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Research Paper

A Study of Tricomplex Numbers: Representation, Subalgebras, Idempotent Forms

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ABSTRACT:*Tricomplex numbers, as natural extensions of complex and bicomplex systems [cf.* ***1–10****], provide a broad algebraic framework enriched with diverse subalgebras and distinctive computational features. This study focuses on three main aspects: the representation of tricomplex numbers, the structure and classification of their subalgebras, and the idempotent decomposition which simplifies calculations and clarifies algebraic behavior. By employing these representations, intricate operations become more tractable, providing insights into the interplay between subalgebras and their idempotent components. The results establish a coherent framework for tricomplex analysis, paving the way for both theoretical exploration and practical applications in advanced mathematical contexts.*

KEYWORDS: *Tricomplex Numbers, Hyperbolic Numbers, Idempotent Elements, Idempotent Representation.*
2010 AMS Subject Classification:*30G35,32A30,16W10, 15A66.*

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I. INTRODUCTION

The set of T**ricomplex numbers** is defined as

$$\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3) = \{x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 : x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{C}_0\}$$

where the imaginary units i_1, i_2, i_3 are mutually commuting elements satisfying

$$i_1^2 = i_2^2 = i_3^2 = -1,$$

and are mutually distinct.

II. SUBALGEBRAS OF $\mathbb{C}(i_1, i_2, i_3)$

To facilitate computations and analysis, we introduce several subalgebrasof $\mathbb{C}(i_1, i_2, i_3)$, categorized by their dimension and algebraic type.

2.11D Subalgebras

$\mathbb{C}_0 = \mathbb{R} =$ The set of Real Numbers

2.22D Subalgebras of $\mathbb{C}(i_1, i_2, i_3)$

(i) Complex-type (unit squares -1):

- $\mathbb{C}(i_1) = \{u + i_1v : u, v \in \mathbb{C}_0\}$

- $\mathbb{C}(i_2) = \{u + i_2 v: u, v \in \mathbb{C}_0\}$
- $\mathbb{C}(i_3) = \{u + i_3 v: u, v \in \mathbb{C}_0\}$
- $\mathbb{C}(i_1 i_2 i_3) = \{u + i_1 i_2 i_3 v: u, v \in \mathbb{C}_0\}$

(ii) Hyperbolic-type (unit squares $+1$):

- $\mathbb{H}(i_1 i_2) = \{u + i_1 i_2 v: u, v \in \mathbb{C}_0\}$
- $\mathbb{H}(i_1 i_3) = \{u + i_1 i_3 v: u, v \in \mathbb{C}_0\}$
- $\mathbb{H}(i_2 i_3) = \{u + i_2 i_3 v: u, v \in \mathbb{C}_0\}$

2.34D Subalgebras (Bicomplex- type, Mixed-type or Hyperbolic-pair type)

(i) Bicomplex-type(both generators square -1):

- $\mathbb{C}(i_1, i_2) = \{x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4: x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$
- $\mathbb{C}(i_1, i_3) = \{x_1 + i_1 x_2 + i_3 x_3 + i_1 i_3 x_4: x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$
- $\mathbb{C}(i_2, i_3) = \{x_1 + i_2 x_2 + i_3 x_3 + i_2 i_3 x_4: x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

(ii) Mixed-type (one generator squares -1, the other $+1$):

- $\mathbb{C}(i_1, i_2 i_3) = \{x_1 + i_1 x_2 + i_2 i_3 x_3 + i_1 i_2 i_3 x_4: x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$
- $\mathbb{C}(i_2, i_1 i_3) = \{x_1 + i_2 x_2 + i_1 i_3 x_3 + i_1 i_2 i_3 x_4: x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$
- $\mathbb{C}(i_3, i_1 i_2) = \{x_1 + i_3 x_2 + i_1 i_2 x_3 + i_1 i_2 i_3 x_4: x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

(iii) Hyperbolic-pair type (both generators square $+1$):

- $\mathbb{H}(i_1 i_2, i_1 i_3) = \{x_1 + i_1 i_2 x_2 + i_1 i_3 x_3 + i_2 i_3 x_4: x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

2.48D Algebra:

The full tricomplex algebra

$$\mathbb{C}(i_1, i_2, i_3) = \{x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8: x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{C}_0\}$$

itself is the full 8D algebra. It contains all the above subalgebras as natural subsystems.

III. Classification of Subalgebras and Subspaces of the Tricomplex Algebra $\mathbb{C}(i_1, i_2, i_3)$:

In the previous sections, we presented the detailed descriptions of the various subalgebras and subspaces of the tricomplex algebra. For clarity and ease of reference, we now summarize this classification in tabular form, highlighting the main structural types, their generators, and their characteristic properties.

SN	Subalgebra / Subspace	Dimension	Algebraic Type	Generators / Squares
1	$\mathbb{C}_0 = \mathbb{R}$	1D	Real	1
2	$\mathbb{C}(i_1)$	2D	Complex-type	$i_1^2 - 1$
3	$\mathbb{C}(i_2)$	2D	Complex-type	$i_2^2 = -1$
4	$\mathbb{C}(i_3)$	2D	Complex-type	$i_3^2 = -1$
5	$\mathbb{C}(i_1 i_2 i_3)$	2D	Complex-type	$(i_1 i_2 i_3)^2 = -1$
6	$\mathbb{H}(i_1 i_2)$	2D	Hyperbolic-type	$(i_1 i_2)^2 = +1$
7	$\mathbb{H}(i_1 i_3)$	2D	Hyperbolic-type	$(i_1 i_3)^2 = +1$
8	$\mathbb{H}(i_2 i_3)$	2D	Hyperbolic-type	$(i_2 i_3)^2 = +1$
9	$\mathbb{C}(i_1, i_2)$	4D	Bicomplex-type	$i_1^2 = -1, i_2^2 = -1$
10	$\mathbb{C}(i_1, i_3)$	4D	Bicomplex-type	$i_1^2 = -1, i_3^2 = -1$
11	$\mathbb{C}(i_2, i_3)$	4D	Bicomplex-type	$i_2^2 = -1, i_3^2 = -1$
12	$\mathbb{C}(i_1, i_2 i_3)$	4D	Mixed-type	$i_1^2 = -1, (i_2 i_3)^2 = +1$
13	$\mathbb{C}(i_2, i_1 i_3)$	4D	Mixed-type	$i_2^2 = -1, (i_1 i_3)^2 = +1$
14	$\mathbb{C}(i_3, i_1 i_2)$	4D	Mixed-type	$i_3^2 = -1, (i_1 i_2)^2 = +1$
15	$\mathbb{H}(i_1 i_2, i_1 i_3)$	4D	Hyperbolic-pair	$(i_1 i_2)^2 = +1, (i_1 i_3)^2 = +1$
16	$\mathbb{C}(i_1, i_2, i_3)$	8D	Full Tricomplex	$i_1^2 = -1, i_2^2 = -1, i_3^2 = -1$

