\![u]\!] - \eta[\![\mathbf{q}]\!]\}\cdot [\![v]\!])ds = \int_{\Gamma_h^b} \lambda uvds, \tag{2.9}$$

By (2.7), (2.8), (2.9) and the definition of numerical flux, $u\in \mathrm{V}_h$, there are

$$\int_{\Omega}(\nabla_h u-\Phi(u))\nabla_h v dx-\int_{\Gamma_h^i}(\{\{\mathbf{q}\}\}-\alpha[\![u]\!]-\eta[\![\mathbf{q}]\!])[\![v]\!]ds+\int_{\Omega}uvdx$$

$$=\int_{\Omega}\big(\nabla_h u-\Phi(u)\big)\big(\nabla_h v-\Phi(v)\big)dx+\int_{\Gamma_h^i}\alpha[\![u]\!][\![v]\!]ds+\int_{\Omega}uvdx \qquad (2.10)$$

therefore

$$a_h(\mathrm{u},v_h):=\int_{\Omega}\big(\nabla_h u-\Phi(u)\big)\big(\nabla_h v-\Phi(v)\big)dx+\int_{\Gamma_h^i}\alpha[\![u]\!][\![v]\!]ds+\int_{\Omega}uvdx=\int_{\Gamma_h^b}\lambda uvdx \qquad (2.11)$$

The finite element approximation of (2.3) is given by: Find$(\lambda_h,u_h)\in C\times S^{\underline{p}}(\mathcal{T}_h)$ and $u_h\neq 0$, such that

$$a_h(u_{\ h},v_h)=\lambda_h<u_h,v_h>,\forall v_h\in S^h. \qquad (2.12)$$

The source problem of (2.3) is given by: Find $w\in H^1(\Omega)$, such that

$$a(w,v)=<f,v>,\quad \forall v\in H^1(\Omega). \qquad (2.13)$$

The local discontinuous finite element approximation of (2.12) is given by: Find $w_h\in V_h$, such that

$$a_h(w_h,v_h)=<f,v_h>,\quad \forall v_h\in V_h. \qquad (2.14)$$

Define the linear bounded operator$T: L^2(\Omega)\to H^1(\partial\Omega)$ satisfying

$$a(Tf,v)=<f,v>,\ \forall f\in L^2(\partial\Omega),\ v\in H^1(\Omega). \qquad (2.15)$$

Then the equivalent operator of (2.4) is the form

$$Tu=\frac{1}{\lambda}u. \qquad (2.16)$$

From (2.13), the corresponding discrete solution operator $T_h: L^2(\partial\Omega)\to V_h$ satisfies

$$a_h(T_hf,v)=<f,v>,\quad \forall f\in L^2(\partial\Omega),\ \forall v\in V_h. \qquad (2.17)$$

The equivalent operator form of (2.12) as follow:

$$T_hu_h=\frac{1}{\lambda_h}u_h. \qquad (2.18)$$

The dual problem of (2.4) is given by: Find$(\lambda^*,u^*)\in C\times H^1(\Omega)$ and $u^*\neq 0$, such that

$$a(v,u^*)=\lambda^*<v,u>,\quad \forall v\in H^1(\Omega). \qquad (2.19)$$

The source problem of (2.18) is given by: Find $w^*\in H^1(\Omega)$, such that

$$a(v,w^*)=<v,g>,\quad \forall v\in H^1(\Omega). \qquad (2.20)$$

Define the linear bounded operator $T^*: L^2(\Omega)\to H^1(\partial\Omega)$ such that

$$a(v,T^*g)=<v,g>,\quad \forall g\in L^2(\partial\Omega),\ v\in H^1(\Omega). \qquad (2.21)$$

The finite element approximation of (2.18) is given by: Find $(\lambda_h^*,u_h^*)\in C\times V_h$ and $u_h^*\neq 0$, such that

$$(v_h, w_h^*) =< v_h, g >, \quad \forall v_h \in V_h.$$

Then the equivalent operator of (2.18) is

$$T^* u^* = \frac{1}{\lambda^*} u^*. \tag{2.22}$$

The finite element approximation of (2.18) is given by:

$$a_h(v_h, u_h^*) = \lambda_h^* < v_h, u_h^* >, \quad \forall v_h \in V^h. \tag{2.23}$$

The local discontinuous finite element approximation of (2.19) is given by: Find $w_h^* \in V_h$, such that

$$a_h(v_h, w_h^*) =< v_h, g >, \quad \forall v_h \in V_h. \tag{2.24}$$

The sum space $V(h) = V_h + H^1(\Omega)$ is introduced which assigns a local discontinuous finite element norm, where the energy norm is:

$$\|v\|_h^2 = \sum_{\kappa \in \mathcal{T}_h} \left(\|\nabla_h v\|_{0,\kappa}^2 + \|v\|_{0,\kappa}^2 \right) + \sum_{e \in \Gamma_h^i} \left\| h^{-\frac{1}{2}} p[\![v]\!] \right\|_{0,\mathcal{E}_i}^2 . \tag{2.25}$$

Galerkin orthogonality is:

$$a_h(w - w_h, v_h) = 0, \forall v_h \in V_h, \tag{2.26}$$

$$a_h(v_h, w^* - w_h^*) = 0, \forall v_h \in V_h, \tag{2.27}$$

Continuity and coercivity of $a_h(u, v)$ as follow:

$$|a_h(u_h, v_h)| \lesssim \|u_h\|_h \|v_h\|_h, \forall u_h, v_h \in V(h), \tag{2.28}$$

$$\|u_h\|_h^2 \lesssim |a_h(u_h, u_h)|, \forall u_h \in V_h. \tag{2.29}$$

Lemma 2.1. Let w be a solution of equation (2.13), $w \in H^{1+r}(\Omega)(r < \frac{1}{2})$, $f \in L^2(\partial\Omega)$, the regularity estimate is as follows

$$\| w \|_{1+r} \leq c_\Omega \| f \|_{0,\partial\Omega}. \tag{2.30}$$

where ψ is the solution of $a_h(v, \psi) = (v, g)$, $\forall v \in H^1(\Omega)$, $g \in L^2(\Omega)$, exists $w \in H^{1+\beta}(\Omega)$ $(\beta > \frac{1}{2})$, we have

$$\| w \|_{1+\beta} \lesssim \| g \|_{0,\Omega}, \tag{2.31}$$

where $\psi^I \in H^{1+\beta}(\Omega)$ is the interpolating function of ψ on $\mathcal{T}_h$.
Lemma 2.2. Refer to Proposition 4.9[11], where $v \in H^{s_\kappa}(\kappa)$
$(s_\kappa \geqslant 1)$, then there exists $\Pi_{p_\kappa}^{h_\kappa} v \in S^{p_\kappa}, p_\kappa = 1,2,\cdots, (0 \leqslant m \leqslant s_\kappa)$ satisfying

$$\| v - \Pi_{p_\kappa}^{h_\kappa} v \|_{m,\kappa} \lesssim h_\kappa^{\min(p_\kappa+1, s_\kappa)-m} p_\kappa^{m-s_\kappa} \|v\|_{s_\kappa,\kappa}, \tag{2.32}$$

$$\| v-\Pi_{p_\kappa}^{h_\kappa} v \|_{0,\partial\kappa} \lesssim h_\kappa^{\min(p_\kappa+1,s_\kappa)-\frac{1}{2}} p_\kappa^{\frac{1}{2}-s_\kappa} \|v\|_{s_\kappa,\kappa}. \tag{2.33}$$

The global discontinuous interpolation operator is: Π_p^h: $H_0^1(\Omega)\to V_h$, such that $\Pi_p^h(u)|_\kappa=\Pi_{p_\kappa}^{h_\kappa}(u|_\kappa)$ for a vector-valued function $\mathrm{r}=(\mathrm{r}_1,\mathrm{r}_2,\cdots,\mathrm{r}_d)$, define $\Pi_p^h(\mathrm{r})|_\kappa=(\Pi_p^h\mathrm{r}_1,\Pi_p^h\mathrm{r}_2,\cdots,\Pi_p^h\mathrm{r}_d)$.

