Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 10 (October 2025) pp: 28-44
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Critical Sobolev Estimate of Various Singular Extension Mappings Hold under the Exponential Weak-Type

Salih Yousuf Mohamed Salih [(1)]and Shawgy Hussein[(2)]
[(1)] *University of Bakht Al-ruda, College of Science, Department of Mathematics*
[(2)] *Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan*

*Received 28 Sep., 2025; **R**evised 06 Oct., 2025; **A**ccepted 08 Oct., 2025*
P**ublished with open access at **www.questjournas.org

Abstract
Given a compact Riemannian manifold $\mathcal{N}$, following the pioneers B. Bulanyi and J. V. Schaftingen [28] we construct for every map u_i in the critical Sobolev space $W^{m/(m+1),m+1}(\mathbb{S}^m,\mathcal{N})$, where $m \in \mathbb{N}\setminus\{0\}$, a map $U_i\colon \mathbb{B}_1^{m+1} \to \mathcal{N}$ whose trace is u_i and which satisfies an exponential weak-type Sobolev estimate. The full hold result carry on to the extensions to a half-space of maps on its boundary hyperplane and to the hyperbolic space of maps on its boundary sphere at infinity.
Keywords: Extension of traces in Sobolev spaces; trace theory; Sobolev embedding theorem; weak-type Marcinkiewicz spaces; Lorentz space.

1. Introduction

It is well-known in Sobolev spaces (see [9]) the classical trace theory states that for every $m \in \mathbb{N}\setminus\{0\}$ and $0 < \epsilon < \infty$, there is a well-defined surjective trace operator $\mathrm{tr}_{\mathbb{S}^m}\colon W^{1,1+\epsilon}(\mathbb{B}_1^{m+1},\mathbb{R}) \to W^{\epsilon/1+\epsilon,1+\epsilon}(\mathbb{S}^m,\mathbb{R})$ that coincides with the restriction on the subsets of continuous functions, where the first-order Sobolev space on the unit ball $\mathbb{B}_1^{m+1} \subset \mathbb{R}^{m+1}$ is defined as

$$W^{1,1+\epsilon}(\mathbb{B}_1^{m+1},\mathbb{R}) := \left\{U_i\colon \mathbb{B}_1^{m+1} \to \mathbb{R} \mid U_i \text{ is weakly differentiable and } \int_{\mathbb{B}_1^{m+1}} \sum_i |\mathrm{D}U_i|^{1+\epsilon} < \infty\right\}$$

and the fractional Sobolev space $W^{1-\epsilon,1+\epsilon}(\mathbb{S}^m,\mathbb{R})$ on the unit sphere $\mathbb{S}^m := \partial\mathbb{B}_1^{m+1}$ is defined for $-1 < \epsilon < 0$ as

$$W^{1-\epsilon,1+\epsilon}(\mathbb{S}^m,\mathbb{R}) := \left\{u_i\colon \mathbb{S}^m \to \mathbb{R} \mid u_i \text{ is measurable and } \iint_{\mathbb{S}^m\times\mathbb{S}^m} \sum_i \frac{|u_i(x)-u_i(x+\epsilon)|^{1+\epsilon}}{|\epsilon|^{m+(1-\epsilon^2)}}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon) < \infty\right\} \quad (1)$$

Overall, the trace operator $\mathrm{tr}_{\mathbb{S}^m}$ has a continuous linear right inverse, so that, for every function $u_i \in W^{\epsilon/1+\epsilon,1+\epsilon}(\mathbb{S}^m,\mathbb{R})$ there exists a function $U_i \in W^{1,1+\epsilon}(\mathbb{B}_1^{m+1},\mathbb{R})$ such that $\mathrm{tr}_{\mathbb{S}^m} U_i = u_i$ and

$$\int_{\mathbb{B}_1^{m+1}} \sum_i |\mathrm{D}U_i|^{1+\epsilon} \le (1+\epsilon) \iint_{\mathbb{S}^m\times\mathbb{S}^m} \sum_i \frac{|u_i(x)-u_i(x+\epsilon)|^{1+\epsilon}}{|\epsilon|^{m+\epsilon}}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon) \quad (2)$$

where the $0 \le \epsilon < \infty$ in (2) depends on m and $(1+\epsilon)$ only.

Given an $\mathcal{N}$ as above, which we can assume to be isometrically embedded into $\mathbb{R}^\nu$ by Nash's isometric embedding theorem (see [16]), we consider the mappings of Sobolev spaces as

$$W^{1,1+\epsilon}(\mathbb{B}_1^{m+1},\mathcal{N}) := \{U_i \in W^{1,1+\epsilon}(\mathbb{B}_1^{m+1},\mathbb{R}^\nu) \mid U_i \in \mathcal{N} \text{ almost everywhere in } \mathbb{B}_1^{m+1}\}$$

and

$$W^{1-\epsilon,1+\epsilon}(\mathbb{S}^m,\mathcal{N}) := \{u_i \in W^{1-\epsilon,1+\epsilon}(\mathbb{S}^m,\mathbb{R}^\nu) \mid u_i \in \mathcal{N} \text{ almost everywhere in } \mathbb{S}^m\}$$

Although it follows immediately from the classical trace theory in [9] that the trace operator is well defined and continuous from $W^{1,1+\epsilon}(\mathbb{B}_1^{m+1},\mathcal{N})$ to $W^{\epsilon/1+\epsilon,1+\epsilon}(\mathbb{S}^m,\mathcal{N})$, the proof of the surjectivity fails to extend to this case: one can just prove that every mapping in $W^{\epsilon/1+\epsilon,1+\epsilon}(\mathbb{S}^m,\mathcal{N})$ can be extended to a function in $W^{1,1+\epsilon}(\mathbb{B}_1^{m+1},\mathbb{R}^\nu)$; since the extension is constructed by convolution, there is no hope that the extension would satisfy the nonlinear manifold constraint in general (see [28]).

In the case of subcritical dimension $m < \epsilon$, maps in the nonlinear Sobolev spaces $W^{\epsilon/1+\epsilon,1+\epsilon}(\mathbb{S}^m, \mathcal{N})$ and $W^{1,1+\epsilon}(\mathbb{B}_1^{m+1}, \mathcal{N})$ can be assumed to be continuous by the Sobolev-Morrey embedding; it turns out that all maps in $W^{\epsilon/1+\epsilon,1+\epsilon}(\mathbb{S}^m, \mathcal{N})$ are traces of maps in $W^{1,1+\epsilon}(\mathbb{B}_1^{m+1}, \mathcal{N})$ if and only if all continuous maps from $\mathbb{S}^m$ to $\mathcal{N}$ are restrictions of continuous mappings, or equivalently, if and only if the m^{th} homotopy group $\pi_m(\mathcal{N})$ is trivial: $\pi_m(\mathcal{N}) \simeq \{0\}$ (see [3, Th. 1]). For the critical dimension $m = \epsilon$, one gets similarly that all mappings in $W^{m/(m+1),m+1}(\mathbb{S}^m, \mathcal{N})$ are traces of maps in $W^{1,m+1}(\mathbb{B}_1^{m+1}, \mathcal{N})$ if and only if $\pi_m(\mathcal{N}) \simeq \{0\}$ (see [3, Th. 2]); this can be explained by the VMO (vanishing mean oscillation) property of maps in $W^{m/(m+1),m+1}(\mathbb{S}^m, \mathcal{N})$ and the possibility to extend homotopy and obstruction theories to such maps (see [7]).

But in the case of supercritical dimension $m > \epsilon$, the situation surely is radically different where it has been proved in a succession of works among other results that the trace operator is surjective from $W^{1,1+\epsilon}(\mathbb{B}_1^{m+1}, \mathcal{N})$ to $W^{1,\epsilon/1+\epsilon}(\mathbb{S}^m, \mathcal{N})$ if and only if the homotopy groups $\pi_1(\mathcal{N}), \ldots, \pi_{\lfloor \epsilon-1 \rfloor}(\mathcal{N})$ are finite and $\pi_{\lfloor \epsilon \rfloor}(\mathcal{N})$ is trivial [2, 3, 11, 15, 27].

For the case of subcritical or critical dimension $m \le \epsilon$, we can wonder whether the linear estimate (2) can still hold for the extension of Sobolev mappings. It has been proved that every map $u_i \in W^{\epsilon/(1+\epsilon),1+\epsilon}(\mathbb{S}^m, \mathcal{N})$ has an extension $U_i \in W^{1,1+\epsilon}(\mathbb{B}_1^{m+1}, \mathcal{N})$ with the estimate (2) when either $m = 1$ and $\pi_1(\mathcal{N}) \simeq \{0\}$ or $m \ge 2, \pi_1(\mathcal{N})$ is finite and $\pi_2(\mathcal{N}) \simeq \cdots \simeq \pi_{\lfloor 1+\epsilon \rfloor}(\mathcal{N}) \simeq \{0\}$, (see [11, 25]); hence, if there is an extension satisfying (2), then the homotopy groups $\pi_1(\mathcal{N}), \ldots, \pi_{\lfloor \epsilon \rfloor}(\mathcal{N})$ have all to be finite (see [2,15]); in the critical case $m = \epsilon$, the additional condition that $\pi_m(\mathcal{N}) \simeq \{0\}$ is necessary for a Sobolev estimate on the extension of continuous mappings that have a continuous extension (see [15]).

In view of the obstructions to linear Sobolev estimates on the extension of the form (2), one can hope to get some nonlinear estimates instead. In the subcritical dimension case, a compactness argument (see [24, Th. 4]) shows that given $m \in \mathbb{N} \setminus \{0\}, m < \epsilon$ and an $\mathcal{N}$ as above, there exists a function $\gamma \in C([0,\infty), [0,\infty))$ such that $\gamma(0) = 0$, and every map $u_i \in W^{\epsilon/(1+\epsilon),1+\epsilon}(\mathbb{S}^m, \mathcal{N})$ that has a continuous extension has an extension $U_i \in W^{1,1+\epsilon}(\mathbb{B}_1^{m+1}, \mathcal{N})$ satisfying

$$\int_{\mathbb{B}_1^{m+1}} \sum_i |DU_i|^{1+\epsilon} \le \gamma\left(\iint_{\mathbb{S}^m \times \mathbb{S}^m} \sum_i \frac{d(u_i(x), u_i(x+\epsilon))^{1+\epsilon}}{|\epsilon|^{m+\epsilon}} \, \mathrm{d}x \, \mathrm{d}(x+\epsilon)\right) \tag{3}$$

where d is the geodesic distance on $\mathcal{N}$ (see [28]).

Because of bubbling phenomena, this estimate (3) does not go to the endpoint $\epsilon = m$ when $\pi_m(\mathcal{N}) \neq \{0\}$. Indeed, any map in $W^{m/(m+1),m+1}(\mathbb{S}^m, \mathcal{N})$ is the weak limit of continuous maps in $W^{m/(m+1),m+1}(\mathbb{S}^m, \mathcal{N})$ that can be extended so that a weak compactness argument would imply that every map in $W^{m/(m+1),m+1}(\mathbb{S}^m, \mathcal{N})$ has an extension $W^{1,m+1}(\mathbb{B}_1^{m+1}, \mathcal{N})$, which cannot be the case when $\pi_m(\mathcal{N}) \not\simeq \{0\}$ (see [23, Prop. 2.8]). In this situation, it has been proved in works by [24] that there is still an extension $U_i \in W^{1,(m+1,\infty)}(\mathbb{B}_1^{m+1}, \mathcal{N})$ satisfying a weak-type Marcinkiewicz-Sobolev estimate (see also [23, 24, Th. 2]): for every $0 \le \epsilon < \infty$,

$$(1+\epsilon)^{m+1}\mathcal{L}^{m+1}(\{x \in \mathbb{B}_1^{m+1} \mid |DU_i(x)| > 1+\epsilon\}) \le \gamma\left(\iint_{\mathbb{S}^m \times \mathbb{S}^m} \sum_i \frac{d(u_i(x), u_i(x+\epsilon))^{m+1}}{|\epsilon|^{2m}} \, \mathrm{d}x \, \mathrm{d}(x+\epsilon)\right) \tag{4}$$

The function γ, appearing in the estimate (4) for a general target manifold $\mathcal{N}$, is a wild double exponential function (see [24, discussion after Th. 2]). Petrache and Rivière get similar estimates with γ being a polynomial when $m = 2$ and $\mathcal{N} = \mathbb{S}^2$ (see [23, Th. C]) (thanks to the Hopf fibration) and an exponential of a power when $m = 3$ and $\mathcal{N} = \mathbb{S}^3$ (see [23, Th. B]).

following B. Bulanyi and J. V. Schaftingen [28] we construct an extension U_i that satisfies (4), where γ can be taken to be an exponential function.