IV. CLASSIFICATION OF REPRESENTATIONS

A tricomplex number

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8,$$

$$\text{where } x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{C}_0$$

can be expressed in several equivalent forms depending on the choice of subalgebra.

(R1) Representation over the Real

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8,$$
$$\text{where } x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{C}_0$$

(R2) Representation in terms of $\mathbb{C}(i_1)$

$$\zeta = (x_1 + i_1 x_2) + i_2(x_3 + i_1 x_5) + i_3(x_4 + i_1 x_6) + i_2 i_3(x_7 + i_1 x_8)$$
$$= z_1 + i_2 z_2 + i_3 z_3 + i_2 i_3 z_4, \text{where} z_1, z_2, z_3, z_4 \in \mathbb{C}(i_1)$$

(R3) Representation in terms of $\mathbb{C}(\boldsymbol{i_2})$

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8$$
$$= (x_1 + i_2 x_3) + i_1(x_2 + i_2 x_5) + i_3(x_4 + i_2 x_7) + i_1 i_3(x_6 + i_2 x_8)$$
$$= w_1 + i_1 w_2 + i_3 w_3 + i_1 i_3 w_4, \text{where} w_1, w_2, w_3, w_4 \in \mathbb{C}(i_2)$$

(R4) Representation in terms of $\mathbb{C}(\boldsymbol{i_3})$

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8$$
$$= (x_1 + i_3 x_4) + i_1(x_2 + i_3 x_6) + i_2(x_3 + i_3 x_7) + i_1 i_2(x_5 + i_3 x_8)$$
$$= \omega_1 + i_1 \omega_2 + i_3 \omega_3 + i_1 i_3 \omega_4, \text{where} \omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{C}(i_3)$$

(R5) Representation in terms of $\mathbb{C}(\boldsymbol{i_1 i_2 i_3})$

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1i_2i_3x_8) + i_1(x_2 - i_1i_2i_3x_7) + i_2(x_3 - i_1i_2i_3x_6) + i_3(x_4 - i_1i_2i_3x_5)$$

$$= \psi_1 + i_1\psi_2 + i_2\psi_3 + i_3\psi_4, \text{where } \psi_1, \psi_2, \psi_3, \psi_4 \in \mathbb{C}(i_1i_2i_3)$$

(R6) Representation in terms of$\mathbb{H}(i_1i_2)$

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1i_2x_5) + i_1(x_2 - i_1i_2x_3) + i_3(x_4 + i_1i_2x_8) + i_1i_3(x_6 - i_1i_2x_7)$$

$$= \phi_1 + i_1\phi_2 + i_3\phi_3 + i_1i_3\phi_4, \text{where}\phi_1, \phi_2, \phi_3, \phi_4 \in \mathbb{H}(i_1i_2)$$

(R7) Representation in terms of$\mathbb{H}(i_1i_3)$

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1i_3x_6) + i_1(x_2 - i_1i_3x_4) + i_2(x_3 + i_1i_3x_8) + i_1i_2(x_5 - i_1i_3x_7)$$

$$= \varphi_1 + i_1\varphi_2 + i_2\varphi_3 + i_1i_2\varphi_4, \text{where}\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{H}(i_1i_3)$$

(R8) Representation in terms of$\mathbb{H}(i_2i_3)$

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2i_3x_7) + i_1(x_2 + i_2i_3x_8) + i_2(x_3 - i_2i_3x_4) + i_1i_2(x_5 - i_2i_3x_6)$$

$$= \chi_1 + i_1\chi_2 + i_2\chi_3 + i_1i_2\chi_4, \text{where } \chi_1, \chi_2, \chi_3, \chi_4 \in \mathbb{H}(i_2i_3)$$

(R9) Representation in terms of$\mathbb{C}(i_1, i_2)$

(i)$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8)$$

$$= \xi + i_3\eta, \text{where}\xi, \eta \in \mathbb{C}(i_1, i_2)$$

(ii)$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_1i_3(x_6 - i_1x_4 + i_2x_8 - i_1i_2x_7)$$

$$= \xi + i_1i_3\eta, \text{where}\xi, \eta \in \mathbb{C}(i_1, i_2)$$

(iii)$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_2i_3(x_7 + i_1x_8 - i_2x_4 - i_1i_2x_6)$$

$$= \xi + i_2i_3\eta, \text{where}\xi, \eta \in \mathbb{C}(i_1, i_2)$$

(iv)$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_1i_2i_3(x_8 - i_1x_7 - i_2x_6 + i_1i_2x_4)$$

