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Research Paper

Derivations on a C^*-Algebra and Its Double Dual — a Slotwise Rank-2 Formulation

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Abreact.
We study extensions building on the pioneering work of J. W. Bunce and W. L. Paschke [15] ; in this paper we extend their framework from rank-1 to slotwise rank-2. Let A be a C^*-algebra and X a Banach A -module. The module action of A on X gives rise to slotwise rank-2 module actions of A^{**} on X^* and X^{**}, and derivations $D: A \to X$ (resp. $D: A \to X^*$) extend to derivations of A^{**} into X^{**} (resp. X^*) compatible with the rank-2 conventions. If A is nuclear and X is a dual Banach A -module with X^* weakly sequentially complete, then every derivation of A into X is inner. Under the same hypothesis on A, the extension to the finite part of A^{**} of any derivation of A into any dual Banach A -module is inner, as are all derivations of A into A^{**}. Every derivation of a semifinite von Neumann algebra into its predual is inner. All definitions, actions and extensions are presented in the slotwise rank-2 form used throughout this paper.

Keywords: *Derivations on a C^*-Algebra; Dual Banach Modules; semifinite von Neumann algebra; Extension of A^{**}.*

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I. Introduction

Under the work of [15] we add a high order a C^*-algebra A is referred to as accessible if all derivations from A to the dual of a Banach A-module are inner and nuclear if the double dual is an injective von Neumann algebra. Connes demonstrated that every accessible C^*-algebra is nuclear in [15] (see also [5]). Some first findings on the converse implication and associated problems are presented in this work. Let X be a Banach A -module and A a C^*-algebra. We suppose that X is unital when A is unital. In Section 1 we demonstrate how the action of A on X extends to actions of A^{**} on X^* and X^{**}, and how a derivation of A into X (resp. X^* extends to a derivation of A^{**} into X^{**} (resp. X^*. These arguments are mainly based on weak compactness results from [2,3], which do not hold when A is assumed only to be a Banach algebra. In Section 3 we show that the derivations of a nuclear C^*-algebra A into certain Banach A -modules are either inner or nearly inner.

If $D: A \to X$ is a derivation into a dual Banach A-module X with X^* weakly sequentially complete and A is nuclear, then D is inner. We show that any derivation of A into A^* is inner, and that the extension of D to the finite part of A^{**} is inner for every derivation D of A into a dual Banach A -module under the same assumptions on A.

The authors of [15] give a this last result raises the question whether every derivation of a von Neumann algebra into its predual is inner. We provide an affirmative answer for semifinite von Neumann algebras in Section 4.

but here [15] prove a much stronger result.

Notation. We have extended the Bunce–Paschke framework from rank-1 to the slotwise (rank-2) setting. Unless otherwise stated, all module actions, dual (transpose) actions, pairings and evaluations are interpreted slotwise whenever elements of second duals or other double dual spaces appear; ordinary (rank-1) readings remain in force for identities written without explicit pairs. We adopt the following case convention: elements of the original algebra A (or R) and of preduals (R_* or A_) are written in lower-case ($a, b, a_{1,}a_2, \ldots;\ f, g, \ldots$); elements of the second dual (R^*, A^*, A^{**}) are written in upper-case ($A, B, U, V, \ldots$). Use an upper-case letter for a unitary in $U(R)$ only when that unitary is explicitly regarded as an element of R^*; otherwise keep the unitary lower-case; any change of membership must be announced explicitly. For constructions used in the proofs we stated them explicitly in slotwise form. Finally, whenever nets and limits in ultrastrong, ultraweak or Mackey topologies are invoked we always specified whether the net elements lie in R or in R^* and applied the above case convention accordingly; a short reminder to "work slotwise" is inserted at the start of each section that makes heavy use of slotwise notation. This paragraph serves as the authoritative reference for any exception or change of reading throughout the paper.

II. Extension of A^{**}

In this subsection we explained in [10, Sect. 1], let A be a C*-algebra and X a Banach A-module. Recall that the definition of A 's action on X* is as follows:

$(x_{1,}x_2, f)(a_{1,}a_2) = f(a_1 x_1 a_2\, x_2)$ and $(f, x_{1,}x_2)(a_{1,}a_2) = f(x_1 a_1 x_2 a_2)\ \ \forall\ a_{1,}a_2 \in A \times A, f \in X^*$, and $x_{1,}x_2 \in X \times X$.We define the elements (x, f) and $[f, x]$ of A^* as follows:

$$(x_{1,}x_2, f)(a_{1,}a_2) = f(a_1 x_1 a_2\, x_2) \qquad (2.1)$$

$$[f, x_{1,}x_2](a_{1,}a_2) = f(x_1 a_1 x_2 a_2). \qquad (2.2)$$

Easy calculations show that the following holds

$$[f, x_{1,}x_2 a] = [f, x_{1,}x_2]a, \qquad (2.3)$$

$$(a x_{1,}x_2, f) = a(x_{1,}x_2, f), \qquad (2.4)$$

$$(x_{1,}x_2 a, f) = (x_{1,}x_2, af), \qquad (2.5)$$

$$[f, a x_{1,}x_2] = [fa, x_{1,}x_2]. \qquad (2.6)$$

Define bidual elements Bf and fB of $X^*\ \forall\ B \in A^{**}$ and $f \in X^*$ by

$$(Bf)(x_{1,}x_2) = \langle [f, x], B\rangle \qquad (2.7)$$

$$(fB)(x_{1,}x_2) = \langle (x_{1,}x_2, f), B\rangle \qquad (2.8)$$

(The pairing between A^* and A^{**} is denoted by angle brackets.) The action previously defined of A on X^* corresponds to b · A. Definitions(7) and (8) are given in [10, pp. 16–17]. For $f \in X^*$ and $B_1, B_2 \in A^{**}$, equation (2.3) shows that $((B_1 B_2)f = B_1(B_2 f)$; hence X^* is a left A^{**}-module. X^* is a right A^{**}-module as shown by equation (2.4). For $a \in A$ and $B \in A^{**}$, equation (2.5) states that $a(fB) = (af)B$, while equation (2.6) states that $B(fa) = (Bf)a$. Furthermore, the actions of A^{**} on X^* are normal in the sense that if $B_\alpha \to B\ \sigma(A^{**}, A^*)$, then $B_\alpha f \to Bf$ and $fB_\alpha \to fB\ \sigma(X^*, X)$. The fact that A is a C^*-algebra has not yet been used in any essential way. However, an Akemann result on C-algebras must be invoked to show that $B_1(fB_2) = (B_1 f)B_2\ \forall\ B_1, B_2 \in A^{**}$.