Theorem 1.1 (see [28]). Let $m \in \mathbb{N} \setminus \{0\}$ and let $\mathcal{N}$ be a compact Riemannian manifold. There exist constants $A, B, \delta \in (0,\infty)$ such that for every $u_i \in W^{m/(m+1),m+1}(\mathbb{S}^m, \mathcal{N})$, there exists a mapping $U_i \in W^{1,1}(\mathbb{B}_1^{m+1}, \mathcal{N})$ such that $\mathrm{tr}_{\mathbb{S}^m} U_i = u_i$ and for every $0 \le \epsilon < \infty$,

$$(1+\epsilon)^{m+1}\mathcal{L}^{m+1}(\{x \in \mathbb{B}_1^{m+1} \mid |DU_i(x)| \ge 1+\epsilon\})$$
$$\le A \exp \sum_i \left(B \iint_{\substack{(x,x+\epsilon) \in \mathbb{S}^m \times \mathbb{S}^m \\ d(u_i(x), u_i(x+\epsilon)) \ge \delta}} \frac{1}{|\epsilon|^{2m}} \, \mathrm{d}x \, \mathrm{d}(x+\epsilon)\right) \iint_{\mathbb{S}^m \times \mathbb{S}^m} \sum_i \frac{d(u_i(x), u_i(x+\epsilon))^{m+1}}{|\epsilon|^{2m}} \, \mathrm{d}x \, \mathrm{d}(x+\epsilon) \tag{5}$$

Moreover, one can take $U_i \in C(\mathbb{B}_1^{m+1} \setminus S, \mathcal{N})$, where the singular set $S \subset \mathbb{B}_1^{m+1}$ is a finite set whose cardinality is controlled by the right-hand side of (5).

The gap potential of the double integral appearing in the exponential in (5)

$$\iint_{\substack{(x,x+\epsilon) \in \mathbb{S}^m \\ d(u_i(x), u_i(x+\epsilon)) \ge \delta}} \sum_i \frac{1}{|\epsilon|^{2m}} \, \mathrm{d}x \, \mathrm{d}(x+\epsilon) \tag{6}$$

first appeared in estimates by Bourgain, Brezis and Mironescu on the topological degree of maps from a sphere to itself (see [4, Open problem 2; 5, Th. 1.1; 19]) (see also [22]) and in estimates on free homotopy decompositions of mappings (see [25]), as well as in estimates on liftings (see [20, Th. 2; 26]); they characterize, in the limit $\delta \to 0$, first-order Sobolev spaces (see [17, 18, 21]) and encompass a property stronger than VMO (see [6]).

Theorem 1.1 implies, in particular, the weak-type estimates of [24]. The improvement of Theorem 1.1 is two-fold: the dependence of the weak-type Sobolev bound on the extension U_i is exponential in the Gagliardo energy of u_i, which is much more reasonable than the double exponential in [24] and the nonlinear part of the estimate, that is the exponential, relies on a gap potential (6) instead of a full fractional Gagliardo energy

$$\iint_{\mathbb{S}^m\times\mathbb{S}^m}\sum_i \frac{d(u_i(x),u_i(x+\epsilon))^{1+\epsilon}}{|\epsilon|^{2m}}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon).$$

The latter controls the former by the immediate estimate

$$\iint_{\substack{(x,x+\epsilon)\in\mathbb{S}^m\times\mathbb{S}^m\\ x(u_i(x),u_i(x+\epsilon))\geq\delta}}\sum_i \frac{1}{|\epsilon|^{2m}}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon)\leq\frac{1}{\delta^{1+\epsilon}}\iint_{\mathbb{S}^m\times\mathbb{S}^m}\sum_i\frac{d(u_i(x),u_i(x+\epsilon))^{1+\epsilon}}{|\epsilon|^{2m}}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon).$$

We also have a counterpart of Theorem 1.1 for the extension on the half-space $\mathbb{R}_+^{m+1}:=\mathbb{R}^m\times(0,\infty)$ of mappings defined on the hyperplane $\mathbb{R}^m\simeq\mathbb{R}^m\times\{0\}$.

Theorem 1.2 (see [28]). Let $m\in\mathbb{N}\setminus\{0\}$ and let $\mathcal{N}$ be a compact Riemannian manifold. There exist constants $A,B,\delta\in(0,\infty)$ such that for every $u_i\in W^{m/(m+1),m+1}(\mathbb{R}^m,\mathcal{N})$, there exists $U_i\in W^{1,1}_{\mathrm{loc}}\left(\overline{\mathbb{R}}_+^{m+1},\mathcal{N}\right)$ such that $\mathrm{tr}_{\mathbb{R}^m}U_i=u_i$ and for every $0\leq\epsilon<\infty$,

$$(1+\epsilon)^{m+1}\mathcal{L}^{m+1}(\{x\in\mathbb{R}_+^{m+1}||\mathrm{D}U_i(x)\ |\geq 1+\epsilon\})$$
$$\leq A\exp\sum_i\left(B\iint_{\substack{(x,x+\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m\\ d(u_i(x),u_i(x+\epsilon))\geq\delta}}\frac{1}{|\epsilon|^{2m}}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon)\right)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\frac{d(u_i(x),u_i(x+\epsilon))^{m+1}}{|\epsilon|^{2m}}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon). \quad (7)$$

Moreover, one can take $U_i\in C(\mathbb{R}_+^{m+1}\setminus S,\mathcal{N})$, where the singular set $S\subset\mathbb{R}_+^{m+1}$ is finite whose cardinality is controlled by the right-hand side of (7).

Here $U_i\in W^{1,1}_{\mathrm{loc}}\left(\overline{\mathbb{R}}_+^{m+1},\mathcal{N}\right)$ means that U_i is weakly differentiable and that for every compact set $K\subset\overline{\mathbb{R}}_+^{m+1}$, $\int_K\sum_i\ |\mathrm{D}U_i|<\infty$. It follows from (7) that for every relatively finite-measure open set $G\subset\overline{\mathbb{R}}_+^{m+1}$ and every $q\in[1,m+1)$, we have $U_i\in W^{1,q}(G,\mathcal{N})$.

Lastly, we consider the extension of maps defined on its boundary sphere $\mathbb{S}^m$ by using the Poincaré ball model and the hyperbolic space $\mathbb{H}^{m+1}$ is the ball $\mathbb{B}_1^{m+1}$, that surely endowed with the metric g_{hyp} defined for $x\in\mathbb{B}_1^{m+1}$ in terms of the Euclidean metric g_{euc} by

$$g_{\mathrm{hyp}}(x)=\frac{4g_{\mathrm{euc}}(x)}{(1-|x|^2)^2},$$

whose boundary $\mathbb{S}^m$ is then considered to be the boundary sphere of $\mathbb{H}^{m+1}$. In this setting, we have a hyperbolic counterpart of Theorem 1.1 and Theorem 1.2.

Theorem 1.3 (see [28]). Let $m\in\mathbb{N}\setminus\{0\}$ and let $\mathcal{N}$ be a compact Riemannian manifold. There exist constants $A,B,\delta\in(0,\infty)$ such that for every $u_i\in W^{m/(m+1),m+1}(\mathbb{S}^m,\mathcal{N})$, there exists $U_i\in W^{1,1}(\mathbb{B}_1^{m+1},\mathcal{N})$ such that $\mathrm{tr}_{\mathbb{S}^m}U_i=u_i$ and for every $0\leq\epsilon<\infty$,

$$\mathcal{H}^{m+1}(\{x\in\mathbb{H}^{m+1}||\mathrm{D}U_i(x)\ |\geq 1+\epsilon\})$$
$$\leq\frac{A}{(1+\epsilon)^{m+1}}\exp\sum_i\left(B\iint_{\substack{(x,x+\epsilon)\in\mathbb{S}^m\times\mathbb{S}^m\\ d(u_i(x),u_i(x+\epsilon))\geq\delta}}\frac{1}{|\epsilon|^{2m}}\,d(x+\epsilon)\right)\iint_{\mathbb{S}^m\times\mathbb{S}^m}\frac{d(u_i(x),u_i(x+\epsilon))^{m+1}}{|\epsilon|^{2m}}\,dx\,d(x+\epsilon) \quad (8)$$

Moreover, one can take $U_i\in C(\mathbb{H}^{m+1}\setminus S,\mathcal{N})$, where the singular set $S\subset\mathbb{H}^{m+1}$ is a finite set whose cardinality is controlled by the right-hand side of (8).

Now we assert that $U_i\in W^{1,1}(\mathbb{B}_1^{m+1},\mathcal{N})$, with the ball $\mathbb{B}_1^{m+1}$ endowed with the Euclidean metric instead of the same ball $\mathbb{H}^{m+1}$ endowed surely with the hyperbolic metric. Indeed, $U_i\in W^{1,1}(\mathbb{B}_1^{m+1},\mathcal{N})$ would translate as

$$\int_{\mathbb{H}^{m+1}}\sum_i\ |\mathrm{D}U_i|=\int_{\mathbb{B}_1^{m+1}}\sum_i\frac{2^m|\mathrm{D}U_i(x)|}{(1-|x|^2)^m}\,\mathrm{d}x<\infty$$

which is neither a consequence of $U_i\in W^{1,1}(\mathbb{B}_1^{m+1},\mathcal{N})$ nor $U_i\in W^{1,1}(\mathbb{H}^{m+1},\mathcal{N})$ and has no reason to be expected.

Although standard conformal transformations between $\mathbb{B}_1^{m+1},\mathbb{H}^{m+1}$ and $\mathbb{R}_+^{m+1}$ preserve the fractional Gagliardo energy of the boundary values in the right-hand sides of (5), (7) and (8), the corresponding gap potentials and the strong-type quantities

$$\int_{\mathbb{B}_1^{m+1}}\sum_i\ |\mathrm{D}U_i|^{m+1},\ \int_{\mathbb{R}_+^{m+1}}\sum_i\ |\mathrm{D}U_i|^{m+1}\ \text{and}\ \int_{\mathbb{H}^{m+1}}\sum_i\ |\mathrm{D}U_i|^{m+1}$$

corresponding to the weak-type quantities in their left-hand side, the left-hand sides as such are not conformally invariant so that Theorems 1.1 to 1.3 are not equivalent to each other. In practice, this explains why we have the same construction of U_i in all three cases, but three different particular estimates on U_i (see [28]).

At the core of the proof of Theorems 1.1 to 1.3 there is a refined understanding of functions in the fractional critical Sobolev space $W^{1-\epsilon,1+\epsilon}(\mathbb{R}^m,\mathbb{R}^\nu)$. We start from the elegant and versatile approach through the VMO property of critical Sobolev maps in [7]: if $(1-\epsilon^2)=m$, it follows from Lebesgue's dominated convergence theorem and from the definition in (1) that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^m}\int_{\mathbb{B}_r^m(x)}\int_{\mathbb{B}_r^m(x)}\sum_i |u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)=0 \tag{9}$$

If $u_i\in\mathcal{N}$ almost everywhere, then we have

$$\operatorname{dist}\left(\int_{\mathbb{B}_r^m(x)}\sum_i u_i,\mathcal{N}\right)^{1+\epsilon}\le\int_{\mathbb{B}_r^m(x)}\int_{\mathbb{B}_r^m(x)}\sum_i |u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon) \tag{10}$$

Thus, by (9) and (10),

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^m}\operatorname{dist}\left(\int_{\mathbb{B}_r^m(x)}\sum_i u_i,\mathcal{N}\right)^{1+\epsilon}=0 \tag{11}$$

Because of the lack of control on the rate of convergence of the limit in (11), we need to refine it to get the quantitative results we are looking for.

