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Research Paper

A Full Description of the Operator-Valued Dyadic Shifts and Theorem$T(1)$

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Abstract
The two pioneers in [18] extend dyadic shifts and representation theorem to an operator-valued setting.They define the boundedness of the operator-valued dyadic shifts, extend the dyadic representation theorem, that is,every scalar-valued Calder ó n–Zygmund operator can be represented as a series of dyadic shifts, paraproducts averaged over randomized dyadic systems, to operator-valued Calderón–Zygmund operators, a proof of the operator-valued, global $T(1)$ theorem. Following and showing the validity of the stream of the full high discussion of their wide usuable integral operators ofR-bounded operator-valued kernels, acting in UMD-spaces,also in Euclidean space equipped with Lebesgue measureand obtain theorems of Boundedness of the dyadic paraproduct, a variant of Pythagoras' theorem for functions of dyadic cubes, and finally a decoupling inequality for martingale differences.

Keywords: *operator-valued, vector-valued, dyadic shift, dyadic representation, paraproduct, $T(1)$, UMD, R-boundedness, decoupling, Pythagoras.*

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I. Introduction

We extend dyadic shifts and the dyadic representation theorem to an operator-valued setting. We work with integral operators that have R- bounded operator-valued kernels and act on functions taking values in UMD-spaces. The domain of the functions is the Euclidean space equipped with the Lebesgue measure.

First, we summarize what is known in the scalar-valued setting. A dyadic shift$S^{ji}$with parameters i and j (and complexitymax$\{j,i\}+1$) is defined by

$$S^{ji}f_r := \sum_{K\in\mathcal{D}}\sum_r D_K^i A_K D_K^i f_r,$$

Whichinvolves the following ingredients:

- the shifted Haar projectionD_K^iassociated with a dyadic cube $K \in$ D is defined by

$$D_K^i f_r := \sum_{I\in\mathcal{D}:I\subseteq K,\ell(I)=2^{-i}\ell(K)}\sum_r D_I f_r,$$

- The Haar projection$D_I$associated with a dyadic cube $I\in\mathcal{D}$is defined by

$$D_I f_r := \sum_{I'\in child(I)}\sum_r(\langle f_r\rangle_{I'}1_{I'}-\langle f_r\rangle_I 1_I) = \sum_{\eta\in\{0,1\}^d\setminus\{0\}}\sum_r \langle f_r, h_I^\eta\rangle h_I^\eta,$$

Wherechild (I) denotes the dyadic children of $I, \langle f_r\rangle_I := \frac{1}{|I|}\int_I \sum_r f_r\,dx$, and $\{h_I^\eta\}_{\eta\in\{0,1\}^d}$ are the Haar functions associated with I,

- the averaging operator$A_K$is definedby

$$A_K f_r(x) := \frac{1_K(x)}{|K|}\int_K \sum_r a_K(x,x')f_r(x')\,dx',$$

Whereit is assumedthat the kernels satisfy$|a_K(x,x')|\le 1$ for all $K\in\mathcal{D}, x\in K,$ and$x'\in K$.
The dyadic paraproductassociated with a function $b:\mathbb{R}^d\to\mathbb{R}$is defined by

$$\Pi_b f_r := \sum_{Q\in\mathcal{D}}\sum_r \quad D_Q b\langle f_r\rangle_Q = \sum_{Q\in\mathcal{D}}\sum_{\eta\in\{0,1\}^d\setminus\{0\}}\sum_r \quad \langle b, h_Q^\eta\rangle\langle f_r\rangle_Q h_Q^\eta \,..$$

Dyadic shifts are bounded on $L^{1+\epsilon}$ .Indeed, by Pythagoras' theorem, they are bounded on L^2, and, by using the Calder $\acute{o}$n − Zygmund decomposition, from L^1 to $L^{1,\infty}$. From the Marcinkiewicz interpolation theorem, it follows that dyadic shifts are bounded on $L^{1+\epsilon}$for $0 \le \epsilon \le \infty$, and hence, by duality, on $L^{1+\epsilon}$for $0 \le \epsilon \le \infty$

The weak-L bound with an exponential dependence on the complexity was proven by [12] and with a linear depence by [9]. It is a classical result that a dyadic paraproduct associated with a function b is bounded on $L^{1+\epsilon}$if and only if b is a BMO function.