$$= \xi + i_1i_2i_3\eta, \text{where}\xi, \eta \in \mathbb{C}(i_1, i_2)$$

(R10) Representation in terms of$\mathbb{C}(i_1, i_3)$

(i) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$$

$$= \xi + i_2\eta, \text{where}\xi, \eta \in \mathbb{C}(i_1, i_3)$$

(ii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_1i_2(x_5 - i_1x_3 + i_3x_8 - i_1i_3x_7)$$

$$= \xi + i_1i_2\eta, \text{where}\xi, \eta \in \mathbb{C}(i_1, i_3)$$

(iii)$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2i_3(x_7 + i_1x_8 - i_3x_3 - i_1i_3x_5)$$

$$= \xi + i_2i_3\eta, \text{where}\xi, \eta \in \mathbb{C}(i_1, i_3)$$

(iv) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_1i_2i_3(x_8 - i_1x_7 - i_3x_5 + i_1i_3x_3)$$

$$= \xi + i_1i_2i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_1, i_3)$$

(R11) Representation in terms of $\mathbb{C}(i_2, i_3)$

(i) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1(x_2 + i_2x_5 + i_3x_6 + i_2i_3x_8)$$

$$= \xi + i_1\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_3)$$

(ii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1i_2(x_5 - i_2x_2 + i_3x_8 - i_2i_3x_6)$$

$$= \xi + i_1i_2\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_3)$$

(iii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1i_3(x_6 + i_2x_8 - i_3x_2 - i_2i_3x_5)$$

$$= \xi + i_1i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_3)$$

(iv) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1i_2i_3(x_8 - i_2x_6 - i_3x_5 + i_2i_3x_2)$$

$$= \xi + i_1i_2i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_3)$$

(R12) Representation in terms of $\mathbb{C}(i_1, i_2i_3)$

(i) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2i_3x_7 + i_1i_2i_3x_8) + i_2(x_3 + i_1x_5 - i_2i_3x_4 - i_1i_2i_3x_6)$$

$$= \xi + i_2\eta, \text{where} \xi, \eta \in \mathbb{C}(i_1, i_2i_3)$$

(ii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2i_3x_7 + i_1i_2i_3x_8) + i_3(x_4 + i_1x_6 - i_2i_3x_3 - i_1i_2i_3x_5)$$

$$= \xi + i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_1, i_2i_3)$$

(iii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2i_3x_7 + i_1i_2i_3x_8) + i_1i_2(x_5 - i_1x_3 - i_2i_3x_6 + i_1i_2i_3x_4)$$

$$= \xi + i_1i_2\eta, \text{where} \xi, \eta \in \mathbb{C}(i_1, i_2i_3)$$

(iv) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1x_2 + i_2i_3x_7 + i_1i_2i_3x_8) + i_1i_3(x_6 - i_1x_4 - i_2i_3x_5 + i_1i_3i_2x_3)$$

$$= \xi + i_1i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_1, i_2i_3)$$

(R13) Representation in terms of $\mathbb{C}(i_2, i_1i_3)$

(i) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_1i_3x_6 + i_1i_2i_3x_8) + i_1(x_2 + i_2x_5 - i_1i_3x_4 - i_1i_2i_3x_7)$$

$$= \xi + i_2\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_1i_3)$$

(ii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_1i_3x_6 + i_1i_2i_3x_8) + i_3(x_4 + i_2x_7 - i_1i_3x_2 - i_1i_2i_3x_5)$$

$$= \xi + i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_1i_3)$$

(iii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_1i_3x_6 + i_1i_2i_3x_8) + i_1i_2(x_5 - i_2x_2 - i_1i_3x_7 + i_1i_2i_3x_4)$$

$$= \xi + i_1i_2\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_1i_3)$$

(iv) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_2x_3 + i_1i_3x_6 + i_1i_2i_3x_8) + i_2i_3(x_7 - i_2x_4 - i_1i_3x_5 + i_1i_2i_3x_2)$$

$$= \xi + i_2i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_2, i_1i_3)$$

(R14) Representation in terms of$\mathbb{C}(i_3, i_1i_2)$

(i) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_3x_4 + i_1i_2x_5 + i_1i_2i_3x_8) + i_1(x_2 + i_3x_6 - i_1i_2x_3 - i_1i_2i_3x_7)$$

$$= \xi + i_1\eta, \text{where} \xi, \eta \in \mathbb{C}(i_3, i_1i_2)$$

(ii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_3x_4 + i_1i_2x_5 + i_1i_2i_3x_8) + i_2(x_3 + i_3x_7 - i_1i_2x_2 - i_1i_2i_3x_6)$$

$$= \xi + i_2\eta, \text{where} \xi, \eta \in \mathbb{C}(i_3, i_1i_2)$$

(iii)$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_3x_4 + i_1i_2x_5 + i_1i_2i_3x_8) + i_1i_3(x_6 - i_3x_2 - i_1i_2x_7 + i_1i_2i_3x_3)$$

$$= \xi + i_1i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_3, i_1i_2)$$

(iv)$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_3x_4 + i_1i_2x_5 + i_1i_2i_3x_8) + i_2i_3(x_7 - i_3x_3 - i_1i_2x_6 - i_1i_2i_3x_2)$$

$$= \xi + i_2i_3\eta, \text{where} \xi, \eta \in \mathbb{C}(i_3, i_1i_2)$$

(R15) Representation in terms of$\mathbb{H}(i_1i_2, i_1i_3)$

(i) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7) + i_1(x_2 - i_1i_2x_3 - i_1i_3x_4 + i_2i_3x_8)$$

$$= \xi + i_1\eta, \text{where} \xi, \eta \in \mathbb{H}(i_1i_2, i_1i_3)$$

(ii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7) + i_2(x_3 - i_1i_2x_2 + i_1i_3x_8 - i_2i_3x_4)$$

$$= \xi + i_2\eta, \text{where} \xi, \eta \in \mathbb{H}(i_1i_2, i_1i_3)$$

(iii) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7) + i_3(x_4 + i_1i_2x_8 - i_1i_3x_2 - i_2i_3x_3)$$

$$= \xi + i_3\eta, \text{where} \xi, \eta \in \mathbb{H}(i_1i_2, i_1i_3)$$

(iv) $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$$= (x_1 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7) + i_1i_2i_3(x_8 + i_1i_2x_4 + i_1i_3x_3 + i_2i_3x_1)$$

$$= \xi + i_1i_2i_3\eta, \text{where} \xi, \eta \in \mathbb{H}(i_1i_2, i_1i_3)$$

V. IDEMPOTENT ELEMENT IN TRICOMPLEX NUMBER $\mathbb{C}(i_1, i_2, i_3)$[CF.9]:

In $\mathbb{C}(i_1, i_2, i_3)$ there are exactly 16 idempotent elements. The complete set is

