Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 1 (2025) pp: 74-81
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

The Two-Grid Scheme of the Interior Penalty Discontinuous Galerkin Finite Element Method for Second-Order Elliptic Eigenvalue Problems

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ABSTRACT: *This paper investigates a class of second-order elliptic eigenvalue problems with variable coefficients and proposes a two-grid discretization scheme based on shifted inverse iteration. First, the corresponding interior penalty discontinuous Galerkin (IPDG) finite element formulation is presented, and the existence and uniqueness of the weak solution for this method are theoretically proven. Second, an a priori error estimate is derived, followed by an error analysis for the proposed scheme. Finally, numerical experiments are conducted to validate the effectiveness of the proposed algorithm.*

KEYWORDS: *Second order elliptic eigenvalues; Interior penalty discontinuous finite element method; Two-grid discrete*

R*eceived 07 Jan., 2025;* ***R****evised 16 Jan., 2025;* ***A****ccepted 18 Jan., 2025*

I. INTRODUCTION

In the field of numerical analysis, the second-order elliptic eigenvalue problem has garnered significant attention due to its broad application background and complex mathematical structure. These problems are commonly encountered in various fields of physics, engineering, and scientific computing, such as vibration mode analysis in elasticity theory and waveguide propagation characteristics in electromagnetics. However, solving second-order elliptic eigenvalue problems with variable coefficients directly often involves challenges such as large computational costs and difficulties in ensuring accuracy. Therefore, designing efficient and stable discretization methods becomes particularly important.The two-grid discretization scheme, as an effective numerical solution, has attracted widespread attention and application since Xu [1] first introduced it for asymmetric and bilinear elliptic problems. Later, Xu and Zhou [2] applied this method to eigenvalue problems for the first time. In [3], Yang and Bi proposed a novel two-grid finite element discretization scheme to solve eigenvalue problems of self-adjoint elliptic differential operators. This scheme combines finite element methods with the shift inverse power method, transforming the eigenvalue problem on the fine grid into an eigenvalue problem on the coarse grid and further into a linear algebraic system on the fine grid, while maintaining asymptotically optimal accuracy. In [4], Yang further applied multi-grid discretization to construct the computational framework for the shift inverse iteration method, and provided a detailed description of the implementation steps of the shift inverse iteration method based on multi-grid discretization, including the initial solution on the coarse grid, correction on the fine grid, and termination criteria for the iteration process.

This paper addresses the second-order elliptic eigenvalue problem with variable coefficients, presenting an interior penalty discontinuous finite element scheme. The theoretical existence and uniqueness of the weak solution, as well as the stability of the method, are proven. Additionally, a two-grid discretization scheme based on shift inverse iteration is proposed, and numerical experiments confirm that the proposed scheme achieves the best convergence rate in practice.

II. BASIC THEORY PREPARATION

Let Ω be a bounded polygon region in R^2, and the boundary $\partial\Omega$ is Lipshitz continuous. Consider the Dirichlet boundary condition eigenvalue problem: find $\lambda \in C$ and $u \in H_0^1(\Omega)$, such that

$$\begin{cases} -\nabla \cdot (\alpha \nabla u) = \lambda u, & in\Omega. \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{2.1}$$

The coefficient $\alpha(x)$ satisfies

$$C_{\alpha_0} \leq \alpha(x) \leq C_{\alpha_1}, x \in \Omega\ , \tag{2.2}$$

where C_{α_0} and C_{α_1} are plus constant.
Define a bilinear form that is continuous

$$a(u, v) = (\alpha \nabla u, \nabla v),\ \ \forall u, v \in H_0^1(\Omega). \tag{2.3}$$

Where

$$(\alpha \nabla u, \nabla v) = \int_\Omega \alpha\, \nabla u \cdot \nabla v dx.$$

There exist two plus constants A and B that are independent of u and v, such that the bilinear form $a(\cdot,\cdot)$ is satisfied

$$\begin{aligned} |a(u, v)| &\leq A \parallel u \parallel_{1,\Omega} \parallel v \parallel_{1,\Omega}, \quad \forall u, v \in H_0^1(\Omega), \\ |a(v, v)| &\geq B \|v\|_{1,\Omega}^2, \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{2.4}$$