Dyadic shifts are dyadic model operators for Calder$\acute{o}$n − Zygmund operators: [14, Lemma 2.1] proved that the Hilbert transform can be represented as a particular dyadic shift averaged over randomized dyadic systems, and [9 , Theorem 4.2] that every Calder$\acute{o}$n − Zygmund operator can be represented as a series of dyadic shifts and paraproducts averaged over randomized dyadic systems. The dyadic representation theorem for Calder$\acute{o}$n − Zygmund operators together with the boundedness of dyadic shifts and paraproducts yields another proof of the global $T1$ theorem for Calder$\acute{o}$n − Zygmund operators. For a detailed proof of the dyadic representation theorem, see the lecture notes on the A_2theorem [8].

The operator-valued setting follows the by-now-usual paradigm of doing Banach-space valued harmonic analysis beyond Hilbert space: Orthogonality of vectors is replaced with unconditionality of martingale differences, and uniform boundedness of operators with R-boundedness. Pioneering examples of this are the result by [2] and [1] that the Hilbert transform is bounded on $L^{1+\epsilon}(E)$if and only if the Banach space E has the UMD property, and the operator-valued Fourier multipliertheorems by [17].

A family of operators $\tau \subseteq \mathcal{L}(E,F)$form a Banach space$(E, |.|_E)$ to a Banach space$(F, |.|_F)$ is said to be R-boundedif there exists a constant $\mathcal{R}_{1+\epsilon}(\mathcal{T})$such that

$$\left(\mathbb{E}\left|\sum_{n=1}^{N}\varepsilon_n T_n e_n\right|_F^{1+\epsilon}\right)^{1/1+\epsilon} \le \mathcal{R}_{1+\epsilon}(\mathcal{T})\left(\mathbb{E}\left|\sum_{n=1}^{N}\varepsilon_n e_n\right|_F^{1+\epsilon}\right)^{1/1+\epsilon}$$

for all choices of operators $(T_n)_{n=1}^N \in \mathcal{T}$, and vectors $(e_n)_{n=1}^N \subseteq E$ where the expectation is taken over independent, unbiased random signs . A Banach space $(E, |.|_E)$is said tobea UMD(unconditional martingale difference) spaceif there exists a constant $\beta_{1+\epsilon}(E)$such that

$$\left\|\sum_{n=1}^{N} e_n d_n\right\|_{L^{1+\epsilon}(E)} \le \beta_{1+\epsilon}(E)\left\|\sum_{n=1}^{N} d_n\right\|_{L^{1+\epsilon}(E)}$$

For all E-valued $L^{1+\epsilon}$-martingale difference sequences$(e_n)_{n=1}^N$and for all choices of signs$(e_n)_{n=1}^N \in \{-1,+1\}^N$. It is well-known that -boundedness andUMD-property are independent (up to the involved constants) of the exponent $0 \le \epsilon \le \infty$; for an exposition on Banach-space-valued martingales, UMD spaces,
And R- boundedness, among other things, see [16].
We conclude by precisely fixing the operator-valued setting and stating the results. We define the operator-valued dyadic shifts and state their boundedness.

Definition 1.1 (see [18])(Operator-valued dyadic shift)**.** Let Ebe a UMD space. An operator-valued dyadic shiftassociated with parameter *j* and *i* is defined by

$$S^{ji}f_r := \sum_K\sum_r \quad D_K^j A_K D_K^i f_r$$

For every locally integrable function$f_r: \mathbb{R}^d \to E$, where, for each$K \in \mathcal{D}$, the averag-ing operator $A_K$associated with an operator-valued kernelis $a_K: \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(E)$ defined by

$$A_K f_r(x) := \frac{1_K(x)}{|K|}\int_K \sum_r a_K(x,x')\, f_r(x')dx.$$

The family of the operator-valued kernels is assumed to be R-bounded so that there exists a positive constant $\mathcal{R}_{1+\epsilon}(\{a\})$such that

$$\mathcal{R}_{1+\epsilon}(\{a_K(x,x') \in \mathcal{L}(E): K \in \mathcal{D}, x \in K, x' \in K\}) \le \mathcal{R}_{1+\epsilon} \quad (\{a\}).$$