To give a glimpse of our core quantitative estimate in a simplified setting, we replace (11) by the integral inequality

$$\begin{aligned}\int_0^\infty \sup_{x\in\mathbb{R}^m}\int_{\mathbb{B}_r^m(x)}\int_{\mathbb{B}_r^m(x)}\sum_i |u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\frac{\mathrm{d}r}{r}\\
\le\frac{1}{\mathcal{L}^m(\mathbb{B}_1^m)^2}\int_0^\infty\int_{\substack{(x+\epsilon,x+2\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m\\|\epsilon|\le 2r}}|u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x\\
+2\epsilon)\frac{\mathrm{d}r}{r^{2m+1}}\\
=\frac{4^m}{2m\mathcal{L}^m(\mathbb{B}_1^m)^2}\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_i\frac{|u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon).\end{aligned} \tag{12}$$

It follows from (12) and the Chebyshev inequality that for every $0<\epsilon<\infty$ there exists $\tau\in(1,1+\epsilon)$ such that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sup_{x\in\mathbb{R}^m}\operatorname{dist}\left(\int_{\mathbb{B}^m_{\tau(1+\epsilon)^{-k}}(x)}\sum_i u_i,\mathcal{N}\right)^{1+\epsilon}\\
\le\sum_{k\in\mathbb{Z}}\sup_{x\in\mathbb{R}^m}\int_{\mathbb{B}^m_{\tau(1+\epsilon)^{-k}}(x)}\int_{\mathbb{B}^m_{\tau(1+\epsilon)^{-k}}(x)}\sum_i |u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\\
\le\frac{4^m}{2m\mathcal{L}^m(\mathbb{B}_1^m)^2\ln(1+\epsilon)}\int_{\mathbb{R}^m\times\mathbb{R}^m}\iint_{|u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}}^{|\epsilon|^{2m}}\sum_i\frac{|u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}}{|\epsilon|^{2m}}\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon).\end{aligned} \tag{13}$$

If we take now an extension u_i by convolution on the half-space $\mathbb{R}_+^{m+1}$, (13) implies that when its right-hand side is small enough, that is, when

$$\ln(1+\epsilon)\simeq\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_i\frac{|u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon), \tag{14}$$

there is a family of hyperplanes whose distance to the boundary is in geometric progression of ratio $(1+\epsilon)$ on which the extension by convolution is close to the manifold. In view of (14), the ratio $(1+\epsilon)$ is controlled by an exponential of the Gagliardo fractional energy.

In order to actually prove the results, we improve the estimate (12) in two directions. First, instead of considering parallel hyperplanes, we consider a decomposition of cubes on the boundary of which we control the MO. Second, we replace the MO by a truncated MO which can be controlled by the gap potential (6).

In comparison, the proof in [24] of (4) relies on the estimate (see [24, Prop. 4.2])

$$\int_0^{1-\epsilon} \sup_{\substack{x\in\mathbb{B}_1^{m+1}\\|x|=r}} \operatorname{dist}\left(\frac{(1-r^2)^m}{\mathcal{H}^m(\mathbb{S}^m)}\int_{\mathbb{S}^m}\sum_i \frac{u_i(x+\epsilon)}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon),\mathcal{N}\right)\frac{\mathrm{d}r}{1-r^2}$$
$$\leq \int_0^{1-\epsilon}\frac{(1-|r|^2)^{2m-1}}{\mathcal{H}^m(\mathbb{S}^m)^2}\sup_{\substack{x\in\mathbb{B}_1^{m+1}\\|x|=r}}\iint_{\mathbb{S}^m\times\mathbb{S}^m}\sum_i\frac{|u_i(x+\epsilon)-u_i(x+2\epsilon)|}{|\epsilon|^{2m}|2\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\,\mathrm{d}r$$
$$\leq (1+\epsilon)\left(\ln\frac{2}{\epsilon}\right)^{\frac{\epsilon}{1+\epsilon}}\left(\iint_{\mathbb{S}^m\times\mathbb{S}^m}\sum_i\frac{|u_i(x+\epsilon)-u_i(x+2\epsilon)|^{1+\epsilon}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\right)^{\frac{1}{1+\epsilon}},$$

with $-1<\epsilon<0$. This allows to find bad balls in the hyperbolic space in the Poincaré ball models outside of which the hyperharmonic extension to $\mathbb{B}_1^{m+1}$ is close to the target manifold and whose radius is controlled by the exponential of the Gagliardo fractional energy. One cannot perform directly a homogeneous extension on the bad balls because they could intersect; through a Besicovitch-type covering argument, the construction is performed on collections of disjoint balls, with a number of collections bounded exponentially; combined with the exponential of the radius appearing in the equivalence between critical Marcinkiewicz-Sobolev quasinorms, this explains the double exponential appearing in the final estimate. Here we avoid this pitfall by working directly with a decomposition into disjoint cubes.

2. MO on $(1+\epsilon)$-Adic Skeletons

2.1. Truncated MO. Our analysis will rely on truncated MOs, which have already been used in estimates on homotopy decompositions in [25] and on liftings over noncompact Riemannian coverings in [26].

Definition 2.1. Given $u_i\colon\mathbb{R}^m\to\mathcal{N}$, we define the function $\mathrm{MO}_{\delta,1+\epsilon}u_i\colon\mathbb{R}_+^{m+1}\to[0,\infty]$ for every $x=(x',x_{m+1})\in\mathbb{R}_+^{m+1}$ by

$$\mathrm{MO}_{\delta,1+\epsilon}u_i(x) := \fint_{\mathbb{B}^m_{x_{m+1}}(x')}\fint_{\mathbb{B}^m_{x_{m+1}}(x')}\sum_i \left(d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta\right)_+^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon) \quad (15)$$

If $\epsilon\geq 0$, one has by Jensen's inequality,

$$\left(\mathrm{MO}_{\delta,1+\epsilon}u_i(x)\right)^{\frac{1}{1+\epsilon}} \leq \left(\mathrm{MO}_{\delta,1+2\epsilon}u_i(x)\right)^{\frac{1}{1+2\epsilon}} \quad (16)$$

whereas if $\delta_0,\delta_1\in[0,\infty)$, one has by the triangle inequality and by convexity,

$$\mathrm{MO}_{\delta_1,1+\epsilon}u_i(x)\leq 2^{\epsilon}\left(\mathrm{MO}_{\delta_0,1+\epsilon}u_i(x)+(\delta_0-\delta_1)_+^{1+\epsilon}\right) \quad (17)$$

The truncated Gagliardo fractional energy can be written in terms of MOs as

$$\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_i\frac{(d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}}{|\epsilon|^{m+(1-\epsilon^2)}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$\simeq \int_{\mathbb{R}_+^{m+1}}\sum_i \mathrm{MO}_{\delta,1+\epsilon}u_i(x)\frac{\mathrm{d}x}{x_{m+1}^{1+(1-\epsilon^2)}}$$

Proposition 2.2 (see 28). Given $\varphi_i\in C^\infty(\mathbb{R}^m,\mathbb{R})$ such that $\int_{\mathbb{R}^m}\sum_i\varphi_i=1$ and $\operatorname{supp}\varphi_i\subseteq\mathbb{B}_1^m$, there exists a constant $0\leq\epsilon<\infty$, depending only on m and φ_i, such that for every $u_i\in L^1_{\mathrm{loc}}(\mathbb{R}^m,\mathbb{R}^\nu)$ and every $Y\subseteq\mathbb{R}^\nu$ satisfying $u_i\in Y$ almost everywhere in $\mathbb{R}^m$, if $V_i\colon\mathbb{R}_+^{m+1}\to\mathbb{R}^\nu$ is defined for every (x',x_{m+1}) as

$$V_i(x) := \frac{1}{x_{m+1}^m}\int_{\mathbb{R}^m}\sum_i u_i(x+2\epsilon)\varphi_i\left(\frac{x'-x-2\epsilon}{x_{m+1}}\right)\mathrm{d}(x+2\epsilon)$$
$$=\int_{\mathbb{R}^m}\sum_i u_i\left(x'-x_{m+1}(x+2\epsilon)\right)\varphi_i(x+2\epsilon)\mathrm{d}(x+2\epsilon) \quad (18)$$

then for every $x=(x',x_{m+1})\in\mathbb{R}_+^{m+1}$, for every $\delta\in[0,\infty)$ and every $0\leq\epsilon<\infty$,

$$\operatorname{dist}(V_i(x),Y)^{1+\epsilon}\leq C^{1+\epsilon}\left(\mathrm{MO}_{\delta,1+\epsilon}u_i(x)+\delta^{1+\epsilon}\right) \quad (19)$$

and

$$|\sum_i \mathrm{D}V_i(x)|^{1+\epsilon}\leq\frac{C^{1+\epsilon}}{x_{m+1}^{1+\epsilon}}\sum_i\left(\mathrm{MO}_{\delta,1+\epsilon}u_i(x)+\delta^{1+\epsilon}\right) \quad (20)$$

Proof. First, we have by (18) and (15),

$$\operatorname{dist}(V_i(x),Y)\leq\int_{\mathbb{B}^m_{x_{m+1}}(x')}\sum_i|V_i(x)-u_i(x+\epsilon)|\mathrm{d}(x+\epsilon)$$

$$\leq \sum_i \|\varphi_i\|_{L^\infty(\mathbb{R}^m)}\mathcal{L}^m(\mathbb{B}_1^m)\int_{\mathbb{B}^m_{x_{m+1}}(x')}\int_{\mathbb{B}^m_{x_{m+1}}(x')}|u_i(x+2\epsilon)-u_i(x+\epsilon)|\mathrm{d}(x+2\epsilon)\mathrm{d}(x+\epsilon)$$
$$\leq \sum_i \|\varphi_i\|_{L^\infty(\mathbb{R}^m)}\mathcal{L}^m(\mathbb{B}_1^m)\mathrm{MO}_{0,1}u_i(x), \quad (21)$$

and the first conclusion (19) follows from (21), (16) and (17).

Next, defining $(\varphi_i)_1: \mathbb{R}^m \to \mathrm{Lin}(\mathbb{R}^{m+1}, \mathbb{R})$ for $(x+2\epsilon) \in \mathbb{R}^m$ and $h = (h', h_{m+1})$ by
$$(\varphi_i)_1(x+2\epsilon)[(h', h_{m+1})] := \mathrm{D}\varphi_i(x+2\epsilon)[h'] - h_{m+1}(m\varphi_i(x+2\epsilon) + \mathrm{D}\varphi_i(x+2\epsilon)[x+2\epsilon])$$
we have for every $x = (x', x_{m+1}) \in \mathbb{R}^{m+1}_+$,

$$\mathrm{D}V_i(x) = \frac{1}{x_{m+1}^{m+1}}\int_{\mathbb{R}^m}\sum_i u_i(x+2\epsilon)(\varphi_i)_1\left(\frac{x'-x-2\epsilon}{x_{m+1}}\right)\mathrm{d}(x+2\epsilon)$$
$$= \frac{1}{x_{m+1}^{m+1}}\int_{\mathbb{B}^m_{x_{m+1}}(x')}\int_{\mathbb{R}^m}\sum_i (u_i(x+2\epsilon)-u_i(x+\epsilon))(\varphi_i)_1\left(\frac{x'-x-2\epsilon}{x_{m+1}}\right)\mathrm{d}(x+2\epsilon)\,\mathrm{d}(x+\epsilon)$$

from which we get

$$|\sum_i \mathrm{D}V_i(x)| \leq \sum_i \|(\varphi_i)_1\|_{L^\infty(\mathbb{R}^m)}\mathcal{L}^m(\mathbb{B}_1^m)\int_{\mathbb{B}^m_{x_{m+1}}(x')}\int_{\mathbb{B}^m_{x_{m+1}}(x')}|u_i(x+2\epsilon)-u_i(x+\epsilon)|\mathrm{d}(x+2\epsilon)\,\mathrm{d}(x+\epsilon)$$

and we conclude as previously.

2.2. $(1+\epsilon)$**-adic cubes and skeletons.** The domain of the extension by convolution V_i given by (18) is the half-space $\mathbb{R}^{m+1}_+$; we will subdivide the latter in $(1+\epsilon)$-adic cubes on which we will perform appropriate constructions.