Proposition 2.1.(see [15]) X^* is a regular Banach A^{**}-module, as defined above, satisfying
$(B_1 f)B_2 = B_1(fB_2)$ for every $f \in X^*, B_1, B_2 \in A^{**}$.

Proof. For $(x_1, x_2) \in X \times X$ and $f \in X^*$, define $\theta_{x_1,x_2,f}: A \to A^*$ by $\theta_{x_1,x_2,f}(a)(a_1, a_2) = f(a_1 a x_1 a_2\ x_2)$. By [2, Corollary II.9], $\theta_{x_1,x_2,f}$ is weakly compact. Let $B_1 \in A^{**}$. A net $\{a_\alpha\}$ in A, bounded by $\|B_1\|$, which converges to B_1 in the $\sigma(A^{**}, A^*)$-topology, is provided by the Kaplansky density theorem. Since $\theta_{x_1,x_2,f}$ sends norm-bounded subsets of A to weakly compact subsets of A^*, we may pass to a subnet so that $\{\theta_{x_1,x_2,f}(a_\alpha)\}$ converges $\sigma(A^*, A^{**})$ to some $g \in A^*$. For $B_2 \in A^{**}$. Then we have

$$\begin{aligned}
(b_1(fB_2))(x_1, x_2) &= \lim_\alpha (a_\alpha(fB_2))(x_1, x_2) \\
&= \lim_\alpha ((a_\alpha f)B_2)(x_1, x_2) \\
&= \lim_\alpha \langle (x_1, x_2, a_\alpha f), b_2 \rangle \\
&= \lim_\alpha \langle \theta_{x_1,x_2,f}(a_\alpha), B_2 \rangle \\
&= \langle g, B_2 \rangle.
\end{aligned}$$

Now for $(a_1, a_2) \in A \times A$

$$\begin{aligned}
(x_1, x_2, B_1 f)(a_1, a_2) &= (B_1 f)(a_1 x_1 a_2\ x_2) \\
&= \langle [f, a_1 x_1 a_2\ x_2], B_1 \rangle \\
&= \lim_\alpha [f, a_1 x_1 a_2\ x_2]](a_\alpha) \\
&= \lim_\alpha f(a_1 x_1 a_2\ x_2] a_\alpha) \\
&= \lim_\alpha (x_1, x_2, a_\alpha f)(a_1, a_2) \\
&= \lim_\alpha \theta_{x_1,x_2,f}(a_\alpha)(a_1, a_2) \\
&= g(a_1, a_2),
\end{aligned}$$

so $(x_1, x_2, B_1 f) = g$. This means that

$$\begin{aligned}
((B_1 f)B_2)(x_1, x_2) &= \langle (x_1, x_2, B_1 f), B_2 \rangle \\
&= \langle g, B_2 \rangle \\
&= (B_1(fB_2))(x_1, x_2)
\end{aligned}$$

According to our previous computations, $(B_1 f)B_2 = B_1(fB_2)$, as required. If $X = A^*$, then the Arens multiplication on A^{**} coincides with the action of A^{**}on X^*.

Since X** is defined as a Banach A-module in Proposition 2.1, it follows that X^{**} is a dual A^{**}-module under the dual action.. This action is denoted by $B * \varphi$ and $\varphi * B(B \in A^{**}, \varphi \in X^{**})$, namely $(B * \varphi)(f) = \varphi(fB)$ and $(\varphi * B)(f) = \varphi(Bf)$ for $f \in X^*$. Another way to make X^{**}into a Banach A^{**}-module is to apply to the A-module X^* the same extension procedure that was used for the A module X. The result is a normal action of A^{**} on X^{**}, which we denote by $B \cdot \varphi$ and $\varphi \cdot B$. In fact, $(B \cdot \varphi)(f) = \langle [\varphi, f], B \rangle$,
where $[\varphi, f] \in A^*$ is defined by $[\varphi, f](a_1, a_2) = \varphi(f, a_1, a_2), (a_1, a_2) \in A \times A$, and similarly for $\varphi \cdot B$.Thus X^{**} carries two compatible actions of A^{**}: the dual action $(*)$and the normal action $(\cdot)$.

It is easily checked that($a_1, a_2) \cdot \varphi = (a_1, a_2) * \varphi$ and $\varphi \cdot (a_1, a_2) = \varphi * (a_1, a_2)\ \forall\ (a_1, a_2) \in A \times A, \varphi \in X^{**}$, but in general the two actions do not coincide on all of A^{**}. For example, let A be the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space H, and let X be the trace-class operators on H. In this case, the action of $A^{**} = B(H)$ on $X^* = B(H)$ given by Eqs. (2.7) and (2.8) is easily seen to be just the action of $B(H)$ on itself by left and right multiplication, so the action $*$ of A^{**} on X^{**} is the usual dual action of $B(H)$ on $B(H)^*$. However, since this action is abnormal, it must differ from the usual action of A^{**} on X^{**}. *We now* show that when X^* is weakly sequentially complete, the two actions coincide.

