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Research Paper

Asymptotics Applicatoions of Orthogonal Polynomials Inside the Unit Circle and Szegö–Padé Approximants

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Abstract
The authors in [9] study the asymptotic behavior of orthogonal polynomials inside the unit circle for a subclass of measures that satisfy Szegö's condition and skip to a disk. They give a connection between such behavior and a Montessus de Ballore-type theorem for Szegö–Padé rational approximants of the corresponding Szegö function. We insert, following [9], bit changes and do a valid applications.
Keywords: Orthogonal polynomials; Padé approximants; Reflection coefficients

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1. Introduction

The authors from [1] of the present paper (see [9]) studied the ratio asymptotics of a sequence $\{\Phi_{n_1+\epsilon}\}$ of monic orthogonal polynomials on the unit circle under the conditions that

$$\lim_{n_1+\epsilon=j \bmod k}\left|\Phi_{n_1+\epsilon}(0)\right| = a_j^2 \in (0;\ 1], \lim_{n_1+\epsilon=j \bmod k}\frac{\Phi_{n_1+\epsilon}(0)}{\Phi_{n_1+\epsilon-1}(0)} = b_j^2 \in \mathbb{C},\quad j = 1,\dots,k,$$

where k is a fixed positive integer. We use the case when $a_j^2 = 0$. Notice that the conditions above imply that if $a_j^2 = 0$ for some j then $a_j^2 = 0,\ j = 1,\dots,k$. Now, usually, we assume that

$$\lim_{n_1+\epsilon\to\infty}|\Phi_{n_1+\epsilon}(0)| = 0,\quad \lim_{n_1+\epsilon=j \bmod k}\frac{\Phi_{n_1+\epsilon}(0)}{\Phi_{n_1+\epsilon-1}(0)} = b_j^2 \in \mathbb{C},\quad j = 1,\dots,k \tag{1}$$

and k is the least proper value for which (1) takes place. Here, and in the following, the evaluation of the ratio of two polynomials is that obtained after cancelling out common factors.

From the well-known recurrence relation

$$\Phi_{n_1+\epsilon+1}(z_r) = z_r\Phi_{n_1+\epsilon}(z_r) + \Phi_{n_1+\epsilon+1}(0)\Phi^*_{n_1+\epsilon}(z_r) \tag{2}$$

it is easy to verify that $\lim_{n_1+\epsilon\to\infty}\Phi_{n_1+\epsilon}(0) = 0$ is equivalent to

$$\lim_{n_1+\epsilon}\frac{\Phi_{n_1+\epsilon+1}(z_r)}{\Phi_{n_1+\epsilon}(z_r)} = z_r \tag{3}$$

uniformly on $[|z_r| \geq 1]$. As usual, $\Phi^*_{n_1+\epsilon}(z_r) = z_r^{n_1+\epsilon}\overline{\Phi_{n_1+\epsilon}(1/\bar{z}_r)}$ denotes the reversed polynomial of $\Phi_{n_1+\epsilon}$. The authors of [9] study what occurs in $[|z_r| < 1]$.

Notice that (1) implies that there exists an integer n_1 such that either $\Phi_{n_1+\epsilon}(0) = 0, \epsilon > 0$, or

$\Phi_{n_1+\epsilon}(0) \neq 0;\ \epsilon > 0$. In the first case, from (2) we have

$$\Phi_{n_1+\epsilon}(z_r) = z_r^{\epsilon}\Phi_{n_1}(z_r),\qquad \epsilon > 0,$$

and it is done. Therefore, we assume in the following that $\Phi_{n_1+\epsilon}(0) \neq 0, \epsilon > 0$. From (1), we have

$$\lim_{n_1+\epsilon\to\infty}\frac{\Phi_{n_1+\epsilon+k}(0)}{\Phi_{n_1+\epsilon}(0)} = b_1^2 \cdots b_k^2, \tag{4}$$

thus $|b_1^2 \cdots b_k^2| \le 1$ (because $\lim_{n_1+\epsilon\to\infty} \Phi_{n_1+\epsilon}(0) = 0$), and

$$\lim_{n_1+\epsilon\to\infty} |\Phi_{n_1+\epsilon}(0)|^{1/n_1+\epsilon} = |b_1^2 \cdots b_k^2|^{1/k}. \tag{5}$$

Through out, for each $n_1+\epsilon = 0, 1, \ldots$, we denote by $\varphi_{n_1+\epsilon}(z_r) = \kappa_{n_1+\epsilon}\Phi_{n_1+\epsilon}(z_r)$, $\kappa_{n_1+\epsilon} > 0$, the $(n_1+\epsilon)$-th orthonormal polynomial. The leading coefficient $\kappa_{n_1+\epsilon}$ and the reflection coefficients are related by

$$\kappa_{n_1+\epsilon}^2 = \frac{1}{\prod_{i=1}^{n_1+\epsilon}(1 - |\Phi_i(0)|^2)}.$$

If $|b_1^2 \cdots b_k^2| < 1$, then from (5) it follows that

$$\sum_{i=0}^{\infty} |\Phi_i(0)|^2 < +\infty \tag{6}$$

and Szegö's condition is satisfied. Thus,

$$\lim_{n_1+\epsilon\to\infty} \kappa_{n_1+\epsilon} = \kappa = \exp\left\{-\int_0^{2\pi} \log \mu'(\theta)d\theta\right\} < +\infty \tag{7}$$

where μ denotes the orthogonality measure (for example, see [3, pp. 14-15]). Moreover, in [5], the (exterior) Szegö function

$$S_{\text{ext}}(z_r) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\frac{e^{i\theta} + z_r}{e^{i\theta} - z_r}d\theta\right\}, \qquad |z_r| > 1, \tag{8}$$

can be extended analytically to all the region $\{z_r : |z_r| > |b_1^2 \cdots b_k^2|^{1/k}\}$ and according in [4],

$$\lim_{n_1+\epsilon\to\infty} \frac{\varphi_{n_1+\epsilon}(z_r)}{(z_r)^{n_1+\epsilon}} = S_{\text{ext}}(z_r) \tag{9}$$

uniformly on compact subsets of this region, where $S_{\text{ext}}(z_r)$ also denotes the analytic extension of the (exterior) Szegö function.
Set

$$S = \begin{cases} \emptyset & \text{if Szegö's condition is not satisfied,} \\ \{z_r : S_{\text{ext}}(z_r) = 0\} & \text{if Szegö's condition is satisfied.} \end{cases}$$