Given $0 < \epsilon < \infty$ and $\tau \in [1, 1+\epsilon]$, we consider for every $k \in \mathbb{Z}$ the set of cubes of $\mathbb{R}^m$
$$\mathcal{Q}_{1+\epsilon,\tau,k} := \{\tau(1+\epsilon)^{-k}([0,1]^m + j) \mid j \in \mathbb{Z}^m\}$$
and the corresponding set of cubes of $\mathbb{R}^{m+1}_+$
$$\mathcal{Q}^+_{1+\epsilon,\tau,k} := \left\{Q \times \left[\frac{\tau(1+\epsilon)^{-k}}{\epsilon}, \frac{\tau(1+\epsilon)^{-(k-1)}}{\epsilon}\right] \middle| \, Q \in \mathcal{Q}_{1+\epsilon,\tau,k}\right\}$$
$$= \{\tau(1+\epsilon)^{-k}([0,1]^{m+1} + (j, (\epsilon)^{-1})) \mid j \in \mathbb{Z}^m\}$$

In particular, we have the decompositions
$$\mathbb{R}^m = \bigcup_{Q \in \mathcal{Q}_{1+\epsilon,\tau,k}} Q \quad \text{and} \quad \mathbb{R}^{m+1}_+ = \bigcup_{k\in\mathbb{Z}}\bigcup_{Q \in \mathcal{Q}^+_{1+\epsilon,\tau,k}} Q$$

Moreover, we consider the part of the boundaries of the cubes in $\mathcal{Q}^+_{1+\epsilon,\tau,k}$ that are parallel to the hyperplane $\mathbb{R}^m \times \{0\}$
$$\mathcal{Q}^{\parallel}_{1+\epsilon,\tau,k} := \left\{Q \times \left\{\frac{\tau(1+\epsilon)^{-k}}{\epsilon}\right\} \middle| \, Q \in \mathcal{Q}_{1+\epsilon,\tau,k}\right\}$$
$$= \{\tau(1+\epsilon)^{-k}([0,1]^m \times \{0\} + (j, (\epsilon)^{-1})) \mid j \in \mathbb{Z}^m\} \quad (22)$$

and those that are normal to the same hyperplane
$$\mathcal{Q}^{\perp}_{1+\epsilon,\tau,k} := \left\{\partial Q \times \left[\frac{\tau(1+\epsilon)^{-k}}{\epsilon}, \frac{\tau(1+\epsilon)^{-(k-1)}}{\epsilon}\right] \middle| \, Q \in \mathcal{Q}_{1+\epsilon,\tau,k}\right\}$$
$$= \{\tau(1+\epsilon)^{-k}((\partial[0,1]^m) \times [0,1] + (j, (\epsilon)^{-1})) \mid j \in \mathbb{Z}^m\} \quad (23)$$

Given $h \in \mathbb{R}^m$, we also define the corresponding translated sets
$$\mathcal{Q}_{1+\epsilon,\tau,k,h} := \{Q + \tau(1+\epsilon)^{-k}h \mid Q \in \mathcal{Q}_{1+\epsilon,\tau,k}\},$$
$$\mathcal{Q}^+_{1+\epsilon,\tau,k,h} := \{Q + \tau(1+\epsilon)^{-k}(h, 0) \mid Q \in \mathcal{Q}^+_{1+\epsilon,\tau,k}\},$$
$$\mathcal{Q}^{\parallel}_{1+\epsilon,\tau,k,h} := \{Q + \tau(1+\epsilon)^{-k}(h, 0) \mid Q \in \mathcal{Q}^{\parallel}_{1+\epsilon,\tau,k}\},$$
$$\mathcal{Q}^{\perp}_{1+\epsilon,\tau,k,h} := \{\Sigma + \tau(1+\epsilon)^{-k}(h, 0) \mid \Sigma \in \mathcal{Q}^{\perp}_{1+\epsilon,\tau,k}\}. \quad (24)$$

2.3. Longitudinal estimate. We first have an estimate on the maximal MO on longitudinal faces of cubes of the $(1+\epsilon)$-adic decomposition of $\mathbb{R}^{m+1}_+$.

Proposition 2.3 (see [28]). For every $m \in \mathbb{N} \setminus \{0\}$, there exists a constant $(1+\epsilon) = C(m) \in (0, \infty)$ such that for every $0 \leq \epsilon < \infty$, for every measurable map $u_i: \mathbb{R}^m \to \mathcal{N}$, for every $0 \leq \epsilon < \infty$ and for every $\delta \in [0, \infty)$, one has

$$\int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_{\Sigma\in\mathcal{Q}^{\parallel}_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x\in\Sigma} \mathrm{MO}_{\delta,1+\epsilon} u_i(x) \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\le (1+\epsilon) \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_i \frac{(d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}}{|\epsilon|^{2m}} \,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon) \qquad (25)$$

Proof. For any $\Sigma \in \mathcal{Q}^{\parallel}_{2+\epsilon,\tau,k,h}$, we can write, in view of (22) and (24),

$$\Sigma = Q \times \left\{\frac{\tau(2+\epsilon)^{-k}}{1+\epsilon}\right\}$$

where $Q \in \mathcal{Q}_{2+\epsilon,\tau,k,h}$. If $x \in \Sigma$ and $(x+\epsilon) \in \mathbb{R}^m$ satisfy $|x'-x-\epsilon| \le \frac{\tau(2+\epsilon)^{-k}}{1+\epsilon}$, then $x' \in Q$ and we have immediately that

$$\mathrm{dist}(x+\epsilon, Q) \le |x'-x-\epsilon| \le \frac{\tau(2+\epsilon)^{-k}}{1+\epsilon} \qquad (26)$$

Since Q is a cube of edge length $\tau(2+\epsilon)^{-k}$, according to (26), we have

$$(x+\epsilon) \in \left(1+\frac{2}{1+\epsilon}\right)Q = \frac{3+\epsilon}{1+\epsilon}Q$$

under the convention that $\frac{3+\epsilon}{1+\epsilon}Q$ is the cube with the same center as Q dilated by a factor $\frac{3+\epsilon}{1+\epsilon}$. Thus, $(x+\epsilon) \in 3Q$, since $\epsilon \ge 0$. One has then, by monotonicity of the integral, for every $x \in \Sigma$,

$$\mathrm{MO}_{\delta,1+\epsilon} u_i(x) = \frac{1}{\mathcal{L}^m(\mathbb{B}_1^m)^2}\left(\frac{(2+\epsilon)^k(1+\epsilon)}{\tau}\right)^{2m} \iint_{\substack{(x+\epsilon,x+2\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m \\ |x+\epsilon-x'|\le\frac{\tau(2+\epsilon)^{-k}}{1+\epsilon} \\ |x+2\epsilon-x'|\le\frac{\tau(2+\epsilon)^{-k}}{1+\epsilon}}} \sum_i (d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$\le \frac{1}{\mathcal{L}^m(\mathbb{B}_1^m)^2}\left(\frac{(2+\epsilon)^k(1+\epsilon)}{\tau}\right)^{2m} \iint_{\substack{(x+\epsilon,x+2\epsilon)\in 3Q\times 3Q \\ |\epsilon|\le\frac{2\tau(2+\epsilon)^{-k}}{1+\epsilon}}} \sum_i (d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon) \qquad (27)$$

Summing (27) over the sets $\Sigma \in \mathcal{Q}^{\parallel}_{2+\epsilon,\tau,k,h}$ and integrating the result over the translations $h \in [0,1]^m$, we get

$$\int_{[0,1]^m} \sum_{\Sigma\in\mathcal{Q}^{\parallel}_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x\in\Sigma} \mathrm{MO}_{\delta,1+\epsilon} u_i(x) \mathrm{d}h$$
$$\le \frac{3^m}{\mathcal{L}^m(\mathbb{B}_1^m)^2}\left(\frac{(2+\epsilon)^k(1+\epsilon)}{\tau}\right)^{2m} \iint_{\substack{(x+\epsilon,x+2\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m \\ |\epsilon|\le\frac{2\tau(2+\epsilon)^{-k}}{1+\epsilon}}} \sum_i (d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon) \qquad (28)$$

Summing (28) over the scales $k \in \mathbb{Z}$ and integrating the result over $\tau \in [1, 2+\epsilon]$, we get

$$\int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_{\Sigma\in\mathcal{Q}^{\parallel}_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x\in\Sigma} \mathrm{MO}_{\delta,1+\epsilon} u_i(x) \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\le \frac{3^m(1+\epsilon)^{2m}}{\mathcal{L}^m(\mathbb{B}_1^m)^2} \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_{k\in\mathbb{Z}} \int_{\tau\in(1,2+\epsilon)} \sum_i \frac{(2+\epsilon)^{2km}(d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}}{\tau^{2m+1}} \,\mathrm{d}\tau\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$\tau \ge \frac{(1+\epsilon)(2+\epsilon)^k|\epsilon|}{2}$$
$$= \frac{3^m(1+\epsilon)^{2m}}{\mathcal{L}^m(\mathbb{B}_1^m)^2} \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_{k\in\mathbb{Z}} \int_{\theta\in((2+\epsilon)^{-k},(2+\epsilon)^{-(k-1)})} \sum_i \frac{(d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}}{\theta^{2m+1}} \,\mathrm{d}\theta\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$\theta \ge \frac{(1+\epsilon)|\epsilon|}{2}$$
$$= \frac{3^m(1+\epsilon)^{2m}}{\mathcal{L}^m(\mathbb{B}_1^m)^2} \iint_{\mathbb{R}^m\times\mathbb{R}^m} \int_{\frac{(1+\epsilon)|\epsilon|}{2}}^{\infty} \sum_i \frac{(d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}}{\theta^{2m+1}} \,\mathrm{d}\theta\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$= \frac{12^m}{2m\mathcal{L}^m(\mathbb{B}_1^m)^2} \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_i \frac{(d(u_i(x+\epsilon),u_i(x+2\epsilon))-\delta)_+^{1+\epsilon}}{|\epsilon|^{2m}} \,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon),$$

which implies the announced conclusion (25) and completes our proof of Proposition 2.3.

2.4. Transversal estimate. We next prove a counterpart of Proposition 2.3, where we estimate the maximal MO on transversal faces of cubes of the $(2+\epsilon)$-adic decomposition of $\mathbb{R}^{m+1}_+$ instead of the longitudinal ones.

Proposition 2.4 (see [28]). For every $m \in \mathbb{N} \setminus \{0\}$ and $0 \leq \epsilon < \infty$, there exists a constant $(1+\epsilon) = C(m, 1+\epsilon) \in (0, \infty)$ such that for every measurable map $u_i: \mathbb{R}^m \to \mathcal{N}$, for every $0 \leq \epsilon < \infty$ and for every $\delta \in [0, \infty)$, one has

$$\int_1^{2+\epsilon} \sum_{k \in \mathbb{Z}} \int_{[0,1]^m} \sum_{\Sigma \in \mathcal{Q}^{\perp}_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x \in \Sigma} \mathrm{MO}_{\delta,1+\epsilon} u_i(x) \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$

$$\leq (1+\epsilon) \iint_{\mathbb{R}^m \times \mathbb{R}^m} \sum_i \frac{\left(d\big(u_i(x+\epsilon), u_i(x+2\epsilon)\big) - \frac{\delta}{2}\right)^{1+\epsilon}}{|\epsilon|^{2m}} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon) \tag{29}$$

Proof. We consider a set $\Sigma \in \mathcal{Q}^{\perp}_{2+\epsilon,\tau,k,h}$, that we can write, in view of (23) and (24), as

$$\Sigma = \partial Q \times \left[\frac{\tau(2+\epsilon)^{-k}}{1+\epsilon}, \frac{\tau(2+\epsilon)^{-(k-1)}}{1+\epsilon}\right] \tag{30}$$

where $Q \in \mathcal{Q}_{2+\epsilon,\tau,k,h}$. We first note that by (15), convexity and the triangle inequality,

$$\begin{aligned}
\mathrm{MO}_{\delta,1+\epsilon} u_i(x) &= \int_{\mathbb{B}^m_{x_{m+1}}(x')} \int_{\mathbb{B}^m_{x_{m+1}}(x')} \sum_i \left(d(u_i(x+\epsilon), u_i(x+2\epsilon)) - \delta\right)_+^{1+\epsilon} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon) \\
&\leq 2^{\epsilon} \Bigg(\int_{\mathbb{B}^m_{x_{m+1}}(x')} \int_{\mathbb{B}^m_{x_{m+1}}(x')} \int_{E_x} \sum_i \left(d\big(u_i(x+\epsilon), u_i(w_i)\big) - \frac{\delta}{2}\right)_+^{1+\epsilon} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon)\, \mathrm{d}w_i \\
&\quad + \int_{\mathbb{B}^m_{x_{m+1}}(x')} \int_{\mathbb{B}^m_{x_{m+1}}(x')} \int_{E_x} \sum_i \left(d\big(u_i(x+\epsilon), u_i(w_i)\big) - \frac{\delta}{2}\right)_+^{1+\epsilon} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon)\, \mathrm{d}w_i \Bigg) \\
&= 2^{1+\epsilon} \int_{\mathbb{B}^m_{x_{m+1}}(x')} \int_{E_x} \sum_i \left(d\big(u_i(x+\epsilon), u_i(x+2\epsilon)\big) - \frac{\delta}{2}\right)_+^{1+\epsilon} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon),
\end{aligned} \tag{31}$$

where

$$E_x := \{(x+2\epsilon) \in \partial Q \,|\, |x+2\epsilon - x'| \leq x_{m+1}\}$$

For every $x = (x', x_{m+1}) \in \Sigma$, since $\epsilon \geq 0$ and Q is a cube of edge length $\tau(2+\epsilon)^{-k}$, we have $\tau(2+\epsilon)^{-k} \geq x_{m+1}$ and thus

$$\mathcal{H}^{m-1}(E_x) \geq \frac{x_{m+1}^{m-1}}{2^{m-1}} \mathcal{L}^{m-1}(\mathbb{B}_1^{m-1}) \tag{32}$$

