Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 11 (November 2025) pp: 14-32
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

A show of Ingham and Chernoff Theorems on Riemannian symmetric spaces of noncompact type

Muslim Elrayh [(1)] and Shawgy Hussein[(2)]
[1]*Sudan University of Science and Technology, Sudan.*
[2]*Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.*

Abstract
A Hilbert spaceL^2 version of the celebrated Denjoy-Carleman theorem regarding quasi-analytic functions was proved by Chernoff on $\mathbb{R}^d$ using iterates of the Laplacian. Ingham used the classical Denjoy-Carleman theorem to relate the decay of Fourier transform and quasi-analyticity of an integrable function on $\mathbb{R}$. The pioneers authors M. Bhowmika, S. Pustia, S. K.Ray [33] prove analogues of the theorems of Chernoff and Ingham for Riemannian symmetric spaces of noncompact type and show that the theorem of Ingham follows from that of Chernoff. We show, following the smooth way of [33], an application on their valid context.

Keywords: *Riemannian symmetric space*, *Quasi-analyticity*, *Ingham's theorem.*

*Received 02 Nov., 2025; **R**evised 09 Nov., 2025; **A**ccepted 11 Nov., 2025* © The author(s) 2025.
*Published with open access at **www.questjournas.org***

I. Introduction

M. Bhowmika, S. Pustia, S. K.Ray [33] study the classical problem of determining the relationship between the rate of decay at infinity of the Fourier transform of an integrable function and the size of the support of the function of a Riemannian symmetric space $X = G^j/K$ of noncompact type. Such that G^j is a connected noncompact semisimple Lie group with finite center and K is a maximal compact subgroup of G^j. They consider the following well-known statement: suppose $f_j \in L^1(X)$ and its Fourier transform satisfies the estimate

$$|\sum_j \tilde{f}_j(\lambda, k)| \leq C \sum_j \widehat{(h_j)_t}(\lambda), \lambda \in \mathrm{a}^*, k \in K,$$

for some $t > 0$, where $(h_j)_t$ is the sequences of the heat kernel for the Laplace-Beltrami operator on X, and C is a positive number. If the sequencef_j vanishes on a nonempty open subset of X then f_j vanishes identically. The Euclidean version of this statement can be easily proved by showing that the very rapid decay of the Fourier transform imposes real analyticity on the function. The proof that $f_j$are still a real analytic functions involves a nontrivial result such as the Kotake-Narasimhan theorem [23] see also [25, p. 237]. For the real line $\mathbb{R}$ it is well-known that under much slower decay of the Fourier transform $\hat{f}_j$, an integrable functions$f_j \in L^1(\mathbb{R})$ vanishes identically if it vanishes on a nonempty open set. There are classical results available and the key idea they depend on is the notion of quasi-analyticity $[20-22,24,28]$. It is a remarkable fact that most of these results provide rather sharp conditions on the decay of the Fourier transform which impose quasi-analyticity on the function. Some of the results cited above have reappeared of the study of the problem of uniqueness of solution of Schrödinger equation on Euclidean spaces [13]. We see these results and explore the possibility of extending them beyond Euclidean spaces. [33] could extend the main result of [24] to Riemannian symmetric spaces X of noncompact type (see [3]) their objective is to do the same with the result of Ingham [21]. In its original version, Ingham's theorem concerns functions defined on the real line. A recent extension to $\mathbb{R}^d$ states the following (see [33]).

Theorem 1.1 ([5], Theorem 2.2). Let $\theta_j: \mathbb{R}^d \to [0, \infty)$ be a radially decreasing functions with $\lim_{\|\xi\| \to \infty} \theta_j(\xi) = 0$ and set

$$I = \int_{\|\xi\| \geq 1} \sum_j \frac{\theta_j(\xi)}{\|\xi\|^d} d\xi. \quad (1.1)$$

(a) Let $f_j \in L^{1+\epsilon}(\mathbb{R}^d), 0 \leq \epsilon \leq 1$, be such that its Fourier transform $\mathcal{F}f_j$ satisfies the estimate

$$|\sum_j \mathcal{F}f_j(\xi)| \leq C \sum_j e^{-\theta_j(\xi)\|\xi\|}, \text{ for almost every } \xi \in \mathbb{R}^d. \quad (1.2)$$

If I is infinite and f_j vanishes on a nonempty open set in $\mathbb{R}^d$ then f_j is the zero functions on $\mathbb{R}^d$.

(b) If I is finite then given any positive number L, there exists a nontrivial radial functions$f_j \in C_c^\infty(\mathbb{R}^d)$ supported in $B(0, L)$, satisfying the estimate (1.2).

Here, the Fourier transform is given by the integral

$$\mathcal{F}f_j(\xi) = \int_{\mathbb{R}^d} \sum_j f_j(x) e^{-2\pi i x \cdot \xi} dx$$

with $x \cdot \xi$ being the Euclidean inner product of the vectors x and ξ.

It is important to observe that there is no assumption on the functionsθ_j which implies that the function $\xi \to \theta_j(\xi)\|\xi\|$ is radially increasing and hence the condition (1.2) does not automatically imply pointwise decay of the Fourier transform. However, the assumption that the integral I diverges would imply that

$$\limsup_{\|\varepsilon\| \to \infty} \sum_j \theta_j(\xi)\|\xi\| = \infty$$

For $d = 1$, using certain assumption on the functionsθ_j, Ingham [21, p. 30] showed that the condition (1.2) together with the divergence of the integral I implies that

$$\sum_{\epsilon=-1}^{\infty} \sum_j \left\| \frac{d^{2+\epsilon} f_j}{dx^{2+\epsilon}} \right\|_\infty^{-\frac{1}{2+\epsilon}} = \infty$$

It then follows from the Denjoy-Carleman theorem ([30, Theorem 19.11]) that f_j is quasi-analytic and hence vanishes identically under the assumption that it vanishes on a nonempty open set. The proof for a general θ_j can then be reduced to the special case mentioned above. It is fairly natural to anticipate that an extension of Ingham's result to $\mathbb{R}^d$ or to a Riemannian symmetric space X will involve a suitable extension of the Denjoy-Carleman theorem to these spaces. One such result was obtained by [6, Theorem 3] which provides an analogue of the Denjoy-Carleman theorem for $\mathbb{R}^d$ involving iterates of the Laplace operator. [9, Theorem 6.1] proved the following important variant of the result of Bochner.

Theorem 1.2 (see [33]). Let $f_j: \mathbb{R}^d \to \mathbb{C}$ be a smooth functions. Suppose that for all $(2 + \epsilon) \in \mathbb{N} \cup \{0\}$, $\Delta_{\mathbb{R}^d}^{2+\epsilon} f_j \in L^2(\mathbb{R}^d)$ and

$$\sum_{(2+\epsilon) \in \mathbb{N}} \sum_j \left\|\Delta_{\mathbb{R}^d}^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} = \infty$$

If f_j and all its partial derivatives vanish at the origin then f_j is identically zero.

In the above $\Delta_{\mathbb{R}^d}$ denotes the usual Laplacian on $\mathbb{R}^d$. It is not hard to show that Theorem 1.2 can be used suitably to prove Ingham's theorem (Theorem 1.1). It is thus natural to look for an analogue of Chernoff's theorem on Riemannian symmetric spaces of noncompact type and then try to use it to prove an analogue of Theorem 1.1. Our first result in this paper is the following weaker version of Theorem 1.2 for a Riemannian symmetric space X of noncompact type. Here and later, Δ denotes the Laplace-Beltrami operator on X.

Theorem 1.3 (see [33]). Let $f_j \in C^\infty(G^j/K)$ be such that $\Delta^{2+\epsilon} f_j \in L^2(G^j/K)$, for all $(2 + \epsilon) \in \mathbb{N} \cup \{0\}$ and

$$\sum_{\epsilon=-1}^{\infty} \sum_j \left\|\Delta^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} = \infty. \quad (1.3)$$

If f_j vanishes on any nonempty open set in G^j/K then f_j is identically zero.
Though the assumption that f_j vanishes on a nonempty open set drastically changes the nature of the theorem it is still sufficient to help us in proving an analogue of Ingham's theorem on symmetric spaces which is the main goal. The main idea of the proof is to suitably use the connection between the Carleman type condition (1.3) and results regarding polynomial approximation proved in [11]. For a discussion on quasianalyticity and polynomial approximations on Lie groups, see [12].

A Riemannian symmetric space X, the vanishing of G^j-invariant differential operators applied to a function at a point of X is analogous to the vanishing of partial derivatives of a function at a point of $\mathbb{R}^d$. For the rank one symmetric spaces one knows that such differential operators are polynomials in the Laplace-Beltrami operator Δ. In [9] Theorem 1.2 is false under the weaker assumption of vanishing of $\Delta_{\mathbb{R}^d}^{2+\epsilon} f_j(0), (2+\epsilon) \in \mathbb{N} \cup \{0\}$ instead of vanishing of all partial derivatives of f_j at zero. Likewise, we also construct an example to show the impossibility of proving an exact analogue of Chernoff's theorem on symmetric spaces if we restrict ourselves only to the class of G^j-invariant differential operators on X.
We show the following variant of the theorem of Ingham on a Riemannian symmetric space X of noncompact type with $\operatorname{rank}(X) = d \geq 1$.
Theorem 1.4 (see [33]). Let $\theta_j\colon [0,\infty) \to [0,\infty)$ be a decreasing function with $\lim_{r\to\infty} \theta_j(r) = 0$ and set

$$I = \int_{\{\lambda \in (\mathfrak{a}^j)^*_+ \mid \|\lambda\|_B \geq 1\}} \sum_j \frac{\theta_j(\|\lambda\|_B)}{\|\lambda\|_B^d} d\lambda. \tag{1.4}$$

(a) Suppose $f_j \in L^1(X)$ and its Fourier transform $\tilde{f}_j$ satisfies the estimate

$$\int_{\mathfrak{a}^* \times K} \sum_j |\tilde{f}_j(\lambda, k)| e^{\theta_j(\|\lambda\|_B)\|\lambda\|_B} |\mathbf{c}(\lambda)|^{-2} d\lambda dk < \infty. \tag{1.5}$$

If f_j vanishes on a nonempty open set in X and I is infinite then f_j is the zero function.
(b) If I is finite then given any positive number L, there exists a nontrivial $f_j \in C_c^\infty(G^j//K)$ supported in $\mathcal{B}(o, L)$ satisfying the estimate (1.5).
Theorem 1.4 is then used to prove Theorem 4.2 which is an exact analogue of Ingham's theorem for Riemannian symmetric spaces of noncompact type.
We will recall the required preliminaries regarding harmonic analysis on Riemannian symmetric spaces of noncompact type.

II. Riemannian symmetric spaces of noncompact type

We describe the necessary preliminaries regarding semisimple Lie groups and harmonic analysis on Riemannian symmetric spaces. See [14,17-19]. We shall gather only those results which will be used throughout.