Let $L^{1+\epsilon}(\mathbb{R}^{1+\epsilon}; E)$denote the Lebesgue–Bochner space, which is equipped with the norm

$$\|f_r\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon};E)} = \left(\int_{\mathbb{R}^d}\sum_r \quad |f_r(x)|_E^{1+\epsilon}dx\right)^{1/1+\epsilon} .$$

We prove the following theorem:
Theorem 1.2 (see [18])(Operator-valued dyadic shifts are bounded)**.** Let $0 < \epsilon < \infty$. Let$E$be a UMD space. LetS^{ij}be a dyadic shift with parameters i and and associatedwith the operator-valued kernelsa_K. Then

$$\|S^{ij}f_r\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon};E)} \le 4(max\{i,j\}+1)\mathcal{R}_{1+\epsilon}(\{a\})\beta_{1+\epsilon}(E)^2\|f_r\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon};E)}$$

for all $f_r \in L^{1+\epsilon}(\mathbb{R}^{1+\epsilon}; E)$

Next, we define the operator-valued Caldeón–Zygmund r operators and state the dyadic representation theorem for them. Following the paradigm of replacing orthogonality by unconditionality of martingale differences and uniform boundedness by R-boundedness, the standard estimates and the weak boundedness property are replaced by the Rademacher standard estimatesand Rademacher weak boundedness property.

Definition 1.3 (see [18])(Rademacher standard estimates)**.** An operator-valued singular kernel $k: \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x.x): x \in \mathbb{R}^d\} \to \mathcal{L}(E)$ satisfies the Rademacher standardestimatesif and only if:

(i) The kernel k satisfies the decay estimate

$$\mathcal{R}(\{k(x,y)|x-y|^d : x \in \mathbb{R}^d, y \in \mathbb{R}^d \text{with } x \neq y\}) \leq \mathcal{R}_{CZ_0}$$

for some constant $\mathcal{R}_{CZ_0}$.

(ii) The kernel k satisfies the Hölder-type estimates

$$\mathcal{R}\left(\left\{\big(k(x,y)-k(x',y)\big)\left(\frac{|x-y|}{|x-y'|}\right)^{1+2\epsilon} |x-y|^d : x \in \mathbb{R}^d, x' \in \mathbb{R}^d, y \in \mathbb{R}^d \; with \; 0 < |x-x'| < \frac{1}{2}|x-y|\right\}\right) \leq \mathcal{R}_{CZ_0}$$

And

$$\mathcal{R}\left(\left\{\big(k(x,y)-k(x,y')\big)\left(\frac{|x-y|}{|y-y'|}\right)^{1+2\epsilon} |x-y|^d : x \in \mathbb{R}^d, y \in \mathbb{R}^d, y' \in \mathbb{R}^d \; with \; 0 < |y-y'| < \frac{1}{2}|x-y|\right\}\right) \leq \mathcal{R}_{CZ_0}$$

for some Hölderexponent $\alpha \in (0,1]$ and for some constant $\mathcal{R}_{CZ_0}$.

Definition 1.4 (see [18])(Rademacher weak boundedness property)**.** An operator T mapping locally integrable E-valued functions to locally integrable E-valued functions satisfies the Rademacher weak boundedness propertyif and only if

$$\mathcal{R}\left(\left\{\frac{1}{|I|}\int_{\mathbb{R}^d} 1_I(x)T(.1_I)(x)dx \in \mathcal{L}(E): I \in \mathcal{D}\right\}\right) \leq \mathcal{R}_{WBP}$$

For some constant $\mathcal{R}_{WBP}$.

The randomized dyadic systems are defined asfollows.Let $\mathcal{D}^0$ designate the

Standard dyadic system. For every parameter $(\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} =: \Omega$ and Every $I \in \mathcal{D}^0$, the translated dyadic cube $I + \omega$ is defined by

$$I + \omega := I + \sum_{j:2^{-j}<l(I)} 2^{-j}\omega_j.$$

For each $\omega \in \Omega$, the translated dyadic system $\mathcal{D}^\omega$ is defined by $\mathcal{D}^\omega := \{I + \omega : I \in \mathcal{D}^\omega\}$.

We equip the parameter set with the natural probability measure: Each component $\omega_j \in \{0,1\}^d$ has an equal probability 2^{-d} of taking any of the 2^d values and all components are stochastically independent.