$\{0,1, e_1, e_1^\dagger, e_2, e_2^\dagger, e_3, e_3^\dagger, e_4, e_4^\dagger, e_5, e_5^\dagger, e_6, e_6^\dagger, e_7, e_7^\dagger\}$.

For clarity, we now present the explicit expressions of all idempotent elements in **tabular form**.

SN	Notation of idempotent element	Value of idempotent element
1	*0*	*0*
2	*1*	*1*
3	e_1	$\frac{1 + i_1i_2}{2}$
4	$e_1^\dagger$	$\frac{1 - i_1i_2}{2}$
5	e_2	$\frac{1 + i_1i_3}{2}$
6	$e_2^\dagger$	$\frac{1 - i_1i_3}{2}$
7	e_3	$\frac{1 + i_2i_3}{2}$

8	$e_3^\dagger$	$\frac{1-i_2i_3}{2}$
9	e_4	$\frac{1}{4}(1+i_1i_2+i_1i_3-i_2i_3)$
10	$e_4^\dagger=1-e_4$	$\frac{1}{4}(3-i_1i_2-i_1i_3+i_2i_3)$
11	e_5	$\frac{1}{4}(1+i_1i_2-i_1i_3+i_2i_3)$
12	$e_5^\dagger=1-e_5$	$\frac{1}{4}(3-i_1i_2+i_1i_3-i_2i_3)$
13	e_6	$\frac{1}{4}(1-i_1i_2+i_1i_3+i_2i_3)$
14	$e_6^\dagger=1-e_6$	$\frac{1}{4}(3+i_1i_2-i_1i_3-i_2i_3)$
15	e_7	$\frac{1}{4}(1-i_1i_2-i_1i_3-i_2i_3)$
16	$e_7^\dagger=1-e_7$	$\frac{1}{4}(3+i_1i_2+i_1i_3+i_2i_3)$

(i) Product relations among idempotent elements

$$e_1e_1^\dagger=e_1e_6=e_1e_7=e_1^\dagger e_4=e_1^\dagger e_5=e_2e_2^\dagger=e_2e_5=e_2e_7=e_2^\dagger e_4=e_2^\dagger e_6=e_3e_3^\dagger=e_3e_4=e_3e_7=e_3^\dagger e_5$$
$$=e_3^\dagger e_6=e_4e_4^\dagger=e_4e_5=e_4e_6=e_4e_7=e_5e_5^\dagger=e_5e_6=e_5e_7=e_6e_6^\dagger=e_6e_7=e_7e_7^\dagger=e_1e_2e_3=0$$

(ii) Addition relations among idempotent elements

$$e_1+e_1^\dagger=e_2+e_2^\dagger=e_3+e_3^\dagger=e_4+e_4^\dagger=e_5+e_5^\dagger=e_6+e_6^\dagger=e_7+e_7^\dagger=e_4+e_5+e_6+e_7=1$$

VI. IDEMPOTENT REPRESENTATIONS OF SUBALGEBRAS OF $\mathbb{C}(i_1,i_2,i_3)$:

In this section, we examine the idempotent representations of all subalgebras of the algebra $\mathbb{C}(i_1,i_2,i_3)$ [**cf. 10**]. By substituting appropriate values into the general idempotent representation of $\mathbb{C}(i_1,i_2,i_3)$, we derive the complete set of idempotent representations for each subalgebra. The exposition is organized hierarchically: beginning with the scalar and two-dimensional subalgebras, proceeding to the intermediate four-dimensional structures, and culminating with the full eight-dimensional algebra. This systematic progression underscores the fundamental role of idempotent elements in decomposing the algebra into direct summands and demonstrates how more intricate subalgebras inherit and extend the decomposition patterns of the simpler ones. The subsequent subsections detail these cases, presenting explicit forms of the idempotents and highlighting their significance for the internal structure of each subalgebra.

6.1Scalar Subalgebra (Dimension 1)

The trivial subalgebraspan $\{1\}$ contains only the elements of the coefficient field.

- **Idempotents:** The only idempotents are 0and 1.
- **Representation:** No nontrivial decomposition is possible.

6.2 Two-Dimensional Subalgebras

Each two-dimensional subalgebra has the form span $\{1, g\}$, where g is the one of the following $\{i_1, i_2, i_3, i_1 i_2, i_1 i_3, i_2 i_3, i_1 i_2 i_3\}$.