Lemma 2.3. Let w and w_h be the solutions of (2.13) and (2.14) respectively, $w|_\kappa\in H^{1+s}(\kappa)$, then there holds

$$\|w-w_h\|_h \lesssim \inf_{v_h\in V_h}\|w-v_h\|_h, \tag{2.34}$$

$$\|w-w_h\|_h \lesssim \sum_{\kappa\in\mathcal{T}_h}(h^{s_\kappa}\ \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}. \tag{2.35}$$

Proof. We first prove (2.34), using (2.29), $v\in S^h$, which yields

$$\| w_h-v \|_{l,h}^2 \leqslant a_h(w_h-v,w_h-v)$$

$$=a_h(w-v,w_h-v)+a_h(w_h,w_h-v)-a_h(w,w_h-v)$$

$$=a_h(w-v,w_h-v)+b_h(f,w_h-v)-a_h(w,w_h-v)$$

When $\| w_h-v \|_{l,h}\neq 0$, from lemma 3.2 [7], we can obtain

$$\| w_h-v \|_h \leqslant \| w-v \|_h+\frac{a_h(w,w_h-v)-b_h(f,w_h-v)}{\| w_h-v \|_h}$$

$$\leqslant \| w-v \|_h+\sum_{\kappa\in\mathcal{T}_h}(h^{s_\kappa+1}\ \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}, \tag{2.36}$$

using the triangle inequality, we get

$$\|w-w_h\|_h \leqslant \|w-v+v-w_h\|_h \leqslant \|w-v\|_h+\|v-w_h\|_h, \tag{2.37}$$

the proof of (2.34) can be obtained by combining (2.36) and (2.37) when h is small enough.

Next, we proof (2.35), from (2.25), let $E_h(w)=w-\Pi_p^h w$, we have

$$\| E_h(w) \|_h^2 \lesssim (\sum_{\kappa\in\mathcal{T}_h}\left(\|\nabla_h E_h(w)\|_{0,\kappa}^2+\|E_h(w)\|_{0,\kappa}^2\right)+\sum_{e\in\Gamma_h^i}\left\|h^{-\frac{1}{2}}[\![E_h(w)]\!]\right\|_{0,\mathcal{E}_i}^2)$$

$$\lesssim \sum_{\kappa\in\mathcal{T}_h}\left(\|\nabla_h E_h(w)\|_{0,\kappa}^2+\|E_h(w)\|_{0,\kappa}^2\right)+\sum_{\kappa\in\mathcal{T}_h}(\sum_{e\in\Gamma_h^i}\left\|h^{-\frac{1}{2}}[\![\ E_h(w)]\!]\right\|_{0,e}^2)$$

$$\lesssim \sum_{\kappa\in\mathcal{T}_h}(h_\kappa^{s_\kappa} \parallel w \parallel_{1+s_\kappa,\kappa})^2 + \sum_{\kappa\in\mathcal{T}_h}(h^s \parallel w \parallel_{1+s_\kappa,\kappa})^2$$

$$\lesssim \sum_{\kappa\in\mathcal{T}_h}(h^s \parallel w \parallel_{1+s_\kappa,\kappa})^2, \tag{2.38}$$

from (2.38), we have

$$\parallel w - \Pi_p^h w \parallel_h \lesssim (\sum_{\kappa\in\mathcal{T}_h} (h^s \parallel w \parallel_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}, \tag{2.39}$$

Using error estimation and interpolation error estimation

$$\inf_{v_h\in V_h} \|w - v_h\| \lesssim \parallel w - \Pi_p^h w \parallel, \tag{2.40}$$

(2.35) can be proved by (2.34), (2.39) and (2.40).

Theorem 2.1. If w and w_h are the solutions of (2.13) and (2.14) respectively and $w\mid_\kappa \in H^{1+s_\kappa}(\kappa)(s_\kappa > \frac{1}{2})$, then there holds

$$\parallel w - w_h \parallel_{0,\Omega} \lesssim h^\beta \parallel w - w_h \parallel_h, \tag{2.41}$$

$$\parallel w - w_h \parallel_{0,\Omega} \lesssim (\sum_{\kappa\in\mathcal{T}_h} (h^{s_\kappa+r} \parallel w \parallel_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}. \tag{2.42}$$

Proof. We first prove (2.41), consider the dual problem of the source problem of (2.1) $a(v,w^*) =< v,g >$, for $g \in L^2(\Omega)$, using the consistency, (2.27) and (2.32), we obtain

$$< w - w_h, g >= a_h(w - w_h, w^*)$$

$$= a_h(w - w_h, w^* - w_h^*)$$

$$\lesssim \parallel w - w_h \parallel_h \parallel w^* - w_h^* \parallel_h. \tag{2.43}$$

Using (2.35) and regularity, let $g = w - w_h$, we have

$$\|w^* - w_h^*\|_h \lesssim h^r \|w^*\|_{1+r,\Omega} \lesssim h^r \|g\|_{0,\Omega}. \tag{2.44}$$

From (2.43) and (2.35)

$$\|w - w_h\|_{0,\Omega} \lesssim \sup_{g\in L^2(\Omega)} \frac{\mid w - w_h, g \mid}{\parallel g \parallel_{0,\Omega}}$$

$$\lesssim \frac{\| w - w_h \|_h, \| w^* - w_h^* \|_h}{\| g \|_{0,\Omega}}$$

$$\lesssim h^r \| w - w_h \|_h. \tag{2.45}$$

therefore, (2.41) is proved.
Next prove (2.42), from (2.53) and (2.41)

$$\| w - w_h \|_{0,\Omega} \lesssim h^r \| w - w_h \|_h \lesssim (\sum_{\kappa\in\mathcal{T}_h} (h^{s_\kappa + r} \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}.$$

the proof is completed.