Theorem 1.5 (see [18])(Operator-valued dyadic representation theorem)**.** LetE be *a* Banach space. Let T be a singular integral operator that satisfies the Rademacher weak boundedness property and whose operator-valued kernel satisfies the Rademacher standard estimates with the Hölder exponent α. Assume that $T: L^{1+\epsilon}(\mathbb{R}^d, E) \to L^{1+\epsilon}(\mathbb{R}^d, E)$ isbounded. Then, for some dyadicshifts $S^{ij}_{\mathcal{D}^\omega}$ and for the dyadic paraproducts $\Pi^{\mathcal{D}^\omega}_{T1}$ and $\Pi^{\mathcal{D}^\omega}_{T*1}$, we have

$$\langle g_r, Tf_r\rangle = \mathbb{E}_\omega \sum_r \left(C_T \sum_{i\geq 0, j\geq 0} \sum_r 2^{(1/\epsilon)} 2^{-(1-\epsilon)(1+2\epsilon)\, max\{i,j\}} \langle g_r, S^{ij}_{\mathcal{D}^\omega} f_r\rangle + \langle g_r, \left(\Pi^{\mathcal{D}^\omega}_{T1} + \left(\Pi^{\mathcal{D}^\omega}_{T*1}\right)^*\right), f_r\rangle \right)$$

for all. $g_r \in C^1_0(\mathbb{R}^d, \mathbb{R}) \otimes E^*$ *and* $f_r \in C^1_0(\mathbb{R}^d, \mathbb{R}) \otimes E$.Moreover,

$$C_T \lesssim_{d,1+2\epsilon} \mathcal{R}_{CZ_0} + \mathcal{R}_{CZ_0} + \mathcal{R}_{WBP}.$$

Remark. Thestatementcontains an auxiliaryparameter ϵ with $0 < \epsilon < 1$. Thefactor $2^{(1/\epsilon)} 2^{-(1-\epsilon)(1+2\epsilon)\, max\{i,j\}}$ can be replaced with the facto $(1 + max\{i,j\})^{\gamma(d+(1+2\epsilon))} 2^{-(1+2\epsilon)\, max\{i,j\}}$. This is achieved by replacing the 'boundary' function

$t \mapsto t^\gamma$ with the function $t \mapsto (1 + (1+2\epsilon)^{-1} \log(t^{-1}))^{-\gamma}$ in the definition of a good dyadic cube, which then results in the decay $2^{-(1+2\epsilon)\, max\{i,j\}}$ in the estimates for the matrix elements. For the details, see the lecture notes on the A_2 theorem [8]. For simplicity, we use the function $t \mapsto t^\gamma$.

For a Banach space $(\tau, |.|_\tau)$, the $BMO_{1+\epsilon}(\mathbb{R}^d;\tau)$-norm is defined by

$$\|b\|_{BMO_{1+\epsilon}(\mathbb{R}^d;\tau)} := \sup_{Q\in\mathcal{D}} \left(\frac{1}{|Q|}\int_Q \left|b(x)-\langle b\rangle_Q\right|_\gamma^{1+\epsilon} dx\right)^{1/1+\epsilon}.$$

The following sufficient condition for the boundedness of the paraproduct$\Pi_b$associated with an operator-valued function *b* was proven by [10] by using interpolation and decoupling of martingale differences. Predecessors of this operatorvalued result (under stronger assumptions) were obtained by [7], based on unpublished ideas of Bourgain recorded by [5] in the case of a scalar-valued function *b*.

Theorem 1.6 (see [18])(Sufficient conditions for the boundedness of a paraproduct)**.** LetE be$a$ UMD space. Let$\tau \subseteq \mathcal{L}(E)$be *a* UMD subspace of$\mathcal{L}(E)$. Then

$$\|\Pi_b f_r\|_{L^{1+\epsilon}(\mathbb{R}^d;E)} \le 6.2^d(1+\epsilon)(1+\epsilon)'\beta_{1+\epsilon}(E)^2\beta_{1+\epsilon}(\tau)\|b\|_{BMO_{1+\epsilon}(\mathbb{R}^d;\tau)}\|f_r\|_{L^{1+\epsilon}(\mathbb{R}^d;E)}$$

for all$b \in BMO_{1+\epsilon}(\mathbb{R}^d;\tau)andf_r \in L^{1+\epsilon}(\mathbb{R}^d;E)$.