ForG^j be a connected, noncompact, real semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. We fix a Cartan involution θ_j of $\mathfrak{g}$ and write $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ where $\mathfrak{l}$ and $\mathfrak{p}$ are +1 and -1 eigenspaces of θ_j respectively. Then $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a linear subspace of $\mathfrak{g}$. The Cartan involution θ_j induces an automorphism Θ of the group G^j and $K = \{g_j \in G^j \mid \Theta(g_j) = g_j\}$ is a maximal compact subgroup of G^j. For a be a maximal subalgebra in $\mathfrak{p}$; then $\mathfrak{a}$ is abelian. We assume that $\dim\mathfrak{a} = d$, called the real rank of G^j. For$B$ denote the Cartan Killing form of $\mathfrak{g}$. It is known that $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite and hence induces an inner product and a norm $\|\cdot\|_B$ on $\mathfrak{p}$. The homogeneous space $X = G^j/K$ is a smooth manifold with $\operatorname{rank}(X) = d$. The tangent space of X at the point $o = eK$ can be naturally identified to $\mathfrak{p}$ and the restriction of B on $\mathfrak{p}$ then induces a G^j-invariant Riemannian metric d on X. For a given $g_j \in G^j$ and a positive number L we define

$$\mathcal{B}(g_jK, L) = \{xK \mid x \in G^j, \mathrm{d}(g_jK, xK) < L\},$$

to be the open ball with center g_jK and radius L. We shall identify a endowed with the inner product induced from $\mathfrak{p}$ with $\mathbb{R}^d$ and let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$. The set of restricted roots of the pair ($\mathfrak{g}, \mathfrak{a}$) is denoted by Σ. It consists of all $\alpha^2 \in \mathfrak{a}^*$ such that

$$\mathfrak{g}_{\alpha^2} = \{X \in \mathfrak{g} \mid [Y, X] = \alpha^2(Y)X, \text{ for all } Y \in \mathfrak{a}\},$$

is nonzero with $m_{\alpha^2} = \dim(\mathfrak{g}_{\alpha^2})$. We choose a system of positive roots $\Sigma_+$and with respect to Σ_+, the positive Weyl chamber $a_+^j = \{X \in a^j \mid \alpha^2(X) > 0, \text{ for all } \alpha^2 \in \Sigma_+\}$. For

$$\mathrm{n} = \bigoplus_{\alpha^2 \in \Sigma_+} \mathfrak{g}_{\alpha^2}$$

Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and we obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. If $N = \exp\ \mathfrak{n}$ and $A = \exp\ \mathfrak{a}$ then N is a nilpotent Lie subgroup and A normalizes N. For the group G^j, we now have the Iwasawa decomposition $G^j = KAN$, that is, every $g_j \in G^j$ can be uniquely written as

$$g_j = \kappa(g_j)\exp\ H(g_j)\eta(g_j), \kappa(g_j) \in K, H(g_j) \in \mathfrak{a}, \eta(g_j) \in N$$

and the map

$$(k, a^j, n) \mapsto k a^j n$$

is a global diffeomorphism of $K \times A \times N$ onto G^j. Let $\rho = \frac{1}{2}\sum_{\alpha^2 \in \Sigma_+} m_{\alpha^2}\alpha^2$ be the half sum of positive roots counted with multiplicity. Let M' and M be the normalizer and centralizer of $\mathfrak{a}$ in K respectively. Then M is a normal subgroup of M' and normalizes N. The quotient group $W = M'/M$ is a finite group, called the Weyl group of the pair ($\mathfrak{g}, \mathfrak{t}$). W acts on $\mathfrak{a}$ by the adjoint action. It is known that W acts as a group of orthogonal transformations (preserving the Cartan-Killing form) on $\mathfrak{a}$. Each $w \in W$ permutes the Weyl chambers and the action of W on the Weyl chambers is simply transitive. Let $A_+ = \exp\ \mathfrak{a}_+$. Since $\exp: \mathfrak{a} \to A$ is an isomorphism we can identify A with $\mathbb{R}^d$. Let $\overline{A_+}$ denote the closure of A_+in G^j. One has the polar decomposition $G^j = KAK$, that is, each $g_j \in G^j$ can be written as

$$g_j = k_1(\exp\ Y)k_2, k_1, k_2 \in K, Y \in \mathfrak{a}$$

In the above decomposition, the A component of $\mathfrak{g}$ is uniquely determined modulo W. In particular, it is well defined in $\overline{A_+}$. The map $(k_1, a^j, k_2) \mapsto k_1 a^j k_2$ of $K \times A \times K$ into G^j induces a diffeomorphism of $K/M \times A_+ \times K$ onto an open dense subset of G^j. It follows that if $g_j K = k_1(\exp\ Y)K \in X$ then

$$\mathrm{d}(o, g_j K) = \|Y\|_B. \tag{2.1}$$

We extend the inner product on $\mathfrak{a}$ induced by B to $\mathfrak{a}^*$ by duality, that is, we set

$$\langle \lambda, \mu \rangle = B(Y_\lambda, Y_\mu), \lambda, \mu \in \mathfrak{a}^*, Y_\lambda, Y_\mu \in \mathfrak{a}$$

where Y_λ is the unique element in a such that

$$\lambda(Y) = B(Y_\lambda, Y), \text{ for all } Y \in \mathfrak{a}$$

This inner product induces a norm, again denoted by $\|\cdot\|_B$, on $(a^j)^*$,

$$\|\lambda\|_B = \langle \lambda, \lambda \rangle^{\frac{1}{2}}, \lambda \in \mathbf{a}^*$$

The elements of the Weyl group W act on $\mathfrak{a}^*$ by the formula

$$sY_\lambda = Y_{s\lambda}, s \in W, \lambda \in \mathrm{a}^*$$

Let $(a^j)^*_{\mathbb{C}}$ denote the complexification of $(a^j)^*$, that is, the set of all complex-valued real linear functionals on $\mathfrak{a}$. If $\lambda: \mathfrak{a} \to \mathbb{C}$ is a real linear functional then $\Re\lambda: \mathfrak{a} \to \mathbb{R}$ and $\Im\lambda: \mathfrak{a} \to \mathbb{R}$, given by

$$\Re\lambda(Y) = \text{ Real part of } \lambda(Y), \text{ for all } Y \in \mathfrak{a},$$

$$\Im\lambda(Y) = \text{ Imaginary part of } \lambda(Y), \text{ for all } Y \in \mathfrak{a},$$

are real-valued linear functionals on $\mathfrak{a}$ and $\lambda = \Re\lambda + i\Im\lambda$. Through the identification of A with $\mathbb{R}^d$, we use the Lebesgue measure on $\mathbb{R}^d$ as the Haar measure da on A. As usual on the compact group K, we fix the normalized Haar measure dk and dn denotes a Haar measure on N. The following integral formulae describe the Haar measure of G^j corresponding to the Iwasawa and Polar decomposition respectively.

$$\int_{G^j}\sum_j f_j(g_j)dg_j = \int_K\int_{a^j}\int_N\sum_j f_j(k\exp\ Yn)e^{2\rho(Y)}dndYdk, f_j \in C_c(G^j)$$

$$= \int_K\int_{A_+}\int_K\sum_j f_j(k_1 a^j k_2)J(a^j)dk_1 da^j dk_2$$

where dY is the Lebesgue measure on $\mathbb{R}^d$ and

$$J(\exp\ Y) = c\prod_{\alpha^2 \in \Sigma_+}(\sinh\ \alpha^2(Y))^{m_{\alpha^2}}, \text{ for } Y \in \overline{a_+^j}$$

c being a normalizing constant. If f_j is a function on $X = G^j/K$ then f_j can be thought of as a function on G^j which is right invariant under the action of K. It follows that on X we have a G^j invariant measure dx such that

$$\int_X \sum_j f_j(x)dx = \int_{K/M} \int_{a_+} \sum_j f_j(k\exp Y)J(\exp Y)dYdk_M$$

where dk_M is the K-invariant measure on K/M. For a sufficiently nice functionsf_j on X, its Fourier transform $\tilde{f}_j$ is defined on $a^*_{\mathbb{C}} \times K$ by the formula

$$\tilde{f}_j(\lambda, k) = \int_{G^j} \sum_j f_j(g_j) e^{(i\lambda-\rho)H(g_j^{-1}k)} dg_j, \lambda \in \mathfrak{a}^*_{\mathbb{C}}, k \in K, \quad (2.2)$$

whenever the integral exists ([19, P. 199]). As M normalizes N the function $k \mapsto \tilde{f}_j(\lambda, k)$ is right M-invariant. It is known that if $f_j \in L^1(X)$ then $\tilde{f}_j(\lambda, k)$ is a continuous function of $\lambda \in a^*$, for almost every $k \in K$ (in fact, holomorphic in λ on a domain containing a^*). If in addition $\tilde{f}_j \in L^1(a^* \times K, |\mathbf{c}(\lambda)|^{-2} d\lambda dk)$ then the following Fourier inversion holds,

$$f_j(g_jK) = \left| W|^{-1} \int_{(a^j)^* \times K} \sum_j \tilde{f}_j(\lambda, k) e^{-(i\lambda+\rho)H(g_j^{-1}k)} \right| \mathbf{c}(\lambda)|^{-2} d\lambda dk, \quad (2.3)$$

for almost every $g_jK \in X$ ([19, Chapter III, Theorem 1.8, Theorem 1.9]). Here $\mathbf{c}(\lambda)$ denotes Harish Chandra's c-function. Moreover, $f_j \mapsto \tilde{f}_j$ extends to an isometry of $L^2(X)$ onto $L^2(a^*_+ \times K, |\mathbf{c}(\lambda)|^{-2} d\lambda dk)$ ([19, Chapter III, Theorem 1.5]).

Remark 2.1 [33]. It is known that [10, P. 117], for $\lambda \in a^*_+$ there exists a positive number C such that

$$|\mathbf{c}(\lambda)|^{-2} \le C(1 + \|\lambda\|_B)^{\dim n}. \quad (2.4)$$

If $\mathrm{rank}(X) = 1$, then a similar lower estimate holds ([2], P. 653); there exist two positive numbers C_1 and C_2 such that for all $\lambda \ge 1$

$$C_1 \lambda^{\dim n} \le |\mathbf{c}(\lambda)|^{-2} \le C_2 \lambda^{\dim n}. \quad (2.5)$$

We now specialize to the case of K-biinvariant functions f_j on G^j. Using the polar decomposition of G^j we may view an integrable or a continuous K-biinvariant functionsf_j on G^j as a function on A_+, or by using the inverse exponential map we may also view f_j as a function on a solely determined by its values on $\mathfrak{a}_+$. Henceforth, we shall denote the set of K-biinvariant functions in $L^{1+\epsilon}(G^j)$ by $L^{1+\epsilon}(G^j//K)$, for $0 \le \epsilon \le \infty$; and K-biinvariant compactly supported smooth functions by $C_c^\infty(G^j//K)$. If $f_j \in L^1(G^j//K)$ then the Fourier transform $\tilde{f}_j$ can also be written as

$$\tilde{f}_j(\lambda, k) = \hat{f}_j(\lambda) = \int_{G^j} \sum_j f_j(g_j) (\phi_j)_{-\lambda}(g_j) dg_j, \quad (2.6)$$

where

$$(\phi_j)_\lambda(g_j) = \int_K \sum_j e^{-(i\lambda+\rho)\left(H(g_j^{-1}k)\right)} dk, \lambda \in \mathbf{a}^*_{\mathbb{C}}, \quad (2.7)$$

is Harish Chandra's elementary spherical function. We now list down some well known properties of the elementary spherical functions which are important for us ([14], Prop 3.1.4 and Chapter 4, §4.6; [19], Lemma 1.18, P. 221).

Theorem 2.2 (see [33]).

(1) $(\phi_j)_\lambda(g_j)$ is K-biinvariant in $g_j \in G^j$ and W-invariant in $\lambda \in (a^j)^*_{\mathbb{C}}$.

(2) $(\phi_j)_\lambda(g_j)$ is C^∞ in $g_j \in G^j$ and holomorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

(3) For all $\lambda \in \overline{a^*_+}$ and $g_j \in G^j$ we have

$$\left| \sum_j (\phi_j)_\lambda(g_j) \right| \le \sum_j (\phi_j)_0(g_j) \le 1. \quad (2.8)$$

(4) For all $Y \in \overline{\mathfrak{a}_+}$and $\lambda \in \overline{\mathfrak{a}^*_+}$

$$0 < \sum_j (\phi_j)_{i\lambda}(\exp Y) \le e^{\lambda(Y)} \sum_j (\phi_j)_0(\exp Y). \quad (2.9)$$

(5) If Δ denotes the Laplace-Beltrami operator on X then

$$\Delta\left(\left(\phi_j\right)_\lambda\right) = -(\|\lambda\|_B^2 + \|\rho\|_B^2)\left(\phi_j\right)_\lambda, \lambda \in \mathfrak{a}^*$$

For K-biinvariant $L^{1+\epsilon}$ functions on G^j, the following Fourier inversion formula is well known ([32], Theorem 3.3 and [26], Theorem 5.4): if $f_j \in L^{1+\epsilon}(G^j//K), 0 \le \epsilon \le 1$ with $\hat{f}_j \in L^1(\mathfrak{a}_+^*, |\mathbf{c}(\lambda)|^{-2}d\lambda)$ then for almost every $g_j \in G^j$,

$$f_j(g_j) = \left| W|^{-1} \int_{(\mathfrak{a}^j)^*} \sum_j \hat{f}_j(\lambda)\left(\phi_j\right)_{-\lambda}(g_j) \right| \mathbf{c}(\lambda)|^{-2} d\lambda dk. \tag{2.10}$$

The spherical Fourier transform and the Euclidean Fourier transform on a are related by the so-called Abel transform. For $f_j \in L^1(G^j//K)$ its Abel transform $\mathcal{A}f_j$ is defined by the integral

$$\mathcal{A}f_j(\exp Y) = e^{\rho(Y)} \int_N \sum_j f_j((\exp Y)n)dn, Y \in \mathfrak{a}$$

([14, P. 107], [16, p.27]). We will need the following theorem regarding Abel transform ([14, Prop 3.3.1, Prop 3.3.2]), which we will refer to as the slice projection theorem.