Theorem 2.2. Let w and w_h are the solutions of (2.13) and (2.14) respectively, $w|_\kappa \in H^{1+s_\kappa}(\kappa)(0 < s_\kappa < \frac{1}{2})$, then there holds

$$\|w - w_h\|_{0,\partial\Omega} \lesssim h^r \|w - w_h\|_h \quad s > \frac{1}{2}, \tag{2.46}$$

$$\| w - w_h \|_{0,\partial\Omega} \lesssim h^{r+s} \| f \|_{0,\partial\Omega}\ r \le s < \frac{1}{2}. \tag{2.47}$$

Proof. We first prove (2.46), considering the dual equation of (2.20), for any fixed $f, g \in L^2(\partial\Omega)$, using regularity and (2.28), we obtain

$$\langle g, w - w_h \rangle = a_h(w - w_h, w^*) = a_h(w - w_h, w^* - w^{*I})$$

$$= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \big(\nabla(w - w_h)\nabla(w^* - w^{*I}) + (w - w_h)(w^* - w^{*I})\big)dx$$

$$+ \sum_{e\in\Gamma_h^i} (\int_e - ([\![w - w_h]\!] \{\{\nabla(w^* - w^{*I})\}\} + \{\{\nabla(w - w_h)\}\}[\![w^* - w^{*I}]\!])ds$$

$$+\eta \int_e [\![w - w_h]\!][\![\nabla(w^* - w^{*I})]\!]$$

$$+[\![\nabla(w - w_h)]\!][\![w^* - w^{*I}]\!]ds + \int_e \alpha[\![w - w_h]\!][\![w^* - w^{*I}]\!])ds$$

$$\lesssim I_1 + I_2 + I_3 \tag{2.48}$$

where

$$I_1 = \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \nabla(w - w_h)\nabla(w^* - w^{*I}) + (w - w_h)(w^* - w^{*I})dx$$

$$+\sum_{e\in\Gamma_h^i}\int_e \alpha[\![w-w_h]\!][\![w^*-w^{*I}]\!])ds$$

$$I_2=\sum_{e\in\Gamma_h^i}\int_e ([\![w-w_h]\!]\{\{\nabla(w^*-w^{*I})\}\}+\{\{\nabla(w-w_h)\}\}[\![w^*-w^{*I}]\!])ds$$

$$I_3=\sum_{e\in\Gamma_h^i}\eta\int_e [\![w-w_h]\!][\![\nabla(w^*-w^{*I})]\!]+[\![\nabla(w-w_h)]\!][\![w^*-w^{*I}]\!]ds$$

When $s>\frac{1}{2}$, for I_1, using (2.35), let $s_\kappa=r$, we get

$$I_1\lesssim\| w-w_h\|_h\| w^*-w^{*I}\|_h\lesssim h^r\ \| w^*\|_{1+\mathrm{r}}\ \|w-w_h\|_h\ \| w^*\|_{1+\mathrm{r}} \tag{2.49}$$

Using inverse estimation, (2.39) and (2.41), there are

$$I_2\lesssim\sum_{\kappa\in\mathcal{T}_h}(\|\ [\![w-w_h]\!]\ \|_{\frac{1}{2}-r,e}\|\ \{\{\nabla(w^*-w^{*I})\}\}\ \|_{r-\frac{1}{2},e}$$

$$+\eta\ \|\ [\![w-w_h]\!]\ \|_{\frac{1}{2}-r,e}\|\ [\![\nabla(w^*-w^{*I}\ )]\!]\ \|_{r-\frac{1}{2},e}$$

$$\lesssim\sum_{\kappa\in\mathcal{T}_h} h^r\ \|\ h^{-\frac{1}{2}}[\![w-w_h]\!]\ \|_{0,e}\|\ w^*\ \|_{1+r} \tag{2.50}$$

Similarly

$$I_3\lesssim\sum_{\kappa\in\mathcal{T}_h}\eta\ \|\ [\![w-w_h]\!]\ \|_{\frac{1}{2}-r,e}\|\ [\![\nabla(w^*-w^{*I}\ )]\!]\ \|_{r-\frac{1}{2},e}$$

$$\lesssim\sum_{e\in\Gamma_h^i}\left(\eta h^{-\frac{1}{2}}p\ \|\ [\![(w-w_h)]\!]\ \|_{0,e}\right)h^r\ \|\ w^*\ \|_{1+r} \tag{2.51}$$

Substituting (2.49), (2.50) and (2.51) into (2.48), the proof of (2.45) is completed.
When $0<s<\frac{1}{2}$, let $s_\kappa=r$, using (2.35) and the regularity

$$I_1\lesssim\| w-w_h\|_h\| w^*-w^{*I}\|_h\lesssim h^r\ \| w^*\|_{1+\mathrm{r}}\ \|w-w_h\|_h \tag{2.52}$$

Using inverse estimates, (2.39), (2.41) and (2.35)

$$I_2=\sum_{e\in\Gamma_h^i}\|\ [\![w-w_h]\!]\ \|_{\frac{1}{2}-r,e}\|\ \{\{\nabla(w^*-w^{*I})\}\}\ \|_{r-\frac{1}{2},e}$$

$$\lesssim h^{r+s}\ \|\ f\ \|_{0,\partial\Omega}\|\ g\ \|_{0,\partial\Omega} \tag{2.53}$$

Similarly

$$I_3 = \sum_{e\in\Gamma_h^i} \eta \parallel [\![w - w_h]\!] \parallel_{\frac{1}{2}-r,e} \parallel [\![\nabla(w^* - w^{*I}\ )]\!] \parallel_{r-\frac{1}{2},e}$$

$$\lesssim \eta h^{r+s} \parallel f \parallel_{0,\partial\Omega} \parallel g \parallel_{0,\partial\Omega} \tag{2.54}$$

Combining the above three formula, we get

$$\parallel w - w_h \parallel_{0,\partial\Omega} \lesssim h^{r+s} \parallel f \parallel_{0,\partial\Omega} \tag{2.55}$$

From (2.35) and the regularity, we have

$$\begin{aligned} \|A_h f\|_h &\lesssim \|A_h f - Af\|_h + \|Af\|_h \\ &\lesssim \|A_h f - Af\|_h + \|Af\|_1 \\ &\lesssim h^r \|Af\|_{1+r} + \|Af\|_1 \\ &\lesssim \|f\|_{0,\partial\Omega} \end{aligned} \tag{2.56}$$

Theorem 2.3 Suppose that M(λ) ⊂ H^(1+r) (Ω) (s > 1/2), using the results from [9], the following inequality holds:

$$|\hat{\lambda}_h - \lambda| \lesssim h^{2\tau} \tag{2.57}$$

$$|\lambda_h - \lambda| \lesssim h^{2\tau/\alpha} \tag{2.58}$$

Let $u_h \in M_h(\lambda)$ be a direct sum of the generalized eigenvector spaces in (2.12), then there exists a characteristic function u of (2.3) such that

$$\|u - u_h\|_{0,\partial\Omega} \lesssim h^{(\tau+r)/\alpha} \tag{2.59}$$

$$\|u - u_h\|_h \lesssim h^{\tau} + h^{(\tau+r)/\alpha} \tag{2.60}$$

$$\|u - u_h\|_0 \lesssim h^r \parallel u - u_h \parallel_h + \|\lambda u - \lambda_h u_h\|_{0,\partial\Omega} \tag{2.61}$$