The weak form of (2.1) is for $(\lambda, u) \in C \times H_0^1(\Omega),\ u \neq 0$,makes the following equation was established

$$a(u, v) = \lambda(u, v), \forall v \in H_0^1(\Omega). \tag{2.5}$$

Let $\mathcal{T}_h = \{\kappa\}$ be the conforming triangular mesh partition of the domain Ω, where the vertices and edges of each element do not lie inside or along the edges of any other element. Let h_e denote the length of the edges of element κ, and h_κ denote the diameter of the circumcircle of element κ. Additionally, define $h = \max_{\kappa \in \Gamma_h} h_\kappa$. Furthermore, define Γ_h^i and Γ_h^b as the sets of internal edges and boundary edges on $\partial\Omega$, respectively. The total set of edges is denoted by Γ_h, which is the union of internal and boundary edges: $\Gamma_h = \Gamma_h^i \cup \Gamma_h^b$.
Define the mean and jump values of v over e:

$$\{v\} = \frac{1}{2}(v^+ + v^-), \quad [[v]] = v^+ n_\kappa^+ + v^- n_\kappa^-,$$

Where $e = \partial\kappa^+ \cap \partial\kappa^-$, $v^+ = v|_{\kappa^+}$, $v^- = v|_{\kappa^-}$, n is the unit external normal vector from κ^+ to κ^-. If $\in \Gamma_h^b$, define the mean and jump values of v over e:

$$\{v\} = v, \quad [[v]] = vn.$$

The fragment function space on partition $\mathcal{T}_h$ is introduced:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega): v|_\kappa \in H^s(\kappa),\ \forall \kappa \in \mathcal{T}_h\},$$

Using $p_\kappa \geq 1$ to represent the degree of the polynomial in unit $\kappa \in \mathcal{T}_h$, denoted by $p = \{p_\kappa\}_{\kappa \in \mathcal{T}_h}$, the hp-finite element space is now defined as:

$$S^p(\mathcal{T}_h) = \{v \in L^2(\Omega): v|_\kappa \in S^{p_\kappa}(\kappa),\ \forall \kappa \in \mathcal{T}_h\}.$$

Define

$$\begin{aligned} a_h(u_h\, , v_h) = \sum_{\kappa \in \Gamma_h} \int_\kappa \alpha\, \nabla u_h \cdot \nabla v_h dx - \sum_{e \in \Gamma_h} \int_e \{\alpha \nabla u_h\} \cdot [[v_h]] ds \\ - \sum_{e \in \Gamma_h} \int_e [[u_h]] \cdot \{\alpha \nabla v_h\} ds + \sum_{e \in \Gamma_h} \eta\, h_e^{-1} \int_e [[u_h]] \cdot [[v_h]] ds. \end{aligned} \tag{2.6}$$

Where η is the penalty parameter.
The finite element approximation of (2.5) is to find $(\lambda_h\, , u_h) \in C \times S^p(\mathcal{T}_h), u_h \neq 0$, such that

$$a_h(u_h\, , v_h) = \lambda_h(u_h\, , v_h), \quad \forall v_h \in S^p(\mathcal{T}_h). \tag{2.7}$$

The source problem of (2.5) is: find $w \in H_0^1(\Omega)$, such that

$$a(w, v) = (f, v), \qquad \forall v \in H_0^1(\Omega). \tag{2.8}$$

The finite element approximation of (2.8) is: find $w_h \in S^p(\Gamma_h)$, such that

$$a_h(w_h\, , v_h) = (f, v_h)\, , \qquad \forall v_h \in S^p(\mathcal{T}_h). \tag{2.9}$$

Define linear bounded operator $T: L^2(\Omega) \to H_0^1(\Omega)$ satisfies

$$a(Tf, v) = (f, v), \quad \forall f \in L^2(\Omega),\ \ v \in H_0^1(\Omega), \tag{2.10}$$

Then (2.5) the equivalent operator form is:

$$Tu = \frac{1}{\lambda} u. \tag{2.11}$$

It is satisfied by (2.7) the corresponding discrete solution operator $T_h: L^2(\Omega) \to S^p(\Gamma_h)$ that can be defined

$$a_h(T_h f, v) = (f, v), \quad \forall f \in L^2(\Omega), \quad \forall v \in S^p(\mathcal{T}_h). \tag{2.12}$$

Then the equivalent operator form of (2.7) is:

$$T_h u_h = \frac{1}{\lambda_h} u_h. \tag{2.13}$$

Introduce a sum space $V(h) = S^p(\Gamma_h) + H_0^1(\Omega)$ endowed with a locally discontinuous finite element norm, where the discontinuous finite element norm is:

$$\| v_h \|_G^2 = \sum_{\kappa\in\mathcal{T}_h} \| \alpha\nabla v_h \|_{0,\kappa}^2 + \sum_{e\in\Gamma_h} h_e^{-1} \|[[v_h]]\|_{0,e}^2 , \tag{2.14}$$

And the h-norm is defined on the fragment function space $H^{1+s}(\mathcal{T}_h)\left(s > \frac{1}{2}\right)$ as:

$$\| v_h \|_h^2 = \| v_h \|_G^2 + \sum_{e\in\Gamma_h} h_e \|\{\alpha\nabla v_h\}\|_{0,e}^2. \tag{2.15}$$

Note that on a discontinuous finite element space $S^p(\mathcal{T}_h)$, $\|\cdot\|_G$ and $\|\cdot\|_{\mathrm{h}}$ are equivalent.