Proposition 2.2(see [15]) The * and $\cdot$ actions of A^{**} on X^{**}coincide if X is a Banach A -module such that $X^*$is weakly sequentially complete, making X^{**} a dual, normal A^{**}-module.

Proof. Take $B \in A^{**}$ and let $\{a_\alpha\}$ be a norm-bounded net in A converging $\sigma(A^{**}, A^*)$ to B. For $f \in X^*$, define $\theta_f: A \to X^*$ by $\theta_f(a) = fa$. Since X* is weakly sequentially complete, θ_f is weakly compact [3], and we may assume that $\{\theta_f(a_\alpha)\}$ converges $\sigma(X^*, X^{**})$ to a functional $g \in X^*$. For $\varphi \in X^{**}$, we have

$$\begin{aligned}(B \cdot \varphi)(f) &= \lim_\alpha (a_\alpha \cdot \varphi)(f) \\ &= \lim_\alpha \varphi(f a_\alpha) \\ &= \varphi(g).\end{aligned}$$

(The *-action is normal, therefore the first equality holds.) However, $(B * \varphi)(f) = \varphi(fB)$ and

$$\begin{aligned}(fB)(x_{1,}x_2) &= \langle (x_{1,}x_2, f), B\rangle \\ &= \lim_\alpha \langle (x_{1,}x_2, f), a_\alpha\rangle \\ &= \lim_\alpha f(a_\alpha, x_{1,}x_2) \\ &= \lim_\alpha (f a_\alpha)(x_{1,}x_2) \\ &= g(x),\end{aligned}$$

so that $fB = g$. Hence $(B * \varphi)(f) = \varphi(g) = (B \cdot \varphi)(f)$, so $B * \varphi = B \cdot \varphi$. An analogous argument shows that $\varphi * b = \varphi \cdot b$, and the proof is complete.

Let X be a Banach A-module and D a derivation of A into X, i.e., $D: A \to X$ is linear and $D(ab) = aD(b) + D(a)b \ \forall\ a, b \in A$. D is automatically norm continuous by [11]. We now show that, with respect to the normal action $\cdot$ of A^{**} on X^{**}**,** the second transpose map $D^{**}: A^{**} \to X^{**}$ is a derivation. If $B \in A^{**}$ and $f \in X^*$, then $D^*(f) = f \circ D, D^{**}(B) = B \circ D^*$, and $D^{**}(B)(f) = \langle f \circ D, B\rangle$. It is immediate that if A and X are identified with their images in A^{**} and X^{**}, then D^{**} extends D, and that D^{**} is $\sigma(A^{**}, A^*)$-to- $\sigma(X^{**}, X^*)$ continuous. Take $(a_{1,}a_2) \in A \times A, B \in A^{**}$; the following computation shows that $D^{**}((a_{1,}a_2)B) = (a_{1,}a_2)D^{**}(B) + D^{**}((a_{1,}a_2)) \cdot B$. Indeed, let $\{a_\alpha\}$ be a net in A converging $\sigma(A^{**}, A^*)$ to B. Then

$$\begin{aligned}D^{**}((a_{1,}a_2)B)(f) &= \langle f \circ D, (a_{1,}a_2)B\rangle \\ &= \langle (f \circ D)(a_{1,}a_2), B\rangle \\ &= \lim_\alpha \langle (f \circ D)(a_{1,}a_2), a_\alpha\rangle \\ &= \lim_\alpha f\left(D((a_{1,}a_2))a_\alpha + (a_{1,}a_2)D(a_\alpha)\right)\end{aligned}$$

By contrast, however

$$\begin{aligned}\left((a_{1,}a_2)D^{**}(B) + D^{**}((a_{1,}a_2)) \cdot B\right)(f) &= D^{**}(B)(f(a_{1,}a_2)) + \left\langle \left(f, D^{**}((a_{1,}a_2))\right), B\right\rangle \\ &= \langle (f(a_{1,}a_2)) \circ D, B\rangle + \left\langle \left(f, D^{**}((a_{1,}a_2))\right), B\right\rangle \\ &= \lim_\alpha \left\{\left\langle (f(a_{1,}a_2)) \circ D, a_\alpha\right\rangle + D^{**}((a_{1,}a_2))(a_\alpha f)\right\} \\ &= \lim_\alpha f\left((a_{1,}a_2)D(a_\alpha) + \left((a_\alpha f) \circ D\right)((a_{1,}a_2))\right) \\ &= \lim_\alpha f\left((a_{1,}a_2)D(a_\alpha) + D((a_{1,}a_2))a_\alpha\right).\end{aligned}$$

By a comparable argument, we obtain that $D^{**}(B(a_{1,}a_2)) = B \cdot D^{**}((a_{1,}a_2)) + D^{**}(B)(a_{1,}a_2)$. Our proof that $D^{**}(BC) = D^{**}(B) \cdot C + B \cdot D^{**}(C) \ \forall\ B, C \in A^{**}$ we again use Akemann's poor compactness result.

Proposition 2.3. (see [15]) $D^{**}: A^{**} \to X^{**}$ is a normal derivation if $D: A \to X$ is a derivation and A^{**} acts properly on X^{**} as described above.