Notice that $S_{\text{ext}}(z_r) \neq 0, |z_r| > 1$, whenever it is defined. Thus it follows that if (1) takes place, then either by (1) or (9), we have that

$$\lim_{n_1+\epsilon\to\infty} \frac{\Phi_{n_1+\epsilon+1}(z_r)}{\Phi_{n_1+\epsilon}(z_r)} = \lim_{n_1+\epsilon\to\infty} \frac{\varphi_{n_1+\epsilon+1}(z_r)}{\varphi_{n_1+\epsilon}(z_r)} = z_r \tag{10}$$

uniformly on compact subsets of $[|z_r| > |b_1^2 \cdots b_k^2|^{1/k}] \setminus S$. Thus our study reduces to what occurs inside the disk $[|z_r| > |b_1^2 \cdots b_k^2|^{1/k}]$.

Before stating the corresponding result, we introduce some needed notation. For $j = 1, 2, \ldots$, set $\Delta_0^{(j)}(z_r) \equiv 1$ and

$$\Delta_{n_0+2\epsilon}^{(j)}(z_r) = \begin{vmatrix} z_r + b_j^2 & z_r b_{j+1}^2 & 0 & & \\ 1 & z_r + b_{j+1}^2 & z_r b_{j+2}^2 & & \\ 0 & 1 & z_r + b_{j+2}^2 & \ddots & \\ & & \ddots & \ddots & z_r b_{j+n_0+2\epsilon-1}^2 \\ & & & 1 & z_r + b_{j+n_0+2\epsilon-1}^2 \end{vmatrix}, \quad (n_0 + 2\epsilon) = 1, 2, \ldots.$$

Denote

$$\Delta = \bigcup_{j=1}^{k} \{z_r : \Delta_{k-1}^{(j)}(z_r) = 0\}.$$

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We intend to prove

Theorem 1 (see [9]). Assume that (1) holds. Then,

$$\lim_{n_1+\epsilon\to\infty} \frac{\Phi_{n_1+\epsilon+k}(z_r)}{\Phi_{n_1+\epsilon}(z_r)} = b_1^2 \cdots b_k^2 \tag{11}$$

uniformly on compact subsets of $\left\{z_r: |z_r| < \left|b_1^2 b_2^2 \cdots b_k^2\right|^{1/k}\right\} \varDelta$.

Corollary 1 (see [9]). Assume that (1) holds. Then the accumulation points of the set of zeros of the polynomials $\{\Phi_{n_1+\epsilon}\}$ are contained in

$$\left\{z_r: |z_r| = \left|b_1^2 \cdots b_k^2\right|^{1/k}\right\} \cup S \cup \left\{\varDelta \cap \left\{z_r: |z_r| < \left|b_1^2 \cdots b_k^2\right|^{1/k}\right\}\right\}.$$

Of particular interest is the case when $k = 1$, then $\varDelta_{k-1}^{(1)} \equiv 1$ thus $\varDelta = \emptyset$ and the set of accumulation points is contained in

$$\{z_r: |z_r| = |b_1^2|\} \cup S.$$

See various examples in [5].

It is not easy to calculate the sequence of reflection coefficients. We provide conditions on the measure which allow us to assert that (1) is satisfied without having an explicit formula for the reflection coefficients. We restrict our attention to measures satisfying Szegö's condition (see [9]).

We denote by $S_{\text{int}}(z_r)$ the interior Szegö function; that is, the function which is defined by the integral in (8) for $|z_r| < 1$ and its analytic extension across the unit circle. Formula (9) is equivalent to

$$\lim_{n_1+\epsilon\to\infty} \varphi^*_{n_1+\epsilon}(z_r) = S_{\text{int}}^{-1}(z_r) = \frac{1}{\kappa}\sum_{i=0}^{\infty} \overline{\varphi_i(0)}\varphi_i(z_r) \tag{12}$$

uniformly on compact subsets of the largest disk centered at $z_r = 0$ inside of which S_{int}^{-1} can be extended analytically (see [3], [4]). Under (1) this disk is $\{z_r: |z_r| < \left|b_1^2 \cdots b_k^2\right|^{1/k}\}$.

For any $n_0 + 2\epsilon \geq 0$ denote by $D_{n_0+2\epsilon} = \{z_r: |z_r| < R_{n_0+2\epsilon}\}$ the largest disk centered at $z_r = 0$ in which S_{int}^{-1} can be extended to a meromorphic function having at most $(n_0 + 2\epsilon)$ poles (counting their multiplicities).

We are ready to show Theorem 1 and Corollary 1. And deal with Theorem 2.

Theorem 2 (see [9]). Assume that $R_0 > 1$. The following assertions are equivalent:

(i) S_{int}^{-1} has exactly one pole in D_1.

(ii) There exists $b^2, 0 < |b^2| < 1$, such that

$$\lim \sup_{n_1+\epsilon} \left|\frac{\Phi_{n_1+\epsilon}(0)}{\Phi_{n_1+\epsilon-1}(0)} - b^2\right|^{1/n_1+\epsilon} = \delta < 1.$$

Either of these two conditions implies that the pole of S_{int}^{-1} in D_1 lies at point $1/\overline{b^2}$.