One has then for every $x \in \Sigma$, by (30), (31), (32) and by monotonicity of the integral,

$$\begin{aligned}
\mathrm{MO}_{\delta,1+\epsilon} u_i(x) &\leq \frac{1+\epsilon}{x_{m+1}^{2m-1}} \iint_{\substack{x+\epsilon, x+2\epsilon \in \mathbb{R}^m \times \partial Q \\ |x+\epsilon - x'| \leq x_{m+1} \\ |x+2\epsilon - x'| \leq x_{m+1}}} \sum_i \left(d\big(u_i(x+\epsilon), u_i(x+2\epsilon)\big) - \frac{\delta}{2}\right)_+^{1+\epsilon} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon) \\
&\leq (1+\epsilon) \iint_{\substack{(x+\epsilon, x+2\epsilon) \in \mathbb{R}^m \times \partial Q \\ |\epsilon| \leq \frac{2\tau(2+\epsilon)^{-(k-1)}}{1+\epsilon}}} \sum_i \frac{\left(d\big(u_i(x+\epsilon), u_i(x+2\epsilon)\big) - \frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m-1}} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon),
\end{aligned} \tag{33}$$

where the constant $0 \leq \epsilon < \infty$ depends only on m and $(1+\epsilon)$. Summing (33) over the sets $\Sigma \in \mathcal{Q}^{\perp}_{2+\epsilon,\tau,k,h}$ and integrating the result with respect to the translations h over $[0,1]^m$, we have

$$\begin{aligned}
&\int_{[0,1]^m} \sum^{\perp}_{\Sigma \in \mathcal{Q}\perp,\tau,k,h} \sum_i \sup_{x \in \Sigma} \mathrm{MO}_{\delta,1+\epsilon} u_i(x) \mathrm{d}h \\
&\leq (1+\epsilon) \int_{[0,1]^m} \sum_{\Sigma \in \mathcal{Q}_{\lambda,\tau,k,h}} \iint_{\substack{(x+\epsilon, x+2\epsilon) \in \mathbb{R}^m \times \partial Q \\ |\epsilon| \leq \frac{2\tau(2+\epsilon)^{-(k-1)}}{1+\epsilon}}} \sum_i \frac{\left(d\big(u_i(x+\epsilon), u_i(x+2\epsilon)\big) - \frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m-1}} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon) \\
&= \frac{(1+\epsilon)(2+\epsilon)^k}{\tau} \iint_{\substack{(x+\epsilon, x+2\epsilon) \in \mathbb{R}^m \times \mathbb{R}^m \\ |\epsilon| \leq \frac{2\tau(2+\epsilon)^{-(k-1)}}{1+\epsilon}}} \sum_i \frac{\left(d\big(u_i(x+\epsilon), u_i(x+2\epsilon)\big) - \frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m-1}} \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon),
\end{aligned} \tag{34}$$

where the constant $0 \leq \epsilon < \infty$ depends only on m and $(1+\epsilon)$. In (34) we have used the fact that for every $f_i: \mathbb{R}^m \to [0, \infty]$, in view of Fubini's theorem and change of variables, it holds

$$\int_{[0,1]^m} \sum_{Q\in \mathcal{Q}_{2+\epsilon,\tau,k,h}} \left(\int_{\partial Q} \sum_i f_i(x+2\epsilon)\mathrm{d}(x+2\epsilon)\right)\mathrm{d}h = (\tau(2+\epsilon)^{-k})^{m-1}\int_{[0,1]^m} \sum_{Q\in \mathcal{Q}_{2+\epsilon,1,0,h}} \left(\int_{\partial Q} \sum_i f_i\big(\tau(2+\epsilon)^{-k}(x+2\epsilon)\big)\mathrm{d}(x+2\epsilon)\right)\mathrm{d}h$$
$$= 2m(\tau(2+\epsilon)^{-k})^{m-1}\int_{\mathbb{R}^m}\sum_i f_i\big(\tau(2+\epsilon)^{-k}(x+2\epsilon)\big)\mathrm{d}(x+2\epsilon)$$
$$= \frac{2m(2+\epsilon)^k}{\tau}\int_{\mathbb{R}^m}\sum_i f_i$$

Thus, summing and integrating (34) over the scales, we get

$$\int_1^{2+\epsilon}\sum_{k\in\mathbb{Z}}\int_{[0,1]^m}\sum_{\Sigma\in \mathcal{QQ}^{\perp}_{2+\epsilon,\tau,k,h}}\sum_i \sup_{x\in\Sigma}\mathrm{MO}_{\delta,1+\epsilon}u_i(x)\mathrm{d}h\frac{\mathrm{d}\tau}{\tau}$$
$$\le (1+\epsilon)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_{k\in\mathbb{Z}}\int_{\substack{\tau\in(1,2+\epsilon)\\ \tau\ge\frac{(2+\epsilon)^k(1+\epsilon)|\epsilon|}{2(2+\epsilon)}}}\sum_i \frac{(2+\epsilon)^k\left(d\big(u_i(x+\epsilon),u_i(x+2\epsilon)\big)-\frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m-1}\tau^2}\,\mathrm{d}\tau\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$= (1+\epsilon)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_{k\in\mathbb{Z}}\int_{\substack{\theta\in((2+\epsilon)^{-k},(2+\epsilon)^{-(k-1)})\\ \theta\ge\frac{(1+\epsilon)|\epsilon|}{2(2+\epsilon)}}}\sum_i \frac{\left(d\big(u_i(x+\epsilon),u_i(x+2\epsilon)\big)-\frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m-1}\theta^2}\,\mathrm{d}\theta\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$= (1+\epsilon)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\int_{\frac{(1+\epsilon)|\epsilon|}{2(2+\epsilon)}}^{\infty}\sum_i \frac{\left(d\big(u_i(x+\epsilon),u_i(x+2\epsilon)\big)-\frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m-1}\theta^2}\,\mathrm{d}\theta\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$
$$= 2(2+\epsilon)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_i \frac{\left(d\big(u_i(x+\epsilon),u_i(x+2\epsilon)\big)-\frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$

so that the conclusion (29) follows, since $\epsilon\ge 0$.

2.5. Combining the estimates. We close this section by summarizing Propositions 2.3 and 2.4 in the following statement.

Proposition 2.5 (see [28]). For every $m\in\mathbb{N}\setminus\{0\}$ and $0\le\epsilon<\infty$, there exists a constant $(1+\epsilon)=C(m,1+\epsilon)\in(0,\infty)$ such that for every measurable map $u_i:\mathbb{R}^m\to\mathcal{N}$, for every $0\le\epsilon<\infty$ and for every $\delta\in[0,\infty)$, one has

$$\int_1^{2+\epsilon}\sum_{k\in\mathbb{Z}}\int_{[0,1]^m}\sum_{Q\in\mathcal{Q}^{+}_{2+\epsilon,\tau,k,h}}\sum_i \sup_{x\in\partial Q}\mathrm{MO}_{\delta,1+\epsilon}u_i(x)\mathrm{d}h\frac{\mathrm{d}\tau}{\tau}$$
$$\le (1+\epsilon)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_i \frac{\left(d\big(u_i(x+\epsilon),u_i(x+2\epsilon)\big)-\frac{\delta}{2}\right)_+^{1+\epsilon}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon).$$

Proof. This follows immediately from Propositions 2.3 and 2.4.

3. Proofs of the Singular Extension Theorems

3.1. Oscillation and gradient estimate on the skeleton. We first estimate the average number of cubes on which the extension V_i of u_i given by (18) is far away from the range of u_i.

Proposition 3.1 (see [28]). Let $m\in\mathbb{N}\setminus\{0\}$. There exist constants $\eta\in(0,1)$ and $0\le\epsilon<\infty$ depending only on m such that for every $\delta\in(0,\infty)$, for every $0\le\epsilon<\infty$, for every measurable function $u_i:\mathbb{R}^m\to\mathbb{R}^\nu$ and every set $Y\subseteq\mathbb{R}^\nu$, if V_i is an extension by convolution given by (18) and if $u_i\in Y$ almost everywhere in $\mathbb{R}^m$, then

$$\int_1^{2+\epsilon}\sum_{k\in\mathbb{Z}}\int_{[0,1]^m}\sum_i \#\left\{Q\in\mathcal{Q}^{+}_{2+\epsilon,\tau,k,h}\mid \sup_{x\in\partial Q}\mathrm{dist}(V_i(x),Y)\ge\delta\right\}\mathrm{d}h\frac{\mathrm{d}\tau}{\tau}$$
$$\le\frac{1+\epsilon}{\delta^{m+1}}\iint_{\mathbb{R}^m\times\mathbb{R}^m}\sum_i \frac{\big(d\big(u_i(x+\epsilon),u_i(x+2\epsilon)\big)-\eta\delta\big)_+^{m+1}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon) \qquad (35)$$

Proof. By (19), we have for every $x\in\mathbb{R}^{m+1}_+$,

$$\mathrm{dist}(V_i(x),Y)^{m+1}\le(1+\epsilon)\big(\mathrm{MO}_{\theta,m+1}u_i(x)+\theta^{m+1}\big) \qquad (36)$$

It is worth noting that, according to (19), the constant $0\le\epsilon<\infty$ depends only on m and the function φ_i in the definition of V_i, since $m=\epsilon$. Thus, fixing the function φ_i in the definition of V_i from now on, we can assume that the constant $(1+\epsilon)$ depends only on m. Hence, taking

$$\eta := \frac{1}{\left((1+2^{m+1})(1+\epsilon)\right)^{\frac{1}{m+1}}},$$

if

$$\mathrm{dist}(V_i(x), Y) \geq \delta \tag{37}$$

the inequality (36) with $\theta = 2\eta\delta$ implies that

$$(\eta\delta)^{m+1} \leq \mathrm{MO}_{2\eta\delta,m+1} u_i(x) \tag{38}$$

We get, by (37), (38) and Proposition [2.5 applied with $m = \epsilon$,

$$\int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_i \#\left\{Q \in \mathcal{Q}^+_{2+\epsilon,\tau,k,h} \mid \sup_{x\in\partial Q} \mathrm{dist}(V_i(x), Y) \geq \delta\right\} \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq \int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_i \#\left\{Q \in \mathcal{Q}^+_{2+\epsilon,\tau,k,h} \mid \sup_{x\in\partial Q} \mathrm{MO}_{2\eta\delta,m+1} u_i(x) \geq (\eta\delta)^{m+1}\right\} \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq \frac{1}{(\eta\delta)^{m+1}} \int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_{Q\in\mathcal{Q}^+_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x\in\partial Q} \mathrm{MO}_{2\eta\delta,m+1} u_i(x) \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq \frac{1+\epsilon}{\delta^{m+1}} \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_i \frac{(d(u_i(x+\epsilon), u_i(x+2\epsilon)) - \eta\delta)_+^{m+1}}{|\epsilon|^{2m}} \,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$

where the constant $0 \leq \epsilon < \infty$ depends only on m. This proves the estimate (35) and completes our proof of Proposition 3.1.

Next, we can prove an average uniform bound on the extension by convolution V_i.

Proposition 3.2 (see [28]). Let $m \in \mathbb{N} \setminus \{0\}$. There exists a constant $(1+\epsilon) = C(m) \in (0, \infty)$ such that for every $0 \leq \epsilon < \infty$, for every measurable function $u_i : \mathbb{R}^m \to \mathbb{R}^\nu$, if V_i is an extension by convolution given by (18), then

$$\int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_{Q\in\mathcal{Q}^+_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \,\mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq (1+\epsilon) \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_i \frac{d(u_i(x+\epsilon), u_i(x+2\epsilon))^{m+1}}{|\epsilon|^{2m}} \,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$

Proof. We proceed similarly as in the proof of Proposition 3.1, where we assume that the function φ_i in the definition of V_i is fixed. Since by (20) applied with $m = \epsilon$,

$$x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \leq (1+\epsilon)\left(\mathrm{MO}_{\theta,m+1} u_i(x) + \theta^{m+1}\right)$$

where we can assume that $0 \leq \epsilon < \infty$ depends only on m, the proof of Proposition 3.2 then follows from Proposition 2.5. Namely,

$$\int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_{Q\in\mathcal{Q}^+_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \,\mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq (1+\epsilon) \int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_{Q\in\mathcal{Q}^+_{2+\epsilon,\tau,k,h}} \sum_i \sup_{x\in\partial Q} \mathrm{MO}_{0,m+1} u_i(x) \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq (1+\epsilon) \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_i \frac{d(u_i(x+\epsilon), u_i(x+2\epsilon)\mathbb{R}^{m+1}}{|\epsilon|^{m}} \,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)$$

where the constant $0 \leq \epsilon < \infty$ depends only on m.

