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**Research Paper**

# On Density and Non-Density of Compactly $C_c^\infty \hookrightarrow W^{2+\epsilon,2+\epsilon}$ on Certain Complete Manifolds

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***Abstract***

Following the pioneers authors in [37] we study the concept of density of compactly supported smooth functions in the Sobolev space $W^{2+\epsilon,2+\epsilon}$ on complete Riemannian manifolds. The related general results in the Hilbertian case, was extended in the wide range $0 \le \epsilon \le 1$. When $\epsilon = 0$ the density under a quadratic Ricci lower bound was obtained. And when $\epsilon > 0$ a suitably controlled growth of the derivatives of Riem of order $(\epsilon - 1)$ was shown. An independent gradient regularity lemma was proved. Now, when $\epsilon > 0$ we give the first counterexample to the density property on manifolds whose sectional curvature is bounded from below by a negative constant. Also when $\epsilon > 0$ the existence of a counterexample to the validity of the Calderón-Zygmund inequality when Sec $\ge 0$, in compact setting was getting. We show the improvement of the impossibility to build a Calderón-Zygmund theory for $\epsilon > 0$ with constants only depending on a bound on the diameter and a lower bound on the Sectional curvature.

**Keywords:** Sobolev space, density, curvature, singular point, Sampson formula, Alexandrov space, RCD space.

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## 1. Introduction

A lot of understanding of Sobolev spaces on noncompact Riemannian manifolds are considered, in the Euclidean spaces one has different equivalent definitions of Sobolev spaces. Once these definitions are transposed on a Riemannian manifold, one would like to know if they remain equivalent or not (see [36]). So, it is useful to know which of the nice properties enjoyed by Sobolev spaces on $\mathbb{R}^{2+\epsilon}$ still hold in the setting of non-compact manifolds.

Consider a complete, $(2+\epsilon)$-dimensional Riemannian manifold without boundary $(M, g)$. For $W^{2+\epsilon,2+\epsilon}(M)$ be the Sobolev space of functions on $M$ all of whose covariant derivatives of order $j$ (in the distributional sense) are tensor fields with finite $L^{2+\epsilon}$-norm, for $0 \le j \le 2+\epsilon$. This turns out to be a Banach space, once endowed with the natural norm

$$\|u_m\|_{W^{2+\epsilon,2+\epsilon}(M)} \doteq \sum_{j=0}^{2+\epsilon} \sum_m \left( \int_M |\nabla^j u_m|^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}}$$

By a generalised Meyers-Serrin-type theorem (see [18]), the inclusion of the set $C^\infty(M) \cap W^{2+\epsilon,2+\epsilon}(M)$ is dense in $W^{2+\epsilon,2+\epsilon}(M)$. However, it is not obvious whether the smaller subset $C_c^\infty(M)$ of compactly supported functions is still dense. Having defined the space $W_0^{2+\epsilon,2+\epsilon}(M) \subseteq W^{2+\epsilon,2+\epsilon}(M)$ as the closure of $C_c^\infty(M)$ with respect to the norm $\|\cdot\|_{W^{2+\epsilon,2+\epsilon}(M)}$, we give a contribution to the following problem:

**Problem 1.1 (see [37]).** Let $\epsilon \ge -1$ be an integer and let $0 \le \epsilon < \infty$. Under which assumptions on $(M, g)$, $(1+\epsilon)$ is it true that

$$W_0^{1+\epsilon,1+\epsilon}(M) = W^{1+\epsilon,1+\epsilon}(M)? \tag{1}$$

Now, we fix a function $\lambda: [0, \infty] \to (0, \infty)$ such that

$$\lambda(t) \doteq t \prod_{j=1}^{K} \ln^{[j]}(t), \text{ for } t \gg 1 \tag{2}$$

where $\ln^{[j]}$ stands for the $j$-th iterated logarithm (e.g. $\ln^{[2]}(t) = \ln \ln t$, etc.) and $K$ is some positive integer. Hereafter, all manifolds considered will have no boundary. Moreover, given a Riemannian manifold $(M, g)$, we denote with $r(x)$ the Riemannian distance from a fixed origin $o \in M$ and by $B_{1+\epsilon}(x)$ the geodesic ball of radius $(1+\epsilon)$ centered at a point $x \in M$. Also, given real-valued functions $(f_m)_1$ and $(f_m)_2$, we write $(f_m)_1 \lesssim (f_m)_2$ to mean that there exists a constant $C > 0$ such that $(f_m)_1 \le C(f_m)_2$.

Problem 1.1 has a long history. It is a standard fact that $W_0^{0,1+\epsilon}(M) = W^{0,1+\epsilon}(M) = L^{1+\epsilon}(M)$, and with a little effort one can also prove that $W_0^{1,1+\epsilon}(M) = W^{1,1+\epsilon}(M)$ for all $0 \le \epsilon < \infty$ on any complete manifold; [2]. Also, it is obvious that $W_0^{1+\epsilon,1+\epsilon}(M) = W^{1+\epsilon,1+\epsilon}(M)$ for all $\epsilon \ge -1$ and $0 \le \epsilon < \infty$ whenever $M$ is compact (see [23]). Concerning the non-trivial case $\epsilon \ge 1$, several partial positive results have been proved:

a non-exhaustive list of contributions include works by [2], [22, 23], [4], [19], [20], [27, 26]. The most general and up-to-date result is the following theorem from [26], which generalizes previously known achievements and goes far beyond the case of constant bounds on the curvature and the specific second order case ( $\epsilon = 0$ ).

**Theorem 1.2 [37] (see Theorem 1.5 and Theorem 1.7 in [26]).** Let ( $M, g$ ) be a complete Riemannian manifold. Define $\lambda$ as in (2). Then,

(i) $W^{2+\epsilon,1+\epsilon}(M) = W_0^{2+\epsilon,1+\epsilon}(M)$ for all $0 \le \epsilon < \infty$ and $\epsilon \ge 0$, if

$$|\nabla^j \mathrm{Ric}_g|(x) \lesssim \lambda(r(x))^{\frac{2+j}{1+\epsilon}}, 0 \le j \le \epsilon$$

and either

$$\mathrm{inj}(x) \gtrsim \lambda(r(x))^{-\frac{1}{1+\epsilon}}, \text{ or } |\mathrm{Riem}_g|(x) \lesssim \lambda(r(x))^{\frac{2}{1+\epsilon}};$$

(ii) $W^{2,2}(M) = W_0^{2,2}(M)$ if

$$\mathrm{Ric}_g(x) \gtrsim -\lambda(r(x))^2 g$$

in the sense of quadratic forms, and $W^{2+\epsilon,2}(M) = W_0^{2+\epsilon,2}(M)$ for $\epsilon > 0$ if

$$|\nabla^j \mathrm{Riem}_g|(x) \lesssim \lambda(r(x))^{\frac{2+j}{1+\epsilon}}, 0 \le j \le \epsilon - 1.$$

Now, for $\epsilon > 0$, the assumptions in (i) and (ii) are skew. A noticeable feature of (ii) is that it requires a control on derivatives only up to the order $\epsilon - 1$ : for instance,

$$W^{3,2}(M) = W_0^{3,2}(M) \text{ provided that } |\mathrm{Riem}_g|(x) \lesssim \lambda(r(x)),$$

and, in particular, if $M$ has bounded sectional curvature (equivalently, bounded curvature operator). Quite surprisingly, for a long-time it remained unknown whether $W_0^{2+\epsilon,1+\epsilon}(M) = W^{2+\epsilon,1+\epsilon}(M)$ on any complete Riemannian manifold or whether any assumption on $(M, g)$ was necessary in order to deduce the result. So, an example has been found proving that $W_0^{2+\epsilon,2+\epsilon}(M) \subsetneq W^{2+\epsilon,2+\epsilon}(M)$ is a proper inclusion on certain manifolds with a very wild geometry, at least for $\epsilon \ge 0$; [36].

We study the validity of the results in (ii) of Theorem 1.2 in the range $0 \le \epsilon < 1$. For second order Sobolev spaces, we prove

**Theorem 1.3 [37].** Let $(M, g)$ be a complete manifold such that

$$\mathrm{Ric}_g(x) \gtrsim -\lambda(r(x))^2 g \tag{3}$$

in the sense of quadratic forms. Then, for all $0 \le \epsilon \le 1$, we have

$$W_0^{2,1+\epsilon}(M) = W^{2,1+\epsilon}(M)$$

**Remark 1.4.** This result still holds with a slightly more general (yet more involved) choice for the function $\lambda$, see [26].

**Remark 1.5 [37].** The curvature of the example in [36] decays to $-\infty$ as $-r(x)^4$, and it seems difficult to refine the construction to make it decay at rate $-r(x)^\alpha$ for $\alpha$ close enough to 2. Therefore, at present, there is a gap between the curvature decays in Theorem 1.3 and in [36]. We anticipate that it would be interesting to produce a counterexample to Theorem [1.3 in the range $0 \le \epsilon \le 1$ when (3) barely fails. A counterexample for $\epsilon > 0$ will be given below.

The proof in [4, 27] for the case $\epsilon = 0$ breaks up into the following steps: (1) in our assumptions, by [5, Corollary 2.3] there exists a family of Laplacian cut-off functions $\chi_{1+\epsilon} \in C_c^\infty(M)$ such that

(a) $\chi_{1+\epsilon} = 1$ on $B_{1+\epsilon}(o)$,
(b) $|\nabla\chi_{1+\epsilon}|(x) \le C\lambda^{-1}(r(x))$,
(c) $|\Delta\chi_{1+\epsilon}| \le C$,
for some constant $C > 0$ independent of $1 + \epsilon$;
(d) the above properties guarantee that $\|f_m \Delta\chi_{1+\epsilon}\|_{L^2(M)} \to 0$ as $\epsilon \to \infty$, for any $f_m \in W^{2,2}(M)$;
(e) using the Bochner formula, the latter step implies that $\|f_m|\nabla^2\chi_{1+\epsilon}|\|_{L^2(M)} \to 0$ as $\epsilon \to \infty$ for any $f_m \in W^{2,2}(M)$
(f) this finally yields that $f_m\chi_{1+\epsilon} \to f_m$ in $\|\cdot\|_{W^{2,2}(M)}$.

The proof we provide here for the case $\epsilon < 0$ follows the same line of thought. In this case one has to control $\|f_m|\nabla^2\chi_{1+\epsilon}|\|_{L^{2+\epsilon}(M)}$. Since the Bochner formula is modelled on $L^2$-norms, by a Hölder inequality we estimate $\|f_m|\nabla^2\chi_{1+\epsilon}|\|_{L^{2+\epsilon}(M)}$ in term of $\||f_m|^{2+\epsilon/2}|\nabla^2\chi_{1+\epsilon}|\|_{L^2(M)}$ and use Bochner formula to control this latter. The main difficulty consists in estimating the remaining term involving $|f_m|^{2+\epsilon/2}$ and its derivatives. To this end, the case $\epsilon = 0$ requires an ad hoc procedure, while for $0 < \epsilon < 1$ we shall need a regularity lemma that, seems to be new. We first define the functional space

$$\widetilde{W}^{2,1+\epsilon}(M) = \{f_m \in L^{1+\epsilon}(M) \colon \Delta f_m \in L^{1+\epsilon}(M) \text{ distributionally }\}$$

endowed with the norm

$$\|f_m\|_{\widetilde{W}^{2,1+\epsilon}(M)} = \|f_m\|_{L^{1+\epsilon}(M)} + \|\Delta f_m\|_{L^{1+\epsilon}(M)}$$

Observe that $\widetilde{W}^{2,1+\epsilon}(M)$ is the domain of the maximal self-adjoint extension of $\Delta\colon C_c^\infty(M) \to C_c^\infty(M)$ in $L^{1+\epsilon}(M)$. Indeed, by a result of [21, Appendix A], there is a unique self-adjoint extension of $\Delta$, equivalently, $C_c^\infty(M)$ is dense in $\widetilde{W}^{2,1+\epsilon}(M)$.

**Lemma 1.6 (see [37]).** Let $M$ be a complete manifold, and fix $0 < \epsilon < \infty$. If $f_m, \nabla f_m, \Delta f_m \in L^{1+\epsilon}(M)$, then $|f_m|^{\frac{1+\epsilon}{2}} \in W^{1,2}(M)$, with the bound

$$\left\|\sum_m \nabla |f_m|^{\frac{1+\epsilon}{2}}\right\|_{L^2(M)}^2 \leq \frac{(1+\epsilon)^2}{4(\epsilon)} \sum_m \|f_m\|_{L^{1+\epsilon}(M)}^{\epsilon} \|\Delta f_m\|_{L^{1+\epsilon}(M)}$$

Moreover, when $0 < \epsilon \leq 1$, if $f_m \in \widetilde{W}^{2,1+\epsilon}(M)$ then $\nabla f_m \in L^{1+\epsilon}(M)$ and

$$\|\sum_m \nabla f_m\|_{L^{1+\epsilon}(M)}^2 \leq \frac{4}{(1+\epsilon)^2} \sum_m \|f_m\|_{L^{1+\epsilon}(M)}^{(1-\epsilon)} \left\|\nabla |f_m|^{\frac{1+\epsilon}{2}}\right\|_{L^2(M)}^2$$

$$\leq \frac{1}{\epsilon} \sum_m \|f_m\|_{L^{1+\epsilon}(M)} \|\Delta f_m\|_{L^{1+\epsilon}(M)} \ \forall f_m \in \widetilde{W}^{2,1+\epsilon}(M) \tag{4}$$

Notably, this lemma in the case $0 < \epsilon \leq 1$ refines the $L^{1+\epsilon}$-gradient estimate found by [14], who showed that, for some constant $C_{1+\epsilon}$,

$$\|\sum_m \nabla f_m\|_{L^{1+\epsilon}(M)}^2 \leq C_{1+\epsilon} \sum_m \|f_m\|_{L^{1+\epsilon}(M)} \|\Delta f_m\|_{L^{1+\epsilon}(M)}, \forall f_m \in C_c^\infty(M) \tag{5}$$

**Remark 1.7 [37].** As pointed out in [14, p.7], a minor modification of the argument in [13, Sec. 5] shows the existence of manifolds (for instance, the connected sum of two copies of $\mathbb{R}^{2(1+\epsilon)}$ ) for which (5) fails for $\epsilon < 0$. Indeed, these examples can be generalized to any $0 < \epsilon \leq 2\epsilon$ reasoning precisely as for Corollary 1.12 below. Note that, by Young's inequality, (4) implies the weaker $L^{2+\epsilon}$-gradient estimate

$$\|\sum_m \nabla f_m\|_{L^{2+\epsilon}(M)} \leq c \sum_m \left(\|f_m\|_{L^{2+\epsilon}(M)} + \|\Delta f_m\|_{L^{2+\epsilon}(M)}\right) \tag{6}$$

Because of [12], (6) is met for each $0 < \epsilon < \infty$ if the Ricci curvature is bounded from below. A direct proof of this latter result can be found in [33, Theorem 8.2]. Sufficient conditions for the validity of (5), more precisely of the stronger

$$\|\sum_m \nabla f_m\|_{L^{1+\epsilon}(M)} \leq C_{1+\epsilon} \sum_m \left\|(-\Delta)^{1/2} f_m\right\|_{L^{1+\epsilon}(M)},$$

have been investigated in [3, 14, 13, 10, 9, 11]. With no assumptions besides the completeness of $M$, the only $L^{1+\epsilon}$-gradient estimate that we are aware of is that in Theorem 2 in [21], where the authors prove the inequality

$$\|\sum_m \nabla f_m\|_{L^{1+\epsilon}(M)}^2 \leq C_{1+\epsilon} \sum_m \|f_m\|_{L^{1+\epsilon}(M)} \left(\|\Delta f_m\|_{L^{1+\epsilon}(M)} + \max\{0, \epsilon - 1\} \|\nabla^2 f_m\|_{L^{1+\epsilon}(M)}\right)$$

for $f_m \in L^{1+\epsilon}(M)$ with $\nabla^2 f_m \in L^{1+\epsilon}(M)$.

We can reproduce the same scheme of proof introduced for $\epsilon = 0$ also for higher orders. The main tool will be a Weitzenböck formula due to J. H. Sampson applied to the totally symmetrized ( $1+\epsilon$ )-th covariant derivative of some special higher order cut-off functions. This point of view has exploited in [26, Section 5] in the case $\epsilon = 1$, finally leading to the result described in Theorem 1.2(ii). Combining this latter technique with our regularity lemma, we are able to deal with the full range $0 \leq \epsilon \leq 1$ and prove the following (see [37])

**Theorem 1.8.** Let $(M, g)$ be a complete manifold such that, for some integer $\epsilon > 0$,

$$\left|\nabla^j \mathrm{Riem}_g\right|(x) \lesssim \lambda(r(x))^{\frac{2+j}{1+\epsilon}}, 0 \leq j \leq \epsilon - 1$$

with $\lambda$ as in (2). Then, for all $0 \leq \epsilon \leq 1$, we have

$$W^{2+\epsilon,1+\epsilon}(M) = W_0^{2+\epsilon,1+\epsilon}(M).$$

The next step is to understand if one could obtain the equality $W_0^{2,1+\epsilon}(M) = W^{2,1+\epsilon}(M)$ under a lower Ricci curvature bound for $\epsilon > 0$. Secondly, we show that the answer is negative also if one assumes a lower bound on the sectional curvature.