[18] give a different proof of Theorem 1.6. This proof is elementary in that neither interpolation nor decoupling of martingale differences is used.

By combining Theorem 1.2, Theorem 1.5, and Theorem 1.6, we obtain a new proof for the following corollary, which is a special case of Hyt$\ddot{o}$nens vector-valued, non-homogeneous, global *Tb* theorem [10, Tb theorem 4]. Earlier results of this type include the first vector-valued *T*1 theorem by [6], and the first operatorvalued *T*1 theorem by [7].

Corollary 1.7 (see [18])(T1 theorem for operator-valued kernels)**.** Let T be *a* singular integral operator that satisfies the Rademacher weak boundedness propertyandwhoseoperator-valued kernel satisfies the Rademacher standardestimates. Assume that$T1 \in BMO_{1+\epsilon}(\mathbb{R}^d;\tau)andT^*1BMO_{1+\epsilon}(\mathbb{R}^d;\tau^*)$for someUMD subspaces$\mathcal{T} \subset \mathcal{L}(E\ \ )$ and$\mathcal{T}^*\mathcal{L}(E^*)$.Then

$$\|T\|_{L^{1+\epsilon}(\mathbb{R}^d;E)\to L^{1+\epsilon}(\mathbb{R}^d;E)}$$
$$\lesssim_{\tau,d,1+\epsilon,1+2\epsilon} \left(\mathcal{R}_{CZ_0}+\mathcal{R}_{CZ_{1+2\epsilon}}+\mathcal{R}_{WBP}+\|T1\|_{BMO_{1+\epsilon}(\mathbb{R}^d;\tau)}+\|T^*1\|_{BMO_{1+\epsilon}(\mathbb{R}^d;\tau^*)}\right)\beta_{1+\epsilon}(E)^2.$$

Here the condition $T^*1 \in BMO_{1+\epsilon}(\mathbb{R}^d;\tau^*)$ is interpreted via duality as follows: There exists $b \in BMO_{1+\epsilon}(\mathbb{R}^d;\tau^*)$ such that $\left(\int_{\mathbb{R}^d} T(.h_I)(x)\ \ dx\right)^* = \int_{\mathbb{R}^d} b(x)h_I(x)\,dx$. This interpretation originates from extracting the paraproducts
as in the equation (6.1) in Section 6.

We compare our results with [15]. They study the question how the operator norm of a general vector-valued Calder$\acute{o}n$ − Zygmun doperator depends on the UMD constant. We prove that this dependence is linear for a large class of Calder$\acute{o}n$ − Zygmun operators. They prove the following estimate for vector-valued dyadic shifts:

Theorem 1.8 (see [18])(Self-adjoint vector-valued dyadic shifts depend linearly on the UMD constant [15])**.** Let $0 < \epsilon < \infty$andlet E be *a* UMD space. Let S^{ij}be *a* self-adjoint dyadic shift with parameters i and j. Then

$$\left\|S^{ij}f_r\right\|_{L^{1+\epsilon}(\mathbb{R}^d;E)\to L^{1+\epsilon}(\mathbb{R}^d;E)} \lesssim \sum_r (\max\{i,j\}+1)2^{\max\{i,j\}/2}\beta_{1+\epsilon}(E)\|f_r\|_{L^{1+\epsilon}(\mathbb{R}^d;E)}$$

for all $f_r \in L^{1+\epsilon}(\mathbb{R}^d;E)$

By the fact that an estimate for dyadic shifts can be transferred to an estimate for Calder$\acute{o}n$ − Zygmun operators by the dyadic representation theorem, their estimate for dyadic shifts then transfers to the following estimate for vector-valued Calder$\acute{o}n$ − Zygmun operators:

Theorem 1.9 (see [18])(Calder$\acute{o}n$ − Zygmun operators that have even kernel with sufficiently smoothness, and vanishing paraproduct depend linearly on the UMD constant [15])**.** Let $0 < \epsilon < \infty$andlet E be *a* UMD space. Let T be a singularintegral operator that satisfies the weak boundedness property and whose kernel satisfies the standard estimates with the Holder-exponent¨ α. Assume that the kernel is even and has smoothnessα >1/2. Assume that T satisfies thevanishing paraproduct condition$T(1) = T^*(1) = 0$. Then

$$\|T\|_{L^{1+\epsilon}(\mathbb{R}^d;E)\to L^{1+\epsilon}(\mathbb{R}^d;E)} \lesssim_{1+2\epsilon,d} C_T\beta_{1+\epsilon}(E),$$

where C_T depends only on the constants in the standard estimates and the weakboundedness property.

