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Research Paper

Validity of Clifford Algebra-Valued Segal-Bargmann Transform and Taylor Isomorphism

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Abstract

The Classical Segal-Bargmann theory studies Hilbert space unitary isomorphisms that describe the wave-particle duality and the configuration space-phase space. S. Eaknipitsari and W. Lewkeeratiyutkul [20] generalized these concepts to Clifford algebra-valued functions. We establish the unitary isomorphisms among the space of Clifford algebra-valued square-integrable functions on $\mathbb{R}^n$ with respect to a Gaussian measure, the space of monogenic square-integrable functions on $\mathbb{R}^{n+1}$ with respect to another Gaussian measure and the space of Clifford algebra-valued linear functionals on symmetric tensor elements of $\mathbb{R}^n$. We follow [20] and show a survey of fixed validity.

Keywords. *Clifford analysis, Segal-Bargmann transform, Fock space.*

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I. Introduction

For $x^r \in \mathbb{R}^n$, let $\rho(x^r) = (2\pi)^{-\frac{n}{2}} e^{-|x^r|^2/2}$. The Segal-Bargmann transform is a map $U : L^2(\mathbb{R}^n, \rho\, dx^r) \to \mathcal{H}L^2(\mathbb{C}^n, \mu\, d(x^r + 2\epsilon))$ defined by

$$U\left(\sum_j \sum_r f_j(x^r + 2\epsilon)\right) = \int_{\mathbb{R}^n} \sum_j \sum_r \rho(2\epsilon) f(x^r) dx^r$$
$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \sum_j \sum_r e^{-|\epsilon|^2} f_j(x^r) dx^r.$$

and $\mathcal{H}L^2(\mathbb{C}^n, \mu\, d(x^r + 2\epsilon))$ is the space of holomorphic square-integrable functions on $\mathbb{C}^n$ with respect to measure $\mu(x^r + 2\epsilon)\, d(x^r + 2\epsilon)$ where $\mu(x^r + 2\epsilon) = \pi^{-n} e^{-|x^r + 2\epsilon|^2}$ and $d(x^r + 2\epsilon)$ is Lebesgue measure on $\mathbb{C}^n$. Segal [17], [18] and Bargmann [1] independently proved that U is a unitary isomorphism. See also [9], [10] for backgrounds and recent developments. The map U is the heat operator $e^{\frac{\Delta}{2}} f_j = \rho * f_j$, followed by the analytic continuation from $\mathbb{R}^n$ to $\mathbb{C}^n$, as in the following commutative diagram:

$$\begin{array}{ccc} & \overset{U}{\nearrow} & \mathcal{H}L^2(\mathbb{C}^n, \mu\, d(x^r + 2\epsilon)) \\ & & \uparrow c \\ L^2(\mathbb{R}^n, \rho\, dx^r) & \xrightarrow[e^{\frac{\Delta}{2}}]{} & \tilde{\mathcal{A}}(\mathbb{R}^n) \end{array} \tag{1.1}$$

and $\mathcal{C}$ denotes the analytic continuation from $\mathbb{R}^n$ to $\mathbb{C}^n$ and e $\tilde{\mathcal{A}}(\mathbb{R}^n)$ is the image of $L^2(\mathbb{R}^n, \rho\, dx^r)$ by the operator $e^{\frac{\Delta}{2}}$.

There is another space, namely the Fock space $\mathcal{F}(\mathbb{C}^n)$ of symmetric tensors over $\mathbb{C}^n$, that is isometrically isomorphic to $\mathcal{H}L^2(\mathbb{C}^n, \mu\, d(x^r + 2\epsilon))$. See [11], [16] for original works. We follow recent developments in [7], [8]. Let X be the complex dual space of $\mathbb{C}^n$ and denote by $X^{\odot k}$ the space of symmetric k-tensors over X. Consider the algebraic direct sum $\sum_{k=0}^{\infty} \ X^{\odot k}$ whose elements are of the form $\alpha_j \ = \sum_{k=0}^{\infty} \ \sum_j \ \alpha_k^j$, where $\alpha_k^j \in X^{\odot k}$ for each k and $\alpha_k^j = 0$ for all but finitely many k. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{C}^n$. Each element $\alpha_k^j \in X^{\odot k}$ has a natural norm given by

$$|\alpha_k^j|^2 = \sum_{|\beta|=k} \sum_j \frac{1}{\beta!} |\alpha_k^j(e^\beta)|^2,$$

where the sum is taken over multi-indices $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$. We use notation $e^\beta = e_1^{\odot \beta_1} \odot \dots \odot e_n^{\odot \beta_n}, |\beta| = \beta_1 + \cdots + \beta_n$ and $\beta! = \beta_1! \dots \beta_n!$. The algebraic direct sum $\sum_{k=0}^{\infty} \ X^{\odot k}$ is equipped with a norm given by

$$\|\alpha^j\| = \left(\sum_{k=0}^{\infty} \sum_j \ |\alpha_k^j|^2 \right)^{\frac{1}{2}}.$$

The Fock space $\mathcal{F}(\mathbb{C}^n)$ is defined to be the Hilbert space completion of the algebraic direct sum with respect to this norm. Thus $\mathcal{F}(\mathbb{C}^n)$ is the set of strong sums $\sum_{k=0}^{\infty} \ \sum_j \ \alpha_k^j$, where $\alpha_k^j \in X^{\odot k}$ for each k, such that $\sum_{k=0}^{\infty} \sum_j \ |\alpha_k^j|^2 < \infty$.