Theorem 2.3 (see [33]). The map $\mathcal{A}: C_c^\infty(G^j//K) \to C_c^\infty(\mathfrak{a})^W$ is a bijection. If $f_j \in C_c^\infty(G^j//K)$ then

$$\mathcal{F}\left(\mathcal{A}f_j\right)(\lambda) = \hat{f}_j(\lambda), \lambda \in \mathfrak{a}^*, \tag{2.11}$$

where $\mathcal{F}(\mathcal{A}f_j)$ denotes the Euclidean Fourier transform of $\mathcal{A}f_j$ on $\mathfrak{a} \cong \mathbb{R}^d$.

Remark 2.4 [33]. It is easy to see that a special case of Theorem 1.4, namely when $f_j \in C_c^\infty(G^j//K)$, can be proved simply by using the slice projection theorem (see [4] for a more general result in this direction). However, this approach cannot be used to proveTheorem 1.4, because if an integrable K-biinvariant functionsf_j vanishes on an open set then it is not necessarily true that $\mathcal{A}f_j$ also vanishes on an open subset of a^j.

We end this section with a short discussion on Opdam hypergeometric functions which will play a crucial role in the next section. Let a_{reg}^j be the subset of regular elements in a, that is,

$$a_{\text{reg}}^{\text{j}} = \cup_{\alpha^2 \in \Sigma} (\ker \alpha^2)^{\text{C}}$$

For $\xi \in$ a, let T_ξ be the Dunkl-Cherednik operator [27, Definition 2.2], defined for $f_j \in C^1(\mathfrak{a})$ and for $x \in \mathfrak{a}_{\text{reg}}$, by

$$T_\xi f_j(x) = \frac{\partial}{\partial \xi} f_j(x) + \sum_{\alpha^2 \in \Sigma_+} \sum_j m_{\alpha^2} \frac{\alpha^2(\xi)}{1 - e^{-\alpha^2(x)}} \left(f_j(x) - f_j(r_{\alpha^2}x)\right) - \rho(\xi) f_j(x),$$

where r_{α^2} is the orthogonal reflection with respect to the hyperplane ker α^2 and $\frac{\partial}{\partial \xi} f_j$ is the directional derivative f_j in the direction of ξ. In particular r_{α^2} is an isometry of $(\ a^j, \|\cdot\|_B\ )$. Let $\{\xi_1, \xi_2, \cdots, \xi_d\}$ be an orthonormal basis of a with respect to the inner product given by B. Then the Heckman-Opdam Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = \sum_{i=1}^{d} T_{\xi_i}^2$$

The restriction of $\mathcal{L}$ to the W-invariant functions on a agrees with the K-invariant part of Δ on X ([15, Theorem 2.2, Remark 2.3]).

Now, suppose that f_j is a K-biinvariant, smooth functions on G^j which vanishes on the ball $\mathcal{B}(o, L)$ for some positive number L. Using the polar decomposition of G^j we can view f_j as a functions on A and hence on the Lie algebra a^j. Using (2.1) it follows that this latter functions (again denoted by f_j) vanishes on the set $\{X \in \text{a} \mid \|X\|_B < L\}$ which we will continue to denote by the same symbol $\mathcal{B}(o, L)$. Since r_{α^2} is an isometry it follows from the expression of the Dunkl-Cherednik operator that in this case $T_\xi f_j$ also vanishes on $\mathcal{B}(o, L)$ for all $\xi \in$ **a**.

The Opdam hypergeometric functions$G_\lambda^j, \lambda \in \mathfrak{a}_\mathbb{C}^*$ is defined to be the unique analytic functions on a such that

$$T_\xi G_\lambda^j = i\lambda(\xi) G_\lambda^j, \qquad \xi \in \mathfrak{a}, \qquad G_\lambda^j(0) = 1, \tag{2.12}$$

(see [27, p. 89]). The elementary spherical functions$(\phi_j)_\lambda$ and the Opdam hypergeometric functionsG_λ^j are related by [27, p. 89]

$$(\phi_j)_\lambda(x) = \frac{1}{|W|}\sum_{w\in W}\sum_j G_\lambda^j(wx),\ \ \lambda \in \mathfrak{a}_{\mathbb{C}}^*,\ \ x \in \mathfrak{a}. \tag{2.13}$$

However, there exists an alternative relation between $(\phi_j)_\lambda$ and G_λ^j which is important for us [27, (4), p. 119]: For all $x \in$ a,

$$(\phi_j)_\lambda(x) = \frac{1}{|W|}\sum_{w\in W}\sum_j g_j(w\lambda)G_{w\lambda}^j(x), \text{ for almost every } \lambda \in \mathrm{a}^*, \tag{2.14}$$

where

$$g_j(\lambda) = \prod_{\alpha^2\in\Sigma_+^0}\left(1 - \frac{\frac{m_{\alpha^2}}{2} + \frac{\frac{m_{\alpha^2}}{2}}{4}}{\lambda_{\alpha^2}}\right), \lambda_{\alpha^2} = \frac{\langle\lambda,\alpha^2\rangle}{\langle\alpha^2,\alpha^2\rangle}, \tag{2.15}$$

and $\Sigma_+^0 = \{\alpha^2 \in \Sigma_+ \mid 2\alpha^2 \notin \Sigma\}$. We now list down a few results regarding the functionsG_λ^j which will be needed:

(1) For $\lambda \in \mathfrak{a}^*$

$$|g_j(\lambda)||\mathbf{c}(\lambda)|^{-1} \le C_1 + C_2\|\lambda\|_B^{1+\epsilon}, \tag{2.16}$$

for some $C_1, C_2, \epsilon \ge -1$ (this follows from [27, equation (8.1)]).

(2) For $\lambda \in$ a*, the function G_λ^j is known to be bounded on a ([27, Proposition 6.1]).

(3) More generally, for any polynomial $(1+\epsilon)$ of degree N there exists a constant $C_{1+\epsilon}$ such that for all $\lambda \in \mathfrak{a}^*, x \in \mathfrak{a}$

$$\left|(1+\epsilon)\left(\frac{\partial}{\partial x}\right)G_\lambda^j(x)\right| \le C_{1+\epsilon}(1+\|\lambda\|_B)^N(\phi_j)_0(x), \tag{2.17}$$

where $\frac{\partial}{\partial x}G_\lambda^j$ is the directional derivative of G_λ^j in the direction x (see [31, Proposition 3.2]).

III. Chernoff's theorem for symmetric spaces

We prove the theorem of Chernoff (Theorem 1.3). The following lemma is just a restatement of [11, Theorem 2.3] in view of the identification of $(a^j)^*$ with $\mathbb{R}^d$.

Lemma 3.1 (see [33]). Let μ be a finite Borel measure on a* such that, for $(2+\epsilon) \in \mathbb{N}$ and $1 \le j_0 \le d$ the quantity $M_{j_0}(2+\epsilon)$, defined by

$$M_{j_0}(2+\epsilon) = \int_{(a^j)^*}|\lambda(\xi_{j_0})|^{2+\epsilon}d\mu(\lambda)$$

is finite, where $\{\xi_1, \xi_2, \cdots, \xi_d\}$ is a basis of a. If for each $j_0 \in \{1, \cdots, d\}$, the sequence $\{M_{j_0}(2(2+\epsilon))\}_{\epsilon=-1}^{\infty}$ satisfies the Carleman's condition

$$\sum_{(2+\epsilon)\in\mathbb{N}} M_{j_0}(2(2+\epsilon))^{-\frac{1}{2(2+\epsilon)}} = \infty, \tag{3.1}$$

then the polynomials in a* are dense in $L^2(\mathrm{a}^*, d\mu)$.

Given a positive number L we consider the following function space

$$G_L^j(\mathfrak{a}^*) = \mathrm{span}\{\chi_x: \mathfrak{a}^* \to \mathbb{C} \mid x \in \mathcal{B}(o,L), \chi_x(\lambda) = G_\lambda^j(x), \lambda \in \mathfrak{a}^*\}$$

Lemma 3.2 (see [33]). Let μ be a finite Borel measure on $(a^j)^*$ such that for each $j_0 \in \{1, \cdots, d\}$ and each $(2+\epsilon) \in \mathbb{N}$ the quantity $M_{j_0}(2+\epsilon)$ (as in Lemma 3.1) is finite. If for each $j_0 \in \{1, \cdots, d\}$ the sequence $M_{j_0}(2(2+\epsilon))$ satisfies the Carleman's condition (3.1) then for each L positive, $G_L^j(\mathrm{a}^*)$ is dense in $L^2(\mathrm{a}^*, d\mu)$.

Proof. We first note that because of the finiteness of μ and boundedness of G_λ^j, for $\lambda \in$ a*, the space $G_L^j(\mathrm{a}^*)$ is a subset of $L^2(\mathrm{a}^*, d\mu)$. Let $f_j \in L^2(\mathrm{a}^*, d\mu)$ be such that

$$\int_{(a^j)^*}\sum_j f_j(\lambda)G_\lambda^j(x)d\mu(\lambda) = 0, \text{ for all } x \in \mathcal{B}(o,L). \tag{3.2}$$

We define a function F_j on a by

$$F_j(x) = \int_{\mathbf{a}^*} \sum_j f_j(\lambda) G_\lambda^j(x) d\mu(\lambda), \text{ for } x \in \mathbf{a}$$

It follows from (3.2) that F_j vanishes on the ball $\mathcal{B}(o, L)$. Estimate (2.17) together with dominated convergence theorem implies that

$$P\left(\frac{\partial}{\partial x}\right) F_j(x) = \int_{(a^j)^*} \sum_j f_j(\lambda) P\left(\frac{\partial}{\partial x}\right) G_\lambda^j(x) d\mu(\lambda)$$

for any polynomial P. Hence, for $\alpha^2 = (\alpha_1^2, \cdots, \alpha_d^2) \in (\mathbb{N} \cup \{0\})^d$ we have from the eigenvalue equation (2.12) that

$$\begin{aligned} T_{\xi_1}^{\alpha_1^2} \cdots T_{\xi_d}^{\alpha_d^2} F_j(x) &= \int_{(a^j)^*} \sum_j f_j(\lambda) T_{\xi_1}^{\alpha_1^2} \cdots T_{\xi_d}^{\alpha_d^2} G_\lambda^j(x) d\mu(\lambda) \\ &= \int_{(a^j)^*} \sum_j f_j(\lambda) \big(i\lambda(\xi_1)\big)^{\alpha_1^2} \cdots \big(i\lambda(\xi_d)\big)^{\alpha_d^2} G_\lambda^j(x) d\mu(\lambda). \end{aligned} \tag{3.3}$$

Since F_j vanishes on the ball $\mathcal{B}(o, L)$ so does $T_{\xi_1}^{\alpha_1^2} \cdots T_{\xi_d}^{\alpha_d^2} F_j$. It now follows from (3.3) by taking $x = 0$, that for all $\alpha^2 = (\alpha_1^2, \cdots \alpha_d^2) \in (\mathbb{N} \cup \{0\})^d$

$$\int_{(a^j)^*} \sum_j \big(\lambda(\xi_1)\big)^{\alpha_1^2} \cdots \big(\lambda(\xi_d)\big)^{\alpha_d^2} f_j(\lambda) d\mu(\lambda) = 0$$

This implies that f_j annihilates all polynomials and hence by Lemma 3.1, f_j is the zero function.

We will also need the following elementary lemma.