Proof. For $f \in X^*$, let $\theta_f: A \to A^*$ be defined by $\theta_f(a)(B) = f(BD(a))$. By [2, Corollary II.9], θ_f is weakly compact. Fix $B \in A^{**}$ and choose a norm-bounded net $\{a_\alpha\}$ in A that converges to B in $\sigma(A^{**}, A^*)$. After passing to a subnet, we may assume $\{\theta_f(a_\alpha)\}$ converges in $\sigma(A^*, A^{**})$ to a functional $\rho \in A^*$. Let C be another fixed element of A^{**}**.**Then

$$\begin{aligned} D^{**}(CB)(f) &= \lim_{\alpha} D^{**}(Ca_\alpha)(f) \\ &= \lim_{\alpha}\big(D^{**}(C)a_\alpha + C\cdot D^{**}(a_\alpha)\big)(f) \\ &= (D^{**}(C)\cdot B)(f) + \lim_{\alpha}\big(C\cdot D^{**}(a_\alpha)\big)(f) \end{aligned}$$

We are aware that $D^{**}(a_\alpha) \to D^{**}(B)$ in the $\sigma(X^{**}, X^*)$ topology, but since the action $\cdot$ is not necessarily dual, we cannot conclude immediately that $\big(C\cdot D^{**}(a_\alpha)\big)(f) \to (C\cdot D^{**}(B))(f)$.However, $\big(C\cdot D^{**}(a_\alpha)\big)(f) = \langle [D^{**}(a_\alpha), f], C\rangle$ and for $(a_{1,}a_2) \in A\times A$, we have

$$\begin{aligned} [D^{**}(a_\alpha), f]((a_{1,}a_2)) &= D^{**}(a_\alpha)(f(a_{1,}a_2)) \\ &= \big\langle (f(a_{1,}a_2))\circ D, a_\alpha\big\rangle \\ &= f\Big((a_{1,}a_2)D(a_\alpha)\Big) \\ &= \theta_f(a_\alpha)((a_{1,}a_2)) \end{aligned}$$

so $[D^{**}(a_\alpha), f] = \theta_f(a_\alpha)$, which converges $\sigma(A^*, A^{**})$ to ρ. Hence

$$\lim_{\alpha}\big(C\cdot D^{**}(a_\alpha)\big)(f) = \langle \rho, C\rangle$$

Now $(C\cdot D^{**}(B))(f) = \langle [D^{**}(B), f], C\rangle$ and for $(a_{1,}a_2) \in A\times A$,

$$\begin{aligned} [D^{**}(B), f]((a_{1,}a_2) &= D^{**}(B)(f(a_{1,}a_2) \\ &= \langle (f(a_{1,}a_2)\circ D, B\rangle \\ &= \lim_{\alpha}\big\langle (f(a_{1,}a_2)\circ D, a_\alpha\big\rangle \\ &= \lim_{\alpha} f\Big((a_{1,}a_2D(a_\alpha)\Big) \\ &= \lim_{\alpha}\theta_f(a_\alpha)((a_{1,}a_2) \\ &= \rho((a_{1,}a_2) \end{aligned}$$

so $[D^{**}(B), f] = \rho$. Hence

$$\begin{aligned} D^{**}(CB)(f) &= (D^{**}(C)\cdot B)(f) + \langle \rho, C\rangle \\ &= (D^{**}(C)\cdot B)(f) + \langle [D^{**}(B), f], C\rangle \\ &= (D^{**}(C)\cdot B + C\cdot D^{**}(B))(f) \end{aligned}$$

The proof is now complete.

If X is a Banach A-module and $D: A \to X^*$ is a derivation, then there is also a way to extend D to a derivation $\tilde{D}: A^{**} \to X^*$. Let $(x_{1,}x_2) \in X\times X$ and define $f_x \in A^*$ by $f_x(a) = D(a)(x)$. Define $\tilde{D}: A^{**} \to X^*$ by

$$\tilde{D}(b)(x) = \langle f_x, b\rangle \qquad (2.9)$$

It is immediate that $\tilde{D}$ is linear and $\sigma(A^{**}, A^*)$-to- $\sigma(X^*, X)$ continuous, and that $\tilde{D}$ extends D.

$\tilde{D}(bc) = b\tilde{D}(c) + \tilde{D}(b)c$ if either b or c is in A and the other is in A, by the weak-continuity of $\tilde{D}$ and the normality of the action of A^{**} on X^*. In spirit, our proof of this equality $\forall\, B, C \in A^{**}$ is comparable to that of Proposition 2.3.

Proposition 2.4 (see [15]) D extends to a normal derivation $\tilde{D}: A^{**} \to X^*$ if $D: A \to X^*$ is a derivation, where the action of of A^{**} on X^* is given by Eqs. (2.7) and (2.8).

Proof. Fix $B, C \in A^{**}$ and $(x_{1,}x_2) \in X\times X$. We must show that

$$\tilde{D}(BC)(x_{1,}x_2) = (B\tilde{D}(C) + \tilde{D}(B)C)(x_{1,}x_2)$$

Define $\theta_{(x_{1,}x_2)}: A \to A^*$ by $\theta_x(a) = ((x_{1,}x_2), D(a_{1,}a_2))$. By [2, Corollary II.9], if $\{a_\alpha\}$ is a norm-bounded net in A that converges to b in $\sigma(A^{**}, A^*)$, then θ_x is weakly compact, we may assume that $\{\theta_x(a_\alpha)\}$ converges $\sigma(A^*, A^{**})$ to a functional $\rho \in A^*$. For $(\,a_{1,}a_2) \in A\times A$, we have

$$\begin{aligned}(x_{1,}x_2, \tilde{D}(b))(a_{1,}a_2) &= \tilde{D}(b)(a_1x_1a_2\, x_2)\\ &= \langle f_{(a_1x_1a_2\, x_2)}, b\rangle\\ &= \lim_\alpha\langle f_{(a_1x_1a_2\, x_2)}, a_\alpha\rangle\\ &= \lim_\alpha D(a_\alpha)(a_1x_1a_2\, x_2)\\ &= \lim_\alpha \left(x_{1,}x_2, D(a_\alpha)\right)(a_{1,}a_2)\\ &= \rho(a_{1,}a_2)\end{aligned}$$

so $\rho = ((x_{1,}x_2), \tilde{D}(B))$. Further,

$$\begin{aligned}(\tilde{D}(B)C)(x_{1,}x_2) &= \langle (x_{1,}x_2, \tilde{D}(B)), C\rangle\\ &= \langle \rho, C\rangle\\ &= \lim_\alpha \left\langle \left(x_{1,}x_2, D(a_\alpha)\right), C\right\rangle\\ &= \lim_\alpha (D(a_\alpha)C)(x_{1,}x_2)\end{aligned}$$

We conclude that

$$\begin{aligned}\tilde{D}(BC)(x_{1,}x_2) &= \lim_\alpha \tilde{D}(a_\alpha C)(x_{1,}x_2)\\ &= \lim_\alpha\left(a_\alpha \tilde{D}(C) + D(a_\alpha)C\right)(x_{1,}x_2)\\ &= \left(B\tilde{D}(C) + \tilde{D}(B)C\right)(x_{1,}x_2)\end{aligned}$$

where the normality of the action of A^{**} on X^* and the weak $*$-continuity of $\tilde{D}$ have been used. The proof is now complete.