The core of the counterexample that we produce is a "block" that can be attached to any smooth manifold. Then, we prove (see [37])

**Theorem 1.9.** For all $\epsilon \geq 0$ and all $\epsilon > 0$ there exists a complete $(2+\epsilon)$-dimensional Riemannian manifold $(M, g)$ with sectional curvature $\mathrm{Sec}_g \geq -1$, with a distinguished relatively compact, open subset $\mathbb{V}$ diffeomorphic to a ball, such that the following holds: for each $(2+\epsilon)$-dimensional smooth manifold $(N, \bar{g})$, and each relatively compact set $\mathbb{V}' \subset N$ that is diffeomorphic to a ball, the connected sum $M\sharp N$ obtained by gluing along $\mathbb{V}$ and $\mathbb{V}'$ and keeping the original metric outside of $\mathbb{V}, \mathbb{V}'$ satisfies

$$W^{2+\epsilon,2(1+\epsilon)}(M\sharp N) \neq W_0^{2+\epsilon,2(1+\epsilon)}(M\sharp N) \quad \text{for each } \epsilon \geq 0.$$

In particular, if $N$ has sectional curvature bounded from below the same holds for $M\sharp N$.

**Remark 1.10 [37].** As we shall see, $M$ is topologically a product $\mathbb{S}^{1+\epsilon} \times \mathbb{R}$ and has finite volume. If now, given $\epsilon \geq 0$ and $\epsilon > 0$, we select a surface $M\sharp N$ as in Theorem 1.9 applied in dimension 2 , and a compact boundaryless manifold $Y$ of dimension $(\epsilon)$, then it is easy to show the following (see [37])

**Corollary 1.11.** For all $\epsilon \geq 0$ and $\epsilon > 0$, the complete, $(2+\epsilon)$-dimensional manifold $Q \doteq (M\sharp N) \times Y$ satisfies

$$W^{2+\epsilon,2+\epsilon}(Q) \neq W_0^{2+\epsilon,2+\epsilon}(Q) \quad \text{for each } \epsilon \geq 0$$

Moreover, if $N$ has sectional curvature bounded from below the same holds for $Q$.

Indeed, this observation allows to extend to the range $\epsilon > 0$ two other counterexamples that were previously known only in the case $\epsilon > 0$. In order to introduce them, hence there is a tight relation between the density of compactly supported functions in $W^{2,2+\epsilon}(M)$ and the validity of a global $L^{2+\epsilon}$-Calderón-Zygmund inequality

$$\left\| \sum_m \nabla^2 f_m \right\|_{L^{2+\epsilon}(M)} \leq C \sum_m \left( \|f_m\|_{L^{2+\epsilon}(M)} + \|\Delta f_m\|_{L^{2+\epsilon}(M)} \right), \forall f_m \in C_c^\infty(M) \qquad (CZ_{2+\epsilon})$$

Indeed, as illustrated in [33, Proposition 4.7],

$$M \text{ supports } \mathrm{CZ}_{2+\epsilon} \implies W_0^{2,2+\epsilon}(M) = W^{2,2+\epsilon}(M) \qquad (7)$$

while the converse is not always true; see Theorem A and the subsequent discussion in [30]. Therefore, counterexamples to the density of $C_c^\infty(M)$ in $W^{2,2+\epsilon}(M)$ have to be searched among those manifolds that do not support $(CZ_{2+\epsilon})$. The existence of such manifolds has first been proved in [20, 29]. However, in these constructions the curvature is not lower bounded. Very recently, the first example of a complete noncompact manifold with non-negative sectional curvature on which $CZ_{2+\epsilon}$ fails for $\epsilon > 0$ has been presented in [30]. Their example confirms a strong indication suggested by [15, Cor. 1.3], where it is proved that for $\epsilon > 0$ it is not possible to construct a Calderón-Zygmund theory on compact manifolds with constants depending only on (a diameter upper bound and) a lower sectional curvature bound. The same trick that allows us to deduce Corollary 1.11 from Theorem 1.9 also enables us to extend the above mentioned results in [30, 15] to the full range $\epsilon > 0$ :

**Corollary 1.12 [37].** For any $\epsilon > 0$, there exists a complete $(2+\epsilon)$-dimensional manifold $Q$ with $\mathrm{Sec} \geq 0$ that does not satisfy $CZ_{2+\epsilon}$. Precisely, if $M^2$ is a 2-dimensional manifold as constructed in [30], and if $Y^\epsilon$ is a compact manifold with $\mathrm{Sec} \geq 0$, then one can take $Q = M^2 \times Y^\epsilon$.

**Remark 1.13 [37].** Clearly, by (7) the counterexample in Corollary 1.11 is also a counterexample to $(CZ_{2+\epsilon})$. The extra information in Corollary 1.12 is the possibility to construct an example with $\mathrm{Sec} \geq 0$, in particular, by the lower volume bound given by Calabi-Yau theorem, each end of $Q$ has infinite volume.

**Corollary 1.14 (see [37]).** Let $(2+\epsilon) \in \mathbb{N}, D \geq 2$ and $\epsilon > 0$. Then there exist sequences of $(2+\epsilon)$-dimensional Riemannian manifolds $(Q_{2+\epsilon}, g_{2+\epsilon})$ with $\mathrm{diam}_{g_{2+\epsilon}}(Q_{2+\epsilon}) \leq D, \mathrm{Sec}_{g_{2+\epsilon}} \geq 0$ and of smooth functions $(f_m)_{2+\epsilon} \in C^\infty(Q_{2+\epsilon})$ such that

$$\|(f_m)_{2+\epsilon}\|_{L^{2+\epsilon}(Q_{2+\epsilon})} + \|\Delta (f_m)_{2+\epsilon}\|_{L^{2+\epsilon}(Q_{2+\epsilon})} = 1$$

but

$$\lim_{\epsilon \to \infty} \|\nabla^2 (f_m)_{2+\epsilon}\|_{L^{2+\epsilon}(Q_{2+\epsilon})} = \infty$$

We now explain the strategy (see [37]) to prove Theorem 1.9. Contradicting $(CZ_{2+\epsilon})$ on $M$ is, in principle, easier than contradicting $W_0^{2,2+\epsilon}(M) = W^{2,2+\epsilon}(M)$. Indeed, for the former, it is enough to prove that, for any given $C > 0$, there exists at least one compactly supported function $f_m$ for which $(CZ_{2+\epsilon})$ with constant $C$ fails. In particular the construction can be localized in a given region of $M$. On the other hand, to disprove

the density of $C_c^\infty(M)$ in $W^{2,2+\epsilon}(M)$ one needs to handle with any possible compactly supported approximation of a given function. A way to overcome this problem has been proposed in [36], which contains the first (and so far unique) example of $W_0^{2,2+\epsilon}(M) \neq W^{2,2+\epsilon}(M)$ whenever $\epsilon \geq 0$. Namely, one can consider a complete manifold $(M, g)$ with two ends $E_+$and $E_-$and finite volume, so that it is possible to choose a function $f_m \in W^{2,2+\epsilon}(M)$ which attains two different constant values (say 1 and -1 ) on each end. Accordingly, to check that $f_m$ has no compactly supported approximations, it is enough to prove that the $W^{2,2+\epsilon}$-norm of any function $F_m$ which is identically -1 and 1 on the two ends cannot be arbitrarily close to zero. In the presence of a constant lower curvature bound, our strategy can roughly be summarized as follows (see [37]). First, we construct a suitable Alexandrov space $( M, d_\infty )$ with finite volume, $\mathrm{Sec} \geq -1$, and a dense set of sharp singular points. We consider an exhaustion $U_j$ of $M$. On each annulus $U_j \setminus U_{j-1}$, inspired by [15, 30] we prove the existence of a family of metrics $\{\sigma_{j,2+\epsilon}\}_{2+\epsilon}$ that GH-converge to $d_\infty$, and then we suitably select a function $(2+\epsilon)\colon \mathbb{N} \to \mathbb{N}$ to produce a global metric $g$ on $M$ that equals $\sigma_{j,k(j)}$ on $U_j \setminus U_{j-1}$ and has the following property: any function $F_m$ with $\|F_m\|_{W^{2,2+\epsilon}(U_{j+1})} \leq 1$ has to be $C^0$-close to a constant on $U_j \setminus U_{j-1}$, in a quantitative way. In particular, if $\|F_m\|_{W^{2,2+\epsilon}(M)}$ is small enough, then $F_m$ cannot attain values $-1,1$ on the two ends, as required.
We conclude this with a list of related questions (see [37]).

- Is it possible to construct a complete manifold with $\mathrm{Sec}_g \geq 0$ for which $W_0^{2,2+\epsilon}(M) \neq W^{2,2+\epsilon}(M)$ for some $\epsilon > 0$ ? What about if we weaken the curvature assumption to $\mathrm{Ric}_g \geq 0$ ? In both of the cases, by Calabi and Yau's theorem all ends have infinite volume, so the construction in Theorem 1.9 cannot be adapted in a straightforward way.
- The manifolds constructed in Corollaries 1.11 and 1.12 are Riemannian products and, in particular, they have nontrivial topology. It would still be interesting to produce counterexamples in the range $\epsilon > 0$ by generalizing, if possible, the technique in [15]. This may lead, for instance, to counterexamples to $(\overline{\mathrm{CZ}}_{2+\epsilon})$ for $\epsilon > 0$ on contractible manifolds.
- Referring to Remark 1.5, are the decay rates assumed for the curvatures considered in Theorem 1.3 and Theorem 1.8 sharp? It seems reasonable to conjecture so, up to lower order terms.
- Does a complete manifold with positive injectivity radius and $\mathrm{Ric}_g \geq -g$ satisfy $W_0^{2,2+\epsilon}(M) = W^{2,2+\epsilon}(M)$ for each $0 \leq \epsilon < \infty$ ? Recall that Theorem 1.2 answers affirmatively under the conditions $|\mathrm{Ric}_g| \lesssim \lambda^2(r)$ and $\mathrm{inj} \gtrsim \lambda(r)^{-1}$. If $|\mathrm{Ric}_g| \lesssim 1$, is a decay assumption on the injectivity radius necessary?

## 2. Density when $0 \leq \epsilon \leq 1$

### 2.1. The regularity lemma.

**Lemma 2.1 (see [37]).** Let $M$ be a complete manifold, and fix $0 < \epsilon < \infty$. Let $o \in M$ be some fixed origin, and let us denote by $B_{1+2\epsilon}$ the geodesic balls of radius $(1+2\epsilon)$ centered at $o$. If $f_m \in W_{\mathrm{loc}}^{2,1+\epsilon}(M)$ then $|f_m|^{\frac{1+\epsilon}{2}} \in W_{\mathrm{loc}}^{1,2}(M)$ and, for each $0 < \epsilon$,

$$\frac{4(\epsilon)}{(1+\epsilon)^2}\left\|\sum_m \nabla|f_m|^{\frac{1+\epsilon}{2}}\right\|_{L^2(B_{1+\epsilon})}^2 \leq \sum_m \|f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})}^{\epsilon}\left(\frac{1}{\epsilon}\|\nabla f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})} + \|\Delta f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})}\right). \quad (8)$$

In particular,

$$f_m, \nabla f_m, \Delta f_m \in L^{1+\epsilon}(M) \implies |f_m|^{\frac{1+\epsilon}{2}} \in W^{1,2}(M) \quad (9)$$

with the bound

$$\left\|\sum_m \nabla|f_m|^{\frac{1+\epsilon}{2}}\right\|_{L^2(M)}^2 \leq \frac{(1+\epsilon)^2}{4(\epsilon)}\sum_m \|f_m\|_{L^{1+\epsilon}(M)}^{\epsilon}\|\Delta f_m\|_{L^{1+\epsilon}(M)}. \quad (10)$$

Moreover, if $0 < \epsilon \leq 1$,

$$f_m \in \widetilde{W}^{2,1+\epsilon}(M) \implies \nabla f_m \in L^{1+\epsilon}(M), |f_m|^{\frac{1+\epsilon}{2}} \in W^{1,2}(M),$$

and

$$\|\sum_m \nabla f_m\|_{L^{1+\epsilon}(M)}^2 \leq \frac{4}{(1+\epsilon)^2}\sum_m \|f_m\|_{L^{1+\epsilon}(M)}^{1-\epsilon}\left\|\nabla|f_m|^{\frac{1+\epsilon}{2}}\right\|_{L^2(M)}^2 \quad (11)$$

$$\leq \frac{1}{\epsilon}\sum_m \|f_m\|_{L^{1+\epsilon}(M)}\|\Delta f_m\|_{L^{1+\epsilon}(M)} \; \forall f_m \in \widetilde{W}^{2,1+\epsilon}(M). \quad (11)$$

**Remark 2.2.** In view of the validity of a local Calderón-Zygmund inequality, we note that the assumption $f_m \in W^{2,1+\epsilon}_{\text{loc}}(M)$ is equivalent to the assumption $f_m, \nabla f_m, \Delta f_m \in L^{1+\epsilon}_{\text{loc}}(M)$.

**Proof.** Let us first assume that $f_m \in C^\infty(M)$. Clearly, $|f_m|^{\frac{1+\epsilon}{2}} \in L^2_{\text{loc}}(M)$. Let $\varphi$ be a linear cut-off function with $\operatorname{supp}\varphi \subset \overline{B_{1+2\epsilon}}, \varphi \equiv 1$ on $B_{1+\epsilon}$ and $|\nabla\varphi| \le 1/(\epsilon)$. For $\varepsilon > 0$, we compute

$$-\int_M \sum_m \left\langle \nabla(f_m^2+\varepsilon)^{\frac{1+\epsilon}{2}}, \nabla\varphi \right\rangle = \int_M \sum_m \varphi\Delta(f_m^2+\varepsilon)^{\frac{1+\epsilon}{2}}$$

$$=(1+\epsilon)\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-1}{2}}[f_m\Delta f_m + |\nabla f_m|^2] + (1+\epsilon)(\epsilon-1)\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-3}{2}} f_m^2|\nabla f_m|^2$$

$$=(1+\epsilon)\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-1}{2}} f_m\Delta f_m + (1+\epsilon)\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-1}{2}}|\nabla f_m|^2$$

$$+(1+\epsilon)(\epsilon-1)\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-3}{2}} f_m^2|\nabla f_m|^2$$

$$\ge(1+\epsilon)\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-1}{2}} f_m\Delta f_m + (1+\epsilon)(\epsilon)\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-3}{2}} f_m^2|\nabla f_m|^2.$$

On the one hand,

$$\left|\int_M \sum_m \left\langle \nabla(f_m^2+\varepsilon)^{\frac{1+\epsilon}{2}}, \nabla\varphi \right\rangle\right| = (1+\epsilon)\left|\int_M \sum_m (f_m^2+\varepsilon)^{\frac{\epsilon-1}{2}} f_m \langle\nabla f_m, \nabla\varphi\rangle\right|$$

$$\le (1+\epsilon)\int_M \sum_m (f_m^2+\varepsilon)^{\frac{\epsilon}{2}}|\nabla f_m||\nabla\varphi|$$

$$\le \frac{1+\epsilon}{\epsilon}\sum_m \left(\int_{B_{1+2\epsilon}} (f_m^2+\varepsilon)^{\frac{1+\epsilon}{2}}\right)^{\frac{\epsilon}{1+\epsilon}} \left(\int_{B_{1+2\epsilon}} |\nabla f_m|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}},$$

on the other hand,

$$\left|\int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon-1}{2}} f_m\Delta f_m\right| \le \int_M \sum_m \varphi(f_m^2+\varepsilon)^{\frac{\epsilon}{2}}|\Delta f_m|$$

$$\le \sum_m \left(\int_{B_{1+2\epsilon}} (f_m^2+\varepsilon)^{\frac{1+\epsilon}{2}}\right)^{\frac{\epsilon}{1+\epsilon}} \left(\int_{B_{1+2\epsilon}} |\Delta f_m|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}}.$$

Summarizing,

$$\frac{4(\epsilon)}{(1+\epsilon)^2}\int_{B_{1+\epsilon}} \sum_m \left|\nabla(f_m^2+\varepsilon)^{\frac{1+\epsilon}{4}}\right|^2 = (\epsilon)\int_{B_{1+\epsilon}} \sum_m (f_m^2+\varepsilon)^{\frac{\epsilon-3}{2}} f_m^2|\nabla f_m|^2$$

$$\le \sum_m \left\|\sqrt{f_m^2+\varepsilon}\right\|^{\epsilon}_{L^{1+\epsilon}(B_{1+2\epsilon})} \left(\frac{1}{\epsilon}\|\nabla f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})} + \|\Delta f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})}\right).$$

Hence, $\left\{(f_m^2+\varepsilon)^{\frac{1+\epsilon}{4}}\right\}$ is uniformly bounded in $W^{1,2}(B_{1+\epsilon})$ and pointwise convergent to $|f_m|^{1+\epsilon/2}$. By a standard result ([16, Lemma 6.2, p.16]), $|f_m|^{1+\epsilon/2} \in W^{1,2}(B_{1+\epsilon})$ and $\nabla(f_m^2+\varepsilon)^{1+\epsilon/4} \rightharpoonup \nabla|f_m|^{1+\epsilon/2}$ weakly on $B_{1+\epsilon}$, thus

$$\frac{4(\epsilon)}{(1+\epsilon)^2}\int_{B_{1+\epsilon}} \sum_m \left|\nabla|f_m|^{\frac{1+\epsilon}{2}}\right|^2 \le \frac{4(\epsilon)}{(1+\epsilon)^2}\liminf_{\varepsilon\to 0}\int_{B_{1+\epsilon}} \sum_m \left|\nabla(f_m^2+\varepsilon)^{\frac{1+\epsilon}{4}}\right|^2$$

$$\le \sum_m \|f_m\|^{\epsilon}_{L^{1+\epsilon}(B_{1+2\epsilon})} \left(\frac{1}{\epsilon}\|\nabla f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})} + \|\Delta f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})}\right).$$

We now claim that (8) holds for $f_m \in W^{2,1+\epsilon}_{\text{loc}}(M)$. Having chosen such $f_m$, by the Meyers-Serrin-type theorem in [18 there exists $\{(f_m)_j\} \subset C^\infty(M)$ such that $(f_m)_j \to f_m$ in $W^{2,1+\epsilon}(B_{1+2\epsilon})$ and pointwise almost everywhere. Applying (8) to $(f_m)_j$ and to $(f_m)_j - (f_m)_i$ shows that $\left\{|(f_m)_j|^{1+\epsilon/2}\right\}$ is uniformly bounded in $W^{1,2}(B_{1+\epsilon})$, and that $\nabla|(f_m)_j|^{1+\epsilon/2}$ is a Cauchy sequence. Hence, by the compactness of $W^{1,2}(B_{1+\epsilon}) \hookrightarrow L^2(B_{1+\epsilon})$, $|(f_m)_j|^{1+\epsilon/2}$ strongly converges in $W^{1,2}(B_{1+\epsilon})$ to its pointwise limit $|f_m|^{1+\epsilon/2}$. Taking limits we get (8) for $f_m \in W^{2,1+\epsilon}_{\text{loc}}(M)$, as claimed.