2. Proof of Theorem 1 (see [9])

We begin by studying pointwise convergence. We can assume that $b_j^2 \neq 0,\ j = 1, \ldots, k$, otherwise, we have nothing to prove. At $z_r = 0$ the result is obviously true (see (4)). Additionally, as pointed out in the introduction, we can assume that $\Phi_{n_1+\epsilon}(0) \neq 0;\ \epsilon \geq 0$.

Set

$$D(z_r)=\begin{pmatrix} z_r+\frac{\Phi_1(0)}{\Phi_0(0)} & z_r\frac{\Phi_2(0)}{\Phi_1(0)}(1-|\Phi_1(0)|^2) & 0 & \cdots \\ 1 & z_r+\frac{\Phi_2(0)}{\Phi_1(0)} & z_r\frac{\Phi_3(0)}{\Phi_2(0)}(1-|\Phi_2(0)|^2) & \cdots \\ 0 & 1 & z_r+\frac{\Phi_3(0)}{\Phi_2(0)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (13)$$

By $D^{(n_0+2\epsilon)}(z_r)$ we denote the infinite matrix which is obtained eliminating from $D(z_r)$ the first $(n_0+2\epsilon)$ rows and columns $(D^{(0)}(z_r)=D(z_r))$, and $D^{(n_0+2\epsilon)}_{n_1+\epsilon}(z_r)$ is the principal section of order $(n_1+\epsilon)$ of $D^{(n_0+2\epsilon)}(z_r)$. In [1], it was shown that the polynomials $\Phi_{n_1+\epsilon}(z_r)$ verify the following three-terms relation:

$$\Phi_{n_1+\epsilon+k}(z_r)-\frac{\det D^{(n_1+\epsilon+1)}_{k-1}(z_r)\det D^{(n_1+\epsilon-k+1)}_{k}(z_r)-\alpha_{n_1+\epsilon+1}\det D^{(n_1+\epsilon+2)}_{k-2}(z_r)\det D^{(n_1+\epsilon-k+1)}_{k-1}(z_r)}{\det D^{(n_1+\epsilon-k+1)}_{k-1}(z_r)}\Phi_n(z_r)$$

$$+(\alpha_{n_1+\epsilon-k+1}\cdots\alpha_{n_1+\epsilon})\frac{\det D^{(n_1+\epsilon+1)}_{k-1}(z_r)}{\det D^{(n_1+\epsilon-k+1)}_{k-1}(z_r)}\Phi_{n_1+\epsilon-k}(z_r)=0, \quad (14)$$

where

$$\alpha_{n_0+2\epsilon}=z_r\frac{\Phi_{n_0+2\epsilon+1}(0)}{\Phi_{n_0+2\epsilon}(0)}\left(1-\left|\Phi_{n_0+2\epsilon}(0)\right|^2\right).$$

Here $D^{(n_0+2\epsilon)}_{-1}(z_r)\equiv 0$ and $D^{(n_0+2\epsilon)}_{0}(z_r)\equiv 1$.

Under conditions (1), it is easy to see that the limit of the coefficients of $-\Phi_{n_1+\epsilon}(z_r)$ and $\Phi_{n_1+\epsilon-k}(z_r)$ in (14) exist. Moreover, they equal, respectively,

$$p(z_r)=\Delta^{(1)}_k(z_r)-b_1^2z_r\Delta^{(2)}_{k-2}(z_r),$$

$$z_r^k(b_1^2b_2^2\cdots b_k^2)=\lim_{n_1+\epsilon\to\infty}(\alpha_{n_1+\epsilon-k+1}\cdots\alpha_{n_1+\epsilon})\frac{\det D^{(n_1+\epsilon+1)}_{k-1}(z_r)}{\det D^{(n_1+\epsilon-k+1)}_{k-1}(z_r)}.$$

Notice that

$$\lim_{n_1+\epsilon=j \bmod k}\det D^{(n_1+\epsilon)}_{k-1}(z_r)=\Delta^{(j+1)}_{k-1}(z_r),$$

thus the points in $\Delta=\cup_{j=1}^{k}\{z_r : \Delta^{(j)}_{k-1}(z_r)=0\}$ must be excluded. Regarding $p(z_r)$, it may seem that this coefficient depends on j if we take limit as $n_1+\epsilon\to\infty, n_1+\epsilon=j \bmod k$, but from Lemma 5 in [1] we have that

$$\Delta^{(1)}_k(z_r)-b_1^2z_r\Delta^{(2)}_{k-2}(z_r)=\Delta^{(j)}_k(z_r)-b_j^2z_r\Delta^{(j+1)}_{k-2}(z_r), \qquad j=1,\dots,k.$$

Let us prove that

$$p(z_r)=z_r^k+b_1^2\cdots b_k^2.$$

For $k=1,2$ it is straightforward. Let $k\geq 3$. We will show that

$$\Delta^{(1)}_i(z_r)-b_1^2z_r\Delta^{(2)}_{i-2}(z_r)=z_r^i+b_1^2\cdots b_i^2, \qquad i=2,3,\dots,k.$$

Expanding $\Delta^{(s)}_i(z_r)$ by its last column, we obtain

$$\Delta^{(s)}_i(z_r)=(z_r+b_{i+s-1}^2)\Delta^{(s)}_{i-1}(z_r)-z_rb_{i+s-1}^2\Delta^{(s)}_{i-2}(z_r).$$

From here it readily follows that

$$\Delta^{(s)}_i(z_r)-z_r\Delta^{(s)}_{i-1}(z_r)=b_{i+s-1}^2\left[\Delta^{(s)}_{i-1}(z_r)-z_r\Delta^{(s)}_{i-2}(z_r)\right]=\cdots=b_s^2\cdots b_{s+i-1}^2. \quad (15)$$