III. Derivations for Nuclear C^*-Algebras

In this section we use the preceding section's results to show that derivations from a nuclear C^*-algebra into certain Banach modules are inner.

Proposition 3.1. (see [15]) Assume A is a nuclear C^*-algebra and X *is* a Banach A-module that is weakly sequentially complete. For each derivation $D: A \to X$ there exists $\varphi \in X^{**}$ such that $D\left(a_{1,}a_2\right) = (a_{1,}a_2)\varphi - \varphi$ $\forall$ $(a_{1,}a_2) \in A \times A$.

Proof. Proposition 2.2 makes X^{**} a dual, normal A^{**}**-**module under the action $\cdot$ defined in the preceding section. Proposition 2.3 shows that D is a normal derivation from A^{**} to X**.** By [7, 10, Proposition 7.6], A^{**}contains an ultraweakly dense, strongly amenable C^*-subalgebra B. *Since* X^{**} is a dual B -module, there exists $\varphi \in X^{**}$ such that $D^{**}((b_{1,}b_2) = (b_{1,}b_2) \cdot \varphi - \varphi \cdot (b_{1,}b_2)$ $\forall$ $(b_{1,}b_2) \in B \times B$. But D^{**} is a normal derivation and the action of A^{**} on X^{**} is normal, so because B is ultraweakly dense in A^{**} we have $D^{**}(\boldsymbol{A}) = \boldsymbol{A} \cdot \varphi - \varphi \cdot \boldsymbol{A}$ $\forall$ $\boldsymbol{A} \in A^{**}$. Given that D^{**} extends D, this establishes the claim.

Theorem 3.2. (see [15]) If A is a nuclear C^*-algebra and X is a dual Banach A-module whose predual $X^*$is weakly sequentially complete, then any derivation $D: A \to X$ is inner.

Proof. Using the previous proposition, we identify $\varphi \in X^{**}$ such that $D(a_{1,}a_2) = (a_{1,}a_2)\varphi - \varphi(a_{1,}a_2)$ $\forall$ $(a_{1,}a_2) \in A \times A$. Let Y be a Banach A -module with $Y^* = X$, where the dual action on Y induces the given action of A on X. The natural inclusion $j: Y \to Y^{**}$ and its transpose $j^*: Y^{***} \to Y^*$ are A-module maps. Identifying Y^* canonically with a subspace of Y^{***}**,** the map j is a projection of Y^{***} onto Y^*. Since $X = Y$, this means j^* is an A -module. Applying j^* to the equation $D(a_{1,}a_2)) = (a_{1,}a_2)\varphi - \varphi(a_{1,}a_2)$, we have $D(a_{1,}a_2) = (a_{1,}a_2)j^*(\varphi) - j^*(\varphi)(a_{1,}a_2)$ $\forall$ $(a_{1,}a_2) \in A \times A$, so that D is implemented by $j^*(\varphi)$. This validates the claim.

We note that if X is a C^*-algebra (resp. a von Neumann algebra) and there exist representations $\pi_1, \pi_2 : A \to X$, then defining the action of on X by $(a_1, a_2).(x_{1,}x_2) = \pi_1(a_1, a_2).(x_{1,}x_2)$ and($x_{1,}x_2)\,(a_1, a_2) = (x_{1,}x_2)\pi_2(a_{1,}a_2)$ yields A -modules X that satisfy the hypotheses of by **[** [15] Proposition 3.1 (resp. Theorem 2.2)]. Such derivations into these A -modules are examined in [4, Sect. 3].

The following theorem shows that the extension $\tilde{D}: A^{**} \to X^*$ (defined by Eq. (2.9) of Section 2) is inner on the finite part of A^{**}*;* however, it remains unclear whether every derivation D of a nuclear C-algebra A into an arbitrary dual module X must be inner. Equations (2.7) and (2.8) of Section 2 describe the action of A^{**} on X^*.

Theorem 3.3. (see [15]) Let p be the largest finite central projection in A^{**}, and let A *be* a nuclear C^*-algebra. If $D: A \to X^*$ is a derivation and X is a Banach A -module, then $\forall\, B \in A^{**}p$ there exists a functional $f \in X^*$ such that $\tilde{D}(B)p = fB - Bf$.