Assume that $f_m, \nabla f_m, \Delta f_m \in L^{1+\epsilon}(M)$. Then, by Remark 2.2 $f_m \in W^{2,1+\epsilon}_{\text{loc}}(M)$ and thus letting $1+2\epsilon = 2(1+\epsilon) \to \infty$ in (8) we readily deduce (10).

Next, we examine the case $f_m \in \widetilde{W}^{2,1+\epsilon}(M)$. We first consider $f_m \in C_c^\infty(M)$, since this latter space is dense in $\widetilde{W}^{2,1+\epsilon}(M)$ by [21, Appendix A]. If $0 < \epsilon \leq 1$, by Hölder's inequality, Stampacchia's theorem and (8) we get

$$\begin{aligned} \|\sum_m \nabla f_m\|^2_{L^{1+\epsilon}(B_{1+\epsilon})} &= \|\sum_m \nabla f_m\|^2_{L^{1+\epsilon}(B_{1+\epsilon}\cap\{|f_m|>0\})} \\ &\leq \sum_m \|f_m\|^{1-\epsilon}_{L^{1+\epsilon}(B_{1+\epsilon}\cap\{|f_m|>0\})} \left\| f_m^{\frac{\epsilon-1}{2}} \nabla f_m \right\|^2_{L^2(B_{1+\epsilon}\cap\{|f_m|>0\})} \\ &= \frac{4}{(1+\epsilon)^2} \sum_m \|f_m\|^{1-\epsilon}_{L^{1+\epsilon}(B_{1+\epsilon})} \left\| \nabla |f_m|^{\frac{1+\epsilon}{2}} \right\|^2_{L^2(B_{1+\epsilon}\cap\{|f_m|>0\})} \\ &\leq \frac{4}{(1+\epsilon)^2} \sum_m \|f_m\|^{1-\epsilon}_{L^{1+\epsilon}(B_{1+2\epsilon})} \left\| \nabla |f_m|^{\frac{1+\epsilon}{2}} \right\|^2_{L^2(B_{1+\epsilon})} \\ &\leq \frac{1}{\epsilon} \sum_m \|f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})} \left( \frac{1}{\epsilon} \|\nabla f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})} + \|\Delta f_m\|_{L^{1+\epsilon}(B_{1+2\epsilon})} \right). \end{aligned} \tag{12}$$

If $f_m \in \widetilde{W}^{2,1+\epsilon}(M)$, take $\{(f_m)_j\} \subset C_c^\infty(M)$ with $(f_m)_j \to f_m$ and $\Delta (f_m)_j \to \Delta f_m$ in $L^{1+\epsilon}(M)$. Applying (12) to $(f_m)_j$ and to $(f_m)_j - (f_m)_i$, letting $\epsilon \to \infty$ and then $j \to \infty$ we deduce that $\nabla (f_m)_j \to \nabla f_m \in L^{1+\epsilon}(M)$. In particular, by (9) the function $f_m$ satisfies (10). The same computations as in (12) can therefore be performed with $\epsilon = \infty$, leading to (11).

**2.2. Density for order 2.**

**Proof of Theorem 1.3 (see [37]).** For $\epsilon \gg 0$, let $\chi_{1+\epsilon} \in C_c^\infty(M)$ be a family of Laplacian cut-off functions such that

- $\chi_{1+\epsilon} = 1$ on $B_{1+\epsilon}(o)$,
- $|\nabla \chi_{1+\epsilon}|(x) \leq C\lambda^{-1}(r(x))$ and $\|\nabla \chi_{1+\epsilon}\|_\infty \leq C\lambda^{-1}(1+\epsilon)$,
- $|\Delta \chi_{1+\epsilon}| \leq C$,

  for some constant $C > 0$ independent of $(1+\epsilon)$. Such a family has been constructed in (the proof of) [26, Corollary 5.2]. As usual, first note that $C^\infty(M) \cap W^{2,1+\epsilon}(M)$ is dense in $W^{2,1+\epsilon}(M)$ (see for instance [18]). Given a smooth $f_m \in W^{2,1+\epsilon}(M)$, define $(f_m)_{1+\epsilon} \doteq \chi_{1+\epsilon} f_m$. We get that

$$\|((f_m)_{1+\epsilon} - f_m)\|_{L^{1+\epsilon}} = \|((1-\chi_{1+\epsilon})f_m)\|_{L^{1+\epsilon}} \tag{13}$$

$$\|\nabla((f_m)_{1+\epsilon} - f_m)\|_{L^{1+\epsilon}} \leq \|f_m \nabla \chi_{1+\epsilon}\|_{L^{1+\epsilon}} + \|(1-\chi_{1+\epsilon})\nabla f_m\|_{L^{1+\epsilon}} \tag{14}$$

$$\|\nabla^2((f_m)_{1+\epsilon} - f_m)\|_{L^{1+\epsilon}} \leq 2\||\nabla \chi_{1+\epsilon}\|\nabla f_m\,|\,\|_{L^{1+\epsilon}} + \|(1-\chi_{1+\epsilon})\nabla^2 f_m\|_{L^{1+\epsilon}} + \|f_m \nabla^2 \chi_{1+\epsilon}\|_{L^{1+\epsilon}} \tag{15}$$

Both $(1-\chi_{1+\epsilon})$ and $\nabla \chi_{1+\epsilon}$ are uniformly bounded and supported in $M \setminus B_{1+\epsilon}(o)$. Since $f_m \in W^{2,1+\epsilon}(M)$ this permits to conclude that all the terms at the RHS of (13), (14) and (15) except the last one tend to 0 as $\epsilon \to \infty$. Concerning $\|f_m \nabla^2 \chi_{1+\epsilon}\|_{L^{1+\epsilon}}$, first observe that $\epsilon \leq 1$ and Hölder's inequality imply

$$\int_M \sum_m |f_m|^{1+\epsilon} |\nabla^2 \chi_{1+\epsilon}|^{1+\epsilon} \leq \sum_m \left( \int_M |f_m|^{1+\epsilon} |\nabla^2 \chi_{1+\epsilon}|^2 \right)^{\frac{1+\epsilon}{2}} \left( \int_M |f_m|^{1+\epsilon} \right)^{\frac{1-\epsilon}{2}} \tag{16}$$

Accordingly, to conclude it is enough to show that

$$\int_M \sum_m |f_m|^{1+\epsilon} |\nabla^2 \chi_{1+\epsilon}|^2 \to 0 \ \text{ as } \epsilon \to \infty$$

Inserting into Bochner formula

$$\frac{1}{2}\text{div}(\nabla |\nabla u_m|^2) = |\nabla^2 u_m|^2 + \text{Ric}_g(\nabla u_m, \nabla u_m) + \langle \nabla \Delta u_m, \nabla u_m \rangle \ \forall u_m \in C^\infty(M)$$

the function $u_m = \chi_{1+\epsilon}$, multiplying by $|f_m|^{1+\epsilon}$ and integrating over $M$ gives

$$\frac{1}{2}\int_M \sum_m |f_m|^{1+\epsilon}\operatorname{div}(\nabla|\nabla\chi_{1+\epsilon}|^2)$$
$$= \int_M \sum_m |f_m|^{1+\epsilon}|\nabla^2\chi_{1+\epsilon}|^2 + \int_M \sum_m |f_m|^{1+\epsilon}\operatorname{Ric}_g(\nabla\chi_{1+\epsilon},\nabla\chi_{1+\epsilon})$$
$$+ \int_M \sum_m |f_m|^{1+\epsilon}\langle\nabla\Delta\chi_{1+\epsilon},\nabla\chi_{1+\epsilon}\rangle.$$

Applying Stokes' theorem to the first and the last integral, we get

$$\int_M \sum_m |f_m|^{1+\epsilon}|\nabla^2\chi_{1+\epsilon}|^2 = -\frac{1}{2}\int_M \sum_m \langle\nabla(|f_m|^{1+\epsilon}),\nabla|\nabla\chi_{1+\epsilon}|^2\rangle - \int_M \sum_m |f_m|^{1+\epsilon}\operatorname{Ric}_g(\nabla\chi_{1+\epsilon},\nabla\chi_{1+\epsilon})$$
$$+ \int_M \sum_m |f_m|^{1+\epsilon}|\Delta\chi_{1+\epsilon}|^2 + \int_M \sum_m \Delta\chi_{1+\epsilon}\langle\nabla(|f_m|^{1+\epsilon}),\nabla\chi_{1+\epsilon}\rangle. \tag{17}$$

First, note that

$$\int_M \sum_m \Delta\chi_{1+\epsilon}\langle\nabla(|f_m|^{1+\epsilon}),\nabla\chi_{1+\epsilon}\rangle \le C\int_{M\setminus B_{1+\epsilon}(o)} \sum_m |\nabla(|f_m|^{1+\epsilon})| \le C_{1+\epsilon}\sum_m \left(\left|\int_{M\setminus B_{1+\epsilon}(o)} |\nabla|f_m\right|^{1+\epsilon}\right)^{1/1+\epsilon}\left(\int_{M\setminus B_{1+\epsilon}(o)} |f_m|^{1+\epsilon}\right)^{(\epsilon)/1+\epsilon} \tag{18}$$

Similarly

$$-\int_M \sum_m |f_m|^{1+\epsilon}\operatorname{Ric}_g(\nabla\chi_{1+\epsilon},\nabla\chi_{1+\epsilon}) \le \int_{M\setminus B_{1+\epsilon}(o)} \sum_m C\lambda^{-2}(1+2\epsilon)\lambda^2(1+2\epsilon)|f_m|^{1+\epsilon} \le C\int_{M\setminus B_{1+\epsilon}(o)} \sum_m |f_m|^{1+\epsilon}$$

and

$$\int_M \sum_m |f_m|^{1+\epsilon}|\Delta\chi_{1+\epsilon}|^2 \le C\int_{M\setminus B_{1+\epsilon}(o)} \sum_m |f_m|^{1+\epsilon}$$

In particular,

$$-\int_M \sum_m |f_m|^{1+\epsilon}\operatorname{Ric}_g(\nabla\chi_{1+\epsilon},\nabla\chi_{1+\epsilon}) + \int_M \sum_m |f_m|^{1+\epsilon}|\Delta\chi_{1+\epsilon}|^2 + \int_M \sum_m \Delta\chi_{1+\epsilon}\langle\nabla(|f_m|^{1+\epsilon}),\nabla\chi_{1+\epsilon}\rangle \to 0 \tag{19}$$

as $\epsilon \to \infty$ for $f_m \in W^{1,1+\epsilon}(M)$. Inserting (19) in (17) we deduce that, in order to prove that

$$\int_M \sum_m |f_m|^{1+\epsilon}|\nabla^2\chi_{1+\epsilon}|^2 \to 0$$

as $\epsilon \to \infty$, it is enough to show that

$$\limsup_{\epsilon\to\infty} -\frac{1}{2}\int_M \sum_m \langle\nabla(|f_m|^{1+\epsilon}),\nabla|\nabla\chi_{1+\epsilon}|^2\rangle - c\int_M \sum_m |f_m|^{1+\epsilon}|\nabla^2\chi_{1+\epsilon}|^2 \le 0 \tag{20}$$

for some $c < 1$ independent of $(1+\epsilon)$. We first suppose that $0 < \epsilon < 1$. By Kato and Young's inequalities we have that

$$-\frac{1}{2}\int_M \sum_m \langle\nabla(|f_m|^{1+\epsilon}),\nabla|\nabla\chi_{1+\epsilon}|^2\rangle \le 2\int_M \sum_m |f_m|^{\frac{1+\epsilon}{2}} \cdot |\nabla f_m|^{\frac{1+\epsilon}{2}}| \cdot |\nabla\chi_{1+\epsilon}| \cdot |\nabla|\nabla\chi_{1+\epsilon}|$$
$$\le \frac{1}{2}\int_M \sum_m |f_m|^{1+\epsilon}|\nabla|\nabla\chi_{1+\epsilon}||^2 + 4\int_{M\setminus B_{1+\epsilon}(o)} \sum_m |\nabla|f_m|^{\frac{1+\epsilon}{2}}|^2 \cdot |\nabla\chi_{1+\epsilon}|^2 \tag{21}$$
$$\le \frac{1}{2}\int_M \sum_m |f_m|^{1+\epsilon}|\nabla^2\chi_{1+\epsilon}|^2 + 4\int_{M\setminus B_{1+\epsilon}(o)} \sum_m |\nabla|f_m|^{\frac{1+\epsilon}{2}}|^2$$

where the last integral is finite and goes to 0 as $\epsilon \to \infty$ due to Lemma 2.1. Hence, (20) holds with $c = 1/2$. In order to deal with the case $\epsilon = 0$, we prove that the first addendum in (20) vanishes as $\epsilon \to \infty$, so (20) holds for every $c > 0$. First, observe that, for each $(1+\epsilon)$,

$$\int_M \sum_m \langle\nabla(|f_m|^{1+\epsilon}),\nabla|\nabla\chi_{1+\epsilon}|^2\rangle = \lim_{\varepsilon\to 0}\Big|^2 \int_M \sum_m \left\langle\nabla\left(\left(f_m^2+\varepsilon\right)^{1/2}\right),\nabla\middle|\nabla\chi_{1+\epsilon}\right\rangle$$

Indeed,

$$|\langle\nabla((f_m^2+\varepsilon)^{1/2}),\nabla|\nabla\chi_{1+\epsilon}|^2\rangle| \le |\nabla|\nabla\chi_{1+\epsilon}|^2\frac{|f_m||\nabla f_m|}{(f_m^2+\varepsilon)^{1/2}} \le ||\nabla|\nabla\chi_{1+\epsilon}|^2|\nabla f_m| \tag{22}$$

so that Lebesgue's dominated convergence theorem applies. Next, for every $g \in C_c^1(M)$

$$\begin{aligned} -\int_M \sum_m \langle \nabla((f_m^2+\varepsilon)^{1/2}), \nabla g\rangle &= \int_M \sum_m \Delta((f_m^2+\varepsilon)^{1/2}) g \\ &= \frac{1}{2}\int_M \sum_m \frac{\Delta(f_m^2+\varepsilon)}{(f_m^2+\varepsilon)^{1/2}} g - \frac{1}{4}\int_M \sum_m \frac{|\nabla(f_m^2+\varepsilon)|^2}{(f_m^2+\varepsilon)^{3/2}} g \\ &= \int_M \sum_m \frac{f_m \Delta f_m}{(f_m^2+\varepsilon)^{1/2}} g + \int_M \sum_m \frac{\varepsilon |\nabla f_m|^2}{(f_m^2+\varepsilon)^{3/2}} g \end{aligned} \tag{23}$$

and rearranging, we get

$$\begin{aligned} \int_M \sum_m \frac{\varepsilon |\nabla f_m|^2}{(f_m^2+\varepsilon)^{3/2}} g &= -\int_M \sum_m \langle \nabla((f_m^2+\varepsilon)^{1/2}), \nabla g\rangle - \int_M \sum_m \frac{f_m \Delta f_m}{(f_m^2+\varepsilon)^{1/2}} g \\ &\le \int_M \sum_m |\nabla f_m||\nabla g| + \int_M \sum_m |\Delta f_m||g|. \end{aligned} \tag{24}$$

Since $|\nabla f_m| \in L^1(M)$, applying (24) with $g = \chi_{1+\epsilon}$ and letting $\epsilon \to \infty$, we get

$$\int_M \sum_m \frac{\varepsilon |\nabla f_m|^2}{(f_m^2+\varepsilon)^{3/2}} \le \int_M \sum_m |\Delta f_m|. \tag{25}$$

On the other hand, applying (23) with $g = |\nabla \chi_{1+\epsilon}|^2$ and using (25) we infer

$$\begin{aligned} -\int_M \sum_m \langle \nabla |f_m|, \nabla |\nabla \chi_{1+\epsilon}|^2 \rangle &= -\lim_{\varepsilon\to 0} \int_M \sum_m \langle \nabla((f_m^2+\varepsilon)^{1/2}), \nabla |\nabla \chi_{1+\epsilon}|^2 \rangle \\ &= \lim_{\varepsilon\to 0} \sum_m \left[ \int_M \frac{f_m \Delta f_m}{(f_m^2+\varepsilon)^{1/2}} |\nabla \chi_{1+\epsilon}|^2 + \int_M \frac{\varepsilon |\nabla f_m|^2}{(f_m^2+\varepsilon)^{3/2}} |\nabla \chi_{1+\epsilon}|^2 \right] \\ &\le 2\|\nabla \chi_{1+\epsilon}\|_\infty^2 \int_M \sum_m |\Delta f_m|, \end{aligned}$$

which vanishes as $\epsilon \to \infty$ because of the properties of $\nabla \chi_{1+\epsilon}$, as claimed.