6.2.1 Idempotent Representation of Complex-type (unit squares -1):

(a) Different idempotent representations of $\zeta = u + i_1 v \in \mathbb{C}(i_1)$ are given as follows

(I) $\mathbb{C}(i_1)$-idempotent representation: $\zeta = u + i_1 v$

(II) $\mathbb{C}(i_2)$-idempotent representation: $\zeta = (u - i_2 v)e_1 + (u + i_2 v)e_1^{\dagger}$

(III) $\mathbb{C}(i_3)$-idempotent representation: $\zeta = (u - i_3 v)e_2 + (u + i_3 v)e_2^{\dagger}$

(IV) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$-idempotent representations:

(i) $\zeta = (u + i_1 v)e_2 + (u - i_2 v)e_5 + (u + i_2 v)e_7$

(ii) $\zeta = (u + i_1 v)e_3 + (u - i_2 v)e_4 + (u + i_2 v)e_7$

(iii) $\zeta = (u + i_1 v)e_2^{\dagger} + (u - i_2 v)e_4 + (u + i_2 v)e_6$

(iv) $\zeta = (u + i_1 v)e_3^{\dagger} + (u - i_2 v)e_5 + (u + i_2 v)e_6$

(V) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u + i_1 v)e_1 + (u - i_3 v)e_6 + (u + i_3 v)e_7$

(ii) $\zeta = (u + i_1 v)e_3 + (u - i_3 v)e_4 + (u + i_3 v)e_7$

(iii) $\zeta = (u + i_1 v)e_1^{\dagger} + (u - i_3 v)e_4 + (u + i_3 v)e_5$

(iv) $\zeta = (u + i_1 v)e_3^{\dagger} + (u + i_3 v)e_5 + (u - i_3 v)e_6$

(VI) Mixed $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u - i_2 v)e_1 + (u - i_3 v)e_6 + (u + i_3 v)e_7$

(ii) $\zeta = (u + i_3 v)e_2^{\dagger} + (u - i_2 v)e_4 + (u + i_2 v)e_6$

(iii) $\zeta = (u + i_2 v)e_1^{\dagger} + (u - i_3 v)e_4 + (u + i_3 v)e_5$

(iv) $\zeta = (u - i_3 v)e_2 + (u - i_2 v)e_5 + (u + i_2 v)e_7$

(b) Different idempotent representations of $\zeta = u + i_2 v \in \mathbb{C}(i_2)$ are given as follows

(I) $\mathbb{C}(i_1)$-idempotent representation: $\zeta = (u - i_1 v)e_1 + (u + i_1 v)e_1^{\dagger}$

(II) $\mathbb{C}(i_2)$-idempotent representation: $\zeta = u + i_2 v$

(III) $\mathbb{C}(i_3)$-idempotent representation: $\zeta = (u - i_3 v)e_3 + (u + i_3 v)e_3^{\dagger}$

(IV) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$-idempotent representations:

(i) $\zeta = (u + i_2 v)e_2^{\dagger} + (u - i_1 v)e_4 + (u + i_1 v)e_6$

(ii) $\zeta = (u + i_2 v)e_3^{\dagger} + (u - i_1 v)e_5 + (u + i_1 v)e_6$

(iii) $\zeta = (u + i_2 v)e_2 + (u - i_1 v)e_5 + (u + i_1 v)e_7$

(iv) $\zeta = (u + i_2 v)e_3 + (u - i_1 v)e_4 + (u + i_1 v)e_7$

(V) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u - i_1 v)e_1 + (u - i_3 v)e_6 + (u + i_3 v)e_7$

(ii) $\zeta = (u + i_3 v)e_3^{\dagger} + (u - i_1 v)e_5 + (u + i_1 v)e_6$

(iii) $\zeta = (u + i_1 v)e_1^{\dagger} + (u + i_3 v)e_4 + (u - i_3 v)e_5$

(iv) $\zeta = (u - i_3v)e_3 + (u - i_1v)e_4 + (u + i_1v)e_7$

(VI) Mixed $\mathbb{C}(i_2)$and$\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u + i_2v)e_1 + (u - i_3v)e_6 + (u + i_3v)e_7$

(ii) $\zeta = (u + i_2v)e_2 + (u - i_3v)e_5 + (u + i_3v)e_7$

(iii) $\zeta = (u + i_2v)e_1^\dagger + (u + i_3v)e_4 + (u - i_3v)e_5$

(iv) $\zeta = (u + i_2v)e_2^\dagger + (u + i_3v)e_4 + (u - i_3v)e_6$

(c) Different idempotent representations of $\zeta = u + i_3v \in \mathbb{C}(i_3)$ are given as follows

(I) $\mathbb{C}(i_1)$-idempotent representation: $\zeta = (u - i_1v)e_2 + (u + i_1v)e_2^\dagger$

(II) $\mathbb{C}(i_2)$-idempotent representation: $\zeta = (u - i_2v)e_3 + (u + i_2v)e_3^\dagger$

(III) $\mathbb{C}(i_3)$-idempotent representation: $\zeta = u + i_3v$

(IV) Mixed $\mathbb{C}(i_1)$and$\mathbb{C}(i_2)$-idempotent representations:

(i) $\zeta = (u - i_1v)e_2 + (u - i_2v)e_5 + (u + i_2v)e_7$

(ii) $\zeta = (u + i_2v)e_3^\dagger + (u + i_1v)e_5 + (u - i_1v)e_6$

(iii) $\zeta = (u + i_1v)e_2^\dagger + (u + i_2v)e_4 + (u - i_2v)e_6$

(iv) $\zeta = (u - i_2v)e_3 + (u - i_1v)e_4 + (u + i_1v)e_7$

(V) Mixed $\mathbb{C}(i_1)$and$\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u + i_3v)e_1^\dagger + (u - i_1v)e_4 + (u + i_1v)e_5$

(ii) $\zeta = (u + i_3v)e_3^\dagger + (u + i_1v)e_5 + (u - i_1v)e_6$

(iii) $\zeta = (u + i_3v)e_1 + (u - i_1v)e_6 + (u + i_1v)e_7$

(iv) $\zeta = (u + i_3v)e_3 + (u - i_1v)e_4 + (u + i_1v)e_7$

(VI) Mixed $\mathbb{C}(i_2)$and$\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u + i_3v)e_1^\dagger + (u + i_2v)e_4 + (u - i_2v)e_5$