If we set $\alpha = 1$, then

$$\|u - u_h\|_{0,\partial\Omega} \lesssim h\|u - u_h\|_h. \tag{2.62}$$

Diagram 1

III. POSTERIORI ERROR ANAIYSIS

3.1 Estimators of Eigenfunctions and Their Reliability

Let (λ_h, u_h) be an eigenpair of (2.12). On each element $\kappa \in \mathcal{T}_h$ and each edge $e \in \mathcal{T}_h$, the element residual and the face residual are defined as follows, respectively.
$\mathcal{R}_\kappa = -\Delta u_h + u_h$,

$$\mathcal{J}_{F,1} = \begin{cases} [\![\nabla u_h]\!], & \forall e \in \Gamma_h^i, \\ \lambda_h u_h - \dfrac{\partial u_h}{\partial \mathbf{n}}, & \forall e \in \Gamma_h^b, \end{cases}$$
$$\mathcal{J}_{F,2} = [\![u_h]\!],\ \forall e \in \Gamma_h^i.$$

Define the local error indicator on each element of $\kappa \in \mathcal{T}_h$

$$\eta_\kappa^2 = h_\kappa^2 \|-\Delta u_h + u_h\|_{0,\kappa}^2 + \sum_{e\in\mathcal{E}_I} h_e \|\mathcal{J}_{F,1}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_D} h_e \|\mathcal{J}_{F,1}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_D} \alpha h_e^{-1} \|\mathcal{J}_{F,2}\|_{0,e}^2. \tag{3.1}$$

The global error indicator is as follow:

$$\eta(u_h) = (\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa^2)^{1/2}. \tag{3.2}$$

In the following, we prove the reliability of the error estimator.
Theorem 3.1 Set(λ, u) and (λ_h, u_h) are the eigenpair of (2.3) and (2.12), respectively, $v \in H_0^1(\Omega)$ (r>1/2), then for any v∈H_0^1 (Ω), the following equation holds

$$\| u - u_h \|_h \lesssim \sup_{v\in H^1(\Omega))} \frac{|(\lambda_h u_h, v) - a_h(u_h, v)|}{\| v \|_h} + \inf_{v\in H_0^1(\Omega)} \| u_h - v \|_h + \|\lambda u - \lambda_h u_h\|_{0,\partial\Omega}. \tag{3.3}$$

Proof: Note that $a(u,v) = a_h(u,v)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$. Let $w \in H^1(\Omega)$ be derived from the ellipticity and continuity of bilinear form

$$\begin{aligned}
\|u-w\|_h^2 &\lesssim |a_h(u-w, u-w)| \lesssim |a_h(u, u-w) - a_h(w, u-w)| \\
&\lesssim |\lambda < u, u-w > - a_h(w, u-w)| \\
&\lesssim |\lambda < u, u-w > - a_h(w, u-w)| \\
&\lesssim |(\lambda < u, u-w >) - a_h(w + u_h - u_h, u-w)| \\
&\lesssim |(< \lambda u + \lambda_h u_h - \lambda_h u_h, u-w >) - a_h(u_h, u-w)| + |a_h(u_h - w, u-w)| \\
&\lesssim |(\lambda_h u_h, u-w) - a_h(u_h, u-w)| + \|\lambda u - \lambda_h u_h\|_{0,\partial\Omega} \|u-w\|_{0,\partial\Omega} + \|u_h - w\|_h \|u-w\|_h,
\end{aligned} \tag{3.4}$$

If we take $v = u - w$, we get

$$\begin{aligned}
\|u-w\|_h &\lesssim \sup_{v\in H^1(\Omega)} \frac{|(\lambda_h u_h, u-w) - a_h(u_h, u-w)|}{\|u-w\|_h} + \|\lambda u - \lambda_h u_h\|_{0,\partial\Omega} + \|u_h - w\|_h \\
&\lesssim \sup_{v\in H^1(\Omega)} \frac{|(\lambda_h u_h, v) - a_h(u_h, v)|}{\|v\|_h} + \|\lambda u - \lambda_h u_h\|_{0,\partial\Omega} + \|u_h - w\|_h,
\end{aligned} \tag{3.5}$$

By the triangle inequality, we get

$$\begin{aligned}
\|u-u_h\|_h &= \|u - w - u_h + w\|_h \lesssim \|u-w\|_h + \|u_h - w\|_h \\
&\lesssim \sup_{v\in H^1(\Omega)} \frac{|(\lambda u, v) - a_h(u_h, v)|}{\|v\|_h} + \|\lambda u - \lambda_h u_h\|_{0,\partial\Omega} + \|u_h - w\|_h,
\end{aligned} \tag{3.6}$$

By the arbitrariness of w, the theorem holds.

Lemma 3.1 Error! Reference source not found.Error! Reference source not found. For any $\varphi \in H^1(\Omega)$, there is a fragment linear interpolation $I^h\varphi \in V_h$ satisfied

$$\|\varphi - I^h\varphi\|_{0,\kappa} + h_\kappa \|\nabla(\varphi - I^h\varphi)\|_{0,\kappa} \lesssim h_\kappa \|\nabla\varphi\|_{0,U_\kappa}, \forall \kappa \in \mathcal{T}_h \tag{3.7}$$

$$\|\varphi - I^h\varphi\|_{0,e} \lesssim h_e^{\frac{1}{2}} \|\nabla\varphi\|_{0,U_e}, \forall e \in \Gamma_h, \tag{3.8}$$

where U_κ is the union of all elements that share at least one node with κ, and U_e is the union of an edge that shares at least one node with edge e.

Theorem 3.2 Set (λ, u) and (λ_h, u_h) are the eigenvalue of (2.4) and (2.12) on, for any $v \in H_0^1(\Omega)$, was established

$$\|u - u_h\|_h \lesssim \eta(u_h) + \|\lambda u - \lambda_h u_h\|_{0,\Omega}. \tag{3.9}$$

Proof: From the interpolation property, we get$[\![v - I^h v]\!] = 0$, which can be obtained by using Green's formula

$$\begin{aligned}
\mathrm{S} &=< \lambda u, v - I^h v > - a_h(u_h, v - I^h v) \\
&= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa (\nabla u_h \nabla v_h + u_h v_h)dx + \sum_{e\in\Gamma_h^i} (\int_e - ([\![u_h]\!]\{\{\nabla v_h\}\} + \{\{\nabla u_h\}\}[\![v_h]\!])ds \\
&+\eta \int_e [\![u_h]\!][\![\nabla v]\!] + [\![\nabla u_h]\!][\![v_h]\!]ds + \int_e \alpha [\![u_h]\!][\![v_h]\!])ds \\
&= \int_{\partial\Omega} \lambda u(v - I^h v)dx - \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \left(\nabla u_h \nabla(v - I^h v) + u_h(v - I^h v)\right)dx \\
&- \sum_{e\in\Gamma_h^i} (- \int_e [\![u_h]\!]\{\{\nabla(v - I^h v\}\}ds + \int_e \{\{\nabla u_h\}\}[\![v - I^h v]\!]ds \\
&+\eta \int_e [\![u_h]\!][\![\nabla(v - I^h v)]\!]ds + \int_e [\![\nabla u_h]\!][\![v - I^h v]\!]ds + \int_e \alpha[\![u_h]\!][\![v - I^h v]\!])ds) \\
&= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa (\Delta u_h - u_h)(v - I^h v)dx + \sum_{e\in\Gamma_h^i} \int_e (\{\{\nabla(v - I^h v)\}\} - \eta[\![\nabla(v - I^h v)]\!])[\![u_h]\!]ds \\
&+ \int_{\partial\Omega} \lambda u(v - I^h v)dx - \int_{\partial\Omega} \frac{\partial u_h}{\partial \mathbf{n}} \cdot (v - I^h v)dx \\
&\equiv \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3,
\end{aligned} \tag{3.10}$$