Lemma 2.1 (Grading-type inequality) For $w_h \in S^p$ there exist constants α, δ and η_1, such that when $\eta \geq \eta_1$, we have

$$a(w_h, w_h) \geq \delta\|w_h\|_h^2 - \alpha\|w_h\|_{0,\kappa}^2. \tag{2.16}$$

Proof By applying the Cauchy-Schwarz inequality and equation (2.2), we obtain

$$\begin{aligned} a(w_h, w_h) \geq C_{\alpha_0} \| \nabla w_h \|_{0,\kappa}^2 - 2\sum_{e\in\Gamma_h} \int_e \alpha \{\nabla w_h\} \cdot [[w_h]]\, ds \\ + \sum_{e\in\Gamma_h} \int_e \eta \cdot h_e^{-1}[[w_h]]\, ds \end{aligned} \tag{2.17}$$

By combining this with the inequality of means, we can obtain

$$\begin{aligned} \sum_{e\in\Gamma_h} \int_e \alpha \{\nabla w_h\} \cdot [[w_h]] ds &\leq C_{\alpha_1}\left(\sum_{\kappa\in\mathcal{T}_h} \| \nabla w_h \|_{1,\kappa}^2\right)^{1/2} \left(\sum_{e\in\Gamma_h} \int_e h_\epsilon^{-1} \left|[[w_h]]\right|^2 ds\right)^{1/2} \\ &\leq 2C_{\alpha_1} \| w_h \|_{1,\kappa}^2 \left(\sum_{e\in\Gamma_h} \int_e \eta \cdot h_e^{-1}\left|[[w_h]]\right|^2 ds\right)^{1/2} \\ &\leq \frac{C_{\alpha_0}}{4} \| w_h \|_{1,\kappa}^2 + \sum_{e\in\Gamma_h} \int_e \eta_1 \cdot h_e^{-1}\left|[[w_h]]\right|^2 ds. \end{aligned} \tag{2.18}$$

Substituting equation (2.18) into equation (2.17), we obtain

$$\begin{aligned} a(w_h, w_h) &\geq C_{\alpha_0} \| \nabla w_h \|_{0,\kappa}^2 - \frac{C_{\alpha_0}}{4} \| w_h \|_{1,\kappa}^2 - \sum_{e\in\Gamma_h} \int_e \eta_1 \cdot h_e^{-1}\left|[[w_h]]\right|^2 ds \\ &+ \sum_{e\in\Gamma_h} \int_e h_e^{-1} \left|[[w_h]]\right|^2 ds = \frac{3C_{\alpha_0}}{4} \| w_h \|_{1,\kappa}^2 - C_{\alpha_0} \| w_h \|_{0,\kappa}^2 \\ &+ \frac{(\eta - \eta_1)}{\eta} \sum_{e\in\Gamma_h} \int_e h_e^{-1} \left|[[w_h]]\right|^2 ds. \end{aligned} \tag{2.19}$$

Let

$$\delta = \min\{\frac{3C_{\alpha_0}}{4}, \min_{e\in\Gamma_h} \left|\frac{\eta - \eta_1}{\eta}\right|\}, \alpha = C_{\alpha_0},$$

By combining equations (2.14) and (2.15), we obtain

$$a(w_h, w_h) \geq \delta \| w_h \|_h^2 - \alpha \| w_h \|_{0,\kappa}^2.$$

The inequality (2.16) is thus proven.

Lemma 2.2 If $\Psi \in H^2(\mathcal{T}_h)$ is the solution to the following discrete variational problem

$$a(\Psi, v_h) = 0$$

then there exists a constant C such that

$$\| \Psi \|_{0,\mathcal{T}_h} \lesssim h \| \Psi \|_h . \tag{2.20}$$

Theorem 2.1 When h is sufficiently small and $\eta \geq \eta_1$, for $f \in L(\Omega)$, the solution u_h to equation (2.9) exists and is unique.