(ii) $\zeta = (u + i_3v)e_2^\dagger + (u + i_2v)e_4 + (u - i_2v)e_6$

(iii) $\zeta = (u + i_3v)e_1 + (u - i_2v)e_6 + (u + i_2v)e_7$

(iv) $\zeta = (u + i_3v)e_2 + (u - i_2v)e_5 + (u + i_2v)e_7$

(d) Different idempotent representations of $\zeta = u + i_1i_2i_3v \in \mathbb{C}(i_1i_2i_3)$ are given as follows

(I) $\mathbb{C}(i_1)$-idempotent representation: $\zeta = (u + i_1v)e_3 + (u - i_1v)e_3^\dagger$

(II) $\mathbb{C}(i_2)$-idempotent representation: $\zeta = (u + i_2v)e_2 + (u - i_2v)e_2^\dagger$

(III) $\mathbb{C}(i_3)$-idempotent representation: $\zeta = (u + i_3v)e_1 + (u - i_3v)e_1^\dagger$

(IV) Mixed $\mathbb{C}(i_1)$and$\mathbb{C}(i_2)$-idempotent representations:

(i) $\zeta = (u - i_2v)e_2^\dagger + (u - i_1v)e_4 + (u + i_1v)e_6$

(ii) $\zeta = (u + i_1v)e_3 + (u + i_2v)e_4 + (u - i_2v)e_7$

(iii) $\zeta = (u + i_2v)e_2 + (u + i_1v)e_5 + (u - i_1s_4)e_7$

(iv) $\zeta = (u - i_1v)e_2^\dagger + (u - i_2v)e_5 + (u + i_2v)e_6$

(V) Mixed $\mathbb{C}(i_1)$and$\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u - i_3 v)e_1^{\dagger} + (u - i_1 v)e_4 + (u + i_1 v)e_5$

(ii) $\zeta = (u + i_1 v)e_3 + (u + i_3 v)e_4 + (u - i_3 v)e_7$

(iii) $\zeta = (u + i_3 v)e_1 + (u + i_1 v)e_6 + (u - i_1 v)e_7$

(iv) $\zeta = (u - i_1 v)e_3^{\dagger} + (u + i_3 v)e_5 + (u - i_3 v)e_6$

(VI) Mixed $\mathbb{C}(i_2)$and$\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (u - i_3 v)e_1^{\dagger} + (u + i_2 v)e_4 + (u - i_2 v)e_5$

(ii) $\zeta = (u + i_2 v)e_2 + (u + i_3 v)e_5 + (u - i_3 v)e_7$

(iii) $\zeta = (u + i_3 v)e_1 + (u + i_2 v)e_6 + (u - i_2 v)e_7$

(iv) $\zeta = (u - i_2 v)e_2^{\dagger} + (u + i_3 v)e_4 + (u - i_3 v)e_6$

6.2.2 Hyperbolic-type (unit squares $+1$)

(a) The idempotent representation of $\zeta = u + i_1 i_2 v \in \mathbb{H}(i_1 i_2)$is given by

$$\zeta = (u + v)e_1 + (u - v)e_1^{\dagger}$$

(b) The idempotent representation of $\zeta = u + i_1 i_3 v \in \mathbb{H}(i_1 i_3)$is given by

$$\zeta = (u + v)e_2 + (u - v)e_2^{\dagger}$$

(c) The idempotent representation of $\zeta = u + i_2 i_3 v \in \mathbb{H}(i_2 i_3)$is given by

$$\zeta = (u + v)e_3 + (u - v)e_3^{\dagger}$$

6.3 Four Dimensional Subalgebras

6.3.1 Bicomplex-type(both generators square -1):

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \in \mathbb{C}(i_1, i_2)$$

(I) $\mathbb{C}(i_1)$-idempotent representation:

$$\zeta = \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_1 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_1^{\dagger}$$

(II) $\mathbb{C}(i_2)$-idempotent representation:

$$\zeta = \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_1 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_1^{\dagger}$$

(III) $\mathbb{C}(i_3)$-idempotent representation:

$$\zeta = \{(x_1 + x_4) - i_3(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_3(x_2 - x_3)\}e_5 + \{(x_1 - x_4) - i_3(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_3(x_2 + x_3)\}e_7$$

(IV) Mixed $\mathbb{C}(i_1)$and$\mathbb{C}(i_2)$-idempotent representations:

(i) $\zeta = \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_4 + \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_5 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_7$

(ii) $\zeta = \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_5 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_7$

(iii) $\zeta = \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_5 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_7$

(iv) $\zeta = \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_4 + \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_5 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_7$

(V) Mixed $\mathbb{C}(\boldsymbol{i_1})and\mathbb{C}(\boldsymbol{i_3})$-idempotent representations:

(i) $\zeta = \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_1 + \{(x_1 - x_4) - i_3(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_3(x_2 + x_3)\}e_7$

(ii) $\zeta = \{(x_1 + x_4) - i_3(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_5$
$+\{(x_1 - x_4) + i_1(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_3(x_2 + x_3)\}e_7$

(iii) $\zeta = \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_1^\dagger + \{(x_1 + x_4) - i_3(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_3(x_2 - x_3)\}e_5$

(iv) $\zeta = \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_3(x_2 - x_3)\}e_5$
$+\{(x_1 - x_4) - i_3(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_7$

(VI) Mixed $\mathbb{C}(\boldsymbol{i_2})and\mathbb{C}(\boldsymbol{i_3})$-idempotent representations:

(i) $\zeta = \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_1 + \{(x_1 - x_4) - i_3(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_3(x_2 + x_3)\}e_7$

(ii) $\zeta = \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_3(x_2 - x_3)\}e_5$
$+\{(x_1 - x_4) + i_2(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_3(x_2 + x_3)\}e_7$

(iii) $\zeta = \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_1^\dagger + \{(x_1 + x_4) - i_3(x_2 - x_3)\}e_4 + \{(x_1 + x_4) + i_3(x_2 - x_3)\}e_5$

(iv) $\zeta = \{(x_1 + x_4) - i_3(x_2 - x_3)\}e_4 + \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_5$
$+\{(x_1 - x_4) - i_3(x_2 + x_3)\}e_6 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_7$

Similarly, we can determine the idempotent representation of $\mathbb{C}(i_1, i_3)$and $\mathbb{C}(i_2, i_3)$.