We describe the unitary isomorphism from $\mathcal{H}L^2(\mathbb{C}^n, \mu\, d(x^r + 2\epsilon))$ onto $\mathcal{F}(\mathbb{C}^n)$. For f_j be the sequence of holomorphic functions on $\mathbb{C}^n$. There is a linear map $D^k\left(\sum_j \ (f_j)\right) : (\mathbb{C}^n)^{\odot k} \to \mathbb{C}$ such that for any $u_1^2, \dots, u_k^2 \in \mathbb{C}^n$,

$$D^k\left(\sum_j \ f_j(u_1^2 \ \odot \cdots \odot \ u_k^2)\right) = \left(\partial_{u_1^2} \dots \partial_{u_k^2}\right)\left(\sum_j \ f_j(0)\right),$$

with $D^0 f_j = f_j(0)$. Here ∂_{v^2} is the directional derivative in the v^2 direction. We identify $D^k\left(\sum_j \ f_j\right)$ as an element of $(X)^{\odot k}$. We write $\sum_{k=0}^{\infty} \ \sum_j \ D^k f_j$ in $\mathcal{F}(\mathbb{C}^n)$ as

$$(1 - D)^{-1}\left(\sum_j \ f_j\right) = \sum_{k=0}^{\infty} \sum_j \ D^k f_j.$$

The map $(1 - D)^{-1}$ is a unitary isomorphism from $\mathcal{H}L^2(\mathbb{C}^n, \mu\, d(x^r + 2\epsilon))$ onto the Fock space $\mathcal{F}(\mathbb{C}^n)$. This map is simply the Taylor series expansion that assigns to each holomorphic function its Taylor coefficients. Hence we will call it the Taylor map.

It can be summarized that the three arrows in the following commutative diagram are unitary isomorphisms. These isomorphisms are used to describe the “wave-particle duality” in quantum field theory.

$$\begin{array}{ccc} L^2(\mathbb{R}^n, \rho\, dx^r) & \xrightarrow{\quad U \quad} & \mathcal{H}L^2(\mathbb{C}^n, \mu\, d(x^r + 2\epsilon)) \\ & \searrow \qquad \swarrow \ (1 - D)^{-1} & \\ & \mathcal{F}(\mathbb{C}^n) & \end{array} \tag{1.2}$$

There is another form of a Segal-Bargmann transform $V : \ L^2(\mathbb{R}^n, dx^r) \to \mathcal{H}L^2(\mathbb{C}^n, \nu\, dx^r \ d(x^r + \epsilon))$ defined by

$$\begin{aligned} V(\sum_j \sum_r \ f_j(x^r + 2\epsilon) &= \int_{\mathbb{R}^n} \sum_j \sum_r \ \rho(2\epsilon) f(x^r) dx^r \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \sum_j \sum_r \ e^{-\frac{|2\epsilon|^2}{2}} f_j(x^r) dx^r, \end{aligned}$$

where $\nu^2(x^r + \epsilon) dx^r \ d(x^r + \epsilon) = \pi^{-\frac{n}{2}} e^{-|x^r + \epsilon|^2} dx^r \ d(x^r + \epsilon)$. The map V is a unitary isomorphism from $L^2(\mathbb{R}^n, dx^r)$ onto $\mathcal{H}L^2(\mathbb{C}^n, \nu^2 \ dx^r \ d(x^r + \epsilon))$. The formula that defines V is the same as that for U, but with different domain and range. However, one does not have a Taylor map to the Fock space in the same way as the U-version of a Segal-Bargmann transform. The main reason is that the monomials are orthogonal with respect to the measure $d\mu$ and not to the measure $d\nu$. The U-version and the V-version both have their advantages, but certainly the existence of this Taylor map onto the Fock space is a significant advantage of the U-version.

The purpose is to generalize the triad (1.2) to the Clifford algebra-valued functions setting. Brackx, Delanghe, and Sommen [2] defined a (left) monogenic function $f_j : \mathbb{R}^n \to \mathbb{C}_n$ as an element in the kernel of a Dirac operator, i.e.,

$$0 = \underline{D}\left(\sum_j \sum_r f_j(\underline{x}^r)\right) = \sum_{j=1}^{n} \sum_r e_j \partial_{e_j} f(\underline{x}^r)$$

where $\mathbb{C}_n$ is the complex Clifford algebra generated by the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Denote by $\mathcal{M}(\mathbb{R}^n)$ and $\mathcal{M}(\mathbb{C}^n)$ the right $\mathbb{C}_n$-modules of monogenic functions on $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. Kirwin, Mourão, Nunes, and Qian [13] used a notion of an $(n+1)$-variable monogenic function, namely a function $f_j : \mathbb{R}^{n+1} \to \mathbb{C}_n$ such that

$$(\partial_{e_0} + \underline{D})\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right) = 0.$$

They obtained a generalized Segal-Bargmann transform on special types of monogenic functions namely, slice monogenic and axial monogenic functions. Mourão, Nunes, and Qian [15] continued their work and generalized the Segal-Bargmann transform to Clifford algebra-valued functions analogous to V as in the following theorem.

Theorem 1.1 ([15]). The map $\tilde{V} : L^2(\mathbb{R}^n, d\underline{x}^r) \otimes \mathbb{C}_n \to \mathcal{M}L^2(\mathbb{R}^{n+1}, \tilde{\nu}^2\, dx_0^r\, d\underline{x}^r)$ given by

$$\tilde{V}\left(\sum_j f_j\right)(x_0^r, \underline{x}^r) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_j \sum_r \left(\int_{\mathbb{R}^n} e^{-\frac{p^2}{2}} e^{i(\underline{p}, -\epsilon)} e^{-ix_0^r \underline{p}}\, d\underline{p}\right) f_j(\underline{x^r + \epsilon})\, d(\underline{x^r + \epsilon}),$$

is a unitary isomorphism. Here $\mathcal{M}L^2(\mathbb{R}^{n+1}, \tilde{\nu}^2\, dx_0^r\, d\underline{x}^r)$ is the Hilbert space of monogenic functions on $\mathbb{R}^{n+1}$ that are square-integrable with respect to measure $\tilde{\nu}^2\, dx_0^r\, d\underline{x}^r$ where $\tilde{\nu}^2(x_0^r) = \frac{1}{\sqrt{\pi}} e^{-(x_0^2)^r}$. Dang, Mourão, Nunes, and Qian [4] also obtained this result for spherical domains. Using the idea of Theorem 1.1 in [15], we can extend the unitary map U in the classical setting as follows (see [20]).