**2.3. Density for higher orders.** We shall first recall a few facts about Sampson's Weitzenböck formula for symmetric tensors. Given a $(2+\epsilon)$-dimensional Riemannian manifold $(M, g)$, consider a tensor bundle $E \to M$ with $(2+\epsilon)$-dimensional fibers endowed with an inner product induced by the metric $g$ and a compatible connection $\nabla$ induced by the Levi-Civita connection on $M$. A Lichnerowicz Laplacian $\Delta_L$ for $E$ is a second order differential operator acting on the space of smooth sections $\Gamma(E)$ of the form

$$\Delta_L = \Delta_B + c\mathfrak{Ric},$$

for $c$ a suitable constant. Here $\Delta_B \doteq -\mathrm{tr}_{12}(\nabla^2) = \nabla^*\nabla$ is the Bochner Laplacian (with $\nabla^*$ denoting the formal $L^2$-adjoint of $\nabla$ ) and $\mathfrak{Ric}$ is a smooth symmetric endomorphism of $\Gamma(E)$ which is called the Weitzenböck curvature operator. As an example, note that when $T$ is a $(0,2+\epsilon)$-tensor then

$$\mathfrak{Ric}(T)(X_1, \dots, X_{2+\epsilon}) = \sum_{i=1}^{2+\epsilon} \sum_j \left(R(E_j, X_i)T\right)(X_1, \dots, E_j, \dots, X_{2+\epsilon}),$$

with $\{E_i\}$ a local orthonormal frame and

$$R(X,Y) \doteq \nabla^2_{X,Y} - \nabla^2_{Y,X} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},$$

which may act on any tensor field. It is important to notice that the Weitzenböck curvature term for $(0,2+\epsilon)$-tensors can actually be estimated in terms of the curvature operator $\mathcal{R}$ of $M$. Indeed, if $\mathcal{R} \ge \alpha$, for some constant $\alpha < 0$ then $g(\mathfrak{Ric}(T), T) \ge \alpha C |T|^2$, where $C$ depends only on $(2+\epsilon)$; see [32, Corollary 9.3.4]. This key feature of Lichnerowicz Laplacians permits to use geometric assumptions in estimation results.

As in [32, Chapter 9] there are several natural Lichnerowicz Laplacians on Riemannian manifolds. A very classical one is the Hodge-Laplacian $\Delta_H$ acting on exterior differential forms, for which the Weitzenböck identity takes the form

$$\Delta_H \omega \doteq (d\delta + \delta d)\omega = \Delta_B \omega + \mathfrak{Ric}(\omega). \tag{26}$$

The Bochner identity which we used in the previous section precisely comes out from this formula evaluated on the skew-symmetric 1 -form $d\chi_{1+\epsilon}$. When considering the higher order case $\epsilon > 0$, one may be tempted to use (26) applied to the ( $1+\epsilon$ )-th covariant derivative of suitable cut-off functions. Unfortunately, this latter is not at all skew-symmetric. However, it is at least almost symmetric, meaning that it can be decomposed as a totally symmetric principal part plus other terms involving derivatives of order at most $(\epsilon)$. This fact led [26] to consider a different Lichnerowicz Laplacian, acting on totally symmetric covariant tensors of any order, which was originally introduced by [35] and that we now recall. Let $T^{(0,2+\epsilon)}(M)$ and $S^{(0,2+\epsilon)}(M)$ be, respectively, the bundle of $(2+\epsilon)$-covariant tensor and its subbundle of totally symmetric ones. Consider the operator $D_S\colon \Gamma S^{(0,1+\epsilon)}(M) \to \Gamma S^{(0,2+\epsilon)}(M)$ acting on $h \in \Gamma S^{(0,1+\epsilon)}(M)$ by

$$(D_S h)(X_0, \dots, X_{1+\epsilon}) \doteq (2+\epsilon) s_{2+\epsilon}(\nabla h)(X_0, \dots, X_{1+\epsilon}),$$

where we are defining by $s_{(2+\epsilon)}$ the symmetrization operator, i.e. the projection of $T^{(0,2+\epsilon)}(M)$ onto $S^{(0,2+\epsilon)}(M)$, that we shortly denote with a superscript $S$. Namely,

$$T^S(X_1, \dots, X_{(2+\epsilon)}) \doteq s_{2+\epsilon}(T)(X_1, \dots, X_{2+\epsilon}) = \frac{1}{(2+\epsilon)!} \sum_{\sigma \in \Pi_{2+\epsilon}} T(X_{\sigma(1)}, \dots, X_{\sigma(2+\epsilon)}) \; \forall T \in T^{(0,2+\epsilon)}(M)$$

Note that

$$|T^S| \leq |T| \tag{27}$$

The formal $L^2$ - adjoint of $D_S$ is $D_S^*\colon \Gamma S^{(0,2+\epsilon)}(M) \to \Gamma S^{(0,1+\epsilon)}(M)$ which acts on $\tilde{h} \in \Gamma S^{(0,2+\epsilon)}(M)$ by

$$(D_S^* \tilde{h})(X_1, \dots, X_{1+\epsilon}) = -\sum_i (\nabla_{E_i} \tilde{h})(E_i, X_1, \dots, X_{1+\epsilon}).$$

We can now define the second order differential operator $\Delta_{\mathrm{Sym}}$ acting on $\Gamma S^{(0,2+\epsilon)}(M)$ via the following Hodge-type decomposition

$$\Delta_{\mathrm{Sym}} \doteq D_S^* D_S - D_S D_S^*$$

By [35] (see also [26, Appendix B] for a proof) we have that

$$\Delta_{\mathrm{Sym}} = \Delta_B - \mathfrak{Ric}, \tag{28}$$

i.e. $\Delta_{\mathrm{Sym}}$ is a Lichnerowicz Laplacian (with the choice $c = -1$ ).

Exploiting (28) we readily deduce the validity of the differential identity

$$\frac{1}{2}\Delta|T^S|^2 = -\langle \Delta_{\mathrm{Sym}} T^S, T^S \rangle - \langle \mathfrak{Ric}(T^S), T^S \rangle + |\nabla T^S|^2, \forall T \in \Gamma T^{(0,2+\epsilon)}(M) \tag{29}$$

**Remark 2.3.** Notice that a totally symmetric 1-tensor $\omega$ is also a skew-symmetric one-form. In this case, $\Delta_{\mathrm{Sym}}\, \omega = 2\Delta_B \omega - \Delta_H \omega$, so that (26) and (28) are equivalent for 1-tensors, as it has to be. However, when deducing the Bochner-type formula, the Weitzenböck curvature term appears with a different sign.

We proof our density result (see [37]).

**Proof of Theorem 1.8.** In our assumptions, we know by [26, Corollary 5.2] that there exists a sequence of cut-off functions $\{\chi_{2+\epsilon}\} \subset C_c^\infty(M)$, and a constant $C > 0$ independent of $(2+\epsilon)$ such that,

$$\begin{aligned} &\chi_{2+\epsilon} = 1 \;\text{ on }\; B_{R_{2+\epsilon}}(o), \;\text{ with }\; R_{2+\epsilon} \doteq C_H^{-1}(\epsilon) \\ &|\Delta \nabla^\epsilon \chi_{2+\epsilon}| \leq C. \end{aligned} \tag{30}$$

These cut-off functions were called in [26] $(2+\epsilon)$-th order rough Laplacian cut-offs. It is important to note that the fact that we are asking only for a control on the trace of the $(2+\epsilon)$-th covariant derivative of the cut-offs (which suffices for our scope) reflects on the weakness of the assumptions we are asking for. Indeed, we are demanding a control on the curvature up to a smaller order than usual (case $\epsilon > 0$ ).

Since smooth functions are dense in $W^{2+\epsilon,1+\epsilon}(M)$, to prove the density result it is sufficient to consider $f_m \in C^\infty(M) \cap W^{2+\epsilon,1+\epsilon}(M)$; see for instance [18]. We want to prove that $\chi_{2+\epsilon} f_m$ converges to $f_m$ in $W^{2+\epsilon,1+\epsilon}(M)$. The lower order terms

$$\int_M \sum_m |\nabla^j(\chi_{2+\epsilon} f_m) - \nabla^j f_m|^{1+\epsilon}, 0 \leq j \leq 1+\epsilon$$

are easily seen to vanish as $\epsilon \to \infty$ by using the Cauchy-Schwarz inequality, Lebesgue convergence theorem and the properties of the cut-off functions. Regarding the $(2+\epsilon)$-th order derivative, we write

$$\begin{aligned} \int_M \sum_m |\nabla^{2+\epsilon}(\chi_{2+\epsilon} f_m) - \nabla^{2+\epsilon} f_m|^{1+\epsilon} &= \int_M \sum_m \left| \left[ \sum_{i=0}^{2+\epsilon} \binom{2+\epsilon}{i} \nabla^{2+\epsilon-i} \chi_{2+\epsilon} \otimes \nabla^i f_m \right] - \nabla^{2+\epsilon} f_m \right|^{1+\epsilon} \\ &\leq C \int_M \sum_m (1-\chi_{2+\epsilon})^{1+\epsilon} |\nabla^{2+\epsilon} f_m|^{1+\epsilon} + \sum_{i=0}^{1+\epsilon} \binom{2+\epsilon}{i} \int_M \sum_m |\nabla^{2+\epsilon-i} \chi_{2+\epsilon}|^{1+\epsilon} |\nabla^i f_m|^{1+\epsilon} \end{aligned}$$

Taking into account the properties of the cut-off functions, all of the addenda vanish as $\epsilon \to \infty$ with the possible exception of the one corresponding to $i = 0$. Applying Hölder inequality as in (16) we deduce that, in order to conclude, it is enough to show that

$$\int_M \sum_m |f_m|^{1+\epsilon} |\nabla^{2+\epsilon} \chi_{2+\epsilon}|^2 \to 0$$

as $\epsilon \to \infty$. Define $h_{2+\epsilon} = \nabla^{1+\epsilon}\chi_{2+\epsilon}$ and its simmetrization $h^S_{2+\epsilon}$. Because of (29),

$$\frac{1}{2}\text{div}(|f_m|^{1+\epsilon}\nabla|h^S_{2+\epsilon}|^2) = |f_m|^{1+\epsilon}[-\langle\Delta_{\text{Sym}}h^S_{2+\epsilon}, h^S_{2+\epsilon}\rangle - \langle\mathfrak{Ric}(h^S_{2+\epsilon}), h^S_{2+\epsilon}\rangle + |\nabla h^S_{2+\epsilon}|^2] + \frac{1}{2}\langle\nabla(|f_m|^{1+\epsilon}), \nabla(|h^S_{2+\epsilon}|^2)\rangle,$$

thus integrating and using [32, Corollary 9.3.4] to control the curvature term we get

$$\begin{aligned}\int_M \sum_m \langle\Delta_{\text{Sym}}h^S_{2+\epsilon}, |f_m|^{1+\epsilon}h^S_{2+\epsilon}\rangle &\le -\int_M \sum_m |f_m|^{1+\epsilon}\langle\mathfrak{Ric}(h^S_{2+\epsilon}), h^S_{2+\epsilon}\rangle + \int_M |f_m|^{1+\epsilon}|\nabla h^S_{2+\epsilon}|^2 \\ &\quad + \frac{1}{2}\sum_m \langle\nabla(|f_m|^{1+\epsilon}), \nabla(|h^S_{2+\epsilon}|^2)\rangle \qquad (31)\\ &\le (-\alpha)C\int_M \sum_m |f_m|^{1+\epsilon}|h^S_{2+\epsilon}|^2 + \int_M \sum_m |f_m|^{1+\epsilon}|\nabla h^S_{2+\epsilon}|^2 \\ &\quad + \frac{1}{2}\sum_m \langle\nabla(|f_m|^{1+\epsilon}), \nabla(|h^S_{2+\epsilon}|^2)\rangle.\end{aligned}$$

Suppose first that $0 < \epsilon \le 1$. By Young's inequality, the regularity Lemma 2.1 and the properties of $h_{2+\epsilon}$,

$$\begin{aligned}\int_M \sum_m \langle\Delta_{\text{Sym}}h^S_{2+\epsilon}, |f_m|^{1+\epsilon}h^S_{2+\epsilon}\rangle &\le (-\alpha)C\int_{M\setminus B_{R_{2+\epsilon}}(o)} \sum_m |f_m|^{1+\epsilon}|h^S_{2+\epsilon}|^2 + \int_M \sum_m |f_m|^{1+\epsilon}|\nabla h^S_{2+\epsilon}|^2 \\ &\quad + \eta\int_M \sum_m |f_m|^{1+\epsilon}|\nabla|h^S_{2+\epsilon}||^2 + \frac{1}{\eta}\int_{M\setminus B_{R_{2+\epsilon}}(o)} \sum_m |\nabla|f_m|^{\frac{1+\epsilon}{2}}|^2, \qquad (32)\end{aligned}$$

for any $\eta > 0$. Notice also that

$$|h^S_{2+\epsilon}| \le |h_{2+\epsilon}| = |\nabla^{1+\epsilon}\chi_{2+\epsilon}| \le C\lambda^{-1}(R_{2+\epsilon}). \qquad (33)$$

By the dominated convergence theorem, the fact that $f_m \in W^{2+\epsilon,1+\epsilon}(M)$ and Lemma 2.1, the first and fourth term in the RHS of (32) vanish as $\epsilon \to \infty$, so using Kato's inequality $|\nabla|h^S_{2+\epsilon}|| \le |\nabla h^S_{2+\epsilon}|$ we obtain

$$\limsup_{\epsilon\to\infty} \sum_m \int_M \langle\Delta_{\text{Sym}}h^S_{2+\epsilon}, |f_m|^{1+\epsilon}h^S_{2+\epsilon}\rangle - (1+\eta)\int_M |f_m|^{1+\epsilon}|\nabla h^S_{2+\epsilon}|^2\Big] \le 0 \qquad (34)$$

Define

$$\mathcal{A}_{2+\epsilon} = \int_M \sum_m \langle D_S^* D_S h^S_{2+\epsilon}, |f_m|^{1+\epsilon}h^S_{2+\epsilon}\rangle, \mathcal{B}_{2+\epsilon} = \int_M \sum_m \langle D_S D_S^* h^S_{2+\epsilon}, |f_m|^{1+\epsilon}h^S_{2+\epsilon}\rangle$$

so that (34) becomes

$$\limsup_{\epsilon\to\infty}\left[\mathcal{A}_{2+\epsilon} - \mathcal{B}_{2+\epsilon} - (1+\eta)\int_M \sum_m |f_m|^{1+\epsilon}|\nabla h^S_{2+\epsilon}|^2\right] \le 0. \qquad (35)$$

By Young's inequality and using (33), for $\delta > 0$ we can estimate $\mathcal{A}_{2+\epsilon}$ as follows:

$$\begin{aligned}\mathcal{A}_{2+\epsilon} &= \int_M \sum_m \langle D_S h^S_{2+\epsilon}, D_S(|f_m|^{1+\epsilon}h^S_{2+\epsilon})\rangle \\ &= \int_M \sum_m |f_m|^{1+\epsilon}|D_S h^S_{2+\epsilon}|^2 + (2+\epsilon)\int_M \sum_m \langle D_S h^S_{2+\epsilon}, 2|f_m|^{\frac{1+\epsilon}{2}} s_{(2+\epsilon)}\left(d|f_m|^{\frac{1+\epsilon}{2}} \otimes h^S_{2+\epsilon}\right)\rangle \qquad (36)\\ &\ge (1-\delta)\int_M \sum_m |f_m|^{1+\epsilon}|D_S h^S_{2+\epsilon}|^2 - \frac{(2+\epsilon)^2}{\delta}\int_M \sum_m \left|\nabla\left(|f_m|^{\frac{1+\epsilon}{2}}\right)\right|^2 |h^S_{2+\epsilon}|^2, \\ &= (1-\delta)\int_M \sum_m |f_m|^{1+\epsilon}|D_S h^S_{2+\epsilon}|^2 + o_{2+\epsilon}(1) \ \text{ as } \epsilon \to \infty,\end{aligned}$$

where the last line follows by the Regularity Lemma and since $h^S_{2+\epsilon}$ is bounded and supported away from $B_{R_{2+\epsilon}}(o)$.