Proof Assume that there exists another solution $\tilde{u}_h \in S^p(\mathcal{T}_h)$ different from u_h that satisfies equation (2.9),

$$a(\tilde{u}_h, v_h) = (f, v_h),\ \forall v_h \in S^p(\mathcal{T}_h). \tag{2.21}$$

By subtracting equation (2.7) from equation (2.21), we obtain $a(u_h - \tilde{u}_h, v_h) = 0, v_h \in S^p(\mathcal{T}_h)$. In particular, when $v_h = u_h - \tilde{u}_h \in S^p(\mathcal{T}_h)$, using Lemma 2.1, we get

$$\delta \| u_h - \bar{u}_h \|_G^2 - \alpha \| u_h - \bar{u}_h \|_{0,\Omega}^2 \leq a(u_h - \bar{u}_h, u_h - \bar{u}_h) = 0. \tag{2.22}$$

By applying Lemma 2.2 and noting that $u_h - \tilde{u}_h \in H^2(\Omega, \mathcal{T}_h)$, we obtain

$$\| u_h - \tilde{u}_h \|_{0,\Omega}^2 \lesssim h^2 \| u_h - \tilde{u}_h \|_h^2 \lesssim h^2 \| u_h - \tilde{u}_h \|_G^2 . \tag{2.23}$$

Substituting equation (2.23) into equation (2.22), we obtain $(\delta-\alpha h^2)\parallel u_h-\tilde{u}_h\parallel_G^2\leq 0$. When h is sufficiently small, we have , $\parallel u_h-\tilde{u}_h\parallel_G^2=0$, thus $u_h=\tilde{u}_h$. This shows that the finite element solution u_h is unique. The proof is complete.

Theorem 2.2 Let w and w_h be the solutions to equations (2.8) and (2.9), respectively. Suppose w satisfies $w|_\kappa\in H^{s_\kappa}(\kappa)$ for all $\kappa\in\mathcal{T}_h$, and for $s\geq 1$, the following inequalities hold:

$$\|w-w_h\|_h\lesssim h_\kappa^{s-1}\|w\|_{s,\kappa}, \tag{2.24}$$

$$\|w-w_h\|_{0,\Omega}\lesssim h^{s+r-1}\|w\|_{s,\Omega}. \tag{2.25}$$

Proof By the triangle inequality, we obtain

$$\|w-w_h\|_h\lesssim\|w-v_h\|_h+\|v_h-w\|_h. \tag{2.26}$$

Let $E_h(w)=w-\Pi_p^h w$ we have

$$\begin{aligned}\|E_h(w)\|_h^2 &\lesssim\sum_{\kappa\in\mathcal{T}_h}\|\alpha\nabla_h E_h(w)\|_{0,\Omega}^2+\sum_{e\in\Gamma_h}\left\|h_e^{-\frac{1}{2}}[[E_h(w)]]\right\|_{0,e}^2+\sum_{e\in\Gamma_h}\left\|h_e^{\frac{1}{2}}\{\alpha\nabla_h E_h(w)\}\right\|_{0,e}^2\\ &\lesssim\sum_{\kappa\in\mathcal{T}_h}\|\alpha\nabla_h E_h(w)\|_{0,\Omega}^2+\sum_{\kappa\in\mathcal{T}_h}\sum_{\epsilon\subset\partial\kappa_\kappa}\left(h_e^{-\frac{1}{2}}\parallel E_h(w)\parallel_{0,e}^2+h_e^{\frac{1}{2}}\parallel\alpha\nabla_h E_h(w)\parallel_{0,e}^2\right)\\ &\lesssim\left(h_\kappa^{s-1}\|w\|_{s,\kappa}\right)^2+\left(h_\kappa^{s-1}\|w\|_{s,\kappa}\right)^2+\left(h_\kappa^{s}\|w\|_{s,\kappa}\right)^2.\end{aligned} \tag{2.27}$$

We obtain the following inequality:

$$\|E_h(w)\|_h\lesssim h_\kappa^{s-1}\|w\|_{s,\kappa}, \tag{2.28}$$

This error estimate is related to the interpolation error formula:

$$\parallel w-v_h\parallel\lesssim\parallel w-\Pi_p^h w\parallel. \tag{2.29}$$

Combining these results, we can prove (2.24).

Now, consider the source problem of the dual problem in equation (2.8), $a(v,w^*)=(v,g),\forall v\in H_0^1(\Omega)$, for any fixed $g\in L^2(\Omega)$. Let $w_h^*=\Pi_p^h w^*$. Using the Galerkin orthogonality, we can derive the following:

$$\begin{aligned}(w-w_h,g)&=a_h(w-w_h,w^*)=a_h(w-w_h,w^*-w_h^*)\\ &\lesssim\parallel w-w_h\parallel_h\parallel w^*-w_h^*\parallel_h,\end{aligned} \tag{2.30}$$

From equation (2.24) and the elliptic regularity estimates, let $g=w-w_h$, we obtain

$$\parallel w^*-w_h^*\parallel_h\lesssim h^r\parallel w^*\parallel_{1+r,\Omega}\lesssim h^r\parallel w-w_h\parallel_{0,\Omega}. \tag{2.31}$$

From equations (2.30) and (2.31), we get

$$\parallel w-w_h\parallel_{0,\Omega}=\sup_{g\in L^2(\Omega)}\frac{|(w-w_h,g)|}{\parallel g\parallel_{0,\Omega}}\lesssim h^r\parallel w-w_h\parallel_h.$$

Thus, we obtain

$$\|w-w_h\|_{0,\Omega}\lesssim h^r\parallel w-w_h\parallel_h\lesssim h^{s+r-1}\|w\|_{s,\Omega}.$$

Therefore, the proof of equation (2.25) is completed.