Theorem 1.2. The map $\widetilde{U}$ given by

$$\widetilde{U}\left(\sum_j f_j\right)(x_0^r, \underline{x}^r) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_j \sum_r \left(\int_{\mathbb{R}^n} e^{-\frac{p^2}{2}} e^{i(\underline{p}^2, -\epsilon)} e^{-ix_0^r \underline{p}^2}\, d\underline{p}^2\right) f_j(\underline{x^r + \epsilon})\, d(\underline{x^r + \epsilon}),$$

is a unitary isomorphism from $L^2(\mathbb{R}^n, \rho\, d\underline{x}^r) \otimes \mathbb{C}_n$ onto $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$, the Hilbert space of monogenic functions on $\mathbb{R}^{n+1}$ that are square-integrable with respect to the measure

$$d\tilde{\mu} = \frac{1}{\pi^{\frac{n+1}{2}}} e^{-(x_0^2)^r - |\underline{x}^r|^2} dx_0^r\, d\underline{x}^r.$$

The map $\widetilde{U}$ can be factorized as in the following diagram:

$$\begin{array}{ccc} & & \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) \\ & \overset{\widetilde{U}}{\nearrow} & \uparrow e^{-x_0\underline{D}} \\ L^2(\mathbb{R}^n, \rho\, d\underline{x}^r) \otimes \mathbb{C}_n & \xrightarrow[e^{\frac{\Delta}{2}}]{} & \tilde{\mathcal{A}}(\mathbb{R}^n) \otimes \mathbb{C}_n \end{array} \tag{1.3}$$

We replace the analytic continuation C by the Cauchy-Kowalevski extension $e^{-x_0^r\underline{D}}$, which will be explained.

Now we turn to the Clifford algebra-valued Fock space. For X be the real dual space of $\mathbb{R}^n$. [20] repeat the construction in the classical case for the Clifford algebra-valued symmetric tensor algebra, which will be identified with $\mathcal{F}(X) \otimes \mathbb{C}_n$ and called the $\mathbb{C}_n$-valued covariant Fock space. An element in $\mathcal{F}(X) \otimes \mathbb{C}_n$ is a strong sum $\sum_{k=0}^{\infty} \sum_j \alpha_k^j$, where each $\alpha_k^j \in X^{\odot k} \otimes \mathbb{C}_n$ and such that

$$\|\alpha_j\|^2 = \sum_{k=0}^{\infty}\sum_j |\alpha_k^j|^2 < \infty.$$

For $f_j : \mathbb{R}^{n+1} \to \mathbb{C}_n$ be a monogenic function. Then there is a linear map $D^k\left(\sum_j f_j\right) : (\mathbb{R}^n)^{\odot k} \to \mathbb{C}_n$ such that for any $u_1^2, \ldots, u_k^2 \in \mathbb{R}^n$,

$$D^k\left(\sum_j f_j(u_1^2 \odot \cdots \odot u_k^2)\right) = \partial_{u_1^2} \ldots \partial_{u_k^2}\left(\sum_j f_j(0, \underline{0})\right),$$

with $D^0 f_j = f_j(0, \underline{0})$. It is natural to write $\sum_{k=0}^{\infty} D^k\left(\sum_j f_j\right) \in \mathcal{F}(X) \otimes \mathbb{C}_n$ as

$$(1 - D)^{-1}\left(\sum_j f_j\right) = \sum_{k=0}^{\infty}\sum_j D^k f_j,$$

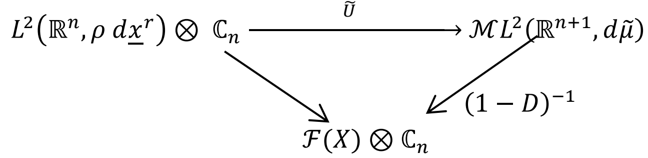
where

$$\left\|(1 - D)^{-1}\left(\sum_j f_j\right)\right\|^2 = \sum_{k=0}^{\infty}\sum_j |D^k f_j|^2 < \infty.$$

We have the second main theorem (see [20]).

Theorem 1.3. The map $(1 - D)^{-1}$ is a unitary isomorphism from the space of square-integrable monogenic functions $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$ onto the Clifford algebra-valued Fock space $\mathcal{F}(X) \otimes \mathbb{C}_n$.

Combining Theorems 1.2 and 1.3 together, we have the following unitary isomorphisms in the Clifford-algebra-valued setting in this diagram:



We recall necessary facts about Clifford algebra and Clifford analysis used. We discuss the Clifford algebra-valued Segal-Bargmann Transform and the Clifford algebra-valued Fock space to prove Theorems 1.2. and 1.3.

II. Preliminaries

2.1. Real and Complex Clifford Algebras

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Define the Clifford algebra $\mathbb{K}_n$ as the $\mathbb{K}-$algebra generated by n elements $e_1, \ldots, e_n$, which can be identified with the canonical basis of $\mathbb{K}^n \subset \mathbb{K}_n$ and satisfy the relations $e_i e_j + e_j e_i = -2\delta_{ij}$, see e.g.[2], [3], [14], [15]. If $\mathbb{K} = \mathbb{R}$ we call $\mathbb{K}_n$ the real Clifford algebra and if $\mathbb{K} = \mathbb{C}$ we call $\mathbb{K}_n$ the complex Clifford algebra.

Note that $\{e_A \mid A \subset \{1, 2, \ldots, n\} = N\}$ is a basis for $\mathbb{K}_n$ where $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$ with $A = \{i_1, i_2, \ldots, i_k\}, 1 \le i_1 < i_2 < \cdots < i_k \le n$, and $e_\phi = 1$.

Thus any $\lambda_j \in \mathbb{K}_n$ can be written as

$$\lambda_j = \sum_{A \subset N}\sum_j \lambda_A^j e_A,$$

where $\lambda_A^j \in \mathbb{K}$. Define the so-called k-vector part of λ_j, for $k = 0, 1, \ldots, n$,
by

$$[\lambda_j]_k = \sum_{|A|=k}\sum_j \lambda_A^j e_A.$$

Now, we focus at $\mathbb{C}_n$. One important operator of $\mathbb{C}_n$, the Hermitian conjugation, is defined by

$$\overline{e_i} = -e_i, \qquad i = 1, 2, \ldots, n,$$
$$\overline{(\lambda_A^j e_A)} = (\lambda^j)_A^c\, \overline{e_A}, \qquad \lambda_A^j \in \mathbb{C}, A \subset N,$$
$$\overline{(\lambda_j \mu)} = \overline{\mu}\, \overline{\lambda}_j, \qquad \lambda_j, \mu \in \mathbb{C}_n,$$

where $(\lambda^j)_A^c$ denotes the complex conjugate of the complex number λ_A^j. This contributes to a Hermitian inner product and its associated norm on $\mathbb{C}_n$, respectively defined by

$$(\lambda_j,\mu) = [\overline{\lambda}_j\mu]_0 \quad \text{and} \quad |\lambda_j|^2 = [\overline{\lambda}_j\lambda_j]_0 = \sum_A \sum_j |\lambda_A^j|^2.$$