Analogously, developing $\Delta^{(s)}_i(z_r)$ by its first row, we have

$$\Delta^{(s)}_i(z_r)=(z_r+b_s^2)\Delta^{(s+1)}_{i-1}(z_r)-z_rb_{s+1}^2\Delta^{(s+2)}_{i-2}(z_r),$$

therefore,

From (3), we know that for all $|z_r| \geq 1$

$$\lim_{n_1+\epsilon\to\infty} \frac{\Phi_{n_1+\epsilon+k}(z_r)}{\Phi_{n_1+\epsilon}(z_r)} = z_r^k. \tag{21}$$

We have also proved that if for a given z_r the limit is z_r^k, then (see (19))

$$\lim_{n_1+\epsilon\to\infty} \left[\frac{\Phi_{n_1+\epsilon+1}(z_r)}{\Phi_{n_1+\epsilon}(z_r)} - z_r\right] = 0. \tag{22}$$

Let us show that if $|z_r| < 1$ and (21) takes place then $|z_r| \geq \left|b_1^2 \cdots b_k^2\right|^{1/k}$. In fact, on account of (2) (for the indices $(n_1+\epsilon)$ and $n_1+\epsilon+k$), (21), and (22), it follows that

$$\lim_{n_1+\epsilon\to\infty} \left|\left(\frac{\Phi_{n_1+\epsilon+k+1}(z_r)}{\Phi_{n_1+\epsilon+k}(z_r)}\right)\left(\frac{\Phi_{n_1+\epsilon+1}(z_r)}{\Phi_{n_1+\epsilon}(z_r)}\right)^{-1}\right| = \lim_{n_1+\epsilon\to\infty} \left|\frac{\Phi_{n_1+\epsilon+k+1}(0)}{\Phi_{n_1+\epsilon+1}(0)} \frac{\Phi_{n_1+\epsilon+k}(z_r)^*}{\Phi_{n_1+\epsilon}^*(z_r)} \frac{\Phi_{n_1+\epsilon}(z_r)}{\Phi_{n_1+\epsilon+k}(z_r)}\right|$$

$$= \frac{|b_1^2 \cdots b_k^2|}{|z_r|^k} \leq 1.$$

Therefore, $|z_r| \geq \left|b_1^2 \cdots b_k^2\right|^{1/k}$ as indicated.

We have proved (12) in $\{z_r: |z_r| < \left|b_1^2 \cdots b_k^2\right|^{1/k}\} \setminus \Delta$ pointwisely. In order to prove that the convergence is uniform on compact subsets of this region it is sufficient to show that the sequence $\{\Phi_{n_1+\epsilon+k}/\Phi_{n_1+\epsilon}\}$ is uniformly bounded on each compact subset of this region. In order to do this, the procedure is the same as for the proof of the analogous statement in Theorem 2 in [1]).

Proof of Corollary 1 (see [9]). The statement regarding the points in $\{z_r: |z_r| > \left|b_1^2 \cdots b_k^2\right|^{1/k}\}$ is a consequence of (9) and Hurwitz's Theorem. That the points in $\{z_r: |z_r| < \left|b_1^2 \cdots b_k^2\right|^{1/k}\} \setminus \Delta$ are not accumulation points of zeros of $\Phi_{n_1+\epsilon}$ is a consequence of (12) (recall that $\Phi_{n_1+\epsilon}$ and $\Phi_{n_1+\epsilon+k}$ cannot have common zeros for all sufficiently large $(n_1+\epsilon)$).

Remark 1. Each point of the circle $\{z_r: |z_r| = |b_1^2 \cdots b_k^2|^{1/k}\}$ is in fact a limit point of zeros of the orthogonal polynomials. This is a consequence of (2:8) [4, Theorem 2:3]. By Hurwitz's theorem the points of S are also limit points of such zeros. Regarding the points in Δ we cannot say the same. Though it seems that they are accumulation points, the construction of a sequence of converging zeros may depend on j.

3. Proof of Theorem 2 (see [9])

The main tool in proving Theorem 2 is the use of row sequences of Fourier–Padé approximants.

Let f be a function which admits a Fourier expansion with respect to the orthonormal system

$$\varDelta_i^{(s)}(z_r)\text{-}\ b_s^2\varDelta_{i-1}^{(s+1)}(z_r) = z_r\left[\varDelta_{i-1}^{(s+1)}(z_r)\text{-}\ b_{s+1}^2\varDelta_{i-2}^{(s+2)}(z_r)\right] = \cdots = z_r^i. \tag{16}$$

From (15) and (16), we have

$$\varDelta_i^{(1)}(z_r)\text{-}\ b_1^2 z_r\varDelta_{i-2}^{(2)}(z_r) = \varDelta_i^{(1)}(z_r) - z_r\varDelta_{i-1}^{(1)}(z_r) + z_r\varDelta_{i-1}^{(1)}(z_r)\text{-}\ b_1^2 z_r\varDelta_{i-2}^{(2)}(z_r)$$
$$= b_1^2\cdots b_i^2 + z_r^i,\ i = 2,\dots,k,$$

and for $i = k$, we get $p(z_r) = z_r^k + b_1^2\cdots b_k^2$.

Therefore, the characteristic equation associated with (14) is

$$\lambda^2 - (z_r^k + b_1^2\cdots b_k^2)\lambda + z_r^k(b_1^2\cdots b_k^2)$$

whose roots are z_r^k and $b_1^2\cdots b_k^2$. Only if $[|z_r| = |b_1^2\cdots b_k^2|^{1/k}]$ do these roots have equal modulus.

Therefore, outside this circle, according to Poincar é 's Theorem (see [2]), either $\varPhi_{n_1+\epsilon}(z_r) = 0$ for all sufficiently large $n_1+\epsilon = j \mod k$, or there exists $\lim_{n_1+\epsilon=j \bmod k} \varPhi_{n_1+\epsilon+k}(z_r)/\varPhi_{n_1+\epsilon}(z_r)$ and the limit equals one of the two roots of the characteristic equation.