III. ERROR ESTIMATION FOR TWO-GRID DISCRETIZATION

3.1 Error Analysis of Two-Grid Discretization for Eigenvalue Problems

In this section, we present a two-grid discretization scheme for solving second-order elliptic eigenvalue problems, along with its error analysis. We introduce Scheme 3.1 and provide a rigorous theoretical analysis, where $S^p(\mathcal{T}_H)\subset S^p(\mathcal{T}_h), h<H$.

Scheme 3.1: Two-Grid Discretization.

Step 1: Solve the problem on the coarse grid $\mathcal{T}_H$: Find $\lambda_H\in R\times V_H$, where $\|u_H\|_H=1$, such that

$$a_H(u_H,v)=\lambda_H(u_H,v),\quad\forall v\in S^p(\mathcal{T}_H).$$

Step 2: Solve the linear problem on the fine grid $\mathcal{T}_H$: Find $u\in S^p(\mathcal{T}_h)$ such that

$$a_h(u,v)-\lambda_H(u,v)=(u_H,v),\quad\forall v\in S^p(\mathcal{T}_h).$$

Set $u_j^h=\frac{u}{\|u\|_h}$.

Step 3: Compute the generalized Rayleigh quotient

$$\lambda_j^h=\frac{a_h\left(u_j^h,u_j^h\right)}{\left(u_j^h,u_j^h\right)}.$$

Lemma 3.1 Let (λ,u) be the eigenpair for problem (2.5), then for any $v_h\in S^p(\mathcal{T}_h)$ with $\sqrt{(v_h,v_h)}\neq 0$, the following holds:

$$\frac{a_h(v_h,v_h)}{(v_h,v_h)}-\lambda=\frac{a_h(u-v_h,u-v_h)}{(v_h,v_h)}-\lambda\frac{(u-v_h,u-v_h)}{(v_h,v_h)}, \tag{3.1}$$

Proof From (2.5) and (2.7), we obtain

$$a_h(u, v_h) = (\lambda u, v_h) = (\lambda_h u, v_h), \quad \forall v \in S^p(\mathcal{T}_h),$$

Therefore, we have

$$\begin{aligned} & a_h(u - v_h, u - v_h) - \lambda(u - v_h, u - v_h) \\ &= a_h(v_h, v_h) - 2a_h(u, v_h) + a_h(u, u) - \lambda(v_h, v_h) + 2\lambda(u, v_h) - \lambda(u, u) \\ &= a_h(v_h, v_h) - 2(\lambda u, v_h) + a_h(u, u) - \lambda(v_h, v_h) + 2\lambda(u, v_h) - \lambda(u, u) \\ &= a_h(v_h, v_h) - \lambda(v_h, v_h) \end{aligned}$$

By dividing both sides by (v_h, v_h), we obtain the desired result in (3.1).

Lemma 3.2 For any non-zero elements u, v in the linear space $(S^p, \|\cdot\|)$, the following inequalities hold:

$$\left\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \right\| \leq 2\frac{\| u - v \|}{\| u \|}, \left\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \right\| \leq 2\frac{\| u - v \|}{\| v \|}$$

The proof can be found in Lemma 3 of [4].

Lemma 3.3 Let (μ_0, u_0) be the j-th approximate eigenpair of (2.5), where μ_0 is not an eigenvalue of T_h, and $, u_0 \in S^p(\mathcal{T}_h)$ with $\| u_0 \|_h = 1$, satisfying the following conditions:

(1) $dist\left(u_0, M_h(\mu_j)\right) \leq \frac{1}{2}$;

(2) $|\mu_0 - \mu_j| \leq \frac{\epsilon}{4}, |\mu_{k,h} - \mu_k| \leq \frac{\epsilon}{4}$, where $k = j - 1, j, j + q (k \neq 0)$, with $\epsilon = \min_{\mu_k \neq \mu} |\mu_k - \mu_j|$ being the constant associated with the eigenvalue μ_j;