2.2. Clifford Analysis

Clifford analysis is a function theory in higher dimensions generalizing complex analysis, see e.g. [2]. We begin by considering the generalized Cauchy-Riemann operator

$$\partial_{e_0} + \underline{D},$$

where

$$\underline{D} = \sum_{j=1}^{n} e_j\partial_{e_j}$$

and $\{e_0, e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^{n+1}$. To make things easier, we also identify $\mathbb{R}^n$ with the subspace of $\mathbb{R}_n$ of 1-vectors

$$\left\{\underline{x}^r = \sum_{j=1}^{n}\sum_r x_j^r e_j : x^r = (x_1^r, \ldots, x_n^r) \in \mathbb{R}^n\right\}.$$

We give a generalized concept of a holomorphic function. A continuously differentiable functions f_j on an open domain $O \subset \mathbb{R}^{n+1}$, taking values in $\mathbb{C}_n$, is called (left) monogenic on O if it satisfies the generalized Cauchy-Riemann equation (see [20]):

$$(\partial_{e_0} + \underline{D})\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right) = 0.$$

Theorem 2.1 ([2]). Let f_j be a $\mathbb{C}_n$-valued analytic functions on $\mathbb{R}^n$. Then there exists a unique $\mathbb{C}_n$-valued monogenic functions F_j on $\mathbb{R}^{n+1}$ such that $F_j(0, \underline{x}^r) = f_j(\underline{x}^r)$.

This extension is called the Cauchy-Kowalevski extension, or simply the C-K extension, of f_j. In [19] and [5], the formula for the C-K extension is given as follows:

Theorem 2.2. Let f_j be a $\mathbb{C}_n$-valued analytic functions on $\mathbb{R}^n$. Then the C-K extension of f_j is given by the formula

$$F_j(x_0^r, \underline{x}^r) = \sum_r e^{-x_0^r\underline{D}}\left(\sum_j f_j(\underline{x}^r)\right) := \left(\sum_{k=0}^{\infty}\sum_j\sum_r (-1)^k \frac{(x^r)_0^k}{k!}\underline{D}^k f_j\right)(\underline{x}^r)$$

where the series converges uniformly on compact subsets.

III. Clifford Algebra-Valued Segal-Bargmann Transform

We introduce the Hilbert space of Clifford algebra-valued square-integrable functions with respect to measure ρ on $\mathbb{R}^n$

$$L^2(\mathbb{R}^n, d\rho\,; \mathbb{C}_n) = \{f_j : \mathbb{R}^n \to \mathbb{C}_n \mid \int_{\mathbb{R}^n}\sum_j\sum_r |f_j(\underline{x}^r)|^2 \rho(x^r)dx^r < \infty\},$$

equipped with the inner product:

$$\langle f_j, g_j\rangle = \int_{\mathbb{R}^n}\sum_j\sum_r \left(f_j(\underline{x}^r), g_j(\underline{x}^r)\right)\rho(x^r)dx^r = \int_{\mathbb{R}^n}\sum_j\sum_r \left[\overline{f_j(\underline{x}^r)}g_j(\underline{x}^r)\right]_0 \rho(x^r)dx^r$$

where

$$\rho(x^r) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|\underline{x}^r|^2}{2}}.$$

We identify $L^2(\mathbb{R}^n, d\rho\,; \mathbb{C}_n)$ with the tensor product $L^2(\mathbb{R}^n, d\rho) \otimes \mathbb{C}_n$. Also, the Hilbert space of Clifford algebra-valued square-integrable functions with respect to measure $\tilde{\mu}$ on $\mathbb{R}^{n+1}$ is given by

$$L^2(\mathbb{R}^{n+1}, d\tilde{\mu}\,; \mathbb{C}_n) = \{F_j : \mathbb{R}^{n+1} \to \mathbb{C}_n \mid \int_{\mathbb{R}^{n+1}}\sum_j\sum_r |F_j(x^r)|^2 d\tilde{\mu} < \infty\},$$

equipped with the inner product:

$$\langle F_j, G_j\rangle = \int_{\mathbb{R}^{n+1}} \sum_j \sum_r (F_j(x^r), G_j(x^r))\, d\tilde{\mu} = \int_{\mathbb{R}^{n+1}} \sum_j \sum_r \left[\overline{F_j(x^r)} G_j(x^r)\right]_0 d\tilde{\mu}$$

where

$$d\tilde{\mu} = \pi^{-\frac{n+1}{2}} e^{-(x^r)_0^2 - |\underline{x}^r|^2} dx_0^r\, d\underline{x}^r.$$

The space $L^2(\mathbb{R}^{n+1}, d\tilde{\mu}; \mathbb{C}_n)$ can be identified with the tensor product $L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) \otimes \mathbb{C}_n$. Denote by $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$ the Hilbert space of monogenic functions on $\mathbb{R}^{n+1}$ that are square-integrable with respect to measure $\tilde{\mu}$.

Following the idea of the proof of Theorem 1.1 in [15], we will prove Theorem 1.2, namely, $\widetilde{U}$ is a unitary isomorphism from $L^2(\mathbb{R}^n, d\rho) \otimes \mathbb{C}_n$ onto $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. The map $\widetilde{U}$ is the heat operator applied to an element in the domain and then followed by the C-K extension. In other words,

$$\widetilde{U}\left(\sum_j (f_j)\right) = \sum_j \sum_r \left(e^{-x_0^r \underline{D}} \circ e^{\frac{\triangle}{2}}\right)(f_j) = \sum_j \sum_r e^{-x_0^r \underline{D}}(\rho * f_j).$$

Note that ρ is analytic on $\mathbb{R}^n$, and so is $\rho * f_j$. Then its C-K extension, $\widetilde{U}(\sum_j f_j)$, exists and is monogenic on $\mathbb{R}^{n+1}$.