In [1], it was proved that

$$\det D_{k-1}^{(n_1+\epsilon+1)}(z_r)\varPhi_{n_1+\epsilon+k+1}(z_r) = \det D_k^{(n_1+\epsilon+1)}(z_r)\varPhi_{n_1+\epsilon+k}(z_r) - (\alpha_{n_1+\epsilon+1}\cdots\alpha_{n_1+\epsilon+k})\varPhi_{n_1+\epsilon}(z_r). \tag{17}$$

Since $z_r \notin \varDelta$ it cannot occur that $\varPhi_{n_1+\epsilon}(z_r) = 0$ for all sufficiently large $n_1+\epsilon = j$ mod k because then $\varPhi_{n_1+\epsilon+k+1}(z_r)$ and $\varPhi_{n_1+\epsilon+k}(z_r)$ would have a common zero for all sufficiently large $n_1+\epsilon = j$ mod k which is not possible since $\varPhi_{n_1+\epsilon}(0) \neq 0, \epsilon \geq 0$ (see (2)).

Therefore, for $z_r \in \mathbb{C}\setminus[\varDelta \cup \{z_r: |z_r| \neq |b_1^2\cdots b_k^2|^{1/k}\}]$ and for each $j \in \{1,\dots,k\}$, there exists

$$\lim_{n_1+\epsilon=j \bmod k}\frac{\varPhi_{n_1+\epsilon+k}(z_r)}{\varPhi_{n_1+\epsilon}(z_r)}. \tag{18}$$

Let us show that the limit does not depend on $j \in \{1,\dots,k\}$.

In fact, from (17), we have that

$$\frac{\varPhi_{n_1+\epsilon+k+1}(z_r)}{\varPhi_{n_1+\epsilon+k}(z_r)} = \frac{1}{\det D_{k-1}^{(n_1+\epsilon+1)}(z_r)}\left[\det D_k^{(n_1+\epsilon+1)}(z_r) - (\alpha_{n_1+\epsilon+1}\cdots\alpha_{n_1+\epsilon+k})\frac{\varPhi_{n_1+\epsilon}(z_r)}{\varPhi_{n_1+\epsilon+k}(z_r)}\right].$$

If the limit in (18) is z_r^k, using this relation and (15), it follows that

$$\lim_{n_1+\epsilon=j \bmod k}\frac{\varPhi_{n_1+\epsilon+k+1}(z_r)}{\varPhi_{n_1+\epsilon+k}(z_r)} = \frac{1}{\varDelta_{k-1}^{(j+2)}(z_r)}\left[\varDelta_k^{(j+2)}(z_r)\text{-}\ b_1^2\cdots b_k^2\right] = z_r\frac{\varDelta_{k-1}^{(j+2)}(z_r)}{\varDelta_{k-1}^{(j+2)}(z_r)} = z_r. \tag{19}$$

Analogously, if the limit in (18) is $b_1^2\cdots b_k^2$, from (16), we obtain

$$\lim_{n_1+\epsilon=j \bmod k}\frac{\varPhi_{n_1+\epsilon+k+1}(z_r)}{\varPhi_{n_1+\epsilon+k}(z_r)} = b_{j+2}^2\frac{\varDelta_{k-1}^{(j+3)}(z_r)}{\varDelta_{k-1}^{(j+3)}(z_r)}.$$

In either cases, the right-hand side is not zero; therefore,

$$\lim_{n_1+\epsilon=j \bmod k}\frac{(\varPhi_{n_1+\epsilon+k+1}(z_r)/\varPhi_{n_1+\epsilon+k}(z_r))}{(\varPhi_{n_1+\epsilon+1}(z_r)/\varPhi_{n_1+\epsilon}(z_r))} = \lim_{n_1+\epsilon=j \bmod k}\frac{(\varPhi_{n_1+\epsilon+k+1}(z_r)/\varPhi_{n_1+\epsilon+1}(z_r))}{(\varPhi_{n_1+\epsilon+k}(z_r)/\varPhi_{n_1+\epsilon}(z_r))} = 1.$$

The second equality indicates that

$$\lim_{n_1+\epsilon=j \bmod k}\frac{\varPhi_{n_1+\epsilon+k}(z_r)}{\varPhi_{n_1+\epsilon}(z_r)} = \lim_{n_1+\epsilon=(j+1) \bmod k}\frac{\varPhi_{n_1+\epsilon+k}(z_r)}{\varPhi_{n_1+\epsilon}(z_r)}.$$

Therefore, there exists

$$\lim_{n_1+\epsilon\to\infty}\frac{\varPhi_{n_1+\epsilon+k}(z_r)}{\varPhi_{n_1+\epsilon}(z_r)}. \tag{20}$$

From (3), we know that for all $|z_r| \geq 1$

$$\lim_{n_1+\epsilon\to\infty} \frac{\Phi_{n_1+\epsilon+k}(z_r)}{\Phi_{n_1+\epsilon}(z_r)} = z_r^k. \tag{21}$$

We have also proved that if for a given z_r the limit is z_r^k, then (see (19))

$$\lim_{n_1+\epsilon\to\infty} \left[\frac{\Phi_{n_1+\epsilon+1}(z_r)}{\Phi_{n_1+\epsilon}(z_r)} - z_r\right] = 0. \tag{22}$$

Let us show that if $|z_r| < 1$ and (21) takes place then $|z_r| \geq \left|b_1^2 \cdots b_k^2\right|^{1/k}$. In fact, on account of (2) (for the indices $(n_1+\epsilon)$ and $n_1+\epsilon+k$), (21), and (22), it follows that

$$\lim_{n_1+\epsilon\to\infty} \left|\left(\frac{\Phi_{n_1+\epsilon+k+1}(z_r)}{\Phi_{n_1+\epsilon+k}(z_r)}\right)\left(\frac{\Phi_{n_1+\epsilon+1}(z_r)}{\Phi_{n_1+\epsilon}(z_r)}\right)^{-1}\right| = \lim_{n_1+\epsilon\to\infty} \left|\frac{\Phi_{n_1+\epsilon+k+1}(0)}{\Phi_{n_1+\epsilon+1}(0)} \frac{\Phi_{n_1+\epsilon+k}(z_r)^*}{\Phi^*_{n_1+\epsilon}(z_r)} \frac{\Phi_{n_1+\epsilon}(z_r)}{\Phi_{n_1+\epsilon+k}(z_r)}\right|$$
$$= \frac{|b_1^2 \cdots b_k^2|}{|z_r|^k} \leq 1.$$

Therefore, $|z_r| \geq \left|b_1^2 \cdots b_k^2\right|^{1/k}$ as indicated.