6.3.2Mixed-type (one generator squares -1, the other $+1$):

The idempotent representation of $\zeta = x_1 + i_1x_2 + i_2i_3x_3 + i_1i_2i_3x_4 \in \mathbb{C}(i_1, i_2i_3)$ is given by

(I) $\mathbb{C}(\boldsymbol{i_1})$-idempotent representation: $\zeta = \{(x_1 + x_3) + i_1(x_2 + x_4)\}e_3 + \{(x_1 - x_3) + i_1(x_2 - x_4)\}e_3^\dagger$

(II) $\mathbb{C}(\boldsymbol{i_2})$-idempotent representation:

$\zeta = \{(x_1 - x_3) - i_2(x_2 - x_4)\}e_4 + \{(x_1 + x_3) - i_2(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) + i_2(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_2(x_2 - x_4)\}e_7$

(III) $\mathbb{C}(\boldsymbol{i_3})$-idempotent representation:

$\zeta = \{(x_1 - x_3) - i_3(x_2 - x_4)\}e_4 + \{(x_1 + x_3) + i_3(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) - i_3(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_3(x_2 - x_4)\}e_7$

(IV) Mixed $\mathbb{C}(\boldsymbol{i_1})and\mathbb{C}(\boldsymbol{i_2})$-idempotent representations:

(i) $\zeta = \{(x_1 - x_3) + i_1(x_2 - x_4)\}e_4 + \{(x_1 + x_3) - i_2(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) + i_1(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_2(x_2 - x_4)\}e_7$

(ii) $\zeta = \{(x_1 + x_3) + i_1(x_2 + x_4)\}e_3 + \{(x_1 - x_3) - i_2(x_2 - x_4)\}e_4 + \{(x_1 - x_3) + i_2(x_2 - x_4)\}e_7$

(iii) $\zeta = \{(x_1 - x_3) - i_2(x_2 - x_4)\}e_4 + \{(x_1 + x_3) + i_1(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) + i_2(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_1(x_2 - x_4)\}e_7$

(iv) $\zeta = \{(x_1 - x_3) + i_1(x_2 - x_4)\}e_3^\dagger + \{(x_1 + x_3) - i_2(x_2 + x_4)\}e_5 + \{(x_1 + x_3) + i_2(x_2 + x_4)\}e_6$

(V) Mixed $\mathbb{C}(\boldsymbol{i_1})and\mathbb{C}(\boldsymbol{i_3})$-idempotent representations:

(i) $\zeta = \{(x_1 - x_3) + i_1(x_2 - x_4)\}e_4 + \{(x_1 + x_3) + i_1(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) - i_3(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_3(x_2 - x_4)\}e_7$

(ii) $\zeta = \{(x_1 + x_3) + i_1(x_2 + x_4)\}e_3 + \{(x_1 - x_3) - i_3(x_2 - x_4)\}e_4 + \{(x_1 - x_3) + i_3(x_2 - x_4)\}e_7$

(iii) $\zeta = \{(x_1 - x_3) - i_3(x_2 - x_4)\}e_4 + \{(x_1 + x_3) + i_3(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) + i_1(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_1(x_2 - x_4)\}e_7$

(iv) $\zeta = \{(x_1 - x_3) + i_1(x_2 - x_4)\}e_3^\dagger + \{(x_1 + x_3) + i_3(x_2 + x_4)\}e_5 + \{(x_1 + x_3) - i_3(x_2 + x_4)\}e_6$

(VI) Mixed $\mathbb{C}(i_2)$and$\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = \{(x_1 - x_3) - i_2(x_2 - x_4)\}e_4 + \{(x_1 + x_3) - i_2(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) - i_3(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_3(x_2 - x_4)\}e_7$

(ii) $\zeta = \{(x_1 - x_3) - i_2(x_2 - x_4)\}e_4 + \{(x_1 + x_3) + i_3(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) + i_2(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_3(x_2 - x_4)\}e_7$

(iii) $\zeta = \{(x_1 - x_3) - i_3(x_2 - x_4)\}e_4 + \{(x_1 + x_3) + i_3(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) + i_2(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_2(x_2 - x_4)\}e_7$

(iv) $\zeta = \{(x_1 - x_3) - i_3(x_2 - x_4)\}e_4 + \{(x_1 + x_3) - i_2(x_2 + x_4)\}e_5$
$+\{(x_1 + x_3) - i_3(x_2 + x_4)\}e_6 + \{(x_1 - x_3) + i_2(x_2 - x_4)\}e_7$

Similarly, we can determine idempotent representations of $\mathbb{C}(i_2, i_1i_3)$and$\mathbb{C}(i_3, i_1i_2)$.

6.3.3Hyperbolic-pair type (both generators square $+1$):

Every element $\zeta = x_1 + i_1i_2x_2 + i_1i_3x_3 + i_2i_3x_4 \in \mathbb{H}(i_1i_2, i_1i_3)$ can be uniquely expressed in the **idempotent decomposition**

$$\zeta = (x_1 + x_2 + x_3 - x_4)e_4 + (x_1 + x_2 - x_3 + x_4)e_5 + (x_1 - x_2 + x_3 + x_4)e_6 + (x_1 - x_2 - x_3 - x_4)e_7$$

Equivalently, by introducing scalar coefficients

$$r_1 = x_1 + x_2 + x_3 - x_4,$$
$$r_2 = x_1 + x_2 - x_3 + x_4,$$
$$r_3 = x_1 - x_2 + x_3 + x_4,$$
$$r_4 = x_1 - x_2 - x_3 - x_4,$$

We obtain the compact representation

$$\zeta = r_1e_4 + r_2e_5 + r_3e_6 + r_4e_7; \quad r_1, r_2, r_3, r_4 \in \mathbb{C}_0.$$

6.3.3.1 Important Observation:

In the bicomplex–type algebra $\mathbb{C}(i_1, i_2), \mathbb{C}(i_1, i_3), \mathbb{C}(i_2, i_3)$ the associated hyperbolic subalgebras $\mathbb{H}(i_1i_2)$, $\mathbb{H}(i_1i_3), \mathbb{H}(i_2, i_3)$, whose idempotent decomposition involves coefficients that are always real.