(3) For $u \in S^p(\mathcal{T}_h)$ and $u_j^h \in S^p(\mathcal{T}_h)$, the following relations hold:

$$(\mu_0 - T_h)u = u_0, \quad u_j^h = \frac{u}{\| u \|_h}, \tag{3.2}$$

Then, it holds that:

$$dist\left(u_j^h, M_h(\mu_j)\right) \leq \frac{4}{\epsilon} \max_{j \leq k \leq j+q-1} |\mu_0 - \mu_{k,h}| dist\left(u_0, M_h(\mu_j)\right).$$

Theorem 3.1 Let $M(\lambda_j) \subset H^{1+\tau}(\Omega) \left(\frac{1}{2} < r \leq 1\right)$, then the following inequality holds

$$| \lambda_j^h - \lambda_j | \leq C(H^{4r-3} + h^{r-1})^2. \tag{3.3}$$

Let (λ_j^h, u_j^h) be the approximate eigenpair of the scheme (4.1), and suppose that H is sufficiently small. Then there exists $\mu_j \in M(\lambda_j)$ such that

$$\| u_j^h - u_j \|_h \leq C(H^{4r-3} + h^{r-1}). \tag{3.4}$$

$$\| u_j^h - u_j \|_{0,\Omega} \leq C(H^{4r-3} + h^{2r-2}). \tag{3.5}$$

Proof We will use Lemma 3.3 to complete the proof. Let $\mu_0 = \frac{1}{\lambda_H}, u_0 = \frac{\lambda_H T_h u_H}{\|\lambda_H T_h u_H\|_h}$, and denote $Tf = w$ and $T_h f = w_h$. By applying the Schwarz inequality, we obtain:

$$\begin{aligned} & a_h\big(T_h(u_H - u), T_h(u_H - u)\big) = \big(u_H - u, T_h(u_H - u)\big) \\ & \leq \| u_H - u \|_{0,\Omega} \| T_h(u_H - u) \|_{0,\Omega} \\ & \leq C(H^{2r-2})^2 \end{aligned}$$

Thus,

$$\| T_h(u_H - u) \|_h \leq C(H^{2r-2})$$

Using the previous result, we get:

$$\begin{aligned} \| \lambda_H T_h u_H - u) \|_h & = \| \lambda_H(T_h u_H - T_h u) + \lambda_H(T_h u - Tu) + (\lambda_H - \lambda)Tu \|_h \\ & \leq C\left(\| T_h(u_H - u) \|_h + \| (T - T_h)_{M(\lambda_j)} \|_h + |\lambda_H - \lambda|\right) \\ & \leq C(H^{2r-2} + h^{r-1} + H^{2r-3}) \\ & \leq C(H^{2r-2} + h^{r-1}) \end{aligned}$$

Let $u' = \frac{\bar{u}}{\|\bar{u}\|_h}$, and by Lemma 3.1, we have

$$\| u_0 - u' \|_h = \| u_0 - \frac{\bar{u}}{\| \bar{u} \|_h} \|_h \leq C \| \lambda_H T_h u_H - \bar{u} \|_h \leq C(H^{2r-2} + h^{r-1}) , \tag{3.6}$$

Let $u_h \in M_h(\lambda_j)$ be such that

$$\| u_h - u' \|_h = \| u_h - \frac{\bar{u}}{\| \bar{u} \|_h} \|_h \leq C(h^{r-1}), \tag{3.7}$$

By the triangle inequality, we can deduce that:

$$dist\left(u_0, M_h(\lambda_j)\right) \leq \| u_0 - u_h \|_h \leq \| u_0 - u' \|_h + \| u_h - u' \|_h \leq C(H^{2r-2} + h^{r-1}). \tag{3.8}$$

Since H is sufficiently small, condition 1 in Lemma 3.3 holds.

Now, we have:

$$\| \mu_0 - \mu_j \|_h = \frac{|\lambda_H - \lambda_j|}{|\lambda_H \lambda_j|} \le C H^{2r-2} \le \frac{\epsilon}{4},$$

$$\| \mu_k - \mu_{k,h} \|_h = \frac{|\lambda_{k,h} - \lambda_k|}{|\lambda_{k,h} \lambda_k|} \le C h^{2r-2} \le \frac{\epsilon}{4}, \quad k = j-1, j, \dots, j+q, k \neq 0.$$

Thus, condition 2 in Lemma 3.3 also holds.
Step 3 of Scheme 4.1 corresponds to the following equation:

$$a_h(u, v) - \lambda_H a_h(T_h u, v) = a_h(T_h u_H, v), \forall v \in S^p(\mathcal{T}_h)$$

and $u_j^h = \frac{u}{\|u\|_h}$.