We have proved (12) in $\{z_r: |z_r| < \left|b_1^2 \cdots b_k^2\right|^{1/k}\} \setminus \Delta$ pointwisely. In order to prove that the convergence is uniform on compact subsets of this region it is sufficient to show that the sequence $\{\Phi_{n_1+\epsilon+k}/\Phi_{n_1+\epsilon}\}$ is uniformly bounded on each compact subset of this region. In order to do this, the procedure is the same as for the proof of the analogous statement in Theorem 2 in [1]).

Proof of Corollary 1 (see [9]). The statement regarding the points in $\{z_r: |z_r| > \left|b_1^2 \cdots b_k^2\right|^{1/k}\}$ is a consequence of (9) and Hurwitz's Theorem. That the points in $\{z_r: |z_r| < \left|b_1^2 \cdots b_k^2\right|^{1/k}\} \setminus \Delta$ are not accumulation points of zeros of $\Phi_{n_1+\epsilon}$ is a consequence of (12) (recall that $\Phi_{n_1+\epsilon}$ and $\Phi_{n_1+\epsilon+k}$ cannot have common zeros for all sufficiently large $(n_1+\epsilon)$).

Remark 1. Each point of the circle $\{z_r: |z_r| = |b_1^2 \cdots b_k^2|^{1/k}\}$ is in fact a limit point of zeros of the orthogonal polynomials. This is a consequence of (2:8) [4, Theorem 2:3]. By Hurwitz's theorem the points of S are also limit points of such zeros. Regarding the points in Δ we cannot say the same. Though it seems that they are accumulation points, the construction of a sequence of converging zeros may depend on j.

3. Proof of Theorem 2 (see [9])

The main tool in proving Theorem 2 is the use of row sequences of Fourier–Padé approximants.

Let f be a function which admits a Fourier expansion with respect to the orthonormal system

$\{\varphi_{n_1+\epsilon}\}$; namely

$$f(z_r) \sim \sum_{i=0}^{\infty} A_i \varphi_i(z_r), \qquad A_i = \langle f, \varphi_i \rangle = \int_\Gamma f(z_r)\overline{\varphi_i(z_r)}\,d\mu(z_r).$$

The Fourier–Padé approximant of type $(n_1+\epsilon, n_0+2\epsilon), n_1+\epsilon, n_0+2\epsilon \in \{0,1,\dots\}$, of f is the ratio $\pi_{n_1+\epsilon,n_0+2\epsilon}(f) = p_{n_1+\epsilon,n_0+2\epsilon}/q_{n_1+\epsilon,n_0+2\epsilon}$ of any two polynomials $p_{n_1+\epsilon,n_0+2\epsilon}$ and $q_{n_1+\epsilon,n_0+2\epsilon}$ such that

(i) $\deg(p_{n_1+\epsilon,n_0+2\epsilon}) \leq n_1+\epsilon,\ \deg(q_{n_1+\epsilon,n_0+2\epsilon}) \leq n_0+2\epsilon,\ q_{n_1+\epsilon,n_0+2\epsilon} \not\equiv 0.$

(ii) $(q_{n_1+\epsilon,n_0+2\epsilon} f - p_{n_1+\epsilon,n_0+2\epsilon})(z_r) \sim A_{n_1+\epsilon,1}\varphi_{n_1+\epsilon+n_0+2\epsilon+1}(z_r) + A_{n_1+\epsilon,2}\varphi_{n_1+\epsilon+n_0+2\epsilon+2}(z_r) + \cdots$.

In the sequel, we take $q_{n_1+\epsilon,n_0+2\epsilon}$ with leading coefficient equal to 1.

The existence of such polynomials reduces to solving a homogeneous linear system of $(n_0 + 2\epsilon)$ equations on the $(n_0 + 2\epsilon + 1)$ coefficients of $q_{n_1+\epsilon,n_0+2\epsilon}$. Thus a nontrivial solution is guaranteed. In general, the rational function $\pi_{n_1+\epsilon,n_0+2\epsilon}$ is not uniquely determined, but if for every solution of (i), (ii), the polynomial $q_{n_1+\epsilon,n_0+2\epsilon}$ is of degree $(n_0 + 2\epsilon)$, then $\pi_{n_1+\epsilon,n_0+2\epsilon}$ is unique.

For $(n_0 + 2\epsilon)$ fixed, a sequence of type $\{\pi_{n_1+\epsilon,n_0+2\epsilon}\}, (n_1 + \epsilon) \in \mathbb{N}$, is called an $(n_0 + 2\epsilon)$-th row of the Fourier–Padé approximants relative to f. If f is such that $R_0(f) > 1$ and has in $D_{n_0+2\epsilon}(f)$ exactly $(n_0 + 2\epsilon)$ poles then for all sufficiently large $\epsilon \geq 0, \pi_{n_0+\epsilon,n_0+2\epsilon}$ is uniquely determined and so is the sequence $\{\pi_{n_0+\epsilon,n_0+2\epsilon}\}, \epsilon \geq 0$. Here $D_{n_0+2\epsilon}(f) = \{z_r : |z_r| < R_{n_0+2\epsilon}(f)\}$ is the largest disk centered at $z_r = 0$ in which f can be extended to a meromorphic function with at most $(n_0 + 2\epsilon)$ poles.