Proof. According to [7], $A^{**}p$ contains an extremely strong C^*-subalgebra B, which is the norm-closed linear span of an amenable group G of unitaries. In this case p is the identity of B, so $UU^* = U^*U = p$ for $U \in G$. Let m be the invariant mean on G. Note that X* need not be a unital B -module. Define F ∈ X*** by

$$F(\varphi) = m_U\{\varphi(\tilde{D}(U)U^*)\}\ (\varphi \in X^{**}). \tag{3.1}$$

Recall that $F * p = F$, where $*$ denotes the dual action of A^{**} on $X^{**}$and the corresponding action on X^{**}. For $U_0 \in G$ we compute (as in the proof of Theorem 2.5 in [10]):

$$\begin{aligned}(U_0^* * F * U_0)(\varphi) &= F(U_0 * \varphi * U_0^*)\\ &= m_U\{(U_0 * \varphi * U_0^*)(\tilde{D}(U)U^*)\}\\ &= m_U\{\varphi(U_0^*\tilde{D}(U)U^*U_0)\}\\ &= m_U\left\{\varphi\left(\left(\tilde{D}(U_0^*U) - \tilde{D}(U_0^*)U\right)(U^*U_0)\right)\right\}\\ &= m_U\{\varphi(\tilde{D}(U_0^*U)(U_0^*U)^*)\} - \varphi(\tilde{D}(U_0^*)U_0)\\ &= F(\varphi) - \varphi(\tilde{D}(U_0^*)U_0)\end{aligned}$$

Then $U_0^* * F * U_0 = F - \tilde{D}(U_0^*)U_0$. It follows that $\tilde{D}(U_0^*)p = F * U_0^* - U_0^* * F * p = F * U_0^* - U_0^* * F$. Since B is the norm-closed linear span of G, we obtain

$$\tilde{D}((b_{1,}b_2)p = F * (b_{1,}b_2) - (b_{1,}b_2) * F\ \ \forall\ (b_{1,}b_2) \in B \times B \tag{3.2}$$

(Since the $*$-action of A^{**} on X^{***} is not normal, we cannot conclude immediately at this point that (3.2) is valid $\forall\, B \in A^{**}p$. Notice also that we have not yet used the finiteness of $A^{**}p$.) Let $j: X \to X^{**}$ be the inclusion map. Define $f \in X^*$ by

$$f(x_{1,}x_2) = m_u\{(\tilde{D}(u)u^*)(x_{1,}x_2)\},\ \text{so that}\ F \circ j = f \tag{3.3}$$

Claim. We have $F(j(x_{1,}x_2) * B) = (Bf)(x_{1,}x_2)$ and $F\left(B * j(x_{1,}x_2)\right) = (fB)(x_{1,}x_2)\ \forall\, B \in A^{**}p$ and $(x_{1,}x_2) \in X \times X$.

Proof of claim. Let $\{a_\alpha\}$ be a norm-bounded net in A converging to Bin the ultrastrong-* topology, fix $(x_{1,}x_2) \in X \times X$and $B \in A^{**}p$, and define the bounded linear map $(A^{**}p)_*$ of $A^{**}p$, θ from B by

$$\theta(d)(Cp) = (Cp\tilde{D}(d))(x_{1,}x_2) \tag{3.4}$$

We have $(d_{1,}d_2) \in B \times B$ and $C \in A^{**}$**.** **(**The action of A^{**} on X^* is normal, so $\theta(d_{1,}d_2)$ is ultraweakly continuous on $A^{**}p$.) By [2, Corollary II.9], θ is weakly compact, hence $\{\theta(U): U \in G\}$ is a relatively weakly compact subset of $(A^{**}p)$. The finiteness *of* $A^{**}p$ implies that $\{|\theta(U)|: U \in G\}$ is likewise relatively weakly compact by [1, Proposition 2] or [12, Theorem 1], and the set $K = \{U\theta(U^*): U \in G\}$ is likewise relatively weakly compact by [12 Theorem 3]. Since $a_\alpha p \to bp = b$ in the Mackey topology [2] and the ultrastrong-topology, $a_\alpha p \to B$ uniformly on the relatively weakly compact set K. Since $\tilde{D}$ is a derivation and $UU^* = p$, we observe that for $U \in G, pU = U$ and $U\tilde{D}(U^*) = \tilde{D}(p) - \tilde{D}(U)U^*$.

Using (3.4), the quantity $U\theta(U^*)(a_\alpha p - b)$, which converges uniformly to zero for $U \in G$ as α increases, can be written as follows:

$$\begin{aligned} U\theta(U^*)(a_\alpha p - B) &= \theta(U^*)(a_\alpha pU - BU) \\ &= \theta(U^*)(a_\alpha U - BU) \\ &= \left(a_\alpha U\tilde{D}(U^*) - bB\tilde{D}(U^*)\right)((x_1,x_2) \\ &= \left(a_\alpha \tilde{D}(p) - b\tilde{D}(p)\right)(x) + \left(B\tilde{D}(U)U^* - a_\alpha \tilde{D}(U)U^*\right)((x_1,x_2) \end{aligned}$$

Since the action of A^{**} on X^{**} is normal, we have $\lim\left(a_\alpha \tilde{D}(p) - B\tilde{D}(p)\right)((x_1,x_2) = 0$. Consequently, $\left(B\tilde{D}(U)U^* - a_\alpha \tilde{D}(U)U^*\right)((x_1,x_2)$ goes to zero uniformly for $U \in G$ as α increases. This implies that

$$m_U\{\left(b\tilde{D}(U)U^*\right)(x)\} = \lim_\alpha m_U\{\left(a_\alpha \tilde{D}(U)U^*\right)((x_1,x_2)\} \qquad (3.5)$$

We now calculate that

$$\begin{aligned} F\left(j\left((x_1,x_2\right) * B\right) &= m_U\{\left(j\left((x_1,x_2\right) * B\right)\left(\tilde{D}(U)U^*\right)\} && \text{(by (3.1))} \\ &= m_U\{\left(b\tilde{D}(u)U^*\right)((x_1,x_2)\} \\ &= \lim_\alpha m_U\{\left(a_\alpha \tilde{D}(u)U^*\right)((x_1,x_2)\} && \text{(by (3.5))} \\ &= \lim_\alpha m_U\{\left(\tilde{D}(u)U^*\right)\left((x_1,x_2)a_\alpha\right)\} \\ &= \lim_\alpha f\left((x_1,x_2)a_\alpha\right) && \text{(by (3.3))} \\ &= \lim_\alpha (a_\alpha f)(x_1,x_2) \\ &= ((b_1,b_2)f)((x_1,x_2)) \end{aligned}$$

A similar argument shows that $F((b_1,b_2) * j(x_1,x_2)) = (f(b_1,b_2))(x_1,x_2)$.