$$(\lambda_H^{-1} - T_h) u = \lambda_H^{-1} T_h u_H,$$

Note that $\lambda_H^{-1} T_h u_H$ and u_0 differ by only a constant. Therefore, step 3 can also be written as

$$(\lambda_H^{-1} - T_h) u = u_0,$$

Thus, all the conditions in Lemma 3.3 are satisfied.
Since the dimension of $M_h(\lambda_j)$ is q, there exists $u^* \in M_h(\lambda_j)$ such that:

$$\| u_j^h - u^* \|_h = dist\left(u_j^h, M_h(\lambda_j)\right), \tag{3.9}$$

where $k = j, j+1, \dots, j+q-1$, and we obtain

$$\begin{aligned} |\mu_0 - \mu_{k,h}| &= \left|\frac{1}{\lambda_H} - \frac{1}{\lambda_{k,h}}\right| \le \left|\frac{\lambda_H - \lambda_{k,h}}{\lambda_H \lambda_{k,h}}\right| \\ &\le C\left(|\lambda_H - \lambda_j| + |\lambda_j - \lambda_{k,h}|\right) \\ &\le C(H^{2r-2} + h^{2r-2}) \le C(H^{2r-2}), \end{aligned} \tag{3.10}$$

Thus, by Lemma 3.3, (3.9), and (3.10), we have

$$\begin{aligned} \| u_j^h - u^* \|_h &= dist\left(u_j^h, M_h(\lambda_j)\right) \\ &\le \frac{C}{\epsilon} \max_{j \le k \le j+q-1} |\mu_0 - \mu_{k,h}| dist\left(u_0, M_h(\lambda_j)\right) \\ &\le C(H^{4r-3} + h^{r-1} H^{2r-2}), \end{aligned} \tag{3.11}$$

There exists $u_j \in M(\lambda_j)$ such that $\| u^* - u_j \|_h = dist\left(u^*, M(\lambda_j)\right)$, and

$$\| u^* - u_j \|_h \le C h^{r-1}, \tag{3.12}$$

Thus, from (3.11) and (3.12), we obtain

$$\| u_j^h - u_j \|_h \le \| u_j^h - u^* \|_h + \| u^* - u_j \|_h \le C(H^{4r-3} + h^{r-1}),$$

which proves (3.4).
Next, we prove (3.5), which gives

$$\| u^* - u_j \|_{0,\Omega} \le C h^{2r-2},$$

Similarly, we have:

$$\| u_j^h - u_j \|_{0,\Omega} \le \| u_j^h - u^* \|_{0,\Omega} + \| u^* - u_j \|_{0,\Omega} \le C(H^{4r-3} + h^{2r-2}),$$

Finally, we prove (3.3). From Step 4 of Scheme 3.1, Lemma 3.1, (3.6), and (3.7), we have:

$$\begin{aligned} |\lambda_j^h - \lambda_j| &= \left| \frac{a_h(u_j^h - u_j, u_j^h - u_j)}{\| u_j^h \|_{0,\Omega}^2} - \lambda_j \frac{(u_j^h - u_j, u_j^h - u_j)}{\| u_j^h \|_{0,\Omega}^2} \right| \\ &\le C\left(\| u_j^h - u_j \|_h^2 + |\lambda_j| \| u_j^h - u_j \|_{0,\Omega}^2\right) \\ &\le C(H^{4r-3} + h^{r-1})^2 \end{aligned}$$

which completes the proof.

3.2 Numerical Experiment

In this section, we will verify the effectiveness of the proposed method through several numerical experiments. Consider the problem (2.1), where the penalty parameter is set as $\eta = 10$ and $\alpha = 1$. We use MATLAB 2017a for solving, and the program is compiled under the iFEM software package. Three test domains are considered in the experiments: the square domain $\Omega_S = (0,1)^2$, the L-shaped domain $\Omega_L = (0,1)^2 \setminus \left(\frac{1}{2}, 1\right)^2$, and the crack structure domain $\Omega_{SL} = (0,1)^2 \setminus \left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 1\right)$. The initial mesh consists of a uniform triangular grid with a side length of $\frac{1}{2}$, and the mesh is uniformly refined by subdividing each triangle into four smaller congruent triangles. Since the exact eigenvalues are unknown, we take the reference eigenvalue $\lambda_1 = 8.3713297112$ for the crack structure domain Ω_{SL}, $\lambda_1 = 9.63972384472$ for the L-shaped domain, and $\lambda_1 = 19.7392088022$ for the square domain. The table below lists the numerical eigenvalue solutions for the square, L-shaped, and crack structure domains Ω_{SL} on both coarse and refined grids, along with the solving time required by this method. The results demonstrate that, as the mesh size increases, the advantages of the shift inverse iteration-based multigrid discretization method become more apparent, further validating the effectiveness of our method, i.e., the obtained solutions can maintain optimal accuracy.