Regarding the term $\mathcal{B}_{2+\epsilon}$, Hölder inequality gives

$$\begin{aligned} \mathcal{B}_{2+\epsilon} &= \int_M \sum_m \langle D_S^* h_{2+\epsilon}^S, D_S^*(|f_m|^{1+\epsilon} h_{2+\epsilon}^S)\rangle \\ &= \int_M \sum_m \left[|f_m|^{1+\epsilon}|D_S^* h_{2+\epsilon}^S|^2 - \left\langle i_{\nabla(|f_m|^{1+\epsilon})} h_{2+\epsilon}^S, D_S^* h_{2+\epsilon}^S\right\rangle\right] \quad (37) \\ &\leq \int_M \sum_m |f_m|^{1+\epsilon}|D_S^* h_{2+\epsilon}^S|^2 + \int_M \sum_m |\nabla(|f_m|^{1+\epsilon})||D_S^* h_{2+\epsilon}^S||h_{2+\epsilon}^S| \\ &\leq \int_M \sum_m |f_m|^{1+\epsilon}|D_S^* h_{2+\epsilon}^S|^2 + (1+\epsilon)\sum_m \left(\int_M |f_m|^{1+\epsilon}|D_S^* h_{2+\epsilon}^S|^{\frac{1+\epsilon}{\epsilon}}\right)^{\frac{\epsilon}{1+\epsilon}} \left(\int_M |\nabla|f_m||^{1+\epsilon}|h_{2+\epsilon}^S|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \end{aligned}$$

By the Ricci identities, a computation (see [26, pp. 31]) shows that

$$\begin{aligned} |D_S^* h_{2+\epsilon}^S|^2 &= |D_S^*(\nabla^{1+\epsilon}\chi_{2+\epsilon})^S|^2 \\ &\leq C\left(|\Delta\nabla^\epsilon \chi_{2+\epsilon}|^2 + |\mathrm{Riem}_g|^2|\nabla^\epsilon\chi_{2+\epsilon}|^2 + \cdots + |\nabla^{\epsilon-1}\mathrm{Riem}_g|^2|\nabla\chi_{2+\epsilon}|^2\right) \\ &\leq \begin{cases} C' & \text{on } M \setminus B_{R_{2+\epsilon}}(o) \text{ by our decay assumptions on Riem and by (30).} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence, (37) and $f_m \in W^{2+\epsilon,1+\epsilon}(M)$ imply that $\limsup_{\epsilon\to\infty} \mathcal{B}_{2+\epsilon} \leq 0$. Note that these estimates for $\mathcal{B}_{2+\epsilon}$ also hold for $\epsilon = 0$, and indeed the fourth line of (37) is unnecessary in such case.

Inserting (36) and (37) into (35) gives

$$\limsup_{\epsilon\to\infty} \int_M \sum_m |f_m|^{1+\epsilon}[(1-\delta)|D_S h_{2+\epsilon}^S|^2 - (1+\eta)|\nabla h_{2+\epsilon}^S|^2] \leq 0 \quad (38)$$

Moreover, by the same reasoning as above and by Young's inequality (see [26, pp.32-33]),

$$\begin{aligned} |\nabla h_{2+\epsilon}^S|^2 &= \left|\frac{1}{(1+\epsilon)!}\nabla s_{1+\epsilon}(\nabla^{1+\epsilon}\chi_{2+\epsilon})\right|^2 \\ &\leq (1+\varepsilon C_{1,2+\epsilon})|\nabla h_{2+\epsilon}|^2 + \frac{C_{1,2+\epsilon}}{\varepsilon}\left(|\mathrm{Riem}_g|^2|\nabla^\epsilon\chi_{2+\epsilon}|^2 + \cdots + |\nabla^{\epsilon-1}\mathrm{Riem}_g|^2|\nabla\chi_{2+\epsilon}|^2\right) \quad (39) \\ |D_S h_{2+\epsilon}^S|^2 &= (2+\epsilon)^2|s_{2+\epsilon}(\nabla s_{1+\epsilon}(\nabla^{1+\epsilon}\chi_{2+\epsilon}))|^2 = (2+\epsilon)^2|s_{2+\epsilon}(\nabla^{2+\epsilon}\chi_{2+\epsilon})|^2 \\ &\geq ((2+\epsilon)^2 - \varepsilon C_{2,2+\epsilon})|\nabla h_{2+\epsilon}|^2 - \frac{C_{2,2+\epsilon}}{\varepsilon}\left(|\mathrm{Riem}_g|^2|\nabla^\epsilon\chi_{2+\epsilon}|^2 + \cdots + |\nabla^{\epsilon-1}\mathrm{Riem}_g|^2|\nabla\chi_{2+\epsilon}|^2\right), \quad (40) \end{aligned}$$

for any $\varepsilon > 0$, and some constants $C_{1,2+\epsilon}, C_{2,2+\epsilon}$. Using (39), (40), the decay assumptions on Riem and $f_m \in L^{1+\epsilon}(M)$, we get that

$$\limsup_{\epsilon\to\infty} \int_M \sum_m |f_m|^{1+\epsilon}[(1-\delta)((2+\epsilon)^2 - \varepsilon C_{2,2+\epsilon}) - (1+\eta)(1+\varepsilon C_{1,2+\epsilon})]|\nabla h_{2+\epsilon}|^2 \leq 0$$

Hence, we can choose $\delta, \eta, \varepsilon$ small enough such that $(1-\delta)((2+\epsilon)^2 - \varepsilon C_{2,2+\epsilon}) - (1+\eta)(1+\varepsilon C_{1,2+\epsilon}) > 0$, which leads to

$$\int_M \sum_m |f_m|^{1+\epsilon}|\nabla h_{2+\epsilon}|^2 \to 0 \text{ as } \epsilon \to \infty,$$

thus concluding the proof for $0 < \epsilon \leq 1$.

We suppose now that $\epsilon = 0$. We first note that

$$\lim_{\epsilon\to\infty} -\frac{1}{2}\int_M \sum_m \langle\nabla|f_m|, \nabla(|h_{2+\epsilon}^S|^2)\rangle = 0. \quad (41)$$

Indeed, by Lebesgue convergence theorem,

$$-\frac{1}{2}\int_M \sum_m \langle\nabla|f_m|, \nabla(|h_{2+\epsilon}^S|^2)\rangle = \lim_{\varepsilon\to 0} -\frac{1}{2}\int_M \sum_m \langle\nabla((f_m^2+\varepsilon)^{1/2}), \nabla(|h_{2+\epsilon}^S|^2)\rangle$$

So performing the same computations as in (22), (23) and (24) we obtain

$$-\frac{1}{2}\int_M \sum_m \langle\nabla((f_m^2+\varepsilon)^{1/2}), \nabla(|h_{2+\epsilon}^S|^2)\rangle \leq \|h_{2+\epsilon}^S\|_\infty^2 \int_M \sum_m |\Delta f_m|.$$

Since the RHS above vanishes as $\epsilon \to \infty$ because of (33), this proves the claimed identity (41). From (31) we therefore deduce

$$\limsup_{\epsilon\to\infty}\left[\mathcal{A}_{2+\epsilon} - \mathcal{B}_{2+\epsilon} - \int_M \sum_m |f_m||\nabla h_{2+\epsilon}^S|^2\right] \leq 0. \quad (42)$$

As the estimate for $\mathcal{B}_{2+\epsilon}$ holds also for $\epsilon = 0$, we only have to deal with $\mathcal{A}_{2+\epsilon}$ :

$$\mathcal{A}_{2+\epsilon} = \int_M \sum_m \langle D_S h^S_{2+\epsilon}, D_S(|f_m| h^S_{2+\epsilon})\rangle$$
$$= \int_M \sum_m |f_m||D_S h^S_{2+\epsilon}|^2 + (2+\epsilon)\int_M \sum_m \langle D_S h^S_{2+\epsilon}, s_{2+\epsilon}(d|f_m| \otimes h^S_{2+\epsilon})\rangle.$$

By Lebesgue convergence theorem,

$$\int_M \sum_m \langle D_S h^S_{2+\epsilon}, s_{(2+\epsilon)}(d|f_m| \otimes h^S_{2+\epsilon})\rangle = \lim_{\varepsilon\to 0}\int_M \sum_m \left\langle D_S h^S_{2+\epsilon}, s_{(2+\epsilon)}\left(d\left(\sqrt{f_m^2+\varepsilon}\right)\otimes h^S_{2+\epsilon}\right)\right\rangle$$

hence we compute

$$\left|\int_M \sum_m \langle D_S h^S_{2+\epsilon}, s_{(2+\epsilon)}(d|f_m| \otimes h^S_{2+\epsilon})\rangle\right| = \left|\lim_{\varepsilon\to 0}\int_M \sum_m \left\langle h^S_{2+\epsilon}, D_S^*\left(s_{(2+\epsilon)}\left(d\left(\sqrt{f_m^2+\varepsilon}\right)\otimes h^S_{2+\epsilon}\right)\right)\right\rangle\right|$$

$$\leq 3\|h^S_{2+\epsilon}\|^2_\infty \int_M \sum_m |\Delta f_m|,$$

$$= \lim_{\varepsilon\to 0}\int_M \sum_m \frac{f_m \Delta f_m}{\sqrt{f_m^2+\varepsilon}}|h^S_{2+\epsilon}|^2 + \frac{f_m}{2\sqrt{f_m^2+\varepsilon}}\langle \nabla f_m, \nabla |h^S_{2+\epsilon}|^2 + \frac{\varepsilon}{(f_m^2+\varepsilon)^{\frac{3}{2}}}|\nabla f_m|h^S_{2+\epsilon}|^2\right]$$

where for the last inequality we reasoned again as in (22), (23) and (24). Summarizing,

$$\mathcal{A}_{2+\epsilon} = \int_M \sum_m |f_m||D_S h^S_{2+\epsilon}|^2 + o_{2+\epsilon}(1) \text{ as } \epsilon \to \infty,$$

and the proof can be concluded as in the case $\epsilon > 0$.

**3. Non-Density when $\epsilon > 0$ : A Counterexample With Curvature Sec $\geq -1$**

We construct a suitable complete, convex hypersurface $(M, g_0) \hookrightarrow \mathbb{H}^{3+\epsilon}$ of finite volume and with two ends. We consider cartesian coordinates $(\mathbf{x}, z) = (x_1, \ldots, x_{2+\epsilon}, z)$ on $\mathbb{R}^{3+\epsilon}$. Let $\mathbb{B}_1 = \{|\mathbf{x}|^2 + z^2 < 1\}$ be the unit ball centered at the origin. Let $h$ be the hyperbolic metric on $\mathbb{B}_1$ induced by the Beltrami-Klein projective model, i.e.

$$h = \frac{\|d\mathbf{y}\|^2}{1-\|\mathbf{y}\|^2} + \frac{(\mathbf{y}\cdot d\mathbf{y})^2}{(1-\|\mathbf{y}\|^2)^2},$$

where $\mathbf{y} \in \mathbb{B}_1$ and $\|\cdot\|$ is the standard Euclidean norm of $\mathbb{R}^{3+\epsilon}$. Define the noncompact hypersurface $M$ by

$$M = \left\{|\mathbf{x}| = -\sqrt{3} + \sqrt{4-z^2} : z \in (-1,1)\right\} \subset \mathbb{B}_1$$

and let $g_0$ be the metric on $M$ induced by $h$. Note that $M$ is the boundary of a domain which is strictly convex in $\mathbb{R}^{3+\epsilon}$, hence also in $(\mathbb{B}_1, h)$ since the Beltrami-Klein model is projective. Thus $\mathrm{Sec}_{g_0} > -1$ by Gauss equations. So, $M$ is invariant by reflection with respect to the plane $z = 0$, and $M \cap \{z \geq 0\}$ can be written as the graph of the strictly concave function

$$f_m : D \doteq \overline{B^{\mathbb{R}^{2+\epsilon}}_{2-\sqrt{3}}(0)} \setminus \{0\} \to [0,\infty), f_m(\mathbf{x}) = \sqrt{1-|\mathbf{x}|^2 - 2\sqrt{3}|\mathbf{x}|} \quad (43)$$

where $B^{\mathbb{R}^{2+\epsilon}}_{2-\sqrt{3}}(0)$ is the Euclidean ball of radius $2-\sqrt{3}$ in $\{z = 0\}$. We will shortly say that $M$ is the bigraph of $f_m$. Denote with $\tilde{f_m} = \mathrm{id} \times f_m$ the graph map. Note that $(M, g_0)$ lies in the interior region of the double cone

$$K = \left\{|\mathbf{x}| = \frac{1-|z|}{\sqrt{3}} : z \in (-1,1)\right\} \overset{i}{\hookrightarrow} (\mathbb{B}_1, h), \quad (44)$$

and that $K$ has finite volume. This can be easily proved by a direct computation, for instance by noticing that each of the two cones forming $K$ is isometric to the half cylinder $\{|\mathbf{x}| = 1, z > 1\}$ in the Poincaré half-space model. Since the orthogonal projection on a convex set of $\mathbb{H}^{3+\epsilon}$ is distance decreasing by the hyperbolic Buseman-Feller theorem [6, II.2.4], we deduce that ( $M, g_0$ ) has finite volume. We fix

$$\mathbb{V} \Subset \text{ bigraph of } f_m \text{ over } \overline{B^{\mathbb{R}^{2+\epsilon}}_{2-\sqrt{3}}(0)} \setminus B^{\mathbb{R}^{2+\epsilon}}_{\frac{1}{8}}(0)$$

whose closure is diffeomorphic to a closed ball (in particular, $\mathbb{V}$ does not disconnect $M$ ) and we define

$$U_0 \doteq \emptyset, U_j \doteq \text{ bigraph of } f_m \text{ over } B^{\mathbb{R}^{2+\epsilon}}_{2-\sqrt{3}}(0) \setminus \overline{B^{\mathbb{R}^{2+\epsilon}}_{\frac{1}{j+8}}(0)} \text{ for } j \geq 1\ldots$$

Roughly speaking, $M$ looks like an American football in vertical position with respect to $\{z = 0\}$, and $U_j$ corresponds to the open set obtained by removing an upper and a lower cap centered at the two vertices.

We begin by constructing, for fixed $j$, a sequence of smooth metrics $\{\sigma_{j,2+\epsilon}\}_{\epsilon=-2}^{\infty}$ on $M$ having $(2+\epsilon)$ "approximated spikes" in $U_j \setminus U_{j-1}$ and converging, as $\epsilon \to \infty$, to an Alexandrov metric that has a dense set of sharp points on $U_j \setminus U_{j-1}$. We recall the notion of sharp singular point, and some basic facts of Alexandrov (more generally, RCD) spaces that will be useful later on. The theory of metric measure spaces $(X, \mathrm{d}, \mathrm{m})$ ( m a Radon measure on $X$ ) that lie in $\mathrm{RCD}(K, 2+\epsilon)$ hugely developed, with a detailed set in [1]. Here, we just point out that $\mathrm{RCD}(K, 2+\epsilon)$ contains all Alexandrov spaces with dimension $(2+\epsilon)$ and curvature bounded from below by $K/(1+\epsilon)$, with m the $(2+\epsilon)$-dimensional Hausdorff measure, as well as the pointed measured Gromov-Hausdorff (mGH) limits of smooth manifolds ( $M_i, g_i, o_i$ ) with $\mathrm{Ric}_g \geq K$, endowed with their Riemannian measure $\mathrm{m}_i$ and reference points $o_i$. For $X \in \mathrm{RCD}(K, 2+\epsilon)$, the Sobolev spaces $W^{1,1+\epsilon}(X)$ can be defined for $0 < \epsilon < \infty$, and $W^{1,2}(X)$ is Hilbert. Given $(X, \mathrm{d}, \mathrm{m}) \in \mathrm{RCD}(K, 2+\epsilon)$ and $x_0 \in X$, the density

$$\vartheta(x_0) \doteq \lim_{\epsilon \to 0} \frac{\mathrm{m}(B_{1+2\epsilon}(x_0))}{(1+2\epsilon)^{2+\epsilon}} \in (0, \infty]$$

does exist. A tangent cone at $x_0$ is, by definition, the mGH limit of some sequence of rescalings

$$\left(X, \frac{\mathrm{d}}{\lambda_i}, \frac{\mathrm{m}}{\lambda_i^{2+\epsilon}}, x_0\right) \text{ where } \lambda_i \to 0^+,$$

and the set of tangent cones is closed under mGH convergence pointed at $x_0$. Under the non-collapsing condition $\vartheta(x_0) < \infty$, every tangent cone at $x_0$ is a metric cone $C(Z)$ over a cross section $Z \in \mathrm{RCD}(1+\epsilon, 2+\epsilon)$ with diameter $\leq \pi$, that is, it can be written as $[0, \infty) \times Z$ with distance

$$\mathrm{d}_{C(Z)}((t,x),(s,y)) = \sqrt{t^2 + s^2 - 2ts\cos(\mathrm{d}_Z(x,y))}.$$

The section is unique for Alexandrov spaces, but this may not be the case in general. Following [15], we say that $x_0 \in X$ is sharp if $\vartheta(x_0) < \infty$ and the cross section of any tangent cone at $x_0$ has diameter $< \pi$.