From the Cauchy-Swartz inequality, equation (3.7) and equation (3.8), there are

$$\begin{aligned}
|\mathrm{S}_1| &\lesssim (\sum_{\kappa\in\mathcal{T}_h} h_\kappa^2 \|\Delta u_h - u_h\|_{0,\kappa}^2)^{\frac{1}{2}} (\sum_{\kappa\in\mathcal{T}_h} \|\nabla v\|_{0,\omega_\kappa}^2)^{\frac{1}{2}} \\
&\lesssim (\sum_{\kappa\in\mathcal{T}_h} h_\kappa^2 \|\Delta u_h - u_h\|_{0,\kappa}^2)^{\frac{1}{2}} \|v\|_h,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
|\mathrm{S}_2| &= \left| \sum_{e\in\Gamma_h^i} \int_e \Phi(u_h)\nabla(v - I^h v)\, ds \right| \lesssim (\sum_{e\in\Gamma_h^i} \left\| h_e^{-\frac{1}{2}} [\![u_h]\!] \right\|_{0,e}^2)^{\frac{1}{2}} (\sum_{\kappa\in\mathcal{T}_h} \|\nabla v\|_{0,\omega_\kappa}^2)^{\frac{1}{2}} \\
&\lesssim (\sum_{e\in\Gamma_h^i} h_e^{-1} \|[\![u_h]\!]\|_{0,e}^2)^{\frac{1}{2}} \|v\|_h,
\end{aligned} \tag{3.12}$$

$$|S_3| = \left| \int_{\partial\Omega} (\lambda u - \frac{\partial u_h}{\partial \mathbf{n}})(v - I^h v) dx \lesssim (\sum_{e\in\Gamma_h^b} h_e \left\| \lambda u - \frac{\partial u_h}{\partial \mathbf{n}} \right\|_{0,e}^2)^{\frac{1}{2}} (\sum_{e\in\Gamma_h^b} h_e^{-1} \| v - I^h v \|_{0,e}^2)^{\frac{1}{2}} \right.$$
$$\lesssim ((\sum_{e\in\Gamma_h^b} h_e \|\lambda u - \lambda_h u_h\|_{0,e}^2)^{\frac{1}{2}} + \left(\sum_{e\in\Gamma_h^b} h_e \left\| \lambda_h u_h - \frac{\partial u_h}{\partial \mathbf{n}} \right\|_{0,e}^2)^{\frac{1}{2}} \right) \|v\|_h, \tag{3.13}$$

Combining (3.11), (3.12), (3.13) we have

$$|S| \lesssim (\eta_\kappa + h^{\frac{1}{2}} \| \lambda u - \lambda_h u_h \|_{0,\partial\Omega}) \|v\|_h. \tag{3.14}$$

And have

$$\begin{aligned} < \lambda_h u_h, v > - a_h(u_h, v) &= < \lambda_h u_h, v - I^h v > - a_h(u_h, v - I^h v) \\ &= < \lambda u, v - I^h v > - a_h(u_h, v - I^h v) + < \lambda_h u_h - \lambda u, v - I^h v > \\ &\lesssim (\eta_\kappa + h^{\frac{1}{2}} \| \lambda u - \lambda_h u_h \|_{0,\partial\Omega}) \|v\|_h + \| \lambda_h u_h - \lambda u \|_{0,\partial\Omega} h^{\frac{1}{2}} \|v\|_h, \end{aligned} \tag{3.15}$$

For any $v \in V_h$, there is a rich operator $E_h: V_h \to V_h \cap H_0^1(\Omega)$ makes [14, 15]

$$\sum_{\kappa\in\mathcal{J}_h} \left(h_\kappa^{-2} \|v - E_h v\|_{0,\kappa}^2 + \|\nabla(v - E_h v)\|_{0,\kappa}^2 \right) \lesssim \sum_{e\in\mathcal{E}_I} h_e^{-1} \| [\![v]\!] \|_{0,e}^2. \tag{3.16}$$

Using (3.3) on the right side of the second (2.27) and (4.15), and pay attention to the $[[E_h u_h]] = 0$, there is

$$\inf_{v\in H^1(\Omega)} \|u_h - v\|_h^2 \lesssim \|E_h u_h - u_h\|_h^2$$
$$= \sum_{\kappa\in\mathcal{J}_h} \left(\|\nabla(E_h u_h - u_h)\|_{0,\kappa}^2 + \|(E_h u_h - u_h)\|_{0,\kappa}^2 \right) + \sum_{e\in\mathcal{E}_I} \alpha h^{-1} \| [\![E_h u_h - u_h]\!] \|_{0,e}^2$$
$$\lesssim \sum_{e\in\mathcal{E}_I} \alpha h_e^{-1} \| [\![u_h]\!] \|_{0,e}^2, \tag{3.17}$$

If (3.3), (3.14) is carried into (3.17), the proof is complete.
By theorem 2.3, when the gradient $\alpha = 1$, we know $\|\lambda u - \lambda_h u_h\|_{0,\Omega}$, and $\|u - u_h\|_{0,\Omega}$ are $\|u - u_h\|_G$ high order small amount, so (3.9) tell us the error estimation indicates $\eta(u_h)$ is one of the upper bound of the local discontinuous finite element energy norm, so the error estimate is reliable.

3.2 THE EFFECTIVENESS OF EIGENFUNCTION ESTIMATOR

To ensure that our estimation method is valid for actual adaptive improvements, our next goal is to show that the local error estimation indicator η_κ provides a local lower bound on the error on κ. By marking $b_\kappa \in H_0^1(\kappa)$ as the standard unit bubble function, $b_e \in H_0^1(U_e)$ as the bubble function on the surface, where U_e is the union of two units κ^+ and κ^- sharing e, we introduce and introduce the following knowledge by using the bubble function technique developed by $Verf\ddot{u}rth$Error! Reference source not found..

Lemma 3.2 For all polynomial functions $v \in P_k(\kappa)$,

$$\|v\|_{0,\kappa} \lesssim \left\| b_\kappa^{1/2} v \right\|_{0,\kappa}, \tag{3.18}$$

For all polynomial functions $w \in P_k(e)$, We have

$$\|w\|_{0,e} \lesssim \left\| b_e^{1/2} w \right\|_{0,e}, \tag{3.19}$$

For each $b_e w$, be extended W_b meet $W_b|_e = b_e w, W_b \in H_0^1(U_e)$

$$\|W_b\|_{0,w_e} \lesssim h_e^{\frac{1}{2}}\|w\|_{0,e}, \tag{3.20}$$

$$\|\nabla W_b\|_{0,w_e} \lesssim h_e^{-1/2}\|w\|_{0,e}. \tag{3.21}$$

According to the above lemma, and using the standard parameters (see Lemma 3.13 in reference [18]), we can show that there are local lower bounds.