Lemma 3.3 (see [33]). Let $\{a_n^j\}$ be a sequence of positive numbers such that the series $\sum_{n\in\mathbb{N}} \sum_j \; a_n^j$ diverges. Then given any $(2+\epsilon) \in \mathbb{N}$,

$$\sum_{n\in\mathbb{N}} \sum_j (a^j)_n^{1+\frac{2+\epsilon}{n}} = \infty$$

Proof. If $\lim\sup a_n^j$ or $\lim\inf a_n^j$ is nonzero then the result follows trivially. Hence, it suffices to prove the result for the case $\lim_{n\to\infty} a_n^j = 0$. Without loss of generality, we can also assume that $a_n^j \in (0,1)$, for all $n \in \mathbb{N}$. Let us define

$$A = \left\{n \in \mathbb{N}: a_n^j \le \frac{1}{n^2}\right\}, B = \left\{n \in \mathbb{N}: a_n^j > \frac{1}{n^2}\right\}$$

As $\sum_{n\in\mathbb{N}} \sum_j \; a_n^j$ diverges, it follows that B is an infinite set. The result now follows by observing that

$$\lim_{n\to\infty, n\in B} \sum_j \frac{(a^j)_n^{1+\frac{2+\epsilon}{n}}}{a_n^j} = \lim_{n\to\infty, n\in B} \sum_j (a^j)_n^{\frac{2+\epsilon}{n}} = \lim_{n\to\infty, n\in B} \sum_j e^{-\frac{2+\epsilon}{n}\log\frac{1}{a_n^j}} = 1,$$

as for $n \in B$

$$1 \le \frac{1}{a_n^j} \le n^2, \text{ and hence } 0 \le \log\frac{1}{a_n^j} \le 2\log n$$

For $f_j \in L^1(X)$, we define the K-biinvariant component $\mathcal{S}f_j$ of f_j by the integral

$$\mathcal{S}f_j(x) = \int_K \sum_j f_j(kx) dk, x \in X, \tag{3.4}$$

and for $g_j \in G^j$, we define the left translation operator l_{g_j} on $L^1(X)$ by

$$l_{g_j} f_j(x) = f_j(g_j x), x \in X$$

Remark 3.4 [33]. The operator l_{g_j} is usually defined as left translation by g_j^{-1}. The reason we have defined l_{g_j} differently because then it follows that $\mathcal{S}\left(l_{g_j} f_j\right) = \mathcal{S}\left(l_{(g_j)_1} f_j\right)$ if $g_j K = (g_j)_1 K$.

For a nonzero integrable functionsf_j, its K-biinvariant component $\mathcal{S}(f_j)$ may be zero. However, the following lemma shows that there always exists $g_j \in G^j$ such that $\mathcal{S}\left(l_{g_j} f_j\right)$ is nonzero.

Lemma 3.5. ([3, Lemma 4.6]) If $f_j \in L^1(X)$ is nonzero then for every L positive, there exists $g_j \in G^j$ with $g_j K \in \mathcal{B}(o, L)$ such that $\mathcal{S}\left(l_{g_j} f_j\right)$ is nonzero.

We now present the proof of Theorem 1.3.

Proof of Theorem 1.3 (see [33]). The following steps will lead to the proof of the theorem.

Step 1: Using translation invariance of Δ, we can assume without loss of generality that $f_j \in C^\infty(G^j/K)$ and vanishes on the ball $\mathcal{B}(0,L)$ for some L positive. We first show that it suffices to prove the result under the additional assumption that f_j is K-biinvariant. To see this, suppose $f_j \in C^\infty(G^j/K)$ vanishes on $\mathcal{B}(0,L)$ and satisfies the hypothesis (1.3). Since Δ is left-translation invariant operator and $\Delta^{2+\epsilon} f_j \in L^2(G^j/K)$, it is easy to check that

$$\Delta^{2+\epsilon}(\mathcal{S}f_j) = \mathcal{S}(\Delta^{2+\epsilon} f_j), \text{ for all } (2+\epsilon) \in \mathbb{N}$$

Therefore

$$\left\|\Delta^{2+\epsilon}(\mathcal{S}f_j)\right\|_2^2 = \int_{G^j} \sum_j \left|\mathcal{S}(\Delta^{2+\epsilon} f_j)(g_j)\right|^2 dg_j \le \int_{G^j}\int_K \sum_j \left|\Delta^{2+\epsilon} f_j(kg_j)\right|^2 dk\,dg_j = \left\|\Delta^{2+\epsilon} f_j\right\|_2^2$$

Hence

$$\sum_{\epsilon=-2}^{\infty}\sum_j \left\|\Delta^{2+\epsilon}(\mathcal{S}f_j)\right\|_2^{-\frac{1}{2(2+\epsilon)}} \ge \sum_{\epsilon=-2}^{\infty}\sum_j \left\|\Delta^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} = \infty$$

If f_j is not identically zero but vanishes on $\mathcal{B}(o,L)$, then by Lemma 3.5, there exists $(g_j)_0 K \in \mathcal{B}(o, L/2)$ such that $\mathcal{S}\left(l_{(g_j)_0} f_j\right)$ is non zero but vanishes on $\mathcal{B}(o, L/2)$. Hence, if the theorem is true for K-biinvariant functions, then $\mathcal{S}\left(l_{(g_j)_0} f_j\right)$ must vanishes identically, which is a contradiction.

So, we assume that $f_j \in C^\infty(G^j//K)$ is such that $\Delta^{2+\epsilon} f_j \in L^2(G^j//K)$ for all $(2+\epsilon) \in \mathbb{N} \cup \{0\}$ and satisfies (1.3). We will show that if f_j vanishes on $\mathcal{B}(o,L)$ then f_j is the zero function.

Step 2: Given such a functionsf_j we define a measure μ on $\mathfrak{a}^*$ by

$$\mu(E) = \int_E \sum_j |\hat{f}_j(\lambda)||\mathbf{c}(\lambda)|^{-2} d\lambda$$

for all Borel subsets E of $\mathbf{a}^*$. Note that the measure μ is W-invariant. We claim that the space $G_L^j(\mathrm{a}^*)$ is dense in $L^2(\mathrm{a}^*, \mu)$ for any given positive number L. Since $\Delta^n f_j \in L^2(\mathrm{a}^*, |\mathbf{c}(\lambda)|^{-2} d\lambda)$ for all $n \in \mathbb{N} \cup \{0\}$ and $|\mathbf{c}(\lambda)|^{-2}$ is of polynomial growth (Remark 2.1) it follows by a simple application of Cauchy-Schwarz inequality that μ is a finite measure and $G_L^j(\mathrm{a}^*)$ is contained in $L^2(\mathrm{a}^*, \mu)$. The same argument also implies that polynomialsare contained in $L^2(\mathrm{a}^*, \mu)$. To prove the claim it suffices, in view of Lemma 3.2, to show that

$$\sum_{\epsilon=-1}^{\infty} M_{j_0}(2(2+\epsilon))^{-\frac{1}{2(2+\epsilon)}} = \infty$$

where $M_{j_0}(2+\epsilon)$ are as in Lemma 3.1. Now, for a large enough $r \in \mathbb{N}$ and for all $(2+\epsilon) \in \mathbb{N}$ we have

$$\begin{aligned} M_{j_0}(2(2+\epsilon)) &\le \int_{(a^j)^*} \sum_j (\|\lambda\|_B^2 + \|\rho\|_B^2)^{2+\epsilon} |\hat{f}_j(\lambda)||\mathbf{c}(\lambda)|^{-2} d\lambda \\ &\le \left(\int_{(a^j)^*} \sum_j (\|\lambda\|_B^2 + \|\rho\|_B^2)^{(2(2+\epsilon)+2r)} |\hat{f}_j(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{(a^j)^*} \frac{|\mathbf{c}(\lambda)|^{-2}}{(\|\lambda\|_B^2 + \|\rho\|_B^2)^{2r}} d\lambda \right)^{\frac{1}{2}} \\ &= A_r \sum_j \left\|\Delta^{2+\epsilon+r} f_j\right\|_2 \end{aligned}$$

where

$$A_r = \left(\int_{(a^j)^*} \frac{|\mathbf{c}(\lambda)|^{-2}}{(\|\lambda\|_B^2 + \|\rho\|_B^2)^{2r}} d\lambda \right)^{\frac{1}{2}}$$

Therefore

$$\left|M_{j_0}(2(2+\epsilon))\right|^{-\frac{1}{2(2+\epsilon)}} \geq A_r^{-\frac{1}{2(2+\epsilon)}} \sum_j \quad \|\Delta^{(2+\epsilon+r)} f_j\|_2^{-\frac{1}{2(2+\epsilon)}} = A_r^{-\frac{1}{2(2+\epsilon)}} \sum_j \quad \left(\|\Delta^{(2+\epsilon+r)} f_j\|_2^{-\frac{1}{2(2+\epsilon+r)}}\right)^{\left(1+\frac{5}{2+\epsilon}\right)}$$

Since $\lim_{\epsilon\to\infty} A_r^{-\frac{1}{2(2+\epsilon)}} = 1$, it follows from Lemma 3.3 and the hypothesis (1.3) that

$$\sum_{\epsilon=-1}^{\infty} M_{j_0}(2(2+\epsilon))^{-\frac{1}{2(2+\epsilon)}} = \infty$$

It follows that for each positive number L the space $G_L^j(\mathrm{a}^*)$ is dense in $L^2(\mathrm{a}^*, \mu)$.

Step 3: Given the relation (2.14) between G_λ^j and $(\phi_j)_\lambda$ one would expect that the previous step implies something regarding the completeness of the elementary spherical functions $(\phi_j)_\lambda$. In this regard, we consider the space

$$\Phi_L(\mathrm{a}^*) = \mathrm{span}\left\{\lambda \mapsto (\phi_j)_\lambda(x) \mid x \in \mathcal{B}(o,L), \lambda \in \mathrm{a}^*\right\}$$

for any given positive number L and claim that $\Phi_L(\mathrm{a}^*)$ is dense in $L^1(\mathrm{a}^*, \mu)^W$. Here $L^1(\mathrm{a}^*, \mu)^W$ is the W-invariant functions in $L^1(\mathrm{a}^*, \mu)$. To prove this we consider a W invariant functions$h_j \in L^\infty(\mathrm{a}^*, \mu)$ such that

$$\int_{\mathrm{a}^*} \sum_j \quad h_j(\lambda)(\phi_j)_\lambda(x) d\mu(\lambda) = 0$$

for all $x \in \mathcal{B}(o, L)$. By (2.14) and the W-invariance of h_j it follows that

$$\int_{\mathrm{a}^*} \sum_j \quad h_j(\lambda) g_j(\lambda) G_\lambda^{\mathrm{j}}(x) d\mu(\lambda) = 0, \tag{3.5}$$

for all $x \in \mathcal{B}(o, L)$. We will now repeatedly apply the operators T_ξ to the integral in (3.5) by viewing this as a function of the variable $x \in \mathrm{a}$. To justify the differentiation under the integral it is necessary to show that for each $n \in \mathbb{N} \cup \{0\}$

$$\int_{(a^j)^*} \sum_j \quad |h_j(\lambda)| \|\lambda\|_B^n |g_j(\lambda)| |G_\lambda^{\mathrm{j}}(x)| d\mu(\lambda) < \infty$$

By using the estimate (2.16) and the boundedness of G_λ^{j}, we get that

$$\begin{aligned}
&\int_{\mathrm{a}^*} \sum_j \quad |h_j(\lambda)| \|\lambda\|_B^n |g_j(\lambda)| |G_\lambda^{\mathrm{j}}(x)| d\mu(\lambda) \\
&\leq C \sum_j \quad \|h_j\|_\infty \int_{\mathrm{a}^*} \|\lambda\|_B^n |g_j(\lambda)| |\hat{f}_{\mathrm{j}}(\lambda)| |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&\leq C \sum_j \quad \|h_j\|_\infty \int_{\mathrm{a}^*} \|\lambda\|_B^n (C_1 + C_2 \|\lambda\|_B^{1+\epsilon}) |\hat{f}_{\mathrm{j}}(\lambda)| |\mathbf{c}(\lambda)|^{-1} d\lambda \\
&\leq C \sum_j \quad \left\| h_j \|_\infty \int_{\mathrm{a}^*} (\|\lambda\|_B^2 + \|\rho\|_B^2)^M \right| \hat{f}_{\mathrm{j}}(\lambda) | |\mathbf{c}(\lambda)|^{-1} d\lambda, M \in \mathbb{N}, M \geq n + 1 + \epsilon \\
&\leq C \sum_j \quad \|h_j\|_\infty \left(\int_{\mathrm{a}^*} (\|\lambda\|_B^2 + \|\rho\|_B^2)^{2M+d+1} |\hat{f}_{\mathrm{j}}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathrm{a}^*} \frac{1}{(\|\lambda\|_B^2 + \|\rho\|_B^2)^{d+1}} d\lambda\right)^{\frac{1}{2}} \\
&< \infty
\end{aligned}$$

where $d = \dim \boldsymbol{a}$. Note that in the last step we have used the assumption $\Delta^{2+\epsilon} f_j \in L^2(G^j/K)$ for all $(2+\epsilon) \in \mathbb{N} \cup \{0\}$. For each $\alpha^2 \in \Sigma_0^+$, we now choose $\xi_{\alpha^2} \in \mathrm{a}$ in such a way that

$$\lambda(\xi_{\alpha^2}) = \lambda_{\alpha^2} = \frac{\lambda(\alpha^2)}{\langle \alpha^2, \alpha^2 \rangle}, \text{ for all } \lambda \in \mathfrak{a}^*$$