**3.1. Construction of the spike metrics $\boldsymbol{\sigma_{j,2+\epsilon}}$.** It is well-known that there exist manifolds ( $M, g_{2+\epsilon}$ ) with $\mathrm{Sec}_{g_{2+\epsilon}} \geq 0$ that converge to an Alexandrov space having a dense set of sharp singular points, [31]. In the next Lemma we will need to localize such a construction, namely, to approximate the singular points in $U_j \setminus U_{j-1}$ without modifying the metric $g_j$ outside. We adapt the construction introduced in [30] to a hyperbolic background. As we shall need more information on the sequence of approximating metrics, the proof of the next result will be done in full detail.

**Lemma 3.1 (see [37]).** For $j \geq 1$, there exists a sequence of smooth metrics $\{\sigma_{j,2+\epsilon}\}_{(2+\epsilon)\in\mathbb{N}}$ on $M$ such that

$$\sigma_{j,2+\epsilon} = g_0 \text{ outside of a compact subset of } U_j \setminus U_{j-1} \text{ (depending on } 2+\epsilon\text{)}, \tag{45}$$

$$\mathrm{Sec}_{\sigma_{j,2+\epsilon}} > -1 \text{ on } M \tag{46}$$

$$\forall (2+\epsilon): \mathbb{N}_{>0} \to \mathbb{N}, \forall S \subset M \text{ Borel}, \sum_{j=1}^{\infty} \mathrm{vol}_{\sigma_{j,k(j)}}\left(S \cap (U_j \setminus U_{j-1})\right) \leq \mathrm{vol}_{i^*h}(K) < \infty \tag{47}$$

$$\exists C_j > 1 \text{ such that } C_j^{-1}\, \mathrm{d}_{g_0}(x,y) \leq \mathrm{d}_{\sigma_{j,2+\epsilon}}(x,y) \leq C_j\, \mathrm{d}_{g_0}(x,y)\ \forall (2+\epsilon) \in \mathbb{N} \cup \{0\}, x, y \in M. \tag{48}$$

Moreover, $(M, \sigma_{j,2+\epsilon}, o) \to M_{j,\infty} \doteq (M, \mathrm{d}_{j,\infty}, o)$ as $\epsilon \to \infty$ in the Gromov-Hausdorff sense, for some $(2+\epsilon)$ dimensional Alexandrov space $M_{j,\infty}$ biLipschitz homeomorphic to $M$, with curvature greater than or equal to -1 , volume $\mathcal{H}^{2+\epsilon}(M_{j,\infty}) \leq \mathrm{vol}_{i^*h}(K) < \infty$, and a dense set of sharp singular points in $U_j \setminus U_{j-1}$.

**Proof.** Define

$$D_0 = \emptyset, D_j = \overline{B_{2-\sqrt{3}}^{\mathbb{R}^{2+\epsilon}}(0)} \setminus \overline{B_{\frac{1}{j+8}}^{\mathbb{R}^{2+\epsilon}}(0)},$$

so $U_j = \tilde{f}_m(D_j) \cup (\widetilde{-f_m})(D_j)$ is the bigraph of $f_m$ over $D_j$. Note that $f_m$ satisfies $f_m(\mathbf{x}) < 1 - \sqrt{3}|\mathbf{x}|$, since the graph of this latter function coincides with $K$ on $\{z \geq 0\}$. Let $\{\mathbf{y}_{m_0}\} \in D_j \setminus \overline{D_{j-1}}$ be a dense sequence. We claim that

there exists a sequence of smooth strictly concave functions $(f_m)_{j,2+\epsilon}: D \to \mathbb{R}, \epsilon \geq 0$, such that

(i) $f_m(\mathbf{x}) \leq (f_m)_{j,1+\epsilon}(\mathbf{x}) < 1 - \sqrt{3}|\mathbf{x}|$ on $D$;

(ii) $(f_m)_{j,1+\epsilon}$ converges uniformly, as $\epsilon \to \infty$, to a concave function $(f_m)_{j,\infty}$, and the graph of $(f_m)_{j,\infty}$ has sharp conical singularities at any $(f_m)_{j,\infty}(\mathbf{y}_{m_0})$;

(iii) $\{\mathbf{x}: (f_m)_{j,1+\epsilon}(\mathbf{x}) \neq f_m(\mathbf{x})\}$ is compactly contained in $D_j \setminus \overline{D_{j-1}}$.

Given the claim, let $(M_{j,1+\epsilon}, \sigma_{j,1+\epsilon})$ be the bigraph of $(f_m)_{j,1+\epsilon}$ with the induced metric. Property (ii) implies the Hausdorff convergence of $M_{j,1+\epsilon}$ to the bigraph $M_{j,\infty}$ of $(f_m)_{j,\infty}$ with the induced intrinsic metric $\mathrm{d}_{j,\infty}$ and it is known that the concavity of $(f_m)_{j,1+\epsilon}$ guarantees the pointed Gromov-Hausdorff convergence $(M_{j,1+\epsilon}, \mathrm{d}_{j,1+\epsilon}, o_{1+\epsilon}) \to (M_{j,\infty}, \mathrm{d}_{j,\infty}, o_\infty)$, with $o_{1+\epsilon}$ being the image of any fixed point in $D_1$. Using again the concavity of $(f_m)_{j,1+\epsilon}$, Gauss' equation implies that $M_{j,1+\epsilon}$ has sectional curvature bounded from below by -1 , and ( $M$, $\mathrm{d}_{j,\infty}$ ) is an Alexandrov space of curvature lower bounded by -1 by Buyalo's theorem, [7]. Next, for $0 \le \epsilon \le \infty$, identify $M$ with $M_{j,1+\epsilon}$ topologically via the map $(\tilde{f_m})_{j,1+\epsilon} \circ \tilde{f}_m^{-1}$, and still denote with $\sigma_{j,1+\epsilon}$ the pulled-back metric on $M$. Note that $\{g_{j,1+\epsilon} \neq g_0\}$ is compactly contained in $U_j \setminus \overline{U_{j-1}}$. The uniform convergence together with the concavity of $(f_m)_{j,1+\epsilon}$ on $D$ guarantee that $\{(f_m)_{j,1+\epsilon}\}_{1+\epsilon}$ are uniformly Lipschitz on $D_j$, hence on the entire $D$ by (iii). In particular, up to identifying the manifolds by means of $(\tilde{f_m})_{j,1+\epsilon} \circ \tilde{f}_m^{-1}$, (48) holds. To conclude, for a given $(1+\epsilon): \mathbb{N}_{>0} \to \mathbb{N}$ we consider the concave function $(f_m)_\infty$ that equals $(f_m)_{j,(1+\epsilon)(j)}$ on $D_j \setminus D_{j-1}$. By the above construction, the bigraph $(M, g_\infty)$ of $(f_m)_\infty$ is the boundary of a convex set in $(\mathbb{B}_1, h)$ contained in $K$, so by the hyperbolic Busemann-Feller theorem the nearest point projection from $K$ to ( $M, g_\infty$ ) is distance decreasing. In particular, for every Borel set $S \subset M$ it holds $\mathrm{vol}_{g_\infty}(S) \le \mathrm{vol}_{i^*h}(K)$, proving (47).

It remains to prove the claim. In [30] it is presented a general procedure to construct a sequence of metrics on a bounded set of a Riemannian manifold which Gromov-Hausdorff converges to an Alexandrov space with a sharp conical singularity at each point of a countable set. For completeness, we reproduce here the construction in our setting. Consider $g: \mathbb{R}^{2+\epsilon} \to \mathbb{R}$ such that

$$\begin{cases} g(\mathbf{x}) = 1 - |\mathbf{x}| - |\mathbf{x}|^2 \text{ for } \mathbf{x} \in B_{1/2}^{\mathbb{R}^{2+\epsilon}} \\ g \in C^\infty\left(B_1^{\mathbb{R}^{2+\epsilon}} \setminus \{0\}\right) \\ \mathrm{supp} g \subseteq B_1^{\mathbb{R}^{2+\epsilon}} \\ g \ge 0. \end{cases}$$

Then, for $\varepsilon > 0$ and $\mathbf{y} \in \mathbb{R}^{2+\epsilon}$ we define $g_{\varepsilon,\mathbf{y}}: \mathbb{R}^{2+\epsilon} \to \mathbb{R}$ as

$$g_{\varepsilon,\mathbf{y}}(x) \doteq g\left(\frac{\mathbf{x}-\mathbf{y}}{\varepsilon}\right)$$

so that $g_{\varepsilon,\mathbf{y}}$ is smooth outside $\mathbf{y}$, non-positive and strictly concave on $B_{\varepsilon/2}^{\mathbb{R}^{2+\epsilon}}(\mathbf{y})$. Let

$$0 < \varepsilon_1 < \mathrm{dist}_{\mathbb{R}^{2+\epsilon}}\left(\mathbf{y}_1, \partial(D_j \setminus D_{j-1})\right)$$

and define

$$\phi_1(\mathbf{x}) \doteq f_m(\mathbf{x}) + \eta_1 g_{\varepsilon_1,\mathbf{y}_1}(\mathbf{x})$$

with $\eta_1 > 0$ small enough so that $\phi_1$ is strictly concave and $\phi_1(\mathbf{x}) < 1 - \sqrt{3}\mathbf{x}$ on $D$. Observe also that $\phi_1$ is smooth on $D \setminus \{\mathbf{y}_1\}$ and its graph has a sharp singular point at $\phi_1(\mathbf{y}_1)$.

Recursively, let $0 < \varepsilon_{(1+\epsilon)} < \mathrm{dist}_{\mathbb{R}^{2+\epsilon}}(\mathbf{y}_{(1+\epsilon)}, \partial(D_j \setminus D_{j-1}) \cup \{\mathbf{y}_1, \dots, \mathbf{y}_\epsilon\})$ and define

$$\phi_{1+\epsilon}(\mathbf{x}) \doteq \phi_\epsilon(\mathbf{x}) + \eta_{1+\epsilon} g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon}}(\mathbf{x}). \tag{49}$$

The function $\phi_{1+\epsilon}$ is smooth on $D \setminus \{\mathbf{y}_1, \dots, \mathbf{y}_{1+\epsilon}\}$, strictly concave and satisfies $\phi_{1+\epsilon}(\mathbf{x}) > \sqrt{3}\mathbf{x} - 1$ provided that $\eta_{1+\epsilon}$ is small enough. Moreover, the graph of $\phi_{1+\epsilon}$ has sharp singularities at $\phi_{1+\epsilon}(\mathbf{y}_1), \dots, \phi_{1+\epsilon}(\mathbf{y}_{1+\epsilon})$. Furthermore, if $\eta_{1+\epsilon}$ are such that $\sum_{1+\epsilon} \eta_{1+\epsilon}$ converges, then $\phi_{1+\epsilon}$ converges uniformly to some $\phi_\infty =: (f_m)_{j,\infty}$ whose graph is convex, has sharp singularities at $\{\tilde{\phi}_\infty(\mathbf{y}_{m_0})\}_{m_0=1}^{\infty}$, coincides with the graph of $f_m$ outside of $D_j \setminus D_{j-1}$ and is contained in the double cone $K$. The sharpness of the singularity at each $\tilde{\phi}_\infty(\mathbf{y}_{m_0})$ can be directly checked, making use of the fact that points of an Alexandrov space have a unique tangent cone.

To define the smooth functions $(f_m)_{j,1+\epsilon}: D \to \mathbb{R}$ approximating $(f_m)_{j,\infty}$, recall that $(f_m)_{j,\infty} = f_m + \sum_{\epsilon=0}^{\infty} \eta_{1+\epsilon} g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon}}$. By a diagonal argument, it is enough to show that each $g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon}}$ can be uniformly approximated by smooth functions which coincide with $g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon}}$ outside $B_{\varepsilon_{1+\epsilon}/2}^{\mathbb{R}^{2+\epsilon}}(\mathbf{y}_{1+\epsilon})$. For $0 < \delta < \varepsilon_{1+\epsilon}/2$, let $g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon},\delta}$ be a smooth function that is strictly concave on $B_{\varepsilon_{1+\epsilon}/2}^{\mathbb{R}^{2+\epsilon}}(\mathbf{y}_{1+\epsilon})$ and coincides with $g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon}}$ outside of $B_\delta^{\mathbb{R}^{2+\epsilon}}(\mathbf{y}_{1+\epsilon})$, see for instance [17, Theorem 2.1]. As $\delta \to 0$, we have that $g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon},\delta} \to g_{\varepsilon_{1+\epsilon},\mathbf{y}_{1+\epsilon}}$ uniformly. This concludes the proof.

Let $E_+, E_-$ be the two connected components of $M \setminus U_1$, respectively contained in $\{z > 0\}$ and in $\{z < 0\}$, and for each $j$ define

$$E_{-,j} \doteq E_- \setminus U_j, E_{+,j} \doteq E_+ \setminus U_j, \tag{50}$$

The metric $g$ on the block $M$ will be constructed from the original metric $g_0$ by prescribing, for each $i \geq 1$, a spike metric $\sigma_{i,k(i)}$ with $k(i)$ approximated spikes on $U_i \setminus U_{i-1}$. The function $k: \mathbb{N}_{>0} \to \mathbb{N}$ will be chosen inductively, by identifying, for each $j \geq 1, k(j)$ depending on $k(1), \ldots, k(j-1)$. Correspondingly, to each $j$ we shall associate a smooth metric $g_j$ on $M$ that corresponds to the choices of $\sigma_{i,k(i)}$ on $U_i \setminus U_{i-1}$ for $1 \leq i \leq j$. In particular, $g_j = g_{j-1}$ outside of $U_j \setminus U_{j-1}$. In the following lemma we summarize the properties of the metrics $g_j$ to be proved.

**Lemma 3.2 (see [37]).** There exists a sequence of metrics $\{g_j\}_{j=1}^{\infty}$ on $M$ with the following properties:

$(P1)$ $\quad \{x: g_j(x) \neq g_{j-1}(x)\}$ is compactly contained in $U_j \setminus \overline{U_{j-1}}$,

$(P2)$ $\quad \mathrm{Sec}_{g_j} \geq -1$

$(P3)$ $\quad \forall S \subset M$ Borel, $\mathrm{vol}_{g_j}(S) \leq \mathrm{vol}_{i\star h}(K) < +\infty$,

$(P4)$ $\quad \exists \bar{C}_j > 1$ such that $\bar{C}_j^{-1}\, \mathrm{d}_{g_0}(x,y) \leq \mathrm{d}_{g_j}(x,y) \leq \bar{C}_j\, \mathrm{d}_{g_0}(x,y)\ \forall x,y \in M$.

where $K$ is the double cone defined in (44), and $\mathrm{d}_{g_j}$ is the distance induced by $g_j$. Furthermore, having defined $E_{\pm,j}$ as in (50), $g_j$ and $g_{j+1}$ satisfy

$$(P5)\ \forall \varphi \in C^\infty(M), \quad \begin{array}{ll} \varphi \leq -1 + 2^{-j} & \text{on } \partial E_{-,j} \\ \varphi \geq 1 - 2^{-j} & \text{on } \partial E_{+,j} \end{array} \Rightarrow \|\varphi\|_{W^{2,1+\epsilon}(U_{j+1}\setminus \mathbb{V}, g_{j+1})} > 1.$$

**Remark 3.3.** About ( $P5$ ), we shall see below that $g_j$ matches the following stronger property: whenever $\varphi$ satisfies the assumptions of ( $P5$ ), the inequality

$$\|\varphi\|_{W^{2,1+\epsilon}(U_{j+1}\setminus \mathbb{V}, \bar{g})} > 1$$

will hold for any choice of $\bar{g}$ that coincides with $g_j$ on $U_j$ and with a spike metric $\sigma_{j+1,m_0}$ on $U_{j+1} \setminus U_j$. In particular, $(P5)$ does not require to have already chosen the integer $k(j+1)$, but holds a-posteriori for every possible choice of it.