3.2. Sobolev and Sobolev-Marcinkiewicz extensions on cubes. The first construction we will perform is a classical extension of the boundary data on cubes with small energy. To simplify the presentation, the result is stated on a ball, which is bi-Lipschitzly equivalent to a cube.

Lemma 3.3 (see [28]). Let $m \in \mathbb{N} \setminus \{0\}$. There exists a constant $(1+\epsilon) = C(m) \in (0, \infty)$ such that for every $w_i \in W^{1,m+1}(\mathbb{S}^m_\rho, \mathbb{R}^\nu)$ there exists a function $W_i \in W^{1,m+1}(\mathbb{B}^{m+1}_\rho, \mathbb{R}^\nu) \cap C(\mathbb{B}^{m+1}_\rho, \mathbb{R}^\nu)$ such that $\mathrm{tr}_{\mathbb{S}^m_\rho} W_i = w_i$,

$$\int_{\mathbb{B}^{m+1}_\rho} \sum_i |\mathrm{D}W_i|^{m+1} \leq (1+\epsilon)\rho \int_{\mathbb{S}^m_\rho} \sum_i |\mathrm{D}w_i|^{m+1} \tag{39}$$

$$\int_{\mathbb{B}_\rho^{m+1}} \int_{\mathbb{S}_\rho^m} \sum_i |W_i(x) - w_i(x+\epsilon)|^{m+1} \, \mathrm{d}x \, \mathrm{d}(x+\epsilon) \le (1+\epsilon)\rho^{2m+2} \int_{\mathbb{S}_\rho^m} \sum_i |\mathrm{D}w_i|^{m+1} \tag{40}$$

and for every $x \in \mathbb{B}_\rho^{m+1}$ and almost every $(x+\epsilon) \in \mathbb{S}_\rho^m$,

$$\sum_i |W_i(x) - w_i(x+\epsilon)|^{m+1} \le (1+\epsilon)\rho \int_{\mathbb{S}_\rho^m} \sum_i |\mathrm{D}w_i|^{m+1} \tag{41}$$

Proof. Let

$$(1+\epsilon) := \frac{(m+1)^2}{m}$$

We have, by the fractional Sobolev-Morrey embedding,

$$\iint_{\mathbb{S}_\rho^m \times \mathbb{S}_\rho^m} \sum_i \frac{|w_i(x+\epsilon) - w_i(x+2\epsilon)|^{1+\epsilon}}{|\epsilon|^{m+\epsilon}} \, \mathrm{d}(x+\epsilon) \, \mathrm{d}(x+2\epsilon) \le (1+\epsilon) \left(\int_{\mathbb{S}_\rho^m} \sum_i |\mathrm{D}w_i|^{m+1} \right)^{\frac{1+\epsilon}{m+1}} \tag{42}$$

Assuming without loss of generality that $\int_{\mathbb{S}_\rho^m} w_i = 0$, Gagliardo's classical trace theory [9] (see also [8, Prop. 17.1; 12, Section 6.9; 13, Th. 9.4; 14, Th. 10.1.1.1]) yields a function $W_i \in W^{1,1+\epsilon}(\mathbb{B}_\rho^{m+1}, \mathbb{R}^\nu)$ such that $\mathrm{tr}_{\mathbb{S}_\rho^m} W_i = w_i$ and

$$\int_{\mathbb{B}_\rho^{m+1}} \sum_i |\mathrm{D}W_i|^{1+\epsilon} + \sum_i \frac{|W_i|^{1+\epsilon}}{\rho^{1+\epsilon}} \le (1+\epsilon) \iint_{\mathbb{S}_\rho^m \times \mathbb{S}_\rho^m} \sum_i \frac{|w_i(x+\epsilon) - w_i(x+2\epsilon)|^{1+\epsilon}}{|\epsilon|^{m+\epsilon}} \, \mathrm{d}(x+\epsilon) \, \mathrm{d}(x+2\epsilon). \tag{43}$$

It follows from Hölder's inequality, (43) and (42), that

$$\int_{\mathbb{B}_\rho^{m+1}} \sum_i |\mathrm{D}W_i|^{m+1} + \sum_i \frac{|W_i|^{m+1}}{\rho^{m+1}} \le (1+\epsilon)\rho \int_{\mathbb{S}_\rho^m} \sum_i |\mathrm{D}w_i|^{m+1} \tag{44}$$

and (39) follows then from (44).

Since for every $x \in \mathbb{B}_\rho^{m+1}$ and $(x+\epsilon) \in \mathbb{S}_\rho^m$,

$$|W_i(x) - w_i(x+\epsilon)|^{m+1} \le 2^m (|W_i(x)|^{m+1} + |w_i(x+\epsilon)|^{m+1})$$

we have

$$\int_{\mathbb{B}_\rho^{m+1}} \int_{\mathbb{S}_\rho^m} \sum_i |W_i(x) - w_i(x+\epsilon)|^{m+1} \, \mathrm{d}x \, \mathrm{d}(x+\epsilon)$$
$$\le (1+\epsilon)\rho^m \int_{\mathbb{B}_\rho^{m+1}} \sum_i |W_i|^{m+1} + (1+\epsilon)\rho^{m+1} \int_{\mathbb{S}_\rho^m} \sum_i |w_i|^{m+1} \tag{45}$$

The estimate (40) follows from the estimates (45), (44) and the Poincaré inequality on the sphere.

Finally, by the Morrey-Sobolev inequality, (43) and (42), we have

$$\sum_i |W_i(x) - w_i(x+\epsilon)| \le (1+\epsilon)|\epsilon|^{1-\frac{m+1}{1+\epsilon}} \left(\int_{\mathbb{B}_\rho^{m+1}} \sum_i |\mathrm{D}W_i|^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}$$
$$\le (1+\epsilon)\rho^{\frac{1}{m+1}} \left(\int_{\mathbb{S}_\rho^m} \sum_i |\mathrm{D}w_i|^{m+1} \right)^{\frac{1}{m+1}}$$

and (41) follows.

When the oscillation is too large on the boundary of cubes, we will perform our construction of the controlled singular extension; the resulting map is quite wild but is sufficiently well controlled to provide an acceptable extension on those cubes.

Lemma 3.4 (see [28]). Let $m \in \mathbb{N} \setminus \{0\}$. If $w_i \in W^{1,m+1}(\mathbb{S}_\rho^m, \mathbb{R}^\nu)$ and if we define $W_i : \mathbb{B}_\rho^{m+1} \to \mathbb{R}^\nu$ for each $x \in \mathbb{B}_\rho^{m+1} \setminus \{0\}$ by

$$W_i(x) := w_i\left(\frac{\rho}{|x|}x\right)$$

then $W_i \in W^{1,1}(\mathbb{B}_\rho^{m+1}, \mathbb{R}^\nu)$ and for every $0 \le \epsilon < \infty$,

$$\mathcal{L}^{m+1}(\{x \in \mathbb{B}_\rho^{m+1} | |\mathrm{D}W_i(x)| > 1+\epsilon\}) \le \frac{\rho}{(m+1)(1+\epsilon)^{m+1}} \int_{\mathbb{S}_\rho^m} \sum_i |\mathrm{D}w_i|^{m+1} \tag{46}$$

Proof. It can be observed immediately that $W_i \in W^{1,m+1}(\mathbb{B}_\rho^{m+1} \setminus \mathbb{B}_\varepsilon^{m+1}, \mathbb{R}^\nu)$ for every $\varepsilon \in (0,\rho)$ and that for every $x \in \mathbb{B}_\rho^{m+1} \setminus \{0\}$, one has

$$|\mathrm{D}W_i(x)| = \frac{\rho}{|x|} \left| \mathrm{D}w_i\left(\frac{\rho}{|x|}x\right) \right|$$

Hence, using Fubini's theorem, we have

$$\mathcal{L}^{m+1}(\{x \in \mathbb{B}_\rho^{m+1} | |\mathrm{D}W_i(x)| > 1+\epsilon\}) = \int_0^\rho \sum_i \mathcal{H}^m\left(\left\{x \in \mathbb{S}_r^m | |\mathrm{D}w_i\left(\frac{\rho}{r}x\right)\right| \geq \frac{(1+\epsilon)r}{\rho}\right\}\right) \mathrm{d}r$$
$$= \int_0^\rho \sum_i \mathcal{H}^m\left(\left\{(x+\epsilon) \in \mathbb{S}_\rho^m | |\mathrm{D}w_i(x+\epsilon)\right| \geq \frac{(1+\epsilon)r}{\rho}\right\}\right)\left(\frac{r}{\rho}\right)^m \mathrm{d}r$$
$$= \frac{\rho}{(1+\epsilon)^{m+1}} \int_0^{1+\epsilon} \sum_i \mathcal{H}^m(\{(x+\epsilon) \in \mathbb{S}_\rho^m | |\mathrm{D}w_i(x+\epsilon)| \geq \tau\})\tau^m \mathrm{d}\tau$$
$$\leq \frac{\rho}{(1+\epsilon)^{m+1}} \int_0^{\infty} \sum_i \mathcal{H}^m(\{(x+\epsilon) \in \mathbb{S}_\rho^m | |\mathrm{D}w_i(x+\epsilon)| \geq \tau\})\tau^m \mathrm{d}\tau$$
$$= \frac{\rho}{(1+\epsilon)^{m+1}} \int_0^{\infty} \sum_i \mathcal{H}^m\left(\left\{(x+\epsilon) \in \mathbb{S}_\rho^m | |\mathrm{D}w_i(x+\epsilon)\right|^{m+1} \geq \tau\right\}\right) \frac{\mathrm{d}\tau}{m+1}$$
$$= \frac{\rho}{(m+1)(1+\epsilon)^{m+1}} \int_{\mathbb{S}_\rho^m} \sum_i |\mathrm{D}w_i(x)|^{m+1} \mathrm{d}x$$

This yields (46) and completes our proof of Lemma 3.4.

3.3. Proofs of the theorems. We first construct and estimate the singular extension on the half-space (Theorem 1.2).

Proof of Theorem 1.2 (see [28]). We fix $\delta_{\mathcal{N}} \in (0,\infty)$ so that the nearest-point retraction $\Pi_{\mathcal{N}}: \mathcal{N} + \mathbb{B}_{\delta_{\mathcal{N}}}^{\nu} \to \mathcal{N}$ is well defined and smooth up to the boundary. We take $V_i: \mathbb{R}_+^{m+1} \to \mathbb{R}^\nu$ to be an extension by convolution of u_i to $\mathbb{R}_+^{m+1}$ as in (18).