Lemma 3.3 Set $(\lambda, u)$and (λ_h, u_h) were (2.4) and (2.12) for jth a an eigenpair, and then we have the following partial lower bound:

(i) For any $\kappa \in \mathcal{T}_h$,

$$h_\kappa\|\Delta u_h - u_h\|_{0,\kappa} \lesssim \|\nabla(u-u_h)\|_{0,\kappa} + h_\kappa\|u-u_h\|_{0,\kappa}.$$

(ii)Set $e \in \Gamma_h^i$,we have

$$h_e^{1/2}\|\mathcal{J}_{F,1}\|_{0,e} \lesssim \|\nabla(u-u_h)\|_{0,\kappa} + h_\kappa\|u-u_h\|_{0,\kappa}.$$

(iii) For each side $e \in \Gamma_h^b$,

$$h_e^{\frac{1}{2}}\|\mathcal{J}_{F,1}\|_{0,e}^2 = \|\nabla(u-u_h)\|_{0,\kappa} + h_\kappa\|u-u_h\|_{0,\kappa} + h_e^{\frac{1}{2}}\|\lambda_h u_h - \lambda u\|_{0,e}.$$

(iv) For each side $e \in \Gamma_h^i$,

$$h_e^{-1}\|\mathcal{J}_{F,2}\|_{0,e}^2 = h_e^{-1}\|[\![u_h]\!]\|_{0,e}^2 \lesssim h_e^{-1}\|u-u_h\|_{0,e}^2.$$

Proof： (i) Set $v_h = \Delta u_h - u_h$ and $v_b = b_\kappa v_h$. Note that $\Delta u - u = 0$ in $L^2(\kappa)$,and $v_b = 0$ on $\partial\kappa$, Using integration by parts, we have

$$\begin{aligned}\left\|b_\kappa^{\frac{1}{2}}v_h\right\|_{0,\kappa}^2 &= \int_\kappa b_\kappa(\Delta u_h - u_h)(\Delta u_h - u_h)dx = \int_\kappa (\Delta u_h - \Delta u + u - u_h)v_b dx \\ &= \int_\kappa \Delta(u_h - u)v_b dx + \int_\kappa (u-u_h)v_b dx \\ &= \int_\kappa \nabla(u-u_h)\nabla v_b dx + \int_{\partial k}\frac{\partial(u_h-u)}{\partial n}v_b + \int_\kappa (u-u_h)v_b dx \\ &= \int_\kappa \nabla(u-u_h)\nabla v_b dx + \int_\kappa (u-u_h)v_b dx,\end{aligned} \tag{3.22}$$

Using formula (3.18) and Cauchy-Swartz inequality, the inverse estimation can be obtained
$h_\kappa\|v_h\|_{0,\kappa} \lesssim \|\nabla(u-u_h)\|_{0,\kappa} + h_\kappa\|u-u_h\|_{0,\kappa}$

Then the proof of (i) is completed.

(ii)For any $e \in \Gamma_h^i$, set $w_h = [\![\nabla u_h]\!]$,$w_b = b_e w_h$, and $W_b \in H_0^1(U_e)$ is content to the extension of (3.20) and (3.21). Notice that W_b, using Green's formula, there is

$$\begin{aligned}\left\|b_e^{\frac{1}{2}}w_h\right\|_{0,e}^2 &= \int_e [\![\nabla \mathrm{u}]\!]\, w_b ds = \int_e ([\![\nabla u_h]\!] - [\![\nabla \mathrm{u}]\!])w_b ds \\ &= \int_e [\![\nabla(u_h-u)]\!]w_b ds = \int_e \frac{\partial(u_h-u)}{\partial n}w_b ds\end{aligned}$$

$$= \sum_{\kappa\in U_e} (\int_\kappa \Delta(u_h - \mathrm{u})W_b dx + \int_\kappa \nabla(u_h - \mathrm{u})\nabla W_b dx + \int_\kappa (\mathrm{u} - \mathrm{u} + u_h - u_h)W_b dx)$$

$$= \sum_{\kappa\in U_e} (\int_\kappa (\Delta u_h - u_h)W_b dx + \int_\kappa \nabla(u_h - \mathrm{u})\nabla W_b dx + \int_\kappa (u_h - \mathrm{u})W_b dx)$$

$$\lesssim \sum_{\kappa\in U_e} (\|\Delta u_h - u_h\|_{0,\kappa}\|W_b\|_{0,\kappa} + \|\nabla(u_h - \mathrm{u})\|_{0,\kappa}\|\nabla W_b\|_{0,\kappa} + \|(u_h - \mathrm{u})\|_{0,\kappa}\|W_b\|_{0,\kappa})$$

(3.23)

As(3.19), (3.20), (3.21), and the conclusions of (i),we can obtain

$$h_e^{1/2}\|w_h\|_{0,e} \lesssim h_e^{\frac{1}{2}}\left\|b_e^{\frac{1}{2}}w_h\right\|_{0,e} \lesssim \sum_{\kappa\in U_e} (h_e\|\Delta u_h - u_h\|_{0,\kappa} + \|\nabla(u_h - u)\|_{0,\kappa} + h_e\|u_h - u\|_{0,\kappa}).$$

(3.24)

Combining $\|\Delta u_h + (\lambda_h - c)u_h - \mathbf{r}\cdot\nabla u_h\|_{0,\kappa}$, the bounds and mesh shapes in (i) are obtained regularly

$$h_e^{1/2}\|[\![\nabla u_h]\!]\|_{0,e} \lesssim \sum_{\kappa\in U_e} \left(\|\nabla(u - u_h)\|_{0,\kappa} + h_\kappa\|u - u_h\|_{0,\kappa}\right),$$

The proof of (i) is completed.

(iii) Let $z_h = \lambda_h u_h - \frac{\partial u_h}{\partial n}$, $z_b = b_e z_h$, $z_b \in H_0^1(U_e)$ is an extension ofz_b satisfying (3.20) and (3.21). Note that $\lambda u - \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ and $\Delta u - u = 0$ on $L^2(\kappa)$ are obtained using integration by parts

$$\left\|b_e^{\frac{1}{2}}z_h\right\|_{0,e}^2 = \int_e z_h z_h b_e ds = \int_e z_h z_b ds = \int_e (\lambda_h u_h - \frac{\partial u_h}{\partial n} - \lambda u + \frac{\partial u}{\partial n})z_b ds$$

$$= \int_e (\frac{\partial u}{\partial n} - \frac{\partial u_h}{\partial n})z_b ds + \int_e \lambda u - \lambda_h u_h)z_b ds$$

$$= \int_e (\frac{\partial u}{\partial n} - \frac{\partial u_h}{\partial n})z_b ds + \int_e \lambda u - \lambda_h u_h)z_b ds$$

$$= \int_\kappa \Delta\mathrm{u}\, Z_b + \nabla\mathrm{u}\nabla Z_b - \Delta u_h Z_b - \nabla u_h \nabla Z_b + (u_h - u_h + u - \mathrm{u})Z_b dx + \int_e \lambda u - \lambda_h u_h)z_b ds$$

$$= \int_\kappa (u_h - \Delta u_h)Z_b + (\nabla\mathrm{u} - \nabla u_h)\nabla Z_b + (u - u_h)Z_b ds + \int_e \lambda u - \lambda_h u_h)z_b ds$$

This can be obtained from lemma 3.2

$$h_e^{\frac{1}{2}}\|z_h\|_{0,e} \lesssim h_e^{\frac{1}{2}}\|b_e^{\frac{1}{2}}z_h\|_{0,e}$$

$$\lesssim h_e\|\Delta u_h - u_h\| + \|\nabla(u - u_h)\|_{0,\kappa} + h_e\|u - u_h\|_{0,\kappa} + h_e^{\frac{1}{2}}\|\lambda_h u_h - \lambda u\|_{0,e}, \tag{3.25}$$

Using the conclusion of (i), the proof of (ii) is obtained.