Proof of Isometry (see [20]). Note that the Schwartz space of $\mathbb{C}_n$-valued functions is identified with the tensor product $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, d\rho)$, it follows that $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$ is dense in $L^2(\mathbb{R}^n, d\rho) \otimes \mathbb{C}_n$. Any $f_j \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$ can be written as $f_j = \sum_{A\subset N} \sum_j (f_j)_A e_A$, where $(f_j)_A \in \mathcal{S}(\mathbb{R}^n)$. Hence the Fourier transform of f_j is given by $\hat{f}_j = \sum_{A\subset N} \sum_j \left(\hat{f}_j\right)_A e_A$. By the density argument, it suffices to show that $\widetilde{U}$ is an isometry on $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$.

The Fourier inversion formula of $f_j \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$ is

$$f_j(x^r) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sum_j \sum_r e^{i\left(\underline{p}, \underline{x}^r\right)} \hat{f}_j\left(\underline{p}^2\right) d\underline{p}^2. \tag{3.1}$$

By applying the operator $e^{-x_0^r \underline{D}} \circ e^{\frac{\triangle}{2}}$ to f_j in (3.1) and pass it inside the integral sign, we see that

$$\widetilde{U}(\sum_j \sum_r (f_j)(x_0^r, \underline{x}^r) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sum_j \sum_r e^{-ix_0^r \underline{p}^2} e^{-\frac{\left|\underline{p}^2\right|^2}{2}} e^{i\left(\underline{p}^2, \underline{x}^r\right)} \hat{f}_j\left(\underline{p}^2\right) d\underline{p}^2. \tag{3.2}$$

Note that since f_j and $\hat{f}_j$, as well as their derivatives, are rapidly decaying (i.e. they are functions in the Schwarz space), this allows passage of the operator inside the integral sign and the interchange of the order of integration. To show the isometry of $\widetilde{U}$, let $f_j, h_j \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$. Then

$$\langle f_j, h_j\rangle = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sum_j \sum_r \left(f_j(\underline{x}^r), h_j(\underline{x}^r)\right) e^{-\frac{|\underline{x}^r|^2}{2}} d\underline{x}^r$$

$$= \frac{1}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{2n}} \sum_j \sum_r \left(e^{i\left(\underline{p}^2, \underline{x}^r\right)} \hat{f}_j\left(\underline{p}^2\right), e^{i\left(\underline{q}^2, \underline{x}^r\right)} \hat{h}_j\left(\underline{q}^2\right)\right) e^{-\frac{|\underline{x}^r|^2}{2}} d\underline{p}^2\, d\underline{q}^2\, d\underline{x}^{\mathrm{r}}$$

$$= \frac{1}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{2n}} \sum_j \left(\hat{f}_j\left(\underline{p}^2\right), \hat{h}_j\left(\underline{q}^2\right)\right) \int_{\mathbb{R}^n} \sum_r e^{i\left(\underline{q}^2 - \underline{p}^2, \underline{x}^r\right)} e^{-\frac{|\underline{x}^r|^2}{2}} d\underline{x}^r\, d\underline{p}^2\, d\underline{q}^2$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \sum_j e^{-\frac{\left|\underline{q}^2 - \underline{p}^2\right|^2}{2}} \left(\hat{f}_j\left(\underline{p}^2\right), \hat{h}_j\left(\underline{q}^2\right)\right) d\underline{p}^2\, d\underline{q}^2.$$

The last equality is obtained by the Fourier transform. On the other hand,

$$\langle \widetilde{U}(f_j), \widetilde{U}(h_j)\rangle = \frac{1}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^{n+1}} \sum_j \sum_r \left(\widetilde{U}(f_j), \widetilde{U}(h_j)\right) e^{-(x^r)_0^2} e^{-|\underline{x}^r|^2} dx_0^r\, d\underline{x}^r \tag{3.3}$$

Note that $\sum_j \sum_r \left(e^{-ix_0^r \underline{p}^2} \hat{f}_j\left(\underline{p}^2\right), e^{-ix_0^r \underline{q}^2} \hat{h}_j\left(\underline{q}^2\right)\right) = \sum_j \sum_r e^{-ix_0^r\left(\underline{p}^2 + \underline{q}^2\right)} \left(\hat{f}_j\left(\underline{p}^2\right), \hat{h}_j\left(\underline{q}^2\right)\right)$ because $\underline{p}^2$ is a 1-vector in $\mathbb{R}_n$, which implies $\overline{i\underline{p}^2} = i\underline{p}^2$. It follows from (3.2) that

$$\left(\widetilde{U}\sum_j (f_j), \widetilde{U}\sum_j (h_j)\right)$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \sum_j \sum_r e^{i\left(\underline{q}^2-\underline{p}^2,\underline{x}^r\right)} e^{-\frac{\left|\underline{p}^2\right|^2+\left|\underline{q}^2\right|^2}{2}} e^{-ix_0^r\left(\underline{p}^2+\underline{q}^2\right)} \left(\hat{f}_j\left(\underline{p}^2\right), \hat{h}_j\left(\underline{q}^2\right)\right) d\underline{p}^2\, d\underline{q}^2.$$

Substitute this into (3.3) and interchange the order of integration by Fubini's theorem. We calculate the integral with respect to $d\underline{x}^r$ and dx_0^r first. The Fourier transform yields the following integrals

$$\int_{\mathbb{R}^n} \sum_r e^{i\left(\underline{q}^2-\underline{p}^2,\underline{x}^r\right)} e^{-\left|\underline{x}^r\right|^2} d\underline{x}^r = \pi^{\frac{n}{2}} e^{-\frac{\left|\underline{q}^2-\underline{p}^2\right|^2}{4}}. \tag{3.4}$$

$$\int_{\mathbb{R}} \sum_r e^{-ix_0^r\left(\underline{p}^2+\underline{q}^2\right)} e^{-(x^r)_0^2}\, dx_0^r = \sqrt{\pi} e^{\frac{\left|\underline{p}^2+\underline{q}^2\right|^2}{4}} \tag{3.5}$$

Putting (3.4) and (3.5) in (3.3) and applying the Parallelogram Law, we have

$$\langle \widetilde{U}(f_j), \widetilde{U}(h_j)\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \sum_j e^{\frac{\left|\underline{p}^2+\underline{q}^2\right|^2-\left|\underline{q}^2-\underline{p}^2\right|^2}{4}} e^{-\frac{\left|\underline{p}^2\right|^2+\left|\underline{q}^2\right|^2}{2}} \left(\hat{f}_j\left(\underline{p}^2\right), \hat{h}_j\left(\underline{q}^2\right)\right) d\underline{p}^2\, d\underline{q}^2$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \sum_j e^{-\frac{\left|\underline{q}^2-\underline{p}^2\right|^2}{2}} \left(\hat{f}_j\left(\underline{p}^2\right), \hat{h}_j\left(\underline{q}^2\right)\right) d\underline{p}^2\, d\underline{q}^2.$$

This establishes the isometry part of $\widetilde{U}$.