Since by assumption $u_i \in \mathcal{N}$ almost everywhere on $\mathbb{R}^m$, by the averaged estimate on the distance to the target (see Proposition [3.1), we have

$$\int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_i \#\left\{Q \in \mathcal{Q}_{2+\epsilon,\tau,k,h}^+ \middle| \sup_{x\in\partial Q} \mathrm{dist}(V_i(x),\mathcal{N}) \geq \delta_{\mathcal{N}}/2\right\} \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq (1+\epsilon) \iint_{\substack{(x+\epsilon,x+2\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m \\ d(u_i|^m),u_i(x+2\epsilon))\geq\frac{\eta\delta_{\mathcal{N}}}{2}}} \sum_i \frac{1}{|x+\epsilon-(x+2\epsilon)m|^{2m}} \mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon), \quad (47)$$

where $\eta > 0$ is the constant of Proposition 3.1 depending only on m, the constant $0 \leq \epsilon < \infty$ depends only on $m, \mathcal{N}$, and we have also used that $\mathcal{N}$ is compact, namely $\mathrm{diam}(\mathcal{N}) < \infty$. We set $\delta := \eta\delta_{\mathcal{N}}/2$. Taking

$$(2+\epsilon) := 1 + \exp\left(2(1+\epsilon) \iint_{\substack{(x+\epsilon,x+2\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m \\ d(u_i(x+\epsilon),u_i(x+2\epsilon))\geq\delta}} \sum_i \frac{1}{|\epsilon|^{2m}} \mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\right) \quad (48)$$

and using (477), we have then

$$\frac{1}{\ln(2+\epsilon)} \int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_i \#\left\{Q \in \mathcal{Q}_{2+\epsilon,\tau,k,h}^+ \mid \sup_{x\in\partial Q} \mathrm{dist}(V_i(x),\mathcal{N}) \geq \delta_{\mathcal{N}}/2\right\} \mathrm{d}h \frac{\mathrm{d}\tau}{\tau} \leq \frac{1}{2} \quad (49)$$

Similarly, by Proposition 3.2,

$$\frac{1}{\ln(2+\epsilon)} \int_1^{2+\epsilon} \sum_{k\in\mathbb{Z}} \int_{[0,1]^m} \sum_{Q\in\mathcal{Q}_{2+\epsilon,\tau,k,h}^+} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |DV_i(x)|^{m+1} \mathrm{d}h \frac{\mathrm{d}\tau}{\tau}$$
$$\leq \frac{1+\epsilon}{\ln(2+\epsilon)} \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_i \frac{d(u_i(x+\epsilon),u_i(x+2\epsilon))^{m+1}}{|\epsilon|^{2m}} \mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon). \quad (50)$$

In view of the estimates (49) and (50), there exists $\tau \in (1,2+\epsilon)$ and, for every $k \in \mathbb{Z}$, $h_k \in [0,1]^m$ such that for every $k \in \mathbb{Z}$, every $Q \in \mathcal{Q}_{2+\epsilon,\tau,k,h_k}^+$ and every $x \in \partial Q$, one has

$$\mathrm{dist}(V_i(x),\mathcal{N}) \leq \frac{\delta_{\mathcal{N}}}{2} \quad (51)$$

and

$$\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_{2+\epsilon,\tau,k,h_k}^+} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |DV_i(x)|^{m+1}$$
$$\leq \frac{1+\epsilon}{\ln(2+\epsilon)} \iint_{\mathbb{R}^m\times\mathbb{R}^m} \sum_i \frac{d(u_i(x+\epsilon),u_i(x+2\epsilon))^{m+1}}{|\epsilon|^{2m}} \mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon). \quad (52)$$

We define now the set of good cubes

$$\mathcal{G} := \left\{ Q \in \mathcal{Q}^{+}_{2+\epsilon,\tau,k,h_k} \left| \sup_{x\in\partial Q} x_{m+1}^{m+1} \left| \mathrm{D}V_i(x) \right|^{m+1} \leq \mu \text{ and } k \in \mathbb{Z} \right. \right\} \tag{53}$$

and the set of bad cubes

$$\mathcal{B} := \left\{ Q \in \mathcal{Q}^{+}_{2+\epsilon,\tau,k,h_k} \left| \sup_{x\in\partial Q} x_{m+1}^{m+1} \left| \mathrm{D}V_i(x) \right|^{m+1} > \mu \text{ and } k \in \mathbb{Z} \right. \right\}, \tag{54}$$

where μ will be chosen in (58). Clearly, any cube is either good or bad and thus

$$\bigcup_{Q\in\mathcal{G}\cup\mathcal{B}} Q = \mathbb{R}^{m+1}_{+}$$

By (52) and (54),

$$\#\mathcal{B} \leq \frac{1}{\mu} \sum_{k\in\mathbb{Z}_Q} \sum_{Q\in\mathcal{Q}^{+}_{2+\epsilon,\tau,k,h_k}} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1}$$

$$\leq \frac{1+\epsilon}{\mu \ln(2+\epsilon)} \iint_{\mathbb{R}^m} \int_{\times\mathbb{R}^m} \sum_i \frac{d(u_i(x+\epsilon), u_i(x+2\epsilon))^{m+1}}{|\epsilon|^{2m}} \, \mathrm{d}(x+\epsilon)\, \mathrm{d}(x+2\epsilon) \tag{55}$$

Notice also that for every $k \in \mathbb{Z}$ and every $Q \in \mathcal{Q}^{+}_{2+\epsilon,\tau,k,h_k}$, we have

$$\tau(2+\epsilon)^{-k} \int_{\partial Q} \sum_i |\mathrm{D}V_i|^{m+1} \leq (1+\epsilon)(\tau(2+\epsilon)^{-k})^{m+1} \sum_i \sup_{x\in\partial Q} |\mathrm{D}V_i(x)|^{m+1}$$

$$\leq (1+\epsilon)^{m+2} \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \tag{56}$$

since for every $x \in Q, x_{m+1} \geq \tau(2+\epsilon)^{-k}/(1+\epsilon)$, in view of (24).

We are now going to define a map $W_i\colon \mathbb{R}^{m+1}_{+} \to \mathcal{N} + \mathbb{B}^{\nu}_{\delta_{\mathcal{N}}}$ separately on $\mathrm{U}_{\mathcal{G}}$ and $\mathrm{U}_{\mathcal{B}}$. For every $Q \in \mathcal{G}$, we apply Lemma 3.3, up to a suitable bi-Lipschitz homeomorphism between a ball and a cube (see [10, Cor. 3]), to define the mapping W_i on Q as an extension of $V_i \restriction_{\partial Q}$. We have, in view of (39) and of (56),

$$\int_Q \sum_i |\mathrm{D}W_i|^{m+1} \leq (1+\epsilon)\tau(2+\epsilon)^{-k} \int_{\partial Q} \sum_i |\mathrm{D}V_i|^{m+1}$$

$$\leq (1+\epsilon)^{m+2} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \tag{57}$$

whereas by the triangle inequality, (51), (41), (56) and (53),

$$\mathrm{dist}(W_i(x), \mathcal{N}) \leq \frac{\delta_{\mathcal{N}}}{2} + (1+\epsilon)\mu^{\frac{1}{m+1}}(1+\epsilon)$$

Hence if

$$\mu = \left(\frac{\delta_{\mathcal{N}}}{2(1+\epsilon)^2} \right)^{m+1} \tag{58}$$

we have for every $x \in Q, W_i(x) \in \mathcal{N} + \mathbb{B}^{\nu}_{\delta_{\mathcal{N}}}$. Using the Chebyshev inequality and (57), we obtain

$$(1+\epsilon)^{m+1} \mathcal{L}^{m+1}(\{x \in Q \,||\mathrm{D}W_i(x)\,| \geq 1+\epsilon\}) \leq (1+\epsilon)^{m+2} \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \tag{59}$$

Next, we apply, up to a suitable bi-Lipschitz homeomorphism between a ball and a cube (see [10, Cor. 3]), Lemma 3.4 on every bad cube $Q \in \mathcal{B}$ to define there W_i by homogeneous extension of $V_i \restriction_{\partial Q}$ with respect to the barycenter of Q. We compute then for such a cube, in view of (46) and (56), for every $0 \leq \epsilon < \infty$,

$$\mathcal{L}^{m+1}(\{x \in Q \,||\mathrm{D}W_i(x)\,| \geq 1+\epsilon\}) \leq \frac{1}{(1+\epsilon)^m} \tau(2+\epsilon)^{-k} \int_{\partial Q} \sum_i |\mathrm{D}V_i|^{m+1}$$

$$\leq (1+\epsilon) \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \tag{60}$$

Combining the estimates (59) and (60), we get

$$(1+\epsilon)^{m+1} \mathcal{L}^{m+1}(\{x \in \mathbb{R}^{m+1}_{+} \,||\mathrm{D}W(x)\,| \geq 1+\epsilon\})$$

$$\leq (1+\epsilon)^{m+2} \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}^{+}_{2+\epsilon,\tau,k,h_k}} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \tag{61}$$

$$\leq (1+\epsilon)\exp\sum_i\left((1+\epsilon)\iint_{\substack{(x+\epsilon,x+2\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m\\ d(u_i(x+\epsilon),u_i(x+2\epsilon))\geq\delta}}\frac{1}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\right)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\frac{d(u_i(x+\epsilon),u_i(x+2\epsilon))^{m+1}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon),$$

in view of (48) and (52), where $(1+\epsilon)$ is positive constants depending only on $m,\mathcal{N}$. Moreover, if S denotes the set of the barycenters of the cubes $Q\in\mathcal{B}$, we have

$$\#S=\#\mathcal{B}\leq(1+\epsilon)\exp\sum_i\left((1+\epsilon)\iint_{\substack{(x+\epsilon,x+2\epsilon)\in\mathbb{R}^m\times\mathbb{R}^m\\ d(u_i(x+\epsilon),u_i(x+2\epsilon))\geq\delta}}\frac{1}{|\epsilon|^{2m}}\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\right)\iint_{\mathbb{R}^m\times\mathbb{R}^m}\frac{d(u_i(x+\epsilon),u_i(x+2\epsilon))^{m+1}}{|\epsilon|^{2m}}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}(x+2\epsilon)\qquad(62)$$

in view of (48), (55) and (58), where $(1+\epsilon)$ is positive constants depending only on $m,\mathcal{N}$.

In order to check that u_i is the trace of W_i, we observe that on the one hand we have, since V_i is an extension by convolution, for every $Q\in\mathcal{G}$, by Poincaré's inequality, by (40) and by (56),

$$\begin{aligned}\int_Q\sum_i|V_i-W_i|^{m+1}&\\ &\leq\frac{1+\epsilon}{(\tau(2+\epsilon)^{-k})^m}\sum_i\left(\int_{\partial Q}\int_Q|V_i(x)-V_i(x+\epsilon)|^{m+1}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}x+\int_{\partial Q}\int_Q|W_i(x)-V_i(x+\epsilon)|^{m+1}\,\mathrm{d}(x+\epsilon)\,\mathrm{d}x\right)\\ &\leq\frac{1+\epsilon}{(\tau(2+\epsilon)^{-k})^m}\int_Q\int_{\partial Q}\sum_i|V_i(x)-V_i(x+\epsilon)|^{m+1}\,\mathrm{d}x\,\mathrm{d}(x+\epsilon)+(1+\epsilon)\big(\tau(2+\epsilon)^{-k}\big)^{m+2}\int_{\partial Q}|\mathrm{D}V_i|^{m+1}\\ &\leq(1+\epsilon)(\tau(2+\epsilon)^{-k})^{m+1}\sum_i\left(\int_Q|\mathrm{D}V_i|^{m+1}+(1+\epsilon)^{m+1}\sup_{x\in\partial Q}x_{m+1}^{m+1}|\mathrm{D}V_i(x)|^{m+1}\right)\end{aligned}\qquad(63)$$

By the classical theory of traces,

$$\int_{\mathbb{R}^{m+1}}\sum_i|\mathrm{D}V_i|^{m+1}<\infty\qquad(64)$$

It follows then from (52), (63) and (64) that

$$\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{G}}\frac{1}{((2+\epsilon)\tau^{-k})^{m+1}}\int_Q\sum_i|V_i-W_i|^{m+1}<\infty\qquad(65)$$

Thus, since the set $\mathcal{B}$ is finite, in view of (65), we have

$$\lim_{k\to\infty}\frac{1}{((2+\epsilon)\tau^{-k})^{m+1}}\sum_{Q\in\mathcal{Q}^+_{2+\epsilon,\tau,k,h_k}}\int_Q\sum_i|V_i-W_i|^{m+1}\,\mathrm{d}x=0$$

This implies that W_i and V_i have the same trace, and hence u_i is the trace of W_i. Finally, we define $U_i:=\Pi_{\mathcal{N}}\circ W_i$. The map U_i also has u_i as the trace on $\mathbb{R}^m\times\{0\}$; the conclusion (77) follows from (61); $U_i\in C(\mathbb{R}^{m+1}_+\setminus S,\mathcal{N})$ with the cardinality of the singular set S being estimated by (62). This completes our proof of Theorem 1.2.

The proof of Theorem 1.2 can be adapted easily to the case of the hyperbolic space (Theorem 1.3).

Proof of Theorem 1.3 (see [28]). We proceed as in the proof of Theorem 1.2. Instead of (60), we proceed using the Poincaré metric on the half-space, (46), (56) and get for every $Q\in\mathcal{Q}^+_{2+\epsilon,\tau,k,h}$ (see (24)),

$$\begin{aligned}\int_{\substack{x\in Q\\ |\mathrm{D}W_i(x)|x_{m+1}\geq1+\epsilon}}\sum_i\frac{1}{x_{m+1}^{m+1}}\,\mathrm{d}x&\leq\sum_i\left(\frac{(1+\epsilon)(2+\epsilon)^k}{\tau}\right)^{m+1}\mathcal{L}^{m+1}\left(\left\{x\in Q\,\middle|\,|\mathrm{D}W_i(x)|\,\frac{\tau(2+\epsilon)^{-(k-1)}}{1+\epsilon}\geq1+\epsilon\right\}\right)\\ &\leq\frac{(2+\epsilon)^{m+1}}{(1+\epsilon)^m}\tau(2+\epsilon)^{-k}\int_{\partial Q}\sum_i|\mathrm{D}V_i|^{m+1}\\ &\leq(1+\epsilon)(\tau(2+\epsilon)^{-k})^{m+1}\sum_i\sup_{x\in\partial Q}|\mathrm{D}V_i(x)|^{m+1}\\ &\leq(1+\epsilon)^{m+2}\sum_i\sup_{x\in\partial Q}x_{m+1}^{m+1}|\mathrm{D}V_i(x)|^{m+1}\end{aligned}$$

where $\epsilon\geq0$, V_i is defined in (18) and, up to a suitable bi-Lipschitz homeomorphism between a ball and a cube (see [10, Cor. 3]), W_i is defined on Q by homogeneous extension of $V_i\restriction_{\partial Q}$ with respect to the barycenter of Q. Observe that the constant $0\leq\epsilon<\infty$ depends only on m. The remainder of the proof is similar.