Applying the composition of the operators $T_{\xi_{\alpha^2}}$, for all $\alpha^2 \in \Sigma_0^+$to both sides of (3.5) it follows that

$$\int_{(a^j)^*}\sum_j h_j(\lambda)g_j(\lambda)\left(\prod_{\alpha^2\in\Sigma_0^+} i\lambda_{\alpha^2}\right)G_\lambda^{\mathrm{j}}(x)d\mu(\lambda)=0, \tag{3.6}$$

for all $x\in\mathcal{B}(o,L)$. From the expression of the functionsg_j given in (2.15), it is easy to see that the functions$g_j(\lambda)\left(\prod_{\alpha^2\in\Sigma_0^+} i\lambda_{\alpha^2}\right)$ is of polynomial growth. Since h_j is a bounded function it follows that the function $h_j(\lambda)g_j(\lambda)\left(\prod_{\alpha^2\in\Sigma_0^+} i\lambda_{\alpha^2}\right)$ is in $L^2(\mathfrak{a}^*,\mu)$. As $G_L^j(\mathfrak{a}^*)$ is dense in $L^2(\mathrm{a}^*,d\mu)$ it follows from (3.6) that $h_j=0$ for almost every λ.

Step 4: By the Fourier inversion (2.10) we have that for all $x\in\mathcal{B}(o,L)$

$$f_j(x)=|W|^{-1}\int_{(a^j)^*}\sum_j \hat{f}_{\mathrm{j}}(\lambda)(\phi_j)_\lambda(x)|\mathbf{c}(\lambda)|^{-2}d\lambda=0$$

This implies that for all $u_j\in\Phi_L(\mathrm{a}^*)$

$$\int_{\mathbf{a}^*}\sum_j \hat{f}_{\mathrm{j}}(\lambda)u_j(\lambda)|\mathbf{c}(\lambda)|^{-2}d\lambda=0. \tag{3.7}$$

As $\hat{f}_{\mathrm{j}}\in L^1(\mathfrak{a}^*,d\mu)^W$ by the completeness of $\Phi_L(\mathfrak{a}^*)$ in $L^1(\mathfrak{a}^*,d\mu)^W$ we can approximate $\overline{\hat{f}}_j$ by the elements of $\Phi_L(\mathrm{a}^*)$, that is, given $\epsilon>0$, there exists $(u_j)_0\in\Phi_L(\mathrm{a}^*)$ such that

$$\left\|\overline{\hat{f}}_{\mathrm{j}}-(u_j)_0\right\|_{L^1((a^j)^*,d\mu)}<\epsilon$$

Therefore

$$\int_{\mathbf{a}^*}\sum_j |\hat{f}_{\mathrm{j}}(\lambda)|^2|\mathbf{c}(\lambda)|^{-2}d\lambda=\int_{\mathbf{a}^*}\sum_j \overline{\hat{f}_{\mathrm{j}}(\lambda)}\hat{f}_{\mathrm{j}}(\lambda)|\mathbf{c}(\lambda)|^{-2}d\lambda$$

$$=\left|\sum_j \int_{\mathbf{a}^*}\left(\overline{\hat{f}_{\mathrm{j}}(\lambda)}-(u_j)_0(\lambda)+(u_j)_0(\lambda)\right)\hat{f}_{\mathrm{j}}(\lambda)\right|\mathbf{c}(\lambda)\Big|^{-2}d\lambda\ |$$

$$\leq\sum_j \int_{\mathbf{a}^*}\left|\overline{\hat{f}_{\mathrm{j}}(\lambda)}-(u_j)_0(\lambda)\right|d\mu(\lambda)+\sum_j \left|\int_{\mathbf{a}^*}\hat{f}_{\mathrm{j}}(\lambda)(u_j)_0(\lambda)\right|\mathbf{c}(\lambda)\Big|^{-2}d\lambda\ |$$

$$<\epsilon$$

It follows that $\hat{f}_j$ is zero and hence so is f_j.

Remark 3.6 [33]. We will like to point out that for rank one symmetric spaces it is possible to prove Theorem 1.3 without appealing to the Dunkl-Cherednik operator T_ξ and the Opdam hypergeometric functions G_λ^{j}. To see this, note that if μ is an even, finite Borel measure on $\mathbb{R}$ and the sequence

$$M(2(2+\epsilon))=\int_{\mathbb{R}}\lambda^{2(2+\epsilon)}d\mu(\lambda),(2+\epsilon)\in\mathbb{N}$$

satisfies the Carleman condition (3.1) then by Lemma 3.1 the polynomials which are even functions form a dense subspace of $L^2(\mathbb{R},\mu)_e$ where

$$L^2(\mathbb{R},\mu)_e=\{f_j\in L^2(\mathbb{R},\mu)\mid f_j(\lambda)=f_j(-\lambda),\text{ for almost every }\lambda\in\mathbb{R}\}$$

We note that given any such polynomial P there exists a polynomial Q such that

$$P(\lambda)=Q(\lambda^2+\rho^2),\ \text{for all }\lambda\in\mathbb{R}$$

The same conclusion is not valid for W-invariant polynomials on $\mathbb{R}^d, d>1$ and this is the main reason why we needed to use the Dunkl-Cherednik operators and the Opdam hypergeometric functions in the proof of Theorem 1.3. Now, if $f_j\in L^2(\mathbb{R},\mu)_e$ is such that for all $n\in\mathbb{N}\cup\{0\}$

$$\int_{\mathbb{R}}\sum_j f_j(\lambda)(\lambda^2+\rho^2)^n d\mu(\lambda)=0$$

then it follows that f_j annihilates all polynomials which are even functions. Consequently, f_j is the zero function. This can be used to prove that the space $\Phi_L(\mathbb{R})$ is dense in $L^2(\mathbb{R},\mu)_e$ (and hence in $L^1(\mathbb{R},\mu)_e$). Precisely, if $f_j\in L^2(\mathbb{R},\mu)_e$ is such that

$$\int_{\mathbb{R}}\sum_j f_j(\lambda)(\phi_j)_\lambda(x)d\mu(\lambda)=0,\ \text{for all }x\in B(o,L)$$

then by defining

$$h_j(x) = \int_{\mathbb{R}} \sum_j f_j(\lambda)(\phi_j)_\lambda(x)d\mu(\lambda), x \in G^j$$

(as in Step 3 of the proof above) and applying the Laplacian Δ repeatedly to h_j and putting $x = e$, we get that for all $n \in \mathbb{N} \cup \{0\}$

$$\int_{\mathbb{R}} \sum_j f_j(\lambda)(\lambda^2 + \rho^2)^n d\mu(\lambda) = 0$$

which implies that f_j is the zero function. The rest of the proof then goes as it is.

It was noted in [9] that Theorem 1.2 fails for $d = 1$ if $\frac{d^{2+\epsilon}(2+\epsilon)}{dx^{2+\epsilon}} f_j(0)$ vanishes only for even natural numbers $(2 + \epsilon)$. An analogous phenomenon occurs for symmetric spaces also and shows that an exact analogue of Theorem 1.2 is not true for X if we restrict ourselves only to the class of G^j-invariant differential operators on X. In the following example, we will illustrate this for the n-dimensional real hyperbolic space $\mathbb{H}^n$ by constructing a nonzero square-integrable functionsf_j on $\mathbb{H}^n$ such that

$$\Delta^{2+\epsilon} f_j(x_0) = 0, \text{ for all } (2 + \epsilon) \in \mathbb{N} \cup \{0\}$$

for some $x_0 \in \mathbb{H}^n$ and satisfies (1.3).

Example 3.7 [33]. We start with some relevant preliminaries on hyperbolic spaces [7]. The real hyperbolic space $\mathbb{H}^n$ is defined by

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid -x_1^2 - x_2^2 - \cdots - x_n^2 + x_{n+1}^2 = 1, x_{n+1} > 0\}$$

This is a rank one symmetric space of noncompact type and $\mathbb{H}^n = \mathrm{SO}(n, 1)/\mathrm{SO}(n)$. In this particular case, we have (see [8, p.212]),

$$m_1 = \dim \mathfrak{g}_{\alpha^2} = n - 1, m_2 = \dim \mathfrak{g}_{2\alpha^2} = 0$$

It is well known that the half sum of positive roots for $\mathbb{H}^n$ is given by

$$\rho := \rho_{\mathrm{H}^n} = \alpha^2 + \beta + 1 = \frac{n-1}{2}$$

Similarly, the half sum of positive roots for $\mathbb{H}^{n+2l}, l \in \mathbb{N}$, is given by

$$\rho_{\mathrm{H}^{n+2l}} = \frac{n + 2l - 1}{2} = \rho + l$$

Let $\Delta_{\mathbb{H}^n}$ be the Laplace Beltrami operator on $\mathbb{H}^n$. The radial part of the operator $\Delta_{\mathbb{H}^n}$ is given by

$$\frac{d^2}{dt^2} + (2n - 1)\coth t \frac{d}{dt}$$

If $\xi \in \mathbb{H}^n$, then using the Cartan decomposition of $\mathrm{SO}(n, 1)$ we can write $\xi = k a_t^{\mathrm{j}} \cdot \xi_0$, where $\xi_0 = (0, \cdots, 1), k \in K/M = S^{n-1}$ and

$$a_t^{\mathrm{j}} = \begin{pmatrix} \cosh t & 0_{1\times n-1} & \sinh t \\ 0_{1\times n-1} & I_{n-1\times n-1} & 0_{1\times n-1} \\ \sinh t & 0_{1\times n-1} & \cosh t \end{pmatrix}$$

In the following, we shall denote the Fourier transform of a function f_j on $\mathbb{H}^n$ by $(\tilde{f}_j)_{\mathbb{H}^n}$ and the spherical Fourier transform of a radial function g_j on $\mathbb{H}^{n+2l}$ by $(\widehat{g_j})_{\mathbb{H}^{n+2l}}$. We will need the following version of the Hecke-Bochner identity on $\mathbb{H}^n$ [7, Proposition 3.3.3]: if

$$f_j(x) = (f_j)_0(r) Y_l(k)$$

for $x = k a_r^{\mathrm{j}} \cdot \xi_0$, where Y_l is spherical harmonic of degree l on $K/M \cong S^{n-1}$ then

$$\begin{aligned}(\tilde{f}_j)_{\mathrm{H}^n}(\lambda, k) &= d_{n,l} Q_l(i\lambda - \rho) \left(\int_0^\infty \sum_j (f_j)_0(r) (\phi_j)_\lambda^{\mathbb{H}^{n+2l}}(r)(\sinh r)^{2\rho + l} dr \right) Y_l(k) \\ &= d_{n,l} Q_l(i\lambda - \rho) \left(\sum_j \frac{\widehat{(f_j)_0}}{(\sinh r)^l} \right)_{\mathrm{H}^{n+2l}} (\lambda) Y_l(k),\end{aligned} \tag{3.8}$$

where $(\phi_j)_\lambda^{\mathbb{H}^{n+2l}}$ is the elementary spherical function on $\mathbb{H}^{n+2l}, d_{n,l}$ is some fixed constant depending only on n, l and $Q_l(i\lambda - \rho)$ is a polynomial in λ given by

$$Q_l(i\lambda-\rho)=\prod_{\epsilon=-2}^{l-1}(i\lambda-\rho-(2+\epsilon)). \tag{3.9}$$