**3.2. Proof of Theorem 1.9 (see [37]).** Let us see how Lemma 3.2 allows to conclude the proof of Theorem 1.9.

Let $g$ be the smooth Riemannian metric on $M$ defined by $g = g_j$ on $U_j$ for $j \geq 0$. It is readily seen by $(P1)$, $(P2), (P3)$ that $\mathrm{Sec}_g \geq -1$ and that $\mathrm{vol}_g(M) \leq \mathrm{vol}_{i^*h}(K) < \infty$. Furthermore, referring to the proof of Lemma 3.1, $(M,g)$ can be realized as the bigraph of a concave function that equals $(f_m)_{j,k(j)}$ on $D_j \setminus D_{j-1}$. Such a bigraph is properly embedded in $(\mathbb{B}_1, h)$, hence $(M,g)$ is complete. Let us glue $N$ to $M$ along $\mathbb{V}'$ and $\mathbb{V}$, by keeping the metric $g$ unchanged outside of $\mathbb{V}$. For convenience, still denote with $\mathbb{V}$ the complement of $M \setminus \mathbb{V}$ inside of $M \sharp N$, and with $g$ the glued metric. Fix a smooth function $F_m: M\sharp N \to \mathbb{R}$ such that

$$F_m \equiv 0 \ \text{ on } \mathbb{V}, F_m \equiv -1 \ \text{ on } E_{-,1}, F_m \equiv 1 \ \text{ on } E_{+,1}.$$

Since $(M,g)$ has finite volume, it is clear that $F_m \in W^{1+\epsilon,1+\epsilon}(M\sharp N)$ for every $(1+\epsilon)$. For each $\epsilon > 0$, we prove that $F_m$ cannot be approximated by compactly supported smooth functions in $W^{2,1+\epsilon}(M\sharp N)$, as the statement for higher $(1+\epsilon)$ is a simple consequence. Suppose by contradiction that there exists a sequence $\{(F_m)_i\}_{i=0}^{\infty} \subset C_c^\infty(M\sharp N)$ such that $\|F_m - (F_m)_i\|_{W^{2,1+\epsilon}(M\sharp N, g)} \to 0$ as $i \to \infty$. In particular, there exists $i$ such that

$$\|F_m - (F_m)_i\|_{W^{2,1+\epsilon}(M\setminus \mathbb{V}, g)} \leq 1/2.$$

Choose $j \geq 1$ so that $(F_m)_i$ has support in $U_j$. Then $F_m - (F_m)_i \equiv -1$ on $E_{-,j}$ and $F_m - (F_m)_i \equiv 1$ on $E_{+,j}$, hence $(P5)$ enables us to conclude that

$$\|F_m - (F_m)_i\|_{W^{2,1+\epsilon}(U_{j+1}\setminus \mathbb{V}, g_{j+1})} > 1.$$

However, since $F_m - (F_m)_i$ is constant outside of $U_j$ and $g = g_{j+1}$ on $U_{j+1}$,

$$\frac{1}{2} \geq \|F_m - (F_m)_i\|_{W^{2,1+\epsilon}(M\setminus \mathbb{V}, g)} \geq \|F_m - (F_m)_i\|_{W^{2,1+\epsilon}(U_{j+1}\setminus \mathbb{V}, g_{j+1})} > 1 \tag{51}$$

contradiction.

**3.3. Proof of Lemma 3.2 (see [37]).** Suppose that $g_{j-1}$ is constructed. Let $\{g_{j,1+\epsilon}\}_{(1+\epsilon)\in\mathbb{N}\cup\{0\}}$ be the sequence of smooth metrics on $M$ being equal to $g_{j-1}$ outside of $U_j \setminus U_{j-1}$ and equal to the spike metric $\sigma_{j,1+\epsilon}$ on $U_j \setminus U_{j-1}$. Then, $g_{j,0} \equiv g_{j-1}$ on $M$ and, denoting with $\mathrm{d}_{j,1+\epsilon}$ the distance induced by $g_{j,1+\epsilon}$, from Lemma 3.1 we easily deduce the following properties:

$$\{x: g_{j,1+\epsilon}(x) \neq g_{j-1}(x)\} \text{ is compactly contained in } U_j \setminus U_{j-1},$$
$$\mathrm{Sec}_{g_{j,1+\epsilon}} \geq -1 \text{ for each } 1+\epsilon,$$
$$\forall S \subset M \text{ Borel, } \mathrm{vol}_{g_{j,1+\epsilon}}(S) \leq \mathrm{vol}_{i^*h}(K) < \infty.$$

For each choice of $k(j)$, the metric $g_j = g_{j,k(j)}$ therefore satisfies *P1*, *P2*, *P3*. To prove *P4* and (P5), for any fixed $1+\epsilon, m_0 \in \mathbb{N}$ we define the smooth metric $g_{j,1+\epsilon,m_0}$ such that

$$g_{j,1+\epsilon,m_0} = \sigma_{j+1,m_0} \text{ on } U_{j+1} \setminus U_j, g_{j,1+\epsilon,m_0} = g_{j,1+\epsilon} \text{ otherwise.}$$

The construction of $g_{j,1+\epsilon,m_0}$ and (iii) in Lemma 3.1 guarantee that there exists a constant $\bar{C}_j > 1$ such that

$$\bar{C}_j^{-1}\, \mathrm{d}_{g_{j-1}}(x,y) \leq \mathrm{d}_{g_{j,1+\epsilon,m_0}}(x,y) \leq \bar{C}_j\, \mathrm{d}_{g_{j-1}}(x,y)\ \forall x,y \in M, 1+\epsilon, m_0 \in \mathbb{N} \cup \{0\} \tag{52}$$

In particular, independently of the possible choice of $k(j)$, $g_j$ also satisfies ($\mathcal{P}_4$). Observe that (52) implies that

$$\exists \nu_j > 0 \text{ such that } \mathrm{vol}_{g_{j,1+\epsilon,m}}\left(B_1^{g_{j,1+\epsilon,m}}(z)\right) \geq \nu_j\ \forall z \in U_j, 1+\epsilon, m_0 \in \mathbb{N}. \tag{53}$$

As anticipated in Remark (3.3, we shall prove the following strengthened version of ($P5$)):

**Claim 1:** there exists $k(j)$ depending on $j$ such that $g_j \doteq g_{j,k(j)}$ satisfies

$$(\mathcal{P}5_j') \qquad \forall \varphi \in C^\infty(M), \quad \begin{matrix} \varphi \leq -1+2^{-j} \text{ on } \partial E_{-,j} \\ \varphi \geq 1-2^{-j} \text{ on } \partial E_{+,j} \end{matrix} \implies \forall m_0, \|\varphi\|_{W^{2,1+\epsilon}(U_{j+1}\setminus\mathbb{V}, g_{j,(1+\epsilon)(j),m})} > 1.$$

Assume, by contradiction, that $(\mathcal{P}5_j')$ does not hold, so that, for $(1+\epsilon)$ large enough, there exists a sequence $\{\varphi_{j,1+\epsilon}\}$ with $\varphi_{j,1+\epsilon} \in C^\infty(M, g_{j,1+\epsilon})$, and a sequence of integers $\{(m_0)_{1+\epsilon}\}$, such that

$$\begin{matrix} \varphi_{j,1+\epsilon} \leq -1+2^{-j} & \text{on } \partial E_{-,j} \\ \varphi_{j,1+\epsilon} \geq 1-2^{-j} & \text{on } \partial E_{+,j} \end{matrix} \quad \text{but } \|\varphi_{j,1+\epsilon}\|_{W^{2,1+\epsilon}(U_{j+1}\setminus\mathbb{V}, g_{j,1+\epsilon,(m_0)_{1+\epsilon}})} \leq 1 \tag{54}$$

We examine the convergence of the sequence $\{\varphi_{j,1+\epsilon}\}_{1+\epsilon}$ on $\overline{U_j} \setminus \overline{\mathbb{V}}$.

**Claim 2:** as $\epsilon \to \infty$, the sequence $\varphi_{j,1+\epsilon}$ converges locally uniformly on $\overline{U_j} \setminus \overline{\mathbb{V}}$ to a function $\varphi_j$ that is locally Hölder continuous on $\overline{U_j} \setminus \overline{\mathbb{V}}$ and locally constant on $U_j \setminus U_{j-1}$ (on $U_1 \setminus \mathbb{V}$, if $j = 1$ ).

We describe how Claim 2 yields to the proof of Claim 1. First, since the convergence is uniform up to the boundary of $U_j$, passing to the limit we obtain

$$\varphi_j \geq 1 \text{ on } \partial E_{+,j}, \varphi_j \leq -1 \text{ on } \partial E_{-,j}. \tag{55}$$

The argument goes then by induction on $j$. If $j = 1, U_1 \setminus \mathbb{V}$ is connected and thus $\varphi_1$ is constant. This contradicts the fact that $\partial E_{+,1} \cup \partial E_{-,1} \subset \partial(U_1 \setminus \mathbb{V})$. Having proved Claim 1 for $j = 1$, and thus having constructed $g_1$ with property $(\mathcal{P}5_1')$, we examine the case $j > 1$. We proceed inductively, that is, we assume to have constructed $g_{j-1}$ in such a way that $(P1)$ , ..., $(\mathcal{P}5_{j-1}')$ hold. If $j > 1$, then $U_j \setminus U_{j-1}$ has at least two connected components, respectively contained in $E_+$and $E_-$. The constancy of $\varphi_j$ on each component, coupled with (55), guarantees that

$$\varphi_j \geq 1 \text{ on } \partial E_{+,j-1}, \varphi_j \leq -1 \text{ on } \partial E_{-,j-1}.$$

Therefore, for $(1+\epsilon)$ large enough,

$$\varphi_{j,1+\epsilon} \geq 1-2^{-j+1} \text{ on } \partial E_{+,j-1}, \varphi_{j,1+\epsilon} \leq -1+2^{-j+1} \text{ on } E_{-,j-1},$$

and thus, by $(\mathcal{P}5_{j-1}')$,

$$\|\varphi_{j,1+\epsilon}\|_{W^{2,1+\epsilon}(U_j\setminus\mathbb{V}, g_{j,1+\epsilon})} > 1.$$

Concluding, since $g_{j,1+\epsilon,(m_0)_{1+\epsilon}} = g_{j,1+\epsilon}$ on $U_j$,

$$1 \geq \|\varphi_{j,1+\epsilon}\|_{W^{2,1+\epsilon}(U_{j+1}\setminus\mathbb{V}, g_{j,1+\epsilon,m_{1+\epsilon}})} \geq \|\varphi_{j,1+\epsilon}\|_{W^{2,1+\epsilon}(U_j\setminus\mathbb{V}, g_{j,1+\epsilon})} > 1,$$

contradicting (54).

It remains to prove Claim 2. The argument is inspired by the recent [15], where the authors study the behaviour of harmonic functions near sharp points of $\mathrm{RCD}(K, 2+\epsilon)$ spaces. Recall that, given a complete metric $\bar{g}$ on $M$ with $\mathrm{Ric}_{\bar{g}} \geq -(1+\epsilon)\bar{g}$, and a geodesic ball $B_{1+\epsilon}(o)$ centered at some fixed origin $o$, there exist constants $C_{\mathcal{D}}, C_{\mathcal{D}}'$ depending on $(2+\epsilon), (1+\epsilon)$ such that

$$\mathrm{vol}_{\bar{g}}\big(B_{2(1+\epsilon)}(z)\big) \leq C_{\mathcal{D}} \mathrm{vol}_{\bar{g}}(B_{1+\epsilon}(z))\ \forall B_{2(1+\epsilon)}(z) \subset B_{1+\epsilon}(o) \tag{56}$$

and, for every $\epsilon \geq 0$ such that $B_s(z) \subset B_{1+\epsilon}(o)$,

$$\frac{\mathrm{vol}_{\bar{g}}(B_{1+\epsilon}(z))}{\mathrm{vol}_{\bar{g}}(B_{1+2\epsilon}(z))} \geq \frac{V_{-1}(1+\epsilon)}{V_{-1}(1+2\epsilon)} \geq C_{\mathcal{D}}'\left(\frac{1+\epsilon}{1+2\epsilon}\right)^{2+\epsilon} \tag{57}$$

where $V_{-1}(t)$ is the volume of a ball of radius $t$ in the $(2+\epsilon)$-dimensional hyperbolic space of curvature -1 . It is a simple consequence of the above two inequalities that there exists $C''_{\mathcal{D}} = C''_{\mathcal{D}}(2+\epsilon, 1+\epsilon)$ such that

$$\text{for each } B'_{1+\epsilon} \subset B_{1+2\epsilon} \text{ geodesic balls in } B_{1+\epsilon}(o), \frac{\mathrm{vol}_{\bar{g}}(B'_{1+\epsilon})}{\mathrm{vol}_{\bar{g}}(B_{1+2\epsilon})} \geq C''_{\mathcal{D}}(2+\epsilon, 1+\epsilon)\left(\frac{1+\epsilon}{1+2\epsilon}\right)^{2+\epsilon} \quad (58)$$

where now $B'_{1+\epsilon}, B_{1+2\epsilon}$ may not be concentric. On the other hand, Buser's isoperimetric inequality [8] (see [34, Th. 5.6.5] or [28, Thm. 1.4.1] for alternative proofs) guarantees the existence, for each $0 \leq \epsilon < \infty$, of a constant $\mathcal{P}_{1+\epsilon} = \mathcal{P}_{1+\epsilon}(2+\epsilon, 1+\epsilon, 1+\epsilon)$ such that

$$\left\{\fint_{B_{1+\epsilon}(x)} |\psi - \bar{\psi}_{B_{1+\epsilon}(x)}|^{1+\epsilon}\right\}^{\frac{1}{1+\epsilon}} \leq (1+\epsilon)\mathcal{P}_{1+\epsilon}\left\{\fint_{B_{1+\epsilon}(x)} |\nabla\psi|^{1+\epsilon}\right\}^{\frac{1}{1+\epsilon}} \quad \forall \psi \in \mathrm{Lip}(B_{1+\epsilon}(o)) \quad (59)$$

where $\bar{\psi}_{B_{1+\epsilon}(x)}$ is the mean value of $\psi$ on $B_{1+\epsilon}(x)$.

Because of Lemma 3.1, up to subsequences $(M, g_{j,1+\epsilon,(m_0)_{1+\epsilon}}, o) \to M_{j,\infty} \doteq (M, \mathrm{d}_{j,\infty}, o)$ as $\epsilon \to 0$ in the Gromov-Hausdorff sense, where $M_\infty$ is an Alexandrov space of curvature not smaller than -1 with a dense set of sharp points in $U_j \setminus U_{j-1}$. Fix a smooth open set $U'_0$ with $\mathbb{V} \Subset U'_0 \Subset U_1$, and such that $U_1 \setminus \overline{U'_0}$ is connected. Choose

$$0 < \varepsilon_j \leq \frac{1}{1000\bar{C}_j^2} \min\left\{ \mathrm{d}_{g_{j-1}}(U_j, \partial U_{j+1}), \mathrm{d}_{g_{j-1}}(U'_0, \mathbb{V})\right\} > 0$$

in such a way that the tubular neighborhood

$$V_j \doteq B_{16}^{g_{j-1}} \bar{C}_j \varepsilon_j (U_j \setminus U'_0) \text{ has smooth boundary.}$$

Hereafter the index $j$ will be fixed, so for notational convenience we omit to write it unless it identifies the sets $U_j$. We also use a superscript or subscript $(1+\epsilon)$ to indicate quantities that refer to the metric $g_{j,1+\epsilon,(m_0)_{1+\epsilon}}$, so for instance we write $|\cdot|_{1+\epsilon}, \mathrm{vol}_{1+\epsilon}$ to denote the norm and volume, and $B_{1+\epsilon}^{1+\epsilon}(z)$ instead of $B_{1+2\epsilon}^{g_{j,1+\epsilon,(m_0)_{1+\epsilon}}}(z)$. Analogously, balls in $M_{j,\infty}$ will be denoted with $B_{1+\epsilon}^{\infty}(z)$. By (52), we have the following inclusions between tubular neighbourhoods:

$$B_{\varepsilon_j}^{1+\epsilon}(U_j \setminus U'_0) \Subset V_j \Subset B_{5\varepsilon_j}^{1+\epsilon}(V_j) \Subset U_{j+1} \setminus \mathbb{V} \; \forall (1+\epsilon) \in \mathbb{N} \quad (60)$$

Again using (52), we can fix $R_j > 0$ such that

$$U_{j+1} \Subset B_{\frac{R_j}{2}-1}^{1+\epsilon}(o) \; \forall (1+\epsilon) \in \mathbb{N}$$

Because $\mathrm{Sec}_{g_{j,1+\epsilon,m_{1+\epsilon}}} \geq -1$ for each $j, 1+\epsilon$, on the balls $B_{R_j}^{1+\epsilon}(o)$ we have the validity of (56), (58) and (59) with constants only depending on $2+\epsilon, 1+\epsilon, R_j$. By using (60), we can apply Morrey's estimates as stated in [24, Thm. 9.2.14] both to $\varphi_{1+\epsilon}$ and to $|\nabla\varphi_{1+\epsilon}|_{1+\epsilon}$, to deduce that for fixed $j$ there exists a constant $C = C(2+\epsilon, 1+\epsilon, R_j)$ such that for each $z \in B_{\varepsilon_i}^{1+\epsilon}(V_j)$ it holds

$$\sup_{x,y \in B_{\varepsilon_j}^{1+\epsilon}(z)} \frac{|\varphi_{1+\epsilon}(x) - \varphi_{1+\epsilon}(y)|}{\mathrm{d}_{1+\epsilon}(x,y)^{1-\frac{2+\epsilon}{1+\epsilon}}} + \frac{|\nabla\varphi_{1+\epsilon}(x)|_{1+\epsilon} - |\nabla\varphi_{1+\epsilon}(y)|_{1+\epsilon}}{\mathrm{d}_{1+\epsilon}(x,y)^{1-\frac{2+\epsilon}{1+\epsilon}}} \leq c(2+\epsilon, 1+\epsilon, R_j)\varepsilon_j^{\frac{2+\epsilon}{1+\epsilon}}\left(\fint_{B_{4\varepsilon_j}^{1+\epsilon}(z)} |\nabla\varphi_{1+\epsilon}|_{1+\epsilon}^{1+\epsilon} + |\nabla^2\varphi_{1+\epsilon}|_{1+\epsilon}^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \quad (61)$$