For any $e \in \Gamma_h^i$, we have $[\![u]\!] = 0$, yielding (iv).

Theorem 3.3 Under the condition that theorem 4.1 holds, the following equation holds

$$\eta_\kappa \lesssim \sum_{\kappa\in w_\kappa} \left(\|\nabla(u - u_h)\|_{0,\kappa} + h_\kappa\|u - u_h\|_{0,\kappa}\right)$$

$$+\sum_{e\in\Gamma_h^i} h_e^{-\frac{1}{2}}\| [\![u-u_h]\!]\|_{0,e}+\sum_{e\in\Gamma_h^b} h_e^{\frac{1}{2}}\|\lambda u-\lambda_h u_h\|_{0,e} \tag{3.26}$$

which is

$$\eta(u_h)\lesssim \|u-u_h\|_h+h\|\lambda u-\lambda_h u_h\|_{0,\Omega} \tag{3.27}$$

Proof: Through η_κ and the definition of lemma 3.3 predominate, (3.26) is available, the reuse of energy norm $\|\cdot\|_h$, the definition of (3.27) can be obtained.

Theorem 3.3 states that the error estimation indicator $\eta(u_h)$ is valid.

3.3 THE RELIABILITY OF EIGENVALUE ERROR ESTIMATION INDICATOR

Lemmon 3.4 (Lemmon 4.6 in [3]) Set (λ,u) and (λ_h,u_h) is then eigenpair of (2.4) and (2.12) respectively, Set (λ^*,u^*) and (λ_h^*,u_h^*) is then eigenpair of (2.19) and (2.23) respectively , $(u_h,u_h^*)\neq 0$, then

$$\lambda-\lambda_h=\lambda\frac{(u-u_h,u^*-u_h^*)}{(u_h,u_h^*)}-\frac{a_h(u-u_h,u^*-u_h^*)}{(u_h,u_h^*)} \tag{3.28}$$

Proof: It can be obtained from formula (2.32) and formula (2.33)

$$a(u,v)=\lambda(u,v),\forall v\in V_h \tag{3.29}$$

$$a(v,u^*)=\lambda(v,u^*),\forall v\in V_h \tag{3.30}$$

It can be obtained from (2.3),(2.12),(3.27) and (3.30)

$$\begin{aligned}&\lambda(u-u_h,u^*-u_h^*)-a_h(u-u_h,u^*-u_h^*)\\&=\lambda(u,u^*)-\lambda(u,u_h^*)-\lambda(u_h,u^*)+\lambda(u_h,u_h^*)\\&-a_h(u,u^*)+a_h(u,u_h^*)+a_h(u_h,u^*)-a_h(u_h,u_h^*)\\&=\lambda(u_h,u_h^*)-a_h(u_h,u_h^*)=(\lambda-\lambda_h)(u_h,u_h^*)\end{aligned} \tag{3.31}$$

divide both sides of the above equation by (u_h,u_h^*) to get (3.30).

Theorem 3.4 Under the condition of lemma 4.4, let the eigenfunction space $M(\lambda),M(\lambda^*)\subset H^{1+r}(\Omega)(1\geqslant r>\frac{1}{2})$,we have

$$|\lambda-\lambda_h|\quad\lesssim\eta(u_h)^2+\eta(u_h^*)^2 \tag{3.32}$$

Proof: theorem 3.1 show $\|u-u_h\|_{0,\Omega}$ than $\|u-u_h\|_h$, higher order $\|u^*-u_h^*\|_{0,\Omega}$ than $\|u^*-u_h^*\|_h$ higher order. Thus, from (3.32), the estimator of u_h (3.9), and the estimator of u_h^*, can be obtained

$$|\lambda-\lambda_h|\lesssim\|u-u_h\|_h\|u^*-u_h^*\|_h\lesssim\eta(u_h)^2+\eta(u_h^*)^2$$

From the above equation, theorem 3.4 is proved.

We can know from theorem 3.2 and theorem 3.3, the characteristic function error $\|u-u_h\|_h^2+\|u^*-u_h^*\|_h^2$ estimates indicate $\eta(u_h)^2+\eta(u_h^*)^2$ is a reliable and efficient, therefore, the adaptive algorithm based on the estimates indicator good gradient mesh can be generated. The approximate eigen function reaches the optimal convergence order $O(dof^{-m})$ in $\|\cdot\|_h^2$. From (3.30) can have $|\lambda-\lambda_h|\leq dof^{-m}$. So can think $\eta(u_h)^2+\eta(u_h^*)^2$can be as the error estimates of lambda _h instructions, numerical experiments of section 4 show that eta $\eta(u_h)^2+\eta(u_h^*)^2$ as error estimation of λ_h instructions is reliable and efficient.

3.4 NUMERICAL EXPERIMENT

In this section, a series of numerical experiments will be conducted to verify the effectiveness of the hp local discontinuous finite element method of Steklov eigenvalue problem by compiling the code under the IFEM package, and the computed results will be sorted in descending order to obtain the data. In this experiment, the test domain are set to be the L-shape domain $\Omega_L = [0,1] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ and square $\Omega_S = [0,1]^2$ respectively.

IV. CONCLUSION

4.1 THE RESULTS OF NUMERICAL EXPERIMENTS

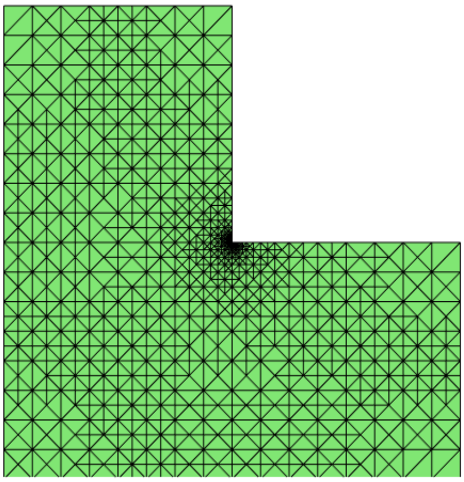
TABLE I. About the region numerical solution results for the first fourth eigenvalues