Let $(h_j)_t$ be the sequence of heat kernel on $\mathbb{H}^{n+2l}$. See [1] for preliminaries regarding the heat kernel on symmetric spaces. Since $(h_j)_t$ is a K-biinvariant functions on $\mathbb{H}^{n+2l}$, using polar decomposition it can be viewed as an even functions on $\mathbb{R}$ and hence as a K-biinvariant functions on $\mathbb{H}^n$. We now choose a spherical harmonic Y_l of degree l such that $Y_l(k_0)=0$, for some $k_0\in K/M$. We define a function f_j on $\mathbb{H}^n$ by

$$f_j(\xi)=(\sinh r)^l\big(h_j\big)_1(r)Y_l(k),\ \text{for}\ \xi=ka_r^{\mathrm{j}}\cdot\xi_0. \tag{3.10}$$

It follows from the pointwise estimate of the heat kernel [1, (3.1)] that $f_j\in L^2(G^j/K)$ and

$$f_j\big(k_0a_r^{\mathrm{j}}\cdot\xi_0\big)=0$$

for all r in $[0,\infty)$. We now claim that

$$\big(\Delta_{\mathbb{H}^n}^{2+\epsilon}f_j\big)\big(k_0a_r^{\mathrm{j}}\cdot\xi_0\big)=0,\ \text{for all}\ r>0$$

To prove this claim we will show that

$$\Delta_{\mathbb{H}^n}^{2+\epsilon}f_j(\xi)=(\sinh r)^l\big(\Delta_{\mathbb{H}^{n+2l}}+\delta\big)^{2+\epsilon}(h_j)_1(r)\,Y_l(k), \tag{3.11}$$

for all $\xi=ka_r^{\mathrm{j}}\cdot\xi_0$, where

$$\delta=(\rho+l)^2-\rho^2=\rho_{\mathrm{H}^{n+2l}}^2-\rho_{\mathrm{H}^n}^2$$

Taking the Fourier transform of the left-hand sides of (3.11) on $\mathbb{H}^n$ and using the Hecke-Bochner identity (3.8) we get

$$\begin{aligned}\big(\Delta_{\mathrm{H}^n}^{2+\epsilon}f_j\big)(\lambda,k)&=\big(-(\lambda^2+\rho^2)\big)^{2+\epsilon}(\tilde{f_j})_{\mathrm{H}^n}(\lambda,k)\\&=\big(-(\lambda^2+\rho^2)\big)^{2+\epsilon}d_{n,l}Q_l(i\lambda-\rho)\big(\widehat{(h_j)_1}\big)\mathrm{H}_{\mathrm{H}^{n+2l}}(\lambda)Y_l(k).\end{aligned} \tag{3.12}$$

On the other hand, using (3.8) we get that the Fourier transform of the right-hand side of (3.11) on $\mathbb{H}^n$ is equal to

$$\begin{aligned}&d_{n,l}Q_l(i\lambda-\rho)\left(\big(\Delta_{\mathbf{H}^{n+2l}}+\delta\big)^{2+\epsilon}(h_j)_1\right)_{\mathbf{H}^{n+2l}}(\lambda)Y_l(k)\\&=d_{n,l}Q_l(i\lambda-\rho)\big(-\big(\lambda^2+\rho_{\mathbf{H}^{n+2l}}^2\big)+\delta\big)^{2+\epsilon}\big(\widehat{(h_j)_1}\big)_{\mathbf{H}^{n+2l}}(\lambda)Y_l(k)\\&=d_{n,l}Q_l(i\lambda-\rho)\left(-\big(\lambda^2+\rho_{\mathbf{H}^n}^2\big)\right)^{2+\epsilon}\left(\widehat{(h_j)_1}\right)_{\mathbb{H}^{n+2l}}(\lambda)Y_l(k)\end{aligned}$$

which proves (3.11). Now, using (3.12) it follows that

$$\begin{aligned}\big\|\Delta_{\mathbb{H}^n}^{2+\epsilon}f_j\big\|_2^2&=\left\|\left(\widehat{\Delta_{\mathbb{H}^n}^{2+\epsilon}f_j}\right)_{\mathbb{H}^n}(\lambda,k)\right\|_{L^2(\mathbb{R}\times K,|\mathbf{c}(\lambda)|^{-2}d\lambda dx)}^2\\&=d_{n,l}^2\int_{\mathbb{R}}\sum_j\quad|Q_l(i\lambda-\rho)|^2(\lambda^2+\rho^2)^{2(2+\epsilon)}\,\Big|\left(\widehat{(h_j)_1}\right)_{\mathbb{H}^{n+2l}}(\lambda)\Big|^2|\mathbf{c}(\lambda)|^{-2}d\lambda\end{aligned}$$

Using (2.4) and (3.9) we have

$$|\mathbf{c}(\lambda)|^{-2}\le C|\lambda|^{n_0},|\lambda|\ge\rho,\ \text{for some}\ n_0$$
$$|Q_l(i\lambda-\rho)|^2\le C(|\lambda|^2+\rho^2)^{p_0},\ \text{for some}\ p_0>0\ \text{and}$$
$$\left(\widehat{(h_j)_1}\right)_{\mathbb{H}^{n+2l}}(\lambda)=e^{-\left(\lambda^2+\rho_{\mathbb{H}^{n+2l}}^2\right)}$$

Therefore

$$\begin{aligned}\big\|\Delta_{\mathbb{H}^n}^{2+\epsilon}f_j\big\|_2^2&\le C\int_{\mathbb{R}}(\lambda^2+\rho^2)^{2(2+\epsilon+p_0)}e^{-2\lambda^2}|\mathbf{c}(\lambda)|^{-2}d\lambda\\&\le C^{2(2+\epsilon+p_0)}+C_1\int_\rho^\infty(2\lambda^2)^{2(2+\epsilon+p_0)}e^{-2\lambda^2}\lambda^{\eta_0}d\lambda\\&\le C^{2(2+\epsilon+p_0)}+C_12^{-(\eta_0+3)/2}\int_0^\infty y^{2(2+\epsilon+p_0)}y^{\frac{n_0-1}{2}}e^{-y}dy\\&=C^{2(2+\epsilon+p_0)}+C_12^{-(\eta_0+3)/2}\Gamma\left(2(2+\epsilon)+2p_0+\frac{n_0+1}{2}\right)\\&\le C_0^{2(2+\epsilon+p_0)}\Gamma\left(2(2+\epsilon)+2p_0+\frac{n_0+1}{2}\right)\end{aligned}$$

Consequently,

$$\sum_{(2+\epsilon)\in N} \left\|\Delta_{\mathrm{H}^n}^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} \geq \sum_{(2+\epsilon)\in N} C_0^{\frac{-2(2+\epsilon+p_0)}{4(2+\epsilon)}} \Gamma\left(2(2+\epsilon)+2p_0+\frac{n_0+1}{2}\right)^{-\frac{1}{4(2+\epsilon)}}$$

Now, using the fact that ([29, p. 30])

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha^2)}{\Gamma(n)n^{\alpha^2}} = 1, \text{ for } \alpha^2 \in \mathbb{C}$$

it follows that

$$\left\|\Delta_{\mathrm{H}^n}^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} \geq C^{\frac{-2(2+\epsilon+p_0)}{4(2+\epsilon)}} (\Gamma(2(2+\epsilon)))^{-\frac{1}{2(2+\epsilon)}} (2(2+\epsilon))^{-\frac{4p_0+n_0+1}{8(2+\epsilon)}}$$

$$\geq C^{\frac{-2(2+\epsilon+p_0)}{4(2+\epsilon)}} (2(2+\epsilon))^{-\frac{1}{2}} (2(2+\epsilon))^{-\frac{4p_0+n_0+1}{8(2+\epsilon)}}$$

Hence, for large $(2+\epsilon)$

$$\left\|\Delta_{\mathrm{H}^n}^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} \geq C(2(2+\epsilon))^{-\frac{4(2+\epsilon)+4p_0+n_0+1}{8(2+\epsilon)}}$$

and

$$(2(2+\epsilon))^{\frac{4(2+\epsilon)+4p_0+n_0+1}{8(2+\epsilon)}} \leq (2(2+\epsilon))^{\frac{8(2+\epsilon)}{8(2+\epsilon)}} = 2(2+\epsilon)$$

Therefore

$$\sum_{2+\epsilon\geq m_0} \left\|\Delta_{\mathrm{H}^n}^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} \geq \sum_{2+\epsilon\geq m_0} \frac{1}{2(2+\epsilon)} = \infty$$

This shows that the functionsf_j satisfies (1.3) and $\Delta_{\mathrm{H}^n}^{2+\epsilon} f_j(k_0 a_r^j.\xi_0)$ is zero for all $(2+\epsilon) \in \mathbb{N} \cup \{0\}$ and r in $[0,\infty]$.

IV. Ingham's theorem for symmetric spaces

We will prove the proposed analogue of Ingham's theorem (Theorem 1.4), using Theorem 1.3.

Proof of Theorem 1.4 (see [33]). We will first prove part (b) by reducing matters to $\mathbb{R}^d$ with the help of Abel transform $\mathcal{A}$. Since the integral in (1.4) is finite, we have

$$\int_1^\infty \sum_j \frac{\theta_j(r)}{r} dr < \infty$$

Since θ_j is decreasing by part (b) of Theorem 1.1 for L positive, there exists a nontrivial radial functions$(h_j)_0 \in C_c^\infty(\mathbb{R}^d)$ with supp $(h_j)_0 \subseteq B(0, L/2)$ such that

$$\left|\sum_j \mathcal{F}(h_j)_0(\xi)\right| \leq C \sum_j e^{-\|\xi\|\theta_j(\|\xi\|)}, \text{ for all } \xi \in \mathbb{R}^d. \tag{4.1}$$

Since $(h_j)_0$ is a radial functions on $\mathbb{R}^d$, it can be thought of as a W-invariant functions on $A \cong \mathbb{R}^d$. Hence by Theorem 2.3, there exists $h_j \in C_c^\infty(G^j//K)$ such that $\mathcal{A}(h_j) = (h_j)_0$ with supp$h_j \subseteq \mathcal{B}(o, L/2)$. For a nontrivial $\phi_j \in C_c^\infty(G^j//K)$ with support contained in $\mathcal{B}(o, L/2)$, we consider the functions$f_j = h_j * \phi_j \in C_c^\infty(G^j//K)$. It follows from Paley-Wiener theorem ([17, Theorem 7.1, Chapter IV]) that the support of f_j is contained in $\mathcal{B}(o, L)$. Using the slice projection theorem (Theorem 2.3) and the estimate (4.1) it follows that

$$\begin{aligned}
&\int_{\mathbf{a}^*} \sum_j |\hat{f}_j(\lambda)| e^{\|\lambda\|_B \theta_j(\|\lambda\|_B)} |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&= \int_{\mathbf{a}^*} \sum_j |\hat{h}_j(\lambda)| |\hat{\phi}_j(\lambda)| \left| e^{\|\lambda\|_B \theta_j(\|\lambda\|_B)} \right| \mathbf{c}(\lambda) \Big|^{-2} d\lambda \\
&= \int_{\mathbf{a}^*} \sum_j |\mathcal{F}(h_j)_0(\lambda)| |\hat{\phi}_j(\lambda)| \left| e^{\|\lambda\|_B \theta_j(\|\lambda\|_B)} \right| \mathbf{c}(\lambda) \Big|^{-2} d\lambda \\
&\leq C \int_{\mathbf{a}^*} \sum_j |\hat{\phi}_j(\lambda)| |\mathbf{c}(\lambda)|^{-2} d\lambda
\end{aligned} \tag{4.2}$$

Since $\hat{\phi}_j$ is a Schwartz function on $\mathbf{a}^*$ it follows from the estimate (2.4) of $|\mathbf{c}(\lambda)|^{-2}$ that the integral in (4.2) is finite and consequently, $\hat{f}_j$ satisfies the condition (1.5). This completes the proof of part (b).

We will prove part (a) under the additional assumptions that f_j is continuous, K biinvariant and vanishes on an open ball centered at o. The general case then can be deduced from this case by mimicking the arguments given in the proof of [3, Theorem 1.2, steps 1-2].