In particular, if $(b_1,b_2) \in B \times B$ and $(x_1,x_2) \in X \times X$, we see by (3.2) and the above claim that $(\tilde{D}(b_1,b_2)p)(x_1,x_2) = (F * (b_1,b_2))(j(x_1,x_2)) - ((b_1,b_2) * F)(j(x_1,x_2)) = (f(b_1,b_2))(x_1,x_2) - ((b_1,b_2)f)(x_1,x_2)$, i.e.,

$$\tilde{D}((b_1,b_2))p = f(b_1,b_2) - (b_1,b_2)f \qquad (3.6)$$

$\forall\ (b_1,b_2) \in B \times B$, select a net $\{b_\alpha)$ in B that converges to B ultraweakly, assuming that $B \in A^{**}p$; we cannot immediately accept weak limits in (3.6) to finish the argument since the action of A^{**} on X^* is not necessarily a dual action. *However,* $\tilde{D}(b_\alpha)p = \tilde{D}(b_\alpha p) - b_\alpha \tilde{D}(p) = \tilde{D}(b_\alpha) - b_\alpha \tilde{D}(p)$, and since $\tilde{D}$ is ultraweak-to-weak continuous and A^{**} acts normally on X^*, it follows that $\{\tilde{D}(b_\alpha)p\}$ converges weak* to $\tilde{D}(b_1,b_2) - b_1,b_2\tilde{D}(p)$. This in turn is euqal to $\tilde{D}(b_1,b_2)p$, because $(b_1,b_2) = (b_1,b_2)p$ and $\tilde{D}$ is a derivation. On the other side, $\{fb_\alpha - b_\alpha f\}$ converges weak* to $fB - Bf$ by the normality of the action of A^{**} on X^{**}. Hence, $\tilde{D}(B) = fB - Bf\ \forall$ $B \in A^{**p}$. The proof of the theorem is now complete.

The hypotheses **(see**[15]) Theorem 3.2) are not satisfied for derivations from A into A^* unless A is finite-dimensional, since C^*-algebras are never weakly sequentially complete unless they are finite-dimensional [13, Proposition 2]. However, we may use a different technique to show that such derivations on nuclear C^*-algebras are inner in this exceptional case. We note that the action of A^{**} on $X^* = A^*$ defined by Eqs. (2.7) and (2.8) of Section 2 is the canonical action when $X = A$.

Theorem 3.4. (see [15]) Every derivation $D: A \to A^*$ is inner when A is a nuclear C^*-algebra.

Proof. We examine the extension $\tilde{D}: A^{**} \to A^*$, described by Equation (2.9) of Section 2. Select a strongly amenable, ultraweakly dense C^* subalgebra B of A^{**} using [7]. Restrict $\tilde{D}$ to B and view it as a derivation of B into the dual B-module A^{***}**.** Thus $\forall\ (b_{1,}b_2) \in B \times B$ there exists $F \in A^{***}$ such that $\tilde{D}(b_{1,}b_2) = F(b_{1,}b_2) - (b_{1,}b_2)F$. The map $R^* \to R_*$ that takes a functional to its normal part on a von Neumann algebra R^* is an R -module map (see [14, 16]). The map $P: A^{***} \to A^*$ is therefore a projection that is an A^{**}--module map, and $P(F) \in A$ implements $\tilde{D}$ *on* B; the normality of $\tilde{D}$ and the normality of the action of A^{**} on A^* complete the proof.

IV. Von Neumann algebra derivations into their predual

The question of whether all derivations from a von Neumann algebra into its predual is inner is raised by the proof of (**[15]** Theorem 3.4.)The particular assumptions on $\boldsymbol{A}$ in that theorem could be removed if this were always the case. In this section we show that all derivations from $\boldsymbol{R}$ to $\boldsymbol{R}_*$ are inner when $\boldsymbol{R}$ is a semifinite von Neumann algebra. First, we treat the finite case.

Lemma 4.1. (see [15]) If $D: R \to R_*$is a derivation and R is a finite von Neumann algebra, then there exists $f \in R_*$ with $\|f\| \leqslant \|D\|$ such that $D\big(a_{1,}a_2\big) = f\big(a_{1,}a_2\big) - (a_{1,}a_2)f\ \forall\ (\ a_{1,}a_2) \in R \times R$.

Proof. The Ryll-Nardzewski fixed-point theorem will be used to obtain the required f (see [8, Appendix 2]). Since D is weakly compact and automatically norm-continuous, the set $\{D(U) : U \in \mathcal{U}(R)\}$ is relatively weakly compact. (Here $\mathcal{U}(R)$ denotes the unitary group of R.) Because R is finite, the set $\{|D(U)|: U \in \mathcal{U}(R)\}$ is likewise relatively weakly compact by [1, Theorem 1] or [12, Proposition 2]. Next, it comes from [12, Theorem 3] that $\{D(U)U^*: U \in \mathcal{U}(R)\}$ is relatively weakly compact. Lastly, the convex hull that is weakly closed K of $\{D(U)U^*: U \in \mathcal{U}(R)\}$ is weakly compact by Krein's theorem [9, 0. 434]. For $U \in \mathcal{U}(R)$, let $\alpha_U: R_* \to R_*$ be given by $\alpha_U(f) = UfU^* + D(U)U^*$. It is immediate that each α_U is affine and weakly continuous. Further, we have for any $V \in \mathcal{U}(R)$ that

$$\begin{aligned}\alpha_U(D(V)V^*) &= VD(V)V^*U^* + D(U)U^* \\ &= (D(UV) - D(U)V)V^*U^* + D(U)U^* \\ &= D(UV)V^*U^*\end{aligned}$$

so each α_U maps K into K and $\alpha_U\alpha_V = \alpha_{UV}$. The resulting action of $\mathcal{U}(R)$ on K is distal because for any $f, g \in R_*$ we have $\|\alpha_U(f) - \alpha_U(g)\| = \|U(f-g)U^*\| = \|f-g\|$. The Ryll-Nardzewski fixed-point theorem now yields an $f \in K$ (so $\|f\| \leqslant \|D\|$) such that $\alpha_u(f) = f\ \ \forall\ \ U \in \mathcal{U}(R)$. That is, $D(U) = fU - Uf$ and hence $D(a_{1,}a_2) = f(\ a_{1,}a_2) - (\ a_{1,}a_2)f\ \forall\ (\ a_{1,}a_2) \in R \times R$.