Analogously, in the tricomplex algebra, the hyperbolic-pair type subalgebras$\mathbb{H}(i_1i_2, i_1i_3)$ exhibit the same property: their idempotent representation

$$\zeta = r_1e_4 + r_2e_5 + r_3e_6 + r_4e_7; \quad r_1, r_2, r_3, r_4 \in \mathbb{C}_0,$$

always features purely real coefficients.

Thus, the tricomplex algebra naturally extends the bicomplex case by preserving the *real-coefficient idempotent decomposition* within its hyperbolic structures.

6.4Note on the Existence and Nature of Idempotent Representations in TricomplexSubalgebras

From the results presented in Sections **6.2** and **6.3**, the following observations can be made regarding the existence and naturality of idempotent representations within various subalgebras of the tricomplex algebra:

1. Complex-type algebras:

For the elements

$\zeta = u + i_1 v \in \mathbb{C}(i_1), \quad \zeta = u + i_2 v \in \mathbb{C}(i_2), \quad \zeta = u + i_3 v \in \mathbb{C}(i_3)$and $\zeta = u + i_1 i_2 i_3 v \in \mathbb{C}(i_1 i_2 i_3)$,

the**idempotent representations are not naturally possible** within their respective algebras. This is because the required idempotent elements$e_k, e_k^\dagger$**do not belong to** these algebras. Such representations become possible **only within the tricomplex algebra**, where these idempotent elements are defined and operate.

2. Hyperbolic-type algebras:

For the elements

$$\zeta = u + i_1 i_2 v \in \mathbb{H}(i_1 i_2), \zeta = u + i_1 i_3 v \in \mathbb{H}(i_1 i_3), \zeta = u + i_2 i_3 v \in \mathbb{H}(i_2 i_3)$$

the**idempotent representations exist naturally** within their respective algebras, since the corresponding idempotent elements are **intrinsic to**$\mathbb{H}(i_1 i_2), \mathbb{H}(i_1 i_3),$ and$\mathbb{H}(i_2 i_3)$ themselves.

3. Bicomplex-type algebras:

For the bicomplexsubalgebras

$$\zeta \in \mathbb{C}(i_1, i_2), \zeta \in \mathbb{C}(i_1, i_3), \quad \zeta \in \mathbb{C}(i_2, i_3)$$

only**two idempotent representations** are naturally possible in each case:

- For $\mathbb{C}(i_1, i_2)$: the $\mathbb{C}(i_1)$- and $\mathbb{C}(i_2)$-idempotent representations.
- For $\mathbb{C}(i_1, i_3)$: the $\mathbb{C}(i_1)$- and $\mathbb{C}(i_3)$-idempotent representations.
- For $\mathbb{C}(i_2, i_3)$: the $\mathbb{C}(i_2)$- and $\mathbb{C}(i_3)$-idempotent representations.

  No other idempotent forms exist naturally within these algebras.

4. Mixed-type algebras:

For the mixed subalgebras

$$\zeta = x_1 + i_1 x_2 + i_2 i_3 x_3 + i_1 i_2 i_3 x_4 \in \mathbb{C}(i_1, i_2 i_3),$$

only the$\mathbb{C}(\boldsymbol{i_1})$**-idempotent representation** is naturally possible.

Similarly,

$$\zeta \in \mathbb{C}(i_2, i_1 i_3)$$

admits only the$\mathbb{C}(\boldsymbol{i_2})$**-idempotent representation**, and

$$\zeta \in \mathbb{C}(i_3, i_1 i_2)$$

admits only the $\mathbb{C}(\boldsymbol{i_3})$**-idempotent representation** as natural.

5. Hyperbolic-pair type algebra:

For

$$\zeta = x_1 + i_1 i_2 x_2 + i_1 i_3 x_3 + i_2 i_3 x_4 \in \mathbb{H}(i_1 i_2, i_1 i_3),$$

only one natural idempotent representation exists, which can be expressed as

$$\zeta = (x_1 + x_2 + x_3 - x_4)e_4 + (x_1 + x_2 - x_3 + x_4)e_5 + (x_1 - x_2 + x_3 + x_4)e_6 + (x_1 - x_2 - x_3 - x_4)e_7.$$

From the above analysis, it is evident that:

Some subalgebras of the tricomplex system admit **natural idempotent representations**, while others do not. The existence of a natural idempotent representation depends on whether the required idempotent elements are contained within the algebra itself.

When such elements are absent, the idempotent representation becomes possible **only within the tricomplex algebra**, which encompasses all these subalgebras and their corresponding idempotent structures.

VII. CONCLUSION

Tricomplex numbers offer a versatile extension of complex and bicomplex systems, characterized by rich algebraic structures and diverse subalgebras. Through this study, we have explored their standard and component-wise representations, examined the structure and classification of subalgebras, and highlighted the utility of idempotent decomposition in simplifying computations. The idempotent framework not only provides clarity in algebraic manipulations but also facilitates a deeper understanding of the interactions between subalgebras. These insights underscore the theoretical significance of tricomplex numbers and open avenues for their application in advanced mathematical analysis and related fields.

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