So, we assume that f_j is K-biinvariant, continuous, integrable functions which vanishes on $\mathcal{B}(o, L)$ and satisfies the hypothesis (1.5)

$$\int_{\mathbf{a}^*} \sum_j |\hat{f}_j(\lambda)| e^{\|\lambda\|_B \theta_j(\|\lambda\|_B)} |\mathbf{c}(\lambda)|^{-2} d\lambda < \infty. \quad (4.3)$$

We observe from (4.3) that $\hat{f}_j \in L^1(\mathbf{a}^*, |\mathbf{c}(\lambda)|^{-2} d\lambda)$. As f_j is an integrable functions, $\hat{f}_j$ is a bounded functions and hence from (4.3) it follows that

$$\int_{\mathbf{a}^*} \sum_j |\hat{f}_j(\lambda)|^2 e^{\theta_j(\|\lambda\|_B)\|\lambda\|_B} |\mathbf{c}(\lambda)|^{-2} d\lambda < \infty. \quad (4.4)$$

We now consider the following two cases as in [21].

Case I: Suppose θ_j satisfies the inequality

$$\theta_j(r) \geq \frac{4}{\sqrt{r}}, \text{ for } r \geq 1. \quad (4.5)$$

From (4.3) and (4.5), it follows that

$$\int_{(a^j)^*} \sum_j |\hat{f}_j(\lambda)| e^{4\sqrt{\|\lambda\|_B}} |\mathbf{c}(\lambda)|^{-2} d\lambda < \infty, \quad (4.6)$$

and hence, in particular

$$\int_{(a^j)^*} \sum_j |\hat{f}_j(\lambda)| \|\lambda\|_B^N |\mathbf{c}(\lambda)|^{-2} d\lambda < \infty$$

for all $N \in \mathbb{N}$. It follows that $f_j \in C^\infty(G^j//K)$. To apply Theorem 1.3 we need to verify condition (1.3). In this regard, an application of (4.4) implies that

$$\begin{aligned} \|\Delta^{2+\epsilon} f_j\|_2 &= \left(\int_{(a^j)^*} \sum_j (\|\lambda\|_B^2 + \|\rho\|_B^2)^{2(2+\epsilon)} |\hat{f}_j(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{\frac{1}{2}} \\ &\leq \sup_{\lambda \in (a^j)^*} (\|\lambda\|_B^2 + \|\rho\|_B^2)^{2+\epsilon} \sum_j e^{-\frac{\|\lambda\|_B \theta_j(\|\lambda\|_B)}{2}} \left(\int_{(a^j)^*} |\hat{f}_j(\lambda)|^2 e^{\|\lambda\|_B \theta_j(\|\lambda\|_B)} |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{\frac{1}{2}} \quad (4.7) \\ &\leq C \sup_{r \in (0,\infty)} \sum_j (\|\rho\|_B^2 + r^2)^{2+\epsilon} e^{-\frac{r\theta_j(r)}{2}} \end{aligned}$$

and the latter quantity is finite by (4.5). In particular, $\Delta^{2+\epsilon} f_j \in L^2(G^j//K)$ for all $(2+\epsilon) \in \mathbb{N} \cup \{0\}$. From now on we shall consider $2+\epsilon \geq \max\{2, \|\rho\|_B\}$. To estimate the L^∞ norm (4.7) let us define

$$(g_j)_{2+\epsilon}(r) = (\|\rho\|_B^2 + r^2)^{2+\epsilon} e^{-\frac{re(r)}{2}}, r \in [0, \infty).$$

Then

$$\|(g_j)_{2+\epsilon}\|_{L^\infty[0,\infty)} \leq \|(g_j)_{2+\epsilon}\|_{L^\infty[0,1]} + \|(g_j)_{2+\epsilon}\|_{L^\infty[1,(2+\epsilon)^4]} + \|(g_j)_{2+\epsilon}\|_{L^\infty((2+\epsilon)^4,\infty)}$$

If $r \in ((2+\epsilon)^4, \infty)$ then by (4.5) we have

$$(g_j)_{2+\epsilon}(r) \leq (\|\rho\|_B^2 + r^2)^{2+\epsilon} e^{-2\sqrt{r}} \leq 2^{2+\epsilon} r^{2(2+\epsilon)} e^{-2\sqrt{r}} := \gamma_{2+\epsilon}(r) \text{ (say)}.$$

The function $\gamma_{2+\epsilon}$ attains its maximum at $r = 4(2+\epsilon)^2$. As $(2+\epsilon)^4 \geq 4(2+\epsilon)^2$ and $\gamma_{2+\epsilon}$ is decreasing on $(4(2+\epsilon)^2, \infty)$ we have

$$\|(g_j)_{2+\epsilon}\|_{L^\infty((2+\epsilon)^4,\infty)} \leq \|\gamma_{2+\epsilon}\|_{L^\infty((2+\epsilon)^4,\infty)} = 2^{2+\epsilon} m^{8(2+\epsilon)} e^{-2(2+\epsilon)^2} = \left(2(2+\epsilon)^8 e^{-2(2+\epsilon)}\right)^{2+\epsilon}. \quad (4.8)$$

Also

$$\|(g_j)_{2+\epsilon}\|_{L^\infty[0,1)} \leq (1 + \|\rho\|_B^2)^{2+\epsilon}. \quad (4.9)$$

For $r \in [1, (2+\epsilon)^4]$, as θ_j is a decreasing function,

$$(g_j)_{2+\epsilon}(r) \leq (\|\rho\|_B^2 + r^2)^{2+\epsilon} e^{-\frac{\theta_j((2+\epsilon)^4)r}{2}} \leq (1+\|\rho\|_B^2)^{2+\epsilon} r^{2(2+\epsilon)} e^{-\frac{\theta_j((2+\epsilon)^4)r}{2}} := \eta_{2+\epsilon}(r)$$

The function $\eta_{2+\epsilon}$ attains its maximum at $r = 4(2+\epsilon)/\theta_j((2+\epsilon)^4)$. As $\epsilon \geq 0$ we have

$$\|\eta_{2+\epsilon}\|_{L^\infty[1,(2+\epsilon)^4]} \leq (1+\|\rho\|_B^2)^{2+\epsilon} \left(\frac{4(2+\epsilon)}{\theta_j((2+\epsilon)^4)}\right)^{2(2+\epsilon)} e^{-2(2+\epsilon)}$$

$$\leq (1+\|\rho\|_B^2)^{2+\epsilon} \left(\frac{4(2+\epsilon)}{\theta_j((2+\epsilon)^4)}\right)^{2(2+\epsilon)} . \quad (4.10)$$

Since the right-hand side of (4.8) goes to zero as $(2+\epsilon)$ goes to infinity and for all large $(2+\epsilon) \in \mathbb{N}$

$$\left(\frac{4(2+\epsilon)}{\theta_j((2+\epsilon)^4)}\right)^{2(2+\epsilon)} \geq \left(\frac{4(2+\epsilon)}{\theta_j(1)}\right)^{2(2+\epsilon)} > 1$$

we have for all large $(2+\epsilon) \in \mathbb{N}$

$$\left\|(g_j)_{2+\epsilon}\right\|_{L^\infty[0,\infty]} \leq 3(1+\|\rho\|_B^2)^{2+\epsilon} \sum_j \left(\frac{4(2+\epsilon)}{\theta_j((2+\epsilon)^4)}\right)^{2(2+\epsilon)}$$

Therefore using inequality (4.7) it follows from above that for all large $(2+\epsilon) \in \mathbb{N}$ and a positive number C

$$\left\|\Delta^{2+\epsilon} f_j\right\|_2 \leq \sum_j \left(\frac{C(2+\epsilon)}{\theta_j((2+\epsilon)^4)}\right)^{2(2+\epsilon)} . \quad (4.11)$$

Applying the change of variable $\|\lambda\|_B = p^4$ in the integral (1.4) defining I, it follows that

$$\int_1^\infty \sum_j \frac{\theta_j(p^4)}{1+\epsilon} d(1+\epsilon) = \infty$$

As θ_j is decreasing in $[0,\infty)$ this, in turn, implies that

$$\sum_{(2+\epsilon)\in\mathbb{N}} \sum_j \frac{\theta_j((2+\epsilon)^4)}{2+\epsilon} = \infty.$$

The inequality (4.11) then implies that

$$\sum_{(2+\epsilon)\in\mathbb{N}} \sum_j \left\|\Delta^{2+\epsilon} f_j\right\|_2^{-\frac{1}{2(2+\epsilon)}} = \infty$$

Since f_j vanishes on $\mathcal{B}(o,L)$, it follows from Theorem 1.3 that f_j vanishes identically. This completes the proof under the assumption (4.5) on the function θ_j.

Case II. We now consider the general case, that is, θ_j is any nonnegative function decreasing to zero at infinity. Again, as in [21], we define

$$(\theta_j)_1(r) = \frac{8}{\sqrt{|r|+1}}, r \in [0,\infty)$$

It is clear that the integral I in (1.4) is finite if θ_j is replaced by $(\theta_j)_1$. Hence, by case (b) there exists a nontrivial $(f_j)_1 \in C_c^\infty(G^j//K)$ such that $\mathrm{supp}(f_j)_1 \subseteq \mathcal{B}(o,L/2)$ and satisfies the estimate

$$\left|(\widehat{f_j})_1(\lambda)\right| \leq C \sum_j e^{-\|\lambda\|_B(\theta_j)_1(\|\lambda\|_B)}, \lambda \in \mathfrak{a}^*. \quad (4.12)$$

We now consider the functions$h_j = f_j * (f_j)_1 \in L^1(G^j//K)$. Since f_j vanishes on $\mathcal{B}(o,L)$, the functionsh_j vanishes on the open set $\mathcal{B}(o,L/2)$. Indeed, if $(g_j)_1K \in \mathcal{B}(o,L/2)$ then for all $g_jK \in \mathcal{B}(o,L/2)$ it follows by using G^j-invariance of the Riemannian metric d that

$$\mathrm{d}(o,(g_j)_1g_jK) \leq \mathrm{d}(o,(g_j)_1K) + \mathrm{d}((g_j)_1K,(g_j)_1g_jK) < L,$$

that is, $(g_j)_1g_jK \in \mathcal{B}(o,L)$. This implies that $f_j((g_j)_1g_j)$ is zero for all $g_jK \in \mathcal{B}(o,L/2) = \mathrm{supp}(f_j)_1$ and hence

$$f_j * (f_j)_1((g_j)_1) = \int_{G^j} \sum_j f_j((g_j)_1g_j)(f_j)_1((g_j)^{-1})dg_j = \int_{\mathrm{supp}(f_j)_1} \sum_j f_j((g_j)_1g_j)(f_j)_1(g_j^{-1})dg_j = 0$$

We observe that

$$\theta_j(r) + (\theta_j)_1(r) \geq \frac{8}{\sqrt{r+1}}, \text{ for } r \geq 1$$

Using the estimate (4.12) and the hypothesis (4.3) it follows that

$$\int_{(a^j)^*} |\hat{h}_j(\lambda)| e^{\|\lambda\|_B\left(\theta_j(\|\lambda\|_B)+(\theta_j)_1(\|\lambda\|_B)\right)} |\mathbf{c}(\lambda)|^{-2} d\lambda < \infty$$

Therefore h_j satisfies all the conditions used in case I and hence is the zero function. This implies that

$$\hat{h}_j(\lambda) = \hat{f}_j(\lambda)\widehat{(f_j)}_1(\lambda) = 0,$$

for almost every $\lambda \in \mathbf{a}^*$. Since $(\hat{f}_j)_1$ is a real analytic functions, it follows that f_j vanishes identically.

Remark 4.1 [33].

(1) It is worth pointing out that Ingham's theorem (Theorem 1.4) can also be proved directly using Lemma 3.2, without appealing to Theorem 1.3. For the sake of completeness, we sketch the line of argument for part (a) of Theorem 1.4. We assume that f_j is K-biinvariant, vanishes on $\mathcal{B}(o, L)$, satisfies the hypothesis (1.5) and the function θ_j satisfies the estimate (4.5) with $I = \infty$. In this case, one can easily show that the measure μ defined in Step 2 of the proof of Theorem 1.3 is again a finite W-invariant measure on $(a^j)^*$. Using (4.3), (4.5) and the arguments used in verifying (1.3), one can show that

$$\sum_{k\in\mathbb{N}} M_{j_0}(2k)^{-\frac{1}{2k}} = \infty, \tag{4.13}$$

where $M_{j_0}(k)$ is as given in Lemma 3.1. By the Fourier inversion (2.3) we get that

$$f_j(x) = |W|^{-1} \int_{(a^j)^*} \sum_j \hat{f}_j(\lambda)\left(\phi_j\right)_\lambda(x) |\mathbf{c}(\lambda)|^{-2} d\lambda = 0, \text{ if } x \in \mathcal{B}(o, L).$$

This implies that for all $u_j \in \Phi_L(\mathbf{a}^*)$

$$\int_{(a^j)^*} \sum_j \hat{f}_j(\lambda) u_j(\lambda) |\mathbf{c}(\lambda)|^{-2} d\lambda = 0. \tag{4.14}$$

As in Step 3 of the proof of Theorem 1.3, it can be shown that $\Phi_L(\mathrm{a}^*)$ is dense in $L^1(\mathrm{a}^*, d\mu)^W$. Therefore we can approximate $\overline{\hat{f}_j}$ by the elements of $\Phi_L(\mathrm{a}^*)$ and hence by (4.14), we get that f_j is identically zero.