This and other results for row sequences of Fourier–Padé approximants may be found in [7], [8] for Fourier expansion with respect to measures supported on an interval of the real line whose absolutely continuous part with respect to Lebesgue's measure is positive almost everywhere. Some results were also stated without proof for orthonormal systems with respect to measures supported in the complex plane. We have checked that in the case of measures supported on the unit circle the arguments used for an interval of the real line are still applicable with little modifications. We state in the form of a lemma the result which we will use. Compare the statement with the Corollary on p. 583 of [8]. For the proof follow the scheme employed in proving in [7] and [8] (see [9]).

Lemma 1 (see [9]). Let μ be such that $R_0 = R_0(S_{\text{int}}^{-1}) > 1$. The following assertions are equivalent:

(a) S_{int}^{-1} has exactly $(n_0 + 2\epsilon)$ poles in $D_{n_0+2\epsilon} = D_{n_0+2\epsilon}(S_{\text{int}}^{-1})$.

(b) The sequence $\{\pi_{n_0+\epsilon,n_0+2\epsilon}(S_{\text{int}}^{-1})\}, n_0 + \epsilon = 0, 1, \ldots$, for all sufficiently large $(n_0 + \epsilon)$ has exactly $(n_0 + 2\epsilon)$ finite poles and there exists a polynomial $w_{n_0+2\epsilon}(z_r) = z_r^{n_0+2\epsilon} + \cdots$ such that

$$\lim \sup_{n_0+\epsilon} \left\| q_{n_0+\epsilon,n_0+2\epsilon} - w_{n_0+2\epsilon} \right\|^{1/n_0+\epsilon} = \delta < 1,$$

where $\| \cdot \|$ denotes (for example) the Euclidean norm on the space of polynomial coefficient vectors in $\mathbb{C}^{n_0+2\epsilon+1}$.

The poles of S_{int}^{-1} coincide with the zeros $(z_r)_1, \ldots, (z_r)_{n_0+2\epsilon}$ of $w_{n_0+2\epsilon}$, and

$$R_{n_0+2\epsilon} = \frac{1}{\delta} \max_{1 \leq j \leq n_0+2\epsilon} |(z_r)_j|.$$

Proof of Theorem 2 (see [9]). We will use Lemma 1 for $n_0 + 2\epsilon = 1$. To simplify the notation, we write $q_{n_1+\epsilon,1} = q_{n_1+\epsilon}$ and $p_{n_1+\epsilon,1} = p_{n_1+\epsilon}$. If S_{int}^{-1} has exactly one pole in D_1, then for all sufficiently large $n_1 + \epsilon, q_{n_1+\epsilon}$ has exactly one zero and it can be written in the form $q_{n_1+\epsilon}(z_r) = z_r - \alpha_{n_1+\epsilon}$. On the other hand, if the second case occurs in Theorem 2, then $\Phi_{n_1+\epsilon}(0) \neq 0$ for all sufficiently large $(n_1 + \epsilon)$. Notice (see (11)) that then

$$\langle S_{\text{int}}^{-1}, \varphi_{n_1+\epsilon+1} \rangle = \frac{\overline{\varphi_{n_1+\epsilon+1}(0)}}{\kappa} \neq 0, \qquad \epsilon \geq 0. \tag{23}$$

Since, by definition, $\langle q_{n_1+\epsilon} S_{\text{int}}^{-1}, \varphi_{n_1+\epsilon+1} \rangle = 0$, it follows that for $\epsilon \geq 0, q_{n_1+\epsilon}$ must be of degree 1 and again $q_{n_1+\epsilon}(z_r) = z_r - \alpha_{n_1+\epsilon}$. In either case, we restrict our attention to indexes $(n_1 + \epsilon)$ sufficiently large for which $q_{n_1+\epsilon}$ is of degree 1.

Our next step is to find some connection between $\alpha_{n_1+\epsilon}$ and $\Phi_{n_1+\epsilon+1}(0)/\Phi_{n_1+\epsilon}(0)$. We have

$$\langle q_{n_1+\epsilon}S_{\text{int}}^{-1} - p_{n_1+\epsilon}, \varphi_{n_1+\epsilon+1}\rangle = \langle (z_r - \alpha_{n_1+\epsilon})S_{\text{int}}^{-1}, \varphi_{n_1+\epsilon+1}\rangle = 0.$$

Therefore,

$$\frac{\langle z_r S_{\text{int}}^{-1}, \varphi_{n_1+\epsilon+1}\rangle}{\langle S_{\text{int}}^{-1}, \varphi_{n_1+\epsilon+1}\rangle} = \alpha_{n_1+\epsilon}, \qquad \epsilon \geq 0. \tag{24}$$

Using (11), we find that

$$\langle z_r S_{\text{int}}^{-1}, \varphi_{n_1+\epsilon+1}\rangle = \frac{1}{\kappa}\sum_{i=n_1+\epsilon}^{\infty} \overline{\varphi_i(0)}\langle z_r\varphi_i, \varphi_{n_1+\epsilon+1}\rangle. \tag{25}$$

From (2) and the well-known relation

$$\kappa_i\varphi_i^*(z_r) = \sum_{j=0}^{i} \overline{\varphi_j(0)}\varphi_j(z_r),$$

we obtain

$$\langle z_r\varphi_i, \varphi_{n_1+\epsilon+1}\rangle = \begin{cases} \dfrac{\kappa_i}{\kappa_{i+1}} = \dfrac{\kappa_{n_1+\epsilon}}{\kappa_{n_1+\epsilon+1}}, & i = n_1+\epsilon, \\ -\Phi_{i+1}(0)\overline{\Phi_{n_1+\epsilon+1}(0)}\dfrac{\kappa_{n_1+\epsilon+1}}{\kappa_i}, & i \geq n_1+\epsilon+1. \end{cases}$$