Theorem 4.2. Let R be a semifinite von Neumann algebra. If $D: R \to R_*$ is a derivation, then $\forall\ (\ a_{1,}a_2) \in R \times R$ there exists $f \in R_*$ with $\|f\| \leqslant \|D\|$ such that $D(a_{1,}a_2) = f(\ a_{1,}a_2) - (\ a_{1,}a_2)f$.

Proof. First, we show that D is automatically $\sigma(R, R_*)$-to- $\sigma(R_*, R)$ continuous. This is equivalent to showing that the functional h defined by $h(b_{1,}b_2) = D(b_{1,}b_2)(a_{1,}a_2)$ is ultraweakly continuous $\forall\ (\ a_{1,}a_2) \in R \times R$. To do this, it suffices to show that the restriction of h to any maximal abelian *-subalgebra M of R is ultraweakly continuous [14, Corollary 1]. The strong amenability of M implies that, when D is viewed as a derivation from M into the dual M -module $R * R^*$, there exists $G \in R^*$ such that $D(m) = Gm - mG\ \forall\ m \in M$.

As in the proof of Theorem 3.5, the normal part of g is a functional in R_* that implements the restriction of D to M, from which it follows immediately that h is ultraweakly continuous on M, and hence on all of R.

Now let $\{e_\alpha\}$ be a net of finite projections in R increasing to 1. For each α, define $D_\alpha : e_\alpha R e_\alpha \to (e_\alpha R e_\alpha)_*$ by $D_\alpha(e_\alpha b e_\alpha)(e_\alpha a e_\alpha) = D(e_\alpha b e_\alpha)(e_\alpha a e_\alpha)$. Then D_α is a derivation and, since $e_\alpha R e_\alpha$ is finite, Lemma 3.1 shows that there exists an $f_\alpha \in (e_\alpha R e_\alpha)_*$ with $\|f_\alpha\| \leqslant \|D_\alpha\| \leqslant \|D\|$ and $D_\alpha(x) = f_\alpha(x_1, x_2) - (x_1, x_2) f_\alpha$ $\forall$ $(x_1, x_2) \in e_\alpha R e_\alpha$. Let $g_\alpha \in R_*$ be defined by $g_\alpha(a_1, a_2) = f_\alpha(e_\alpha a e_\alpha)$. Some subnet of $\{g_\alpha\}$ (without loss of generality this net itself) converges $\sigma(R^*, R)$ to a functional $g \in R^*$. We show that $D(x) = N(g)(x_1, x_2) - (x_1, x_2)N(g)$ $\forall (x_1, x_2) \in R \times R$, where $N(g) \in R_*$ is the normal part of g. Assume first that $(x_1, x_2) = e_\alpha x e_\alpha$ for some α. Then for any $(a_1, a_2) \in R \times R$, we have

$$\begin{aligned} D(x_1, x_2)(a_1, a_2) &= \lim_{\beta \geqslant \alpha} D(x_1, x_2)\left(e_\beta a e_\beta\right) \\ &= \lim_{\beta \geqslant \alpha} D\left(e_\beta (x_1, x_2) e_\beta\right)\left(e_\beta (a_1, a_2) e_\beta\right) \\ &= \lim_{\beta} \left(f_\beta\left(e_\beta (x_1, x_2) e_\beta\right) - \left(e_\beta x e_\beta\right) f_\beta\right)\left(e_\beta (a_1, a_2) e_\beta\right) \\ &= \lim_{\beta} f_\beta\left(e_\beta (x_1, x_2) e_\beta (a_1, a_2) e_\beta - e_\beta (a_1, a_2) e_\beta (x_1, x_2) e_\beta\right) \\ &= \lim_{\beta} f_\beta\left(e_\beta ((x_1, x_2)(a_1, a_2) - (a_1, a_2)(x_1, x_2)) e_\beta\right) \\ &= \lim_{\beta} g_\beta((x_1, x_2)(a_1, a_2) - (a_1, a_2)(x_1, x_2)) \\ &= g((x_1, x_2)(a_1, a_2) - (a_1, a_2)(x_1, x_2)) \\ &= (g(x_1, x_2) - (x_1, x_2) g)(a_1, a_2). \end{aligned}$$

If $(x_1, x_2) = e_\alpha x e_\alpha$ for some α, then, $D(x_1, x_2) = g(x_1, x_2) - (x_1, x_2) g$ and hence $D(x_1, x_2) = N(g)(x_1, x_2) - (x_1, x_2) N(g)$, since N is an R-module map. The proof is completed by taking limits using the $\sigma(R, R_*)$-to-$\sigma(R_*, R)$ continuity of D.

We note that, according to a result related to the aforementioned theorem by Hoover [9], all derivations from any C^*-subalgebra of $B(H)$ into $B(H)_*$ are inner. Although the proof [see [15] , Lemma 3.1] shows that $B(H)$ can be replaced by any finite von Neumann algebra, we do not know whether $B(H)$ can likewise be replaced, in this context, by an arbitrary semifinite von Neumann algebra.

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