Proof of Surjectivity (see [20]). Let $\{H_{\beta^2} : \beta^2 \in \mathbb{N}_0^n\}$ denote the orthogonal basis of $L^2(\mathbb{R}^n, d\rho)$ consisting of n-dimensional Hermite polynomials, each of which is a product of 1-dimensional Hermite polynomials, i.e. for

$$H_{\beta^2}(\underline{x}^r) = H_{\beta_1^2}(x_1^r)\ldots H_{\beta_n^2}(x_n^r)$$

for $\beta^2 = (\beta_1^2,\ldots,\beta_n^2) \in \mathbb{N}_0^n$. Morever, $\left\|H_{\beta^2}\right\|^2 = \beta^2! = \beta_1^2!\ldots\beta_n^2!$. Details about Hermite polynomials can be found in standard literatures, e.g. [6]. It can be directly computed that

$$e^{\frac{\triangle}{2}} H_{\beta^2} = (\underline{x}^r)^{\beta^2} = (x^r)_1^{\beta_1^2}\ldots(x^r)_n^{\beta_n^2}. \tag{3.6}$$

Let $G_j \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. Then $g_j(\underline{x}^r) = G_j(0,\underline{x}^r)$ is an analytic functions and hence it has a Taylor expansion with infinite radius of convergence

$$g_j(\underline{x}^r) = \sum_A \sum_{\beta^2\in\mathbb{N}_0^n} \sum_j \sum_r \alpha_{\beta^2,A}^j\, (\underline{x}^r)^{\beta^2}\, e_A.$$

Take

$$f_j = \sum_A \sum_{\beta^2\in\mathbb{N}_0^n} \sum_j \alpha_{\beta^2,A}^j\, H_{\beta^2}\, e_A.$$

where the series is taken in the L^2-norm sense. Then $f_j \in L^2(\mathbb{R}^n, d\rho) \otimes \mathbb{C}_n$. Since $\widetilde{U}$ is bounded, we can pass it inside the summations:

$$\widetilde{U}\left(\sum_j (f_j)\right) = \sum_A \sum_{\beta^2\in\mathbb{N}_0^n} \sum_j \alpha_{\beta^2,A}^j\, \widetilde{U}\left(H_{\beta^2}\right) e_A.$$

Since $\widetilde{U}\left(\sum_j (f_j)\right)$ is monogenic, we can evaluate its value at $x_0^r = 0$.

$$\widetilde{U}\left(\sum_j \sum_r (f_j)(0,\underline{x}^r)\right) = \sum_A \sum_{\beta^2\in\mathbb{N}_0^n} \sum_j \alpha_{\beta^2,A}^j\, e^{\frac{\triangle}{2}}\left(H_{\beta^2}\right) e_A = \sum_A \sum_{\beta^2\in\mathbb{N}_0^n} \sum_j \sum_r \alpha_{\beta^2,A}^j \left(\underline{x}^r\right)^{\beta^2} e_A.$$

Hence $\widetilde{U}\left(\sum_j \sum_r (f_j)(0,\underline{x}^r)\right) = g_j(\underline{x}^r)$, which implies $\widetilde{U}\left(\sum_j (f_j)\right) = G_j$. We have established that $\widetilde{U}$ is a unitary map from $L^2(\mathbb{R}^n, d\rho) \otimes \mathbb{C}_n$ onto $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$.

Since $\widetilde{U}$ is a unitary map, it follows that $\left\{\widetilde{U}\left(H_{\beta^2}\right) : \beta^2 \in \mathbb{N}_0^n\right\}$ is an orthogonal basis for $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. Define

$$P_{\beta^2}(x_0^r, \underline{x}^r) = \widetilde{U}(H_{\beta^2})(x_0^r, \underline{x}^r) = e^{-x_0^r \underline{D}} \left(\underline{x}^r\right)^{\beta^2}. \quad (3.7)$$

Moreover, $\|P_{\beta^2}\|^2 = \|\widetilde{U}(H_{\beta^2})\|^2 = \|H_{\beta^2}\|^2 = \beta^2!$. We put it into the following Corollary.

Corollary 3.1. $\{P_{\beta^2} : \beta^2 \in \mathbb{N}_0^n\}$ is an orthogonal basis for $\mathcal{ML}^2(\mathbb{R}^{n+1}, d\tilde{\mu})$ and $\|P_{\beta^2}\|^2 = \beta^2!$ for each $\beta^2 \in \mathbb{N}_0^n$.

IV. Clifford Algebra-Valued Fock Space

For $X = (\mathbb{R}^n)^*$, the real dual space of $\mathbb{R}^n$. Denote by $X^{\odot k}$ the algebraic symmetric k-tensor product of X. We will write $\mathrm{Sym}(X)$ for the algebraic symmetric tensor algebra over X, i.e. $\mathrm{Sym}(X)$ is the weak direct sum $\sum_{k=0}^{\infty} X^{\odot k}$ consisting of elements of the form $\sum_{k=0}^{\infty} \sum_j \alpha_k^j$, where each $\alpha_k^j \in X^{\odot k}$ and $(\alpha_j)_k = 0$ for all but finitely many k. Each $\alpha_k^j \in X^{\odot k}$ has a natural norm given by

$$\left|(\alpha_j)_k\right|^2 = \sum_{\substack{0 \le \beta_1^2, \ldots, \beta_n^2 \le k \\ \beta_1^2 + \cdots + \beta_n^2 = k}} \sum_j \frac{1}{\beta_1^2! \ldots \beta_n^2!} \left|\alpha_k^j \left(e_1^{\odot \beta_1^2} \odot \ldots \odot e_n^{\odot \beta_n^2}\right)\right|^2 = \sum_{\substack{|\beta^2| = k \\ \beta^2 \in \mathbb{N}_0^n}} \sum_j \frac{1}{\beta^2!} \left|\alpha_k^j \left(e^{\beta^2}\right)\right|^2,$$

where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$. We define $\mathcal{F}(X)$ to be the Hilbert space completion of $\mathrm{Sym}(X)$ with respect to the norm

$$\|\alpha_j\| = \left(\sum_{k=0}^{\infty} \sum_j \left|\alpha_k^j\right|^2\right)^{\frac{1}{2}}$$

and call it the covariant Fock space.