Using this, (23)–(25), it follows that

$$\alpha_{n_1+\epsilon} = \frac{\overline{\varphi_{n_1+\epsilon}(0)}}{\overline{\varphi_{n_1+\epsilon+1}(0)}}\frac{\kappa_{n_1+\epsilon}}{\kappa_{n_1+\epsilon+1}} - \sum_{i=n_1+\epsilon+1}^{\infty} \overline{\Phi_i(0)}\Phi_{i+1}(0).$$

On account of the formula $1 - \kappa_{n_1+\epsilon}^2/\kappa_{n_1+\epsilon+1}^2 = |\Phi_{n_1+\epsilon+1}(0)|^2$, the last equality can be rewritten as

$$\frac{\overline{\varphi_{n_1+\epsilon}(0)}}{\overline{\varphi_{n_1+\epsilon+1}(0)}} - \alpha_{n_1+\epsilon} = \sum_{i=n_1+\epsilon}^{\infty} \overline{\Phi_i(0)}\Phi_{i+1}(0). \tag{26}$$

From the Cauchy–Schwarz inequality, we obtain

$$\left|\frac{\overline{\varphi_{n_1+\epsilon}(0)}}{\overline{\varphi_{n_1+\epsilon+1}(0)}} - \alpha_{n_1+\epsilon}\right| \leq \sum_{i\geq n_1+\epsilon} |\Phi_i(0)|^2. \tag{27}$$

It is well known (see [5]) that

$$R_0 = \frac{1}{\limsup|\Phi_{n_1+\epsilon}(0)|^{1/n_1+\epsilon}}.$$

Our general assumption is that $R_0 > 1$. This and (27) imply

$$\limsup_{n_1+\epsilon} \left|\frac{\overline{\varphi_{n_1+\epsilon}(0)}}{\overline{\varphi_{n_1+\epsilon+1}(0)}} - \alpha_{n_1+\epsilon}\right|^{1/n_1+\epsilon} \leq \frac{1}{R_0} < 1. \tag{28}$$

From (28) and the triangular inequality, it follows that

$$\limsup_{n_1+\epsilon} |\alpha_{n_1+\epsilon} - \alpha|^{1/n_1+\epsilon} = \varrho_1 < 1$$

if and only if

$$\limsup_{n_1+\epsilon} \left|\frac{\overline{\varphi_{n_1+\epsilon}(0)}}{\overline{\varphi_{n_1+\epsilon+1}(0)}} - \alpha_{n_1+\epsilon}\right|^{1/n_1+\epsilon} = \varrho_1 < 1.$$

Assume that S_{int}^{-1} has exactly one pole in D_1 (and $R_0 > 1$). From Lemma 1, we have that

$$\lim\sup_{n_1+\epsilon} |\alpha_{n_1+\epsilon} - \alpha|^{1/n_1+\epsilon} = \delta < 1,$$

where $\alpha, 1 < |\alpha| < \infty$, is the unique pole which S_{int}^{-1} has in D_1. Therefore,

$$\lim\sup_{n_1+\epsilon} \left|\frac{\overline{\varphi_{n_1+\epsilon}(0)}}{\varphi_{n_1+\epsilon+1}(0)} - \alpha_{n_1+\epsilon}\right|^{1/n_1+\epsilon} = \varrho_1 < 1.$$

Since $1 < |\alpha| < \infty$, we obtain

$$\lim\sup_{n_1+\epsilon} \left|\frac{\varphi_{n_1+\epsilon+1}(0)}{\varphi_{n_1+\epsilon}(0)} - \frac{1}{\overline{\alpha}}\right|^{1/n_1+\epsilon} = \varrho_1 < 1.$$

Thus the first assertion in Theorem 2 implies the second one with $b^2 = 1/\overline{\alpha}$.
Reciprocally, assume that the second assertion takes place. Since $0 < |b^2| < 1$, we get

$$\lim\sup_{n_1+\epsilon} \left|\frac{\overline{\varphi_{n_1+\epsilon}(0)}}{\varphi_{n_1+\epsilon+1}(0)} - \frac{1}{\overline{b^2}}\right|^{1/n_1+\epsilon} = \delta < 1.$$

Thus,

$$\lim\sup_{n_1+\epsilon} \left|\alpha_{n_1+\epsilon} - \frac{1}{\overline{b^2}}\right|^{1/n_1+\epsilon} = \varrho_1 < 1.$$

This is equivalent to the second part of Lemma 1 which in turn implies that S_{int}^{-1} has exactly one pole in D_1 at $\alpha = 1/\overline{b^2}$.

The following example illustrates that ϱ_1 and ϱ_2 (in the notation used in the proof of Theorem 2) need not be equal. Therefore, we cannot obtain a formula for R_1 similar to the one displayed in Lemma 1 in terms of the rate of convergence of the sequence $\{\Phi_{n_1+\epsilon}(0)/\Phi_{n_1+\epsilon-1}(0)\}$ to b^2. In fact, take $\Phi_{n_1+\epsilon}(0) = (a^2)^{n_1+\epsilon}$, $(n_1+\epsilon) \in \mathbb{N}$, where $0 < |a^2| < 1$. In this case, $\Phi_{n_1+\epsilon+1}(0)/\Phi_{n_1+\epsilon}(0) = a^2$ for all $(n_1+\epsilon)$; therefore,

$$\lim_{n} \left|\frac{\varphi_{n_1+\epsilon+1}(0)}{\varphi_{n_1+\epsilon}(0)} - a^2\right|^{1/n_1+\epsilon} = 0\,.$$

On the other hand, formula (26) gives us

$$\frac{1}{\overline{a^2}} - \alpha_{n_1+\epsilon} = a^2 \sum_{i=n_1+\epsilon}^{\infty} |a^2|^{2i} = a^2 \frac{|a^2|^{2(n_1+\epsilon)}}{1 - |a^2|^2}.$$

From here, we obtain

$$\lim_{n} \left|\frac{1}{\overline{a^2}} - \alpha_{n_1+\epsilon}\right|^{1/n_1+\epsilon} = |a^2|^2.$$

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