Table 1 When $\alpha = 1$, the numerical solution results of primary eigenvalues for region Ω_{SL}

j	H	h	dof	$\lambda_{j,H}$	λ_j^h	*CPU(s)*
1	$\sqrt{2}/8$	$\sqrt{2}/128$	393216	8.502687494284409	8.374534203816751	4.014843
2	$\sqrt{2}/16$	$\sqrt{2}/256$	1572864	8.416472112406419	8.372841483445075	23.062553
3	$\sqrt{2}/32$	$\sqrt{2}/128$	393216	8.388412704353046	8.374534086185649	4.578289
4	$\sqrt{2}/32$	$\sqrt{2}/256$	1572864	8.388412704353046	8.372841482347949	22.451584
5	$\sqrt{2}/64$	$\sqrt{2}/128$	393216	8.378456441200685	8.374534086159587	5.380091
6	$\sqrt{2}/64$	$\sqrt{2}/256$	1572864	8.378456441200685	8.372841482366800	24.042532

Table 2 When $\alpha = 1$, the numerical solution results of primary eigenvalues for region Ω_L

j	H	h	dof	$\lambda_{j,H}$	λ_j^h	*CPU(s)*
1	$\sqrt{2}/16$	$\sqrt{2}/128$	294912	9.695763084936109	9.641613039316191	4.323964
2	$\sqrt{2}/16$	$\sqrt{2}/256$	1179648	9.695763084936109	9.640392599570252	18.067086
3	$\sqrt{2}/32$	$\sqrt{2}/256$	1179648	9.656978546370258	9.640392598630628	18.294710
4	$\sqrt{2}/32$	$\sqrt{2}/512$	4718592	9.656978546370258	9.639968945625661	493.690383
5	$\sqrt{2}/64$	$\sqrt{2}/256$	1179648	9.645296254148009	9.640392598628596	18.917496
6	$\sqrt{2}/64$	$\sqrt{2}/512$	4718592	9.645296254148009	9.639968945688045	476.549280

Table 3 When $\alpha = 1$, the numerical solution results of primary eigenvalues for region Ω_S

j	H	h	dof	$\lambda_{j,H}$	λ_j^h	*CPU(s)*
1	$\sqrt{2}/8$	$\sqrt{2}/128$	393216	19.874442070843379	19.739760007228625	5.161880
2	$\sqrt{2}/16$	$\sqrt{2}/256$	1572864	19.773825622934325	19.739346781234033	37.911447
3	$\sqrt{2}/32$	$\sqrt{2}/128$	393216	19.747958750767381	19.739760004506429	5.815663
4	$\sqrt{2}/32$	$\sqrt{2}/256$	1572864	19.747958750767381	19.739346781232186	35.770067
5	$\sqrt{2}/64$	$\sqrt{2}/128$	393216	19.741407900178068	19.739760004587787	6.833049
6	$\sqrt{2}/64$	$\sqrt{2}/256$	1572864	19.741407900178068	19.739346781389482	34.177723

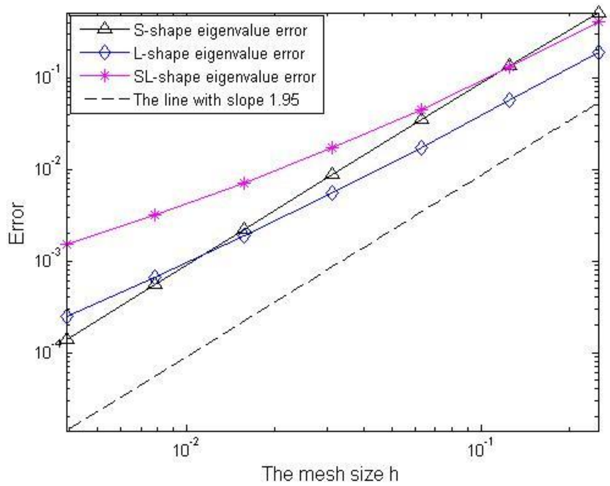

Figure1: When $\alpha = 1$, the error curve of the primary eigenvalues

In Tables 1, 2, and 3, we present the numerical solutions of eigenvalues computed using the two-grid discretization method outlined in Scheme 3.1. Additionally, we plot the eigenvalue error curve for the primary elements in the figure. Both the numerical results shown in the figure and the tables clearly demonstrate that our method achieves the optimal convergence rate for the eigenvalues and provides the best-order error estimation for the eigenvalue functions. The numerical experiments further validate the effectiveness of the proposed algorithm.

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