We can repeat the construction above for the Clifford algebra-valued symmetric tensor algebra, which will be identified with $\mathcal{F}(X) \otimes \mathbb{C}_n$ and called the $\mathbb{C}_n$-valued covariant Fock space. An element in $\mathcal{F}(X) \otimes \mathbb{C}_n$ is a strong sum $\sum_{k=0}^{\infty} \sum_j \alpha_k^j$, where each $\alpha_k^j \in X^{\odot k} \otimes \mathbb{C}_n$ and such that

$$\|\alpha_j\|^2 = \sum_{k=0}^{\infty} \sum_j \left|\alpha_k^j\right|^2 < \infty.$$

Definition 4.1. Let $f_j : \mathbb{R}^{n+1} \to \mathbb{C}_n$ be a monogenic function. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. For each $(x_0^r, \underline{x}^r) \in \mathbb{R}^{n+1}$, define the directional derivative operator $D\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right)$ to be a linear map on $\mathbb{R}^n$ such that $D\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)(e_i)\right) = \partial_{e_i}\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right)$ for each i. More generally, for each $k \in \mathbb{N}$, define a k-linear map $D^k(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)$ on $(\mathbb{R}^n)^k$ by

$$D^k\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)(e_{i_1}, \ldots, e_{i_k})\right) = \left(\partial_{e_{i_1}} \ldots \partial_{e_{i_k}}\right)\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right).$$

Since f_j is smooth, $D^k(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)$ is a symmetric k-linear map on $(\mathbb{R}^n)^k$.

This induces a $\mathbb{C}_n$-valued linear map on $(\mathbb{R}^n)^{\odot k}$ which is also denoted by $D^k(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)$. Thus for any $u_1^2, \ldots, u_l^2 \in \mathbb{R}^n$,

$$D^k(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)(u_1^2 \odot \cdots \odot u_l^2) = \begin{cases} \left(\partial_{u_1^2} \ldots \partial_{u_k^2}\right)\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right) & \text{if } l = k; \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for each $k \in \mathbb{N}_0$, the norm of $D^k\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right)$ is

$$\left|D^k\left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right)\right|^2 = \sum_{\substack{|\beta^2| = k \\ \beta^2 \in \mathbb{N}_0^n}} \sum_j \sum_r \frac{1}{\beta^2!} \left|\partial^{\beta^2} f_j(x_0^r, \underline{x}^r)\right|^2,$$

where $\partial^{\beta^2} = \partial_1^{\beta_1^2} \dots \partial_n^{\beta_n^2}$ for $\beta^2 = (\beta_1^2, \dots, \beta_n^2)$. Next, we identify $D^k \left(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)\right)$ as an element of $(X)^{\odot k} \otimes \mathbb{C}_n$. With $D^0(\sum_j \sum_r f_j(x_0^r, \underline{x}^r)$ defined as $f_j(x_0^r, \underline{x}^r)$, it is natural to write

$$(1 - D)^{-1}_{(x_0^r, \underline{x}^r)}\left(\sum_j f_j\right) = \sum_{k=0}^{\infty} \sum_j \sum_r D^k f_j(x_0^r, \underline{x}^r).$$

Then $(1 - D)^{-1}_{(x_0^r, \underline{x}^r)}\left(\sum_j f_j\right) \in \mathcal{F}(X) \otimes \mathbb{C}_n$ if

$$\left\|(1 - D)^{-1}_{(x_0^r, \underline{x}^r)}\left(\sum_j f_j\right)\right\|^2 := \sum_{k=0}^{\infty} \sum_j \sum_r \left|D^k f_j(x_0^r, \underline{x}^r)\right|^2 < \infty.$$

For simplicity, we write $(1-D)^{-1}$ instead of $(1-D)^{-1}_{(0,\underline{0})}$. We will prove Theorem 1.3, which states that the map $(1-D)^{-1}$ is a unitary isomorphism from $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$ onto $\mathcal{F}(X) \otimes \mathbb{C}_n$.

Proof of Theorem 1.3 (see [20]). First, we show that $(1 - D)^{-1}$ is an isometry. Let $F \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. By Corollary 3.1, the orthogonality of $\{P_{\beta^2}\}$ implies

$$F_j = \sum_{\beta^2 \in \mathbb{N}_0^n} \sum_j \omega_{\beta^2}^j P_{\beta^2} \quad \text{and} \quad \left\|F_j\right\|^2 = \sum_{\beta \in \mathbb{N}_0^n} \sum_j \beta^2! \left|\omega_{\beta^2}^j\right|^2$$

where $\omega_{\beta^2}^j \in \mathbb{C}_n$ for each β and the first sum converges in $L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) \otimes \mathbb{C}_n$ sense and also converges uniformly on compact sets by monogenicity of F_j. Since $P_{\beta^2}(x_0^r, \underline{x}^r) = e^{-x_0^r \underline{D}} \left(\underline{x}^r\right)^{\beta^2}$ and any partial differential operator commutes with $\underline{D}$ and hence with $e^{-x_0^r \underline{D}}$, it follows that $\partial^{\alpha_j} P_{\beta^2}(0, \underline{0}) = \beta^2! \, \delta_{\alpha_j \beta^2}$. Thus

$$\left|D^k\left(\sum_j F_j(0, \underline{0})\right)\right|^2 = \sum_{|\beta^2|=k} \sum_j \frac{1}{\beta^2!}\left|\partial^{\beta^2} F_j(0, \underline{0})\right|^2 = \sum_{|\beta^2|=k} \sum_j \beta^2! \left|\omega_{\beta^2}^j\right|^2$$