h	P	dof	λ_1	λ_2	λ_3	λ_4
1/4	1	24	0.241985120396731	1.557082494331800	1.567981965130290	2.766574268815750
1/4	2	48	0.240079724549933	1.496947672301450	1.499428975046800	2.082787917163070
1/4	3	80	0.240079112305256	1.492322541686640	1.492334691413000	2.082653128244820
1/8	1	96	0.240603412309877	1.515427100614430	1.519205201504950	2.252320633910260
1/8	2	192	0.240079142099497	1.492630666254280	1.492794826780140	2.082659643988760
1/8	3	320	0.240079085865350	1.492303508371800	1.492303745768920	2.082647158468340
1/16	1	384	0.240215724296756	1.498834475702410	1.500067866013550	2.125958682038990
1/16	2	768	0.240079089485034	1.492324336253270	1.492334931269400	2.082647939864130
1/16	3	1280	0.240079085433790	1.492303140728370	1.492303144658960	2.082647055702750
1/32	1	1536	0.240113813327580	1.494002217106130	1.494356189460460	2.093634847726540
1/32	2	3072	0.240079085696021	1.492304473625110	1.492305148150550	2.082647112024300
1/32	3	5120	0.240079085425170 + 0.00000000000000i	1.492303134470372 + 0.00000000000000i	1.492303134692798 + 0.000000000000000i	2.082647053903782 + 0.000000000000000i
1/64	1	6144	0.240087824313781	1.492733483605300	1.492827836095860	2.085414539962150
1/64	2	12288	0.240079085442566	1.492303218530770	1.492303261109540	2.082647057751940
1/64	3	20480	0.240079085419030 + 0.000000000000000i	0.768274571229386 - 0.842808219666222i	0.768274571229386 + 0.842808219666222i	-1.157380518255964 +0.000000000000000i

TABLE II. About the region numerical solution results for the first fourth eigenvalues

h	P	dof	λ_1	λ_2	λ_3	λ_4
1/2	1	18	0.184370548231648	0.980232299404417	1.82906530795754	4.13808322383428
1/2	2	36	0.182980279501687	0.918525278982131	1.699156800429130	3.275523798631990
1/2	3	60	0.182966075088768	0.900855678534774	1.689327741901200	3.220087676768690
1/4	1	72	0.183358727346188	0.931174919280224	1.73919672226841	3.56578249426117
1/4	1	144	0.182966480133728	0.900805354854160	1.689905958347010	3.223623216518960
1/4	2	240	0.182964513803436	0.896396671029967	1.688714315870450	3.217910283979370
1/8	1	288	0.183069281417831	0.908401968715363	1.70330292648774	3.32131530982049
1/8	1	576	0.182964567302139	0.896233781046824	1.688763930472870	3.218267571713410
1/8	2	960	0.182964279921273	0.894722861609305	1.688617360337970	3.217860903736770
1/16	1	1152	0.182991309607356	0.899267392795164	1.692530493140910	3.245737724595320
1/16	2	2304	0.182964287235214	0.894643891263804	1.688622521043840	3.217886398047420
1/16	3	3840	0.182964243597903	0.894070589896526	1.688603059166600	3.217859838784520
1/32	1	4608	0.182971105542950	0.895785659995388	1.689615048411320	3.225034155288780
1/32	2	9216	0.182964244673090	0.894037759700364	1.688603653469660	3.217861497450140

1/32	3	15360	0.182964237949173 +0.00000000000000i	0.893813548474596 +0.000000000000000i	1.688600895976938 +0.000000000000000i	-2.349206210258648 -1.759265727335553i



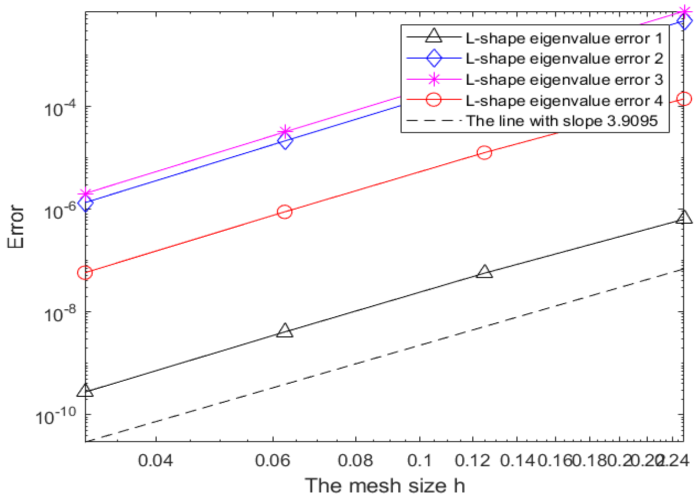
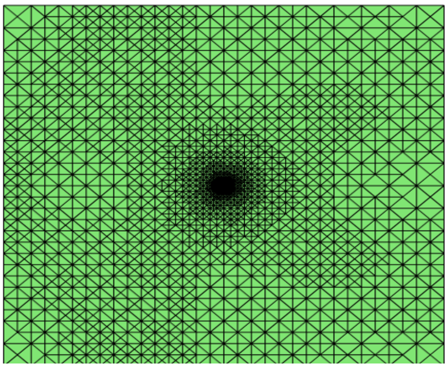


Figure. 1:The error curves of the approximation for the first fourth eigenvalues on Ω_S with P = 2.

4.2 THE RESULTS OF NUMERICALEXPERIMENTS



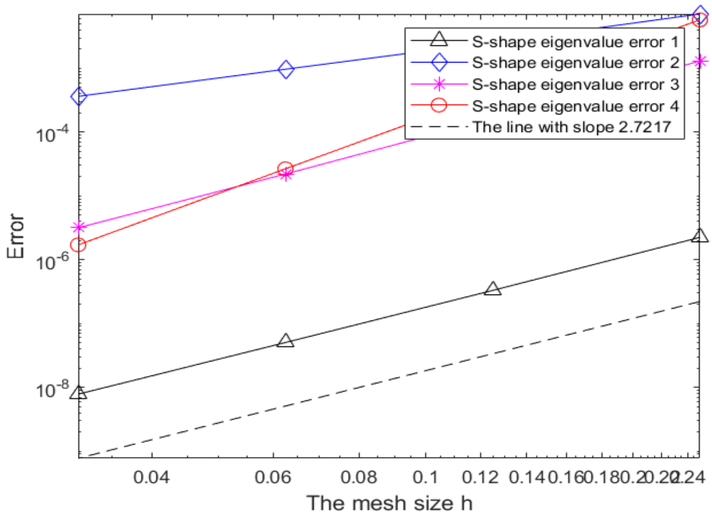


Figure 2: The error curves of the approximation for the first fourth eigenvalues on Ω_L with P= 2.

Steklov eigenvalue problem is widely used in physics. In this paper, a local discontinuity Galerkin method based on h is used to obtain the optimal order of convergence, and the optimal order error is estimated in the L-shaped and square regions respectively under the iFEM package. The convergence of the Dirichlet operator is superior on the continuous Ω field, which shows that the numerical experiment is effective and feasible, so it has good application value. We list the eigenvalue numerical solution results obtained by adaptive calculation in Table 1 and Table 2, and describe the adaptive grid and error curve in the figure. In FIG. 1 and FIG. 2, we can see that the error curve results of quadratic discontinuity elements show that the adaptive algorithm can achieve the optimal order of convergence, and from the error curve, it can also be seen that under the same degree of freedom (dof), the approximation obtained by the adaptive algorithm is more accurate than that calculated by the uniform grid.

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