(2) It is easy to see that part (a) of Theorem 1.4 remains true if the integral I in (1.2) is replaced by the integral

$$\int_{\{\lambda \in \mathfrak{a}_+^* \mid \|\lambda\|_B \geq 1\}} \sum_j \frac{\theta_j(\|\lambda\|_B)}{\|\lambda\|_B^\eta} |\mathbf{c}(\lambda)|^{-2} d\lambda,$$

where $\eta = d + \mathrm{dimn}$, is the dimension of the symmetric space X. This follows from the estimate (2.4) of $|\mathbf{c}(\lambda)|^{-2}$ as

$$\begin{aligned}\int_1^\infty \sum_j \frac{\theta_j(r)}{r} dr &= C \int_{\{\lambda \in (a^j)_+^* \mid \|\lambda\|_B \geq 1\}} \sum_j \frac{\theta_j(\|\lambda\|_B)}{\|\lambda\|_B^d} d\lambda \\ &= C \int_{\{\lambda \in \mathfrak{a}_+^* \mid \|\lambda\|_B \geq 1\}} \sum_j \frac{\theta_j(\|\lambda\|_B)}{\|\lambda\|_B^\eta} \|\lambda\|_B^{\mathrm{dimn}} d\lambda \\ &\geq C \int_{\{\lambda \in \mathfrak{a}_+^* \mid \|\lambda\|_B \geq 1\}} \sum_j \frac{\theta_j(\|\lambda\|_B)}{\|\lambda\|_B^\eta} |\mathbf{c}(\lambda)|^{-2} d\lambda = \infty.\end{aligned}$$

Moreover, because of the estimate (2.5), part (b) of Theorem 1.4 also remains true in this case if the real rank of G^j is one.

An $L^{1+\epsilon}$ version of Theorem 1.4 can be proved by using Theorem 1.4 itself. To illustrate this we prove an L^∞ version of the above theorem which can be thought of as an exact analogue of Theorem 1.1.

Theorem 4.2 (see [33]). Let θ_j and I be as in Theorem 1.4.

(a) Suppose $f_j \in L^1(X)$ and the Fourier transform $\tilde{f}_j$ satisfies the estimate

$$\left|\sum_j \tilde{f}_j(\lambda, k)\right| \leq C \sum_j e^{-\|\lambda\|_B \theta_j(\|\lambda\|_B)}, \text{ for all } \lambda \in \mathfrak{a}^*, k \in K. \tag{4.15}$$

If f_j vanishes on a nonempty open subset of X and I is infinite, then $f_j = 0$.

(b) If I is finite then given any $L > 0$, there exists a nontrivial $f_j \in C_c^\infty(G^j//K)$ supported in $\mathcal{B}(o, L)$ satisfying the estimate (4.15).

Proof. As in Theorem 1.4, it suffices to prove the theorem for $f_j \in L^1(G^j//K)$ vanishing on an open ball of the form $\mathcal{B}(o, L)$ such that $\hat{f}_j$ satisfies the estimate

$$|\sum_j \hat{f}_j(\lambda)| \leq C \sum_j e^{-\|\lambda\|_B \theta_j(\|\lambda\|_B)}, \text{ for all } \lambda \in \mathfrak{a}_+^*.$$

We choose a nonzero $\phi_j \in C_c^\infty(G^j//K)$ with supp$\phi_j \subseteq \mathcal{B}(o, L/2)$ and consider the functions$f_j * \phi_j$. Since f_j vanishes on $\mathcal{B}(o, L)$ and the support of the functionsϕ_j is contained in $\mathcal{B}(o, L/2)$ it follows as before that $f_j * \phi_j$ vanishes on $\mathcal{B}(o, L/2)$. Now,

$$\int_{\mathbf{a}^*} \sum_j |\widehat{f_j * \phi_j}(\lambda)| e^{\|\lambda\|_B \theta_j(\|\lambda\|_B)} |\mathbf{c}(\lambda)|^{-2} d\lambda$$
$$= \int_{\mathbf{a}^*} \sum_j |\hat{\phi}_j(\lambda)| |\hat{f}_j(\lambda)| e^{\|\lambda\|_B \theta_j(\|\lambda\|_B)} |\mathbf{c}(\lambda)|^{-2} d\lambda$$
$$\leq C \int_{\mathbf{a}^*} \sum_j |\hat{\phi}_j(\lambda)| |\mathbf{c}(\lambda)|^{-2} d\lambda < \infty$$

It now follows from Theorem 1.4 that $f_j * \phi_j$ is zero almost everywhere. Since $\hat{\phi}_j$ is nonzero almost everywhere we conclude that $\hat{f}_j$ vanishes almost everywhere on $\mathfrak{a}^*$ and so does f_j.

To prove part (b) we observe that if I is finite then the functionsh_j constructed in the proof of Theorem 1.4, (b) satisfies the estimate (4.15).

Remark 4.3 [33]. Theorem 1.3 and Theorem 1.4 can be proved in the context of Damek-Ricci spaces [2] by similar arguments. It will be interesting to see whether both the theorems hold for hypergeometric transforms associated to root systems [26,31].

References

[1] Jean-Philippe Anker, Patrick Ostellari, The heat kernel on noncompact symmetric spaces, in: Lie Groups and Symmetric Spaces, in: Amer. Math. Soc. Transl. Ser. 2, 210, Adv. Math. Sci., vol. 54, Amer. Math. Soc.,
Providence, RI, 2003, pp. 27-46, MR2018351 (2005b:58031).

[2] Jean-Philippe Anker, Ewa Damek, Chokri Yacoub, Spherical analysis on harmonic AN groups, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 23 (4) (1996) 643-679 (1997). MR1469569 (99a:22014).

[3] M. Bhowmik, Swagato K. Ray, A theorem of Levinson for Riemannian symmetric spaces of noncompact type, Int. Math. Res. Not. (2019), https://doi.org/10.1093/imrn/rnz165.

[4] M. Bhowmik, S. Sen, Uncertainty principles of Ingham and Paley-Wiener on semisimple Lie groups, Isr. J. Math. 225 (1) (2018) 193-221, MR3805646.

[5] M. Bhowmik, Swagato K. Ray, S. Sen, Around theorems of Ingham-type regarding decay of Fourier transform on $\mathbb{R}^n, \mathrm{T}^n$ and two step nilpotent Lie groups, Bull. Sci. Math. 155 (2019) 33-73, MR3944135.

[6] S. Bochner, Quasi-analytic functions, Laplace operator, positive kernels, Ann. Math. (2) 51 (1950) 68-91, MR0032708 (11,334g).

[7] William O. Bray, Aspects of Harmonic Analysis on Real Hyperbolic Space. Fourier Analysis, Lecture Notes in Pure and Appl. Math., vol. 157, Dekker, New York, 1994, pp. 77-102, MR1277819.

[8] William O. Bray, Generalized spectral projections on symmetric spaces of noncompact type: PaleyWiener theorems, J. Funct. Anal. 135 (1) (1996) 206-232, MR1367630.

[9] Paul R. Chernoff, Quasi-analytic vectors and quasi-analytic functions, Bull. Am. Math. Soc. 81 (1975) 637-646, MR0636276 (58#30432).

[10] M. Cowling, S. Giulini, S. Meda, $L^p - L^q$ estimates for functions of the Laplace-Beltrami operator on noncompact symmetric spaces I, Duke Math. J. 72 (1) (1993) 109-150, MR1242882 (95b:22031).

[11] Marcel de Jeu, Determinate multidimensional measures, the extended Carleman theorem and quasianalytic weights, Ann. Probab. 31 (3) (2003) 1205-1227, MR1988469 (2004f:44006).

[12] Marcel de Jeu, Subspaces with equal closure, Constr. Approx. 20 (1) (2004) 93-157, MR2025416 (2005e:41083).

[13] L. Escauriaza, C.E. Kenig, G. Ponce, L. Vega, Uniqueness properties of solutions to Schrödinger equations, Bull. Am. Math. Soc. (N.S.) 49 (3) (2012) 415-442, MR2917065.

[14] R. Gangolli, V.S. Varadarajan, Harmonic Analysis of Spherical Functions on Real Reductive Groups, Springer-Verlag, Berlin, 1988, MR954385 (89m:22015).

[15] G.J. Heckman, Dunkl operators, in: Séminaire Bourbaki, vol. 1996/97, Astérisque 245 (1997), Exp. No. 828, 4, 223-246. MR1627113.

[16] S. Helgason, A duality for symmetric spaces with applications to group representations, Adv. Math. 5 (1970) 1-154, MR0263988 (41 #8587).

[17] S. Helgason, Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators, and Spherical Functions, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, Providence, RI, 2000, MR1790156 (2001h:22001).

[18] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, MR1834454 (2002b:53081).

[19] S. Helgason, Geometric Analysis on Symmetric Spaces, Mathematical Surveys and Monographs, vol. 39, American Mathematical Society, Providence, RI, 2008, MR2463854.

[20] I.I. Hirschman, On the behaviour of Fourier transforms at infinity and on quasi-analytic classes of functions, Am. J. Math. 72 (1950) 200-213, MR0032816 (11,350f).

[21] A.E. Ingham, A note on Fourier transforms, J. Lond. Math. Soc. S1-9 (1) (1934) 29-32, MR1574706.

[22] P. Koosis, The Logarithmic Integral I, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, Cambridge, 1998, xvii + 606 pp. MR1670244 (99j:30001).M. Bhoumik et al. / Journal of Functional Analysis 279 (2020) 108760

[23] Takeshi Kotake, Mudumbai S. Narasimhan, Regularity theorems for fractional powers of a linear elliptic operator, Bull. Soc. Math. Fr. 90 (1962) 449-471, MR0149329.

[24] N. Levinson, On a class of non-vanishing functions, Proc. Lond. Math. Soc. (2) 41 (5) (1936) 393-407, MR1576177.

[25] P. Mohanty, K. Ray Swagato, P. Sarkar Rudra, A. Sitaram, The Helgason-Fourier transform for symmetric spaces. II, J. Lie Theory 14 (1) (2004) 227-242.

[26] E.K. Narayanan, A. Pasquale, S. Pusti, Asymptotics of Harish-Chandra expansions, bounded hypergeometric functions associated with root systems, and applications, Adv. Math. 252 (2014) 227-259, MR3144230.

[27] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, Acta Math. 175 (1) (1995) 75-121, MR1353018.

[28] R.E.A.C. Paley, N. Wiener, Notes on the theory and application of Fourier transforms. I, II, Trans. Am. Math. Soc. 35 (2) (1933) 348-355, MR1501688.

[29] R.B. Paris, D. Kaminski, Asymptotics and Mellin-Barnes Integrals, Encyclopedia of Mathematics and Its Applications, vol. 85, Cambridge University Press, Cambridge, 2001, MR1854469.

[30] W. Rudin, Real and Complex Analysis, third edition, McGraw-Hill Book Co., New York, 1987, MR0924157 (88k:00002).

[31] Bruno Schapira, Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel, Geom. Funct. Anal. 18 (1) (2008) 222-250, MR2399102 (2009f:33019).

[32] Robert J. Stanton, Peter A. Tomas, Pointwise inversion of the spherical transform on $L^P(G/K)$, $1 \leq p < 2$, Proc. Am. Math. Soc. 73 (3) (1979) 398-404, MR0518528.

[33] MithunBhowmika, Sanjoy Pusti, Swagato K. Ray, Theorems of Ingham and Chernoff on Riemannian symmetric spaces of noncompact type, Journal of Functional Analysis 279 (2020) 1-30.