Hence

$$\left\|(1 - D)^{-1}\left(\sum_j F_j\right)\right\|^2 = \sum_{k=0}^{\infty} \sum_j \left|D^k F_j(0, \underline{0})\right|^2 = \sum_{\beta^2 \in \mathbb{N}_0^n} \sum_j \beta^2! \left|\omega_{\beta^2}^j\right|^2 = \left\|F_j\right\|^2.$$

This establishes the isometry of $(1-D)^{-1}$. Next, we show that $(1-D)^{-1}$ is surjective. Let $\alpha_j \in \mathcal{F}(X) \otimes \mathbb{C}_n$. Then $\alpha_j = \sum_{k=0}^{\infty} \sum_j \alpha_k^j$ where $\alpha_k^j \in X^{\odot k} \otimes \mathbb{C}_n$ and $\left\|\alpha_j\right\|^2 = \sum_{k=0}^{\infty} \sum_j \left|\alpha_k^j\right|^2 < \infty$. For each $\underline{x}^r \in \mathbb{R}^n$, define $\exp_k (\underline{x}^r) \in (\mathbb{R}^n)^{\odot k}$ by

$$\exp_k (\underline{x}^r) = \sum_{|\beta^2|=k} \sum_r \frac{1}{\beta^2!}(x_1^r e_1)^{\odot \beta_1^2} \odot \dots \odot (x_n^r e_n)^{\odot \beta_n^2} = \sum_{|\beta^2|=k} \sum_r \frac{\left(\underline{x}^r\right)^{\beta^2}}{\beta^2!} e^{\beta^2}.$$

where $e^{\beta^2} = e_1^{\odot \beta_1^2} \odot \dots \odot e_n^{\odot \beta_n^2}$. For each $k \in \mathbb{N}_0$, let $\left(f_j\right)_k(\underline{x}^r) = \alpha_k^j(\exp_k (\underline{x}^r))$ and $f_j(\underline{x}^r) = \sum_{k=0}^{\infty} \sum_j \sum_r \left(f_j\right)_k(\underline{x}^r)$. Then each $\left(f_j\right)_k$ is analytic, which implies $f_j(\underline{x}^r)$ is analytic.

Define $F_j(x_0^r, \underline{x}^r) = e^{-x_0^r \underline{D}} f_j(\underline{x}^r)$ to be the $C-K$ extension of f_j. By (3.7), we have

$$F_j(x_0^r, \underline{x}^r) = \sum_{k=0}^{\infty} \sum_{|\beta^2|=k} \sum_j \frac{1}{\beta^2!} \alpha_k^j(e^{\beta^2}) P_{\beta^2}.$$

It follows from Corollary 3.1 that

$$\left\|F_j\right\|^2 = \sum_{k=0}^{\infty} \sum_{|\beta^2|=k} \sum_j \frac{1}{\beta^2!}\left|\alpha_k^j(e^{\beta^2})\right|^2 = \sum_{k=0}^{\infty} \sum_j \left|\alpha_k^j\right|^2 = \left\|\alpha_j\right\|^2 < \infty.$$

To show that $(1 - D)^{-1}\left(\sum_j F_j\right) = \alpha_j$, let $m \in \mathbb{N}_0$. For $e^{\gamma^2} = e_1^{\odot \gamma_1^2} \odot e_2^{\odot \gamma_2^2} \odot \dots \odot e_n^{\odot \gamma_n^2}$ with $|\gamma^2| = m$, we have

$$[D^m\left(\sum_j F_j(0,\underline{0})\right)](e^{\gamma^2}) = \partial_{e_1}^{\gamma_1^2}\dots\partial_{e_n}^{\gamma_n^2}\left[\sum_r e^{-x_0^r\underline{D}}\left(\sum_{k=0}^{\infty}\sum_j \alpha_k^j\left(\exp_k\ (\underline{x}^r)\right)\right)\right](0,\underline{0})$$

$$= \partial_{e_1}^{\gamma_1^2}\dots\partial_{e_n}^{\gamma_n^2}\left[\left(\sum_{k=0}^{\infty}\sum_j\sum_r \alpha_k^j\left(\exp_k\ (\underline{x}^r)\right)\right)\right](\underline{0})$$

$$=\sum_{k=0}^{\infty}\sum_j\sum_r\left[\left(\partial_{e_1}^{\gamma_1^2}\dots\partial_{e_n}^{\gamma_n^2}\right)\alpha_k^j\left(\exp_k\ (\underline{x}^r)\right)\right](\underline{0}).$$

We can differentiate term-by-term inside the power series because $F_j(0,\underline{x}^r)$ is analytic. Since $\left(\partial_{e_1}^{\gamma_1^2}\dots\partial_{e_n}^{\gamma_n^2}\right)((\underline{x}^r)^{\beta^2}) = \beta^2!\ \delta_{\beta^2\gamma^2}$, evaluating at $\underline{x}^r = \underline{0}$ gives

$$\left(\partial_{e_1}^{\gamma_1^2}\dots\partial_{e_n}^{\gamma_n^2}\right)\left(\exp_k\ (\underline{x}^r)\right) = \begin{cases} e_1^{\odot\gamma_1^2}\odot e_2^{\odot\gamma_2^2}\odot\dots\odot e_n^{\odot\gamma_n^2}, & \text{for } k = m;\\ 0, & \text{for } k \neq m.\end{cases}$$

It follows that $D^m\left(\sum_j F_j(0,\underline{0})(e^{\gamma^2})\right) = \sum_j \alpha_m^j(e^{\gamma^2})$ for any $\gamma^2 \in \mathbb{N}_0^n$ with $|\gamma^2| = m$. Since $\{e^{\gamma^2} : \gamma^2 \in \mathbb{N}_0^n, |\gamma^2| = m\}$ forms an orthogonal basis for $(\mathbb{R}^n)^{\odot m}$, we conclude that $D^m\left(\sum_j F_j(0,\underline{0})\right) = \sum_j \alpha_m^j$ and that

$$(1 - D)^{-1}\left(\sum_j F_j\right) = \sum_{m=0}^{\infty}\sum_j D^m F_j(0,\underline{0}) = \sum_{m=0}^{\infty}\sum_j \alpha_m^j = \alpha_j.$$

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