Using (58), (53) and (54), we get

$$\varepsilon_j^{\frac{2+\epsilon}{1+\epsilon}}\left(\fint_{B_{4\varepsilon_j}^{1+\epsilon}(z)} |\nabla\varphi_{1+\epsilon}|_{1+\epsilon}^{1+\epsilon} + |\nabla^2\varphi_{1+\epsilon}|_{1+\epsilon}^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \leq C\varepsilon_j^{\frac{2+\epsilon}{1+\epsilon}}\left(\frac{1}{\varepsilon_j^{2+\epsilon}\mathrm{vol}_{1+\epsilon}(B_1^{1+\epsilon}(z))}\int_{B_{4\varepsilon_j}^{1+\epsilon}(z)} |\nabla\varphi_{1+\epsilon}|_{1+\epsilon}^{1+\epsilon} + |\nabla^2\varphi_{1+\epsilon}|_{1+\epsilon}^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \leq C'$$

Thus (61) gives

$$\sup_{x,y \in B_{\varepsilon_j}^{1+\epsilon}(z)} \frac{|\varphi_{1+\epsilon}(x) - \varphi_{1+\epsilon}(y)|}{\mathrm{d}_{1+\epsilon}(x,y)^{1-\frac{2+\epsilon}{1+\epsilon}}} + \frac{|\nabla\varphi_{1+\epsilon}(x)|_{1+\epsilon} - |\nabla\varphi_{1+\epsilon}(y)|_{1+\epsilon}}{\mathrm{d}_{1+\epsilon}(x,y)^{1-\frac{2+\epsilon}{1+\epsilon}}} \leq C''(2+\epsilon, 1+\epsilon, R_j) \; \forall z \in B_{\varepsilon_j}^{1+\epsilon}(V_j) \quad (62)$$

A simple chain argument using (48) then allows to extend the uniform Hölder estimates in (62) to $x, y \in B_{\varepsilon_j}^{1+\epsilon}(U_j \setminus U'_0)$. Briefly, since $V_j$ has smooth boundary we can fix a constant $\hat{C}_j$ such that, for each $x, y \in V_j$, there exists a curve $\gamma_{xy} \subset V_j$ joining $x$ to $y$ whose length is at most $\hat{C}_j \, \mathrm{d}_{g_{j-1}}(x,y)$. Restricting to $x, y \in$

$B_{\varepsilon_j}^{1+\epsilon}(U_j \setminus U'_0)$, choose points $\{x_i\}_{i=1}^{s}$ along $\gamma_{xy}$ in such a way that $x_0 = x, x_s = y$ and the length of each subsegment $\gamma_{x_i x_{i+1}}$ with respect to $g_{j-1}$ does not exceed $\varepsilon_j / (2\hat{C}_j \bar{C}_j)$. By (52), there exists $\tilde{C}_j$ such that

$$x_i \in B_{\varepsilon_j}^{1+\epsilon}(x_{i-1}) \; \forall i \in I, (1+\epsilon) \in \mathbb{N}, \sum_i \mathrm{d}_{1+\epsilon}(x_i, x_{i+1}) \leq \tilde{C}_j \, \mathrm{d}_{1+\epsilon}(x,y)$$

Applying (62) with $z = y = x_i$ and $x = x_{i+1}$, and summing up, we get

$$|\varphi_{1+\epsilon}(x) - \varphi_{1+\epsilon}(y)| + ||\nabla\varphi_{1+\epsilon}(x)|_{1+\epsilon} - |\nabla\varphi_{1+\epsilon}(y)|_{1+\epsilon}| \leq C'''(2+\epsilon, 1+\epsilon, R_j)\mathrm{d}_{1+\epsilon}(x,y)^{1-\frac{2+\epsilon}{1+\epsilon}}\ \forall x, y \in B^{1+\epsilon}_{\varepsilon_j}(U_j \setminus U'_0) \tag{63}$$

Next, by (54) and since $M \setminus U'_0$ is connected while $M \setminus \overline{U_j}$ is not, each curve in $M \setminus U'_0$ joining two points $x \in \partial E_{-,j}, y \in \partial E_{+,j}$ shall contain a point $x_{1+\epsilon} \in U_j \setminus U'_0$ for which $\varphi_{1+\epsilon}(x_{1+\epsilon}) = 0$. Hence, $\{\varphi_{1+\epsilon}\}$ is equibounded on $B^{1+\epsilon}_{\varepsilon_j}(U_j \setminus U'_0)$ and subconverges, by Ascoli-Arzelá theorem, pointwise to some $\varphi: B^{\infty}_{\varepsilon_j}(U_j \setminus U'_0) \to \mathbb{R}$ that, because of (63), is uniformly continuous on $B^{\infty}_{\varepsilon_j}(U_j \setminus U'_0)$. Furthermore, by [25, Prop. 3.19] and up to subsequences, $\varphi_{1+\epsilon} \to \varphi L^2$-weakly on each ball $B^{1+\epsilon}_{\varepsilon_j}(z) \subset B^{1+\epsilon}_{\varepsilon_j}(U_j \setminus U'_0)$, see also [25, Rem. 3.8]. By Hölder inequality, (60) and since $(M, g_{j,1+\epsilon,(m_0)_{1+\epsilon}})$ has uniformly bounded volume,

$$\limsup_{1+\epsilon} \|\varphi_{1+\epsilon}\|_{W^{1,2}\left(B^{1+\epsilon}_{\varepsilon_j}(z), g_{j,1+\epsilon,(m_0)_{1+\epsilon}}\right)} < \infty$$

and $\varphi_{1+\epsilon} \to \varphi L^2$-strongly on $B^{\infty}_{\varepsilon_j}(z)$. By [25, Thm. 1.3], $\varphi \in W^{1,2}(B^{\infty}_{1+\epsilon}(z), \mathrm{d}_\infty)$ for each $1+\epsilon < \varepsilon_j, \varphi$ is in the domain of the Laplacian $\mathcal{D}^2\left(\Delta, B^{\infty}_{\varepsilon_j}(z)\right)$ on $M_{j,\infty}$ and

$$\begin{aligned} \Delta\varphi_{1+\epsilon} \to \Delta\varphi \quad & L^2 \text{ weakly on } B^{\infty}_{\varepsilon_j}(z) \\ \nabla\varphi_{1+\epsilon} \to \nabla\varphi \quad & L^2 \text{ strongly on } B^{\infty}_{1+\epsilon}(z), \text{ for each } 1+\epsilon < \varepsilon_j \end{aligned} \tag{64}$$

In particular, by [25, Thm. 3.28], $|\nabla\varphi_{1+\epsilon}| \to |\nabla\varphi| L^2$ strongly on $B^{\infty}_{1+\epsilon}(z)$, hence pointwise a.e by [25, Prop. 3.32]. Passing to the limit in (63), $\varphi$ and $|\nabla\varphi|$ are uniformly continuous on $\overline{U_j \setminus U_0}$. If $z$ is a sharp point we apply [15, Proposition 2.5] to infer the existence of $\delta_0 = \delta_0(2+\epsilon, z)$ and $\varepsilon'_j = \varepsilon'(2+\epsilon, z, \varepsilon_j) \in (0, \varepsilon_j)$ such that

$$\fint_{B^{\infty}_{1+\epsilon/2}(z)} |\nabla\varphi|^2 \leq (1-\delta_0)\fint_{B^{\infty}_{1+\epsilon}(z)} |\nabla\varphi|^2 + (1+\epsilon)^2 C(2+\epsilon, z, \varepsilon_j)\fint_{B^{\infty}_{1+\epsilon}(z)} (\Delta\varphi)^2\ \forall (1+\epsilon) \leq \varepsilon'_j$$

Using [25, Thm. 3.29] and (64) we deduce that, for every $1+\epsilon \leq \varepsilon'_j$,

$$\|\Delta\varphi\|_{L^2(B^{\infty}_{1+\epsilon}(z))} \leq \liminf_{1+\epsilon} \|\Delta\varphi_{1+\epsilon}\|_{L^2(B^{1+\epsilon}_{1+\epsilon}(z))}$$

hence by Hölder inequality and (57) we deduce

$$\begin{aligned} (1+\epsilon)^2 \fint_{B^{1+\epsilon}_{1+\epsilon}(z)} |\Delta\varphi_{1+\epsilon}|^2 &\leq (1+\epsilon)^2 \mathrm{vol}_{1+\epsilon}(B^{1+\epsilon}_{1+\epsilon}(z))^{\frac{-2}{1+\epsilon}} \left(\int_{B^{1+\epsilon}_{1+\epsilon}(z)} |\Delta\varphi_{1+\epsilon}|^{1+\epsilon}\right)^{\frac{2}{1+\epsilon}} \\ &\leq (1+\epsilon)^2 \mathrm{vol}_{1+\epsilon}(B^{1+\epsilon}_{1+\epsilon}(z))^{-\frac{2}{1+\epsilon}} \leq (C'_D)^{-\frac{2}{1+\epsilon}}(1+\epsilon)^2\left(\frac{\varepsilon_j}{1+\epsilon}\right)^{\frac{2(2+\epsilon)}{1+\epsilon}} \mathrm{vol}_{1+\epsilon}\left(B^{1+\epsilon}_{\varepsilon_j}(z)\right)^{-\frac{2}{1+\epsilon}} \\ &\leq C(2+\epsilon, 1+\epsilon, R_j, \varepsilon_j, v_j)(1+\epsilon)^{2\frac{-1}{1+\epsilon}} \end{aligned}$$

where, in the last step, we used again (53). Inserting into (65) we eventually obtain

$$\fint_{B^{\infty}_{1+\epsilon/2}(z)} |\nabla\varphi|^2 \leq (1-\delta_0)\fint_{B^{\infty}_{1+\epsilon}(z)} |\nabla\varphi|^2 + C(2+\epsilon, 1+\epsilon, R_j, v_j, z, \varepsilon_j)(1+\epsilon)^{\frac{2(-1)}{1+\epsilon}}\ \forall (1+\epsilon) \leq \varepsilon'_j$$

Consequently,

$$\lim_{\epsilon\to 0} \fint_{B^{\infty}_{1+\epsilon/2}(z)} |\nabla\varphi|^2 = 0 \text{ for every sharp point } z$$

From the uniform continuity of $|\nabla\varphi|$ and the density of the set of sharp points in $U_j \setminus U_{j-1}$, we conclude that $|\nabla\varphi| = 0$ on $\overline{U_j \setminus U_{j-1}}$ (on $U_1 \setminus \mathbb{V}$, if $j = 1$ ), as claimed. This concludes the proof of Lemma 3.2.

**3.4. Proof of Corollaries 1.11, 1.12 and 1.14 (see [37]).** All of them are based on the following simple observation: let $X, Y$ be Riemannian manifolds, with $Y$ compact, and consider a (say, smooth) function $\varphi \in W^{2,1+\epsilon}(X \times Y)$. For every $y \in Y$ fixed, define $\varphi_y: X \to \mathbb{R}$ by $\varphi_y(x) = \varphi(x, y)$. Denote with $\nabla, \bar{\nabla}, \Delta, \bar{\Delta}$, the Levi-Civita connections and the Laplace operator of $X$ and $X \times Y$ respectively. From

$$|\bar{\nabla}\varphi(x,y)| \geq |\nabla\varphi_y(x)|, |\bar{\nabla}^2\varphi(x,y)| \geq |\nabla^2\varphi_y(x)|$$

it holds

$$\|\varphi\|^{1+\epsilon}_{L^{1+\epsilon}(X\times Y)} + \|\bar{\nabla}\varphi\|^{1+\epsilon}_{L^{1+\epsilon}(X\times Y)} + \|\bar{\nabla}^2\varphi\|^{1+\epsilon}_{L^{1+\epsilon}(X\times Y)} \geq \int_Y \left\{\|\varphi_y\|^{1+\epsilon}_{L^{1+\epsilon}(X)} + \|\nabla\varphi_y\|^{1+\epsilon}_{L^{1+\epsilon}(X)} + \|\nabla^2\varphi_y\|^{1+\epsilon}_{L^{1+\epsilon}(X)}\right\} \mathrm{d}y$$

with equality if $\varphi$ just depends on $y$. Hence, by the definition of $W^{2,1+\epsilon}$ norm, there exists a constant $C_{1+\epsilon} > 0$ only depending on $(1+\epsilon)$ such that

$$\|\varphi\|^{1+\epsilon}_{W^{2,1+\epsilon}(X\times Y)} \geq C_{1+\epsilon}\int_Y \|\varphi_y\|^{1+\epsilon}_{W^{2,1+\epsilon}(X)} \mathrm{d}y\ \forall \varphi \in C^\infty(X\times Y) \cap W^{2,1+\epsilon}(X\times Y) \tag{66}$$

Conversely, let $\pi: X \times Y \to X$ be the projection onto the first factor and for any $\psi \in W^{2,1+\epsilon}(X)$ define $\bar{\psi} \doteq \psi \circ \pi \in W^{2,1+\epsilon}(X \times Y)$. Then

$$\|\bar{\psi}\|_{L^{1+\epsilon}(X\times Y)}^{1+\epsilon} = \mathrm{vol}(Y)\|\psi\|_{L^{1+\epsilon}(X)}^{1+\epsilon}, \qquad \|\bar{\nabla}\bar{\psi}\|_{L^{1+\epsilon}(X\times Y)}^{1+\epsilon} = \mathrm{vol}(Y)\|\nabla\psi\|_{L^{1+\epsilon}(X)}^{1+\epsilon},$$

$$\|\bar{\nabla}^2\bar{\psi}\|_{L^{1+\epsilon}(X\times Y)}^{1+\epsilon} = \mathrm{vol}(Y)\|\nabla^2\psi\|_{L^{1+\epsilon}(X)}^{1+\epsilon}, \quad \|\bar{\Delta}\bar{\psi}\|_{L^{1+\epsilon}(X\times Y)}^{1+\epsilon} = \mathrm{vol}(Y)\|\Delta\psi\|_{L^{1+\epsilon}(X)}^{1+\epsilon}$$

Regarding Corollary 1.11, for fixed $\epsilon \geq 0$, and $\epsilon > 0$, consider a surface $M\sharp N$ and the smooth function $F_m \in W^{1+\epsilon,2+\epsilon}(M\sharp N)$ (for each $(1+\epsilon) \in \mathbb{N}$ ) constructed in Theorem 1.9 for dimension 2 . In particular,

$$\|v_m - F_m\|_{W^{2,2+\epsilon}(M\sharp N)} \geq 1 \text{ for every } v_m \in C_c^\infty(M\sharp N).$$

Consider a compact, boundaryless manifold $Y$ of dimension $(\epsilon)$, let $\pi: Q = (M\sharp N) \times Y \to Y$ be the projection onto the second factor, and define $\bar{F}_m \doteq F_m \circ \pi \in W^{2,2+\epsilon}(Q)$. Then, from (66), for every $u_m \in C_c^\infty(Q)$ it holds

$$\|u_m - \bar{F}_m\|_{W^{1+\epsilon,2+\epsilon}(Q)}^{2+\epsilon} \geq \|u_m - \bar{F}_m\|_{W^{2,2+\epsilon}(Q)}^{2+\epsilon} \geq C_{2+\epsilon}\int_Y \|(u_m)_y - F_m\|_{W^{2,2+\epsilon}(M\sharp N)}^{2+\epsilon}\,\mathrm{d}y \geq C_{2+\epsilon}\mathrm{vol}(Y)$$

Hence $F_m \notin W_0^{1+\epsilon,2+\epsilon}(M)$ and

$$W_0^{1+\epsilon,2+\epsilon}(Q) \neq W^{1+\epsilon,2+\epsilon}(Q)$$

as claimed.

As for Corollary 1.12, given $\epsilon > 0$, let $(M^2, g)$ be a complete surface with $\mathrm{Sec}_g \geq 0$ constructed in 30, so that there exists a sequence $\{(F_m)_{1+\epsilon}\} \subset C_c^\infty(M)$ with $\|(F_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M)} + \|\Delta(F_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M)} = 1$ but $\|\nabla^2(F_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M)} \to \infty$. Fix a compact manifold $Y^\epsilon$ with $\mathrm{Sec} \geq 0$, and define as above $(\bar{F}_m)_{1+\epsilon} = (F_m)_{1+\epsilon} \circ \pi \in C_c^\infty(M \times X)$. It is immediate to deduce that

$$\|(\bar{F}_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M\times X)} + \|\bar{\Delta}(\bar{F}_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M\times X)} = \mathrm{vol}(Y)^{1/2+\epsilon}, \text{ but } \|\bar{\nabla}^2(\bar{F}_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M)} \to \infty$$

Corollary 1.14 can be proved in a very similar way, starting from a sequence of compact 2-dimensional positively curved manifolds $M_{1+\epsilon}$ and a sequence of functions $(F_m)_{1+\epsilon} \in C^\infty(M_{1+\epsilon})$ which verify

$$\|(F_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M_{1+\epsilon})} + \|\Delta(F_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M_{1+\epsilon})} = \mathrm{vol}(Y)^{-\frac{1}{2+\epsilon}}, \text{ but } \|\nabla^2(F_m)_{1+\epsilon}\|_{L^{2+\epsilon}(M_{1+\epsilon})} \to \infty$$

the existence of these sequences is guaranteed by [15].

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