The case of the singular extension to the ball is slightly more complicated, as we will rely on the parameterization of $\mathbb{B}^{m+1}_1$ by $\mathbb{R}^{m+1}_+$ through a classical suitable conformal mapping.

Proof of Theorem 1.1 (see [28]). We recall that the map

$$\Psi(x) := \frac{4(x+e)}{|x+e|^2} - 2e$$

where $e := (0, \dots, 0, 1) \in \mathbb{R}_+^{m+1}$, defines a diffeomorphism from $\mathbb{B}_1^{m+1}$ to $\mathbb{R}_+^{m+1}$. Indeed, if $x \in \overline{\mathbb{B}}_1^{m+1} \setminus \{e\}$, then

$$e \cdot \Psi(x) = \frac{2 - 2|x|^2}{|x+e|^2}$$

Moreover, we have

$$\Psi^{-1}(x) = \frac{4(x+2e)}{|x+2e|^2} - e$$

In particular, the ball $\mathbb{B}_1^{m+1}$ is isometric to the half-space $\mathbb{R}_+^{m+1}$ endowed with the metric g defined for $x \in \mathbb{R}_+^{m+1}$ and $v \in \mathbb{R}^{m+1}$ by

$$g(x)[v, v] = \frac{16|v|^2}{|x+2e|^4}$$

Moreover, since Ψ is a conformal map and $u_i : \mathbb{S}^m \to \mathcal{N}$, one has

$$\iint_{\mathbb{S}^m \times \mathbb{S}^m} \sum_i \frac{d(u_i(x), u_i(x+\epsilon))^{1+\epsilon}}{|\epsilon|^{2m}} \,\mathrm{d}x\, \mathrm{d}(x+\epsilon) = \iint_{\mathbb{R}^m \times \mathbb{R}^m} \sum_i \frac{d\left(u_i(\Psi^{-1}(x)), u_i(\Psi^{-1}(x+\epsilon))\right)^{1+\epsilon}}{|\epsilon|^{2m}} \,\mathrm{d}x\, \mathrm{d}(x+\epsilon)$$

and

$$\iint_{\substack{x, x+\epsilon \in \mathbb{S}^m \\ d(u_i(x), u_i(x+\epsilon)) \geq \delta}} \sum_i \frac{d(u_i(x), u_i(x+\epsilon))^{1+\epsilon}}{|\epsilon|^{2m}} \,\mathrm{d}x\, \mathrm{d}(x+\epsilon)$$

$$= \iint_{\substack{x, x+\epsilon \in \mathbb{R}^m \\ d\left(u_i(\Psi^{-1}(x)), u_i(\Psi^{-1}(x+\epsilon))\right) \geq \delta}} \sum_i \frac{d\left(u_i(\Psi^{-1}(x)), u_i(\Psi^{-1}(x+\epsilon))\right)^{1+\epsilon}}{|\epsilon|^{2m}} \,\mathrm{d}x\, \mathrm{d}(x+\epsilon)$$

(the reader may also consult [1, discussion after I, (15)]). We proceed then as in the proof of Theorem 1.2, using $u_i \circ \Psi^{-1}$ instead of u_i. In order to replace the estimate (60), we proceed as follows.

Given $Q \in \mathcal{Q}^+_{2+\epsilon,\tau,k,h}$ and setting

$$m_Q = \inf_{x \in Q} \frac{4}{|x+2e|^2} \quad \text{and} \quad M_Q = \sup_{x \in Q} \frac{4}{|x+2e|^2}$$

we have

$$\int_{\substack{x \in Q \\ |DW_i(x)| \geq 4(1+\epsilon)/|x+2e|^2}} \sum_i \frac{4^{m+1}}{|x+2e|^{2m+2}} \,\mathrm{d}x \leq \sum_i M_Q^{m+1} \mathcal{L}^{m+1}(\{x \in Q \,||\mathrm{D}W(x)| \geq (1+\epsilon) m_Q\})$$

$$\leq \frac{1}{(1+\epsilon)^m} \left(\frac{M_Q}{m_Q}\right)^{m+1} \tau(2+\epsilon)^{-k} \int_{\partial Q} \sum_i |\mathrm{D}V_i|^{m+1}$$

$$\leq \frac{1}{(1+\epsilon)^m} \left(\frac{M_Q}{m_Q}\right)^{m+1} (\tau(2+\epsilon)^{-k})^{m+1} \sum_i \sup_{x \in \partial Q} |\mathrm{D}V_i(x)|^{m+1}$$

$$\leq (1+\epsilon) \left(\frac{M_Q}{m_Q}\right)^{m+1} \sum_i \sup_{x \in \partial Q} x_{m+1}^{m+1} |\mathrm{D}V_i(x)|^{m+1} \tag{66}$$

where $\epsilon \geq 0$, V_i is defined in (18) and, up to a suitable bi-Lipschitz homeomorphism between a ball and a cube (see [10, Cor. 3]), W_i is defined on Q by homogeneous extension of $V_i \restriction_{\partial Q}$ with respect to the barycenter of Q. It is worth noting that the constant $0 \leq \epsilon < \infty$ depends only on m. We observe now that if $x, x+\epsilon \in Q$, then

$$\frac{|x+2e|}{|x+\epsilon+2e|} \leq 1 + \frac{|\epsilon|}{|x+\epsilon+2e|} \leq 1 + \frac{\tau(2+\epsilon)^{-k}\sqrt{m+1}}{2 + \frac{\tau(2+\epsilon)-k}{1+\epsilon}} \leq 1 + (1+\epsilon)\sqrt{m+1}, \tag{67}$$

and we have thus by (66) and (67),

$$\int_{\substack{x\in Q\\ |DW_i(x)|\geq 4(1+\epsilon)/|x+2e|^2}} \sum_i \frac{4^{m+1}}{|x+2e|^{2m+2}}\, dx$$

$$\leq (1+\epsilon)(1+(1+\epsilon)\sqrt{m+1})^{2m+2} \sum_i \sup_{x\in\partial Q} x_{m+1}^{m+1} |DV_i(x)|^{m+1}$$

The rest of the proof is similar to the proof of Theorem 1.2,

References

[1] L. V. Ahlfors, Möbius transformations in several dimensions, Ordway Professorship Lectures in Mathematics, University of Minnesota, School of Mathematics, Minneapolis, MN, 1981. ↑20

[2] F. Bethuel, A new obstruction to the extension problem for Sobolev maps between manifolds, J. Fixed Point Theory Appl. 15 (2014), no. 1, 155-183, doi:10.1007/s11784-014-0185-0. ↑ 2

[3] F. Bethuel and F. Demengel, Extensions for Sobolev mappings between manifolds, Calc. Var. Partial Differential Equations 3 (1995), no. 4, 475-491, doi:10.1007/BF01187897. ↑ 2

[4] J. Bourgain, H. Brezis, and P. Mironescu, Lifting, degree, and distributional Jacobian revisited, Comm. Pure Appl. Math. 58 (2005), no. 4, 529-551, doi:10.1002/cpa.20063. ↑ 3

[5] J. Bourgain, H. Brezis, and Nguyen H.-M., A new estimate for the topological degree, C. R. Math. Acad. Sci. Paris 340 (2005), no. 11, 787-791, doi:10.1016/j.crma.2005.04.007. ↑ 3

[6] H. Brezis and Nguyen H.-M., On a new class of functions related to VMO, C. R. Math. Acad. Sci. Paris 349 (2011), no. 3-4, 157-160, doi:10.1016/j.crma.2010.11.026. ↑ 4

[7] H. Brezis and L. Nirenberg, Degree theory and BMO. I. Compact manifolds without boundaries, Selecta Math. (N.S.) 1 (1995), no. 2, 197-263, doi:10.1007/BF01671566 ↑ 2,5

[8] E. DiBenedetto, Real analysis, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser/Springer, New York, 2016, doi:10.1007/978-1-4939-4005-9. ↑ 14

[9] E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, Rend. Sem. Mat. Univ. Padova 27 (1957), 284-305. ↑ 1,14

[10] J. A. Griepentrog, W. Höppner, H.-C. Kaiser, and J. Rehberg, A bilipschitz continuous, volume preserving map from the unit ball onto a cube, Note Mat. 28 (2008), no. 1, 177-193, doi:10.1285/i15900932v28n1p177. ↑ 17,19,20

[11] R. Hardt and F.-H. Lin, Mappings minimizing the L^p norm of the gradient, Comm. Pure Appl. Math. 40 (1987), no. 5, 555-588, doi:10.1002/cpa.3160400503. ↑ 2

[12] A. Kufner, O. John, and S. Fučik, Function spaces, Noordhoff, Leyden, 1977. ↑ 14

[13] G. Leoni, A first course in fractional Sobolev spaces, Graduate Studies in Mathematics, vol. 229, American Mathematical Society, Providence, R.I., 2023, doi:10.1090/gsm/229. ↑ 14

[14] V. Maz'ya, Sobolev spaces with applications to elliptic partial differential equations, Second, revised and augmented edition, Grundlehren der Mathematischen Wissenschaften, vol. 342, Springer, Heidelberg, 2011, doi:10.1007/978-3-642-15564-2 ↑ 14

[15] P. Mironescu and J. Van Schaftingen, Trace theory for Sobolev mappings into a manifold, Ann. Fac. Sci. Toulouse Math. (6) 30 (2021), no. 2, 281-299, doi:10.5802/afst.1675. ↑ 2

[16] J. Nash, The imbedding problem for Riemannian manifolds, Ann. of Math. (2) 63 (1956), 20-63, doi:10.2307/1969989 ↑ 2

[17] Nguyen H.-M., Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006), no. 2, 689-720, doi:10.1016/j.jfa.2006.04.001 ↑ 4

[18] , Γ-convergence and Sobolev norms, C. R. Math. Acad. Sci. Paris 345 (2007), no. 12, 679-684, doi:10.1016/j.crma.2007.11.005 ↑ 4

[19] , Optimal constant in a new estimate for the degree, J. Anal. Math. 101 (2007), 367-395, doi:10.1007/s11854-007-0014-0. ↑ 3

[20] , Inequalities related to liftings and applications, C. R. Math. Acad. Sci. Paris 346 (2008), no. 17-18, 957-962, doi:10.1016/j.crma.2008.07.026. ↑ 4

[21] , Further characterizations of Sobolev spaces, J. Eur. Math. Soc. (JEMS) 10 (2008), no. 1, 191-229, doi:10.4171/JEMS/108 ↑ 4

[22] , Estimates for the topological degree and related topics, J. Fixed Point Theory Appl. 15 (2014), no. 1, 185-215, doi:10.1007/s11784-014-0182-3. ↑ 3

[23] M. Petrache and T. Rivière, Global gauges and global extensions in optimal spaces, Anal. PDE 7 (2014), no. 8, 1851-1899, doi:10.2140/apde.2014.7.1851. ↑ 3

[24] M. Petrache and J. Van Schaftingen, Controlled singular extension of critical trace Sobolev maps from spheres to compact manifolds, Int. Math. Res. Not. IMRN 12 (2017), 3647-3683, doi:10.1093/imrn/rnw109, †3, 4, 6
[25] J. Van Schaftingen, Estimates by gap potentials of free homotopy decompositions of critical Sobolev maps, Adv. Nonlinear Anal. 9 (2020), no. 1, 1214-1250, doi:10.1515/anona-2020-0047. ↑ 2,3,7
[26] , Lifting of fractional Sobolev mappings to noncompact covering spaces, Ann. Inst. H. Poincaré Anal. Non Linéaire, doi:10.4171/AIHPC/98. ↑ 4,7
[27] , The extension of traces for Sobolev mappings, available at https://arxiv.org/abs/2403.18738 ↑ 2
[28] Bohdan Bulanyi and Jean Van Schaftingen, Singular Extension of Critical Sobolev Mappings Under an Exponential Weak-Type Estimate, Journal of Functional Analysis, 288 (2025), no. 1, 1-21.