Now, we compare our estimate for dyadic shifts with Pott and Stoica's estimate. We note that the dependence on the complexity dictates whether the series in the dyadic representation theorem converges. On the one hand, our estimate depends linearly on the complexity, whereas theirs exponentially, which then translates into the smoothness condition $\alpha > 1/2$ in their estimate for Calderón − Zygmun operators. On the other hand, their estimate depends linearly on the UMD constant, whereas ours depends quadratically. We remark that by interpolating between our estimate and theirs (by multiplying the inequalities $\|S\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{R}\ ;E)\to L^{1+\epsilon}(\mathbb{R}\ ;E)} \lesssim k^{-\epsilon}\beta_{1+\epsilon}(E)^{2(-\epsilon)}$ and $\|S\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{R}\ ;E)\to L^{1+\epsilon}(\mathbb{R}\ ;E)} \lesssim 2^{(1+\epsilon)k/2}\beta_{1+\epsilon}(E)^{1+\epsilon}$), we obtain that

$$\|S\|_{L^{1+\epsilon}(\mathbb{R}\ ;E)\to L^{1+\epsilon}(\mathbb{R}\ ;E)} \lesssim \beta_{1+\epsilon}(E)^{1-\epsilon}k^{-\epsilon}2^{(1+\epsilon)k/2},$$

which then transfers to:

Corollary 1.10 (see [18])(Calderón-Zygmunoperators that have even kernel and vanishing paraproduct depend subquadratically on the UMD constant)**.** Let $0 < \epsilon < \infty$. Let E be *a* UMDspace. Let T be a singular integral operator thatsatisfies the weak boundedness property and whose kernel satisfies the standard estimates with the Hölder-exponent¨ α. Assume that the kernel is even. Assume that T satisfies the vanishing paraproduct conditio.$T(1) = T^*(1) = 0$Then

$$\|T\|_{L^{1+\epsilon}(\mathbb{R}\ ;E)\to L^{1+\epsilon}(\mathbb{R}\ ;E)} \lesssim_{1+2\epsilon,d} C_T \begin{cases} \frac{1}{(-\epsilon)^c}\beta_{1+\epsilon}(E)^{2(-\epsilon)} & for\ \epsilon \le -1/4, \\ \beta_{1+\epsilon}(E) & for\ \epsilon > -1/4, \end{cases}$$

for every $(1+\epsilon)$ with $\epsilon \ge 0$. Here C_T depends only on the constants in the standard estimates and the weak boundedness property.

We prove the estimate for dyadic shifts by using a martingale decoupling equality, whereas Pott and Stoica prove theirs by using the Bellman function method. At the moment, we do not know how to reproduce their result by our method nor our result by their method. A more complete understanding of both methods could yield interesting further results.

II. Preliminaries

2.1.Sum of stochastically independent conditional expectations.

Lemma 2.1 (see [18])(Sum of stochastically independent conditional expectations)**.** Let $(X_n, \mathcal{F}_n, \mu_n)$ be a probability space for each $n = 1, \ldots, N.$ denote the product probability space$(\Pi_{n=1}^N X_n, \times_{n=1}^N \mathcal{F}_n, \times_{n=1}^N \mu_n)$. Let $0 \le \epsilon \le \infty.$ Assume that $(f_r)_n \in L^{1+\epsilon}(X_n, \mathcal{F}_n, \mu_n; E)$ and that $\mathcal{G}_n$ is a sup $-\sigma-$ algebra of $\mathcal{F}_n$ for each $n = 1, \ldots, N$.

Then

$$\|\textstyle\sum_{n=1}^N \sum_r \mathbb{E}[(f_r)_n|\mathcal{G}_n]\|_{L^{1+\epsilon}(X,\mathcal{F},\mu,E)} \le \sum_r \|\sum_{n=1}^N (f_r)_n\|_{L^{1+\epsilon}(X,\mathcal{F},\mu,E)}.$$

Proof. We prove that $\mathbb{E}[(f_r)_n|\mathcal{G}_n] = \mathbb{E}[(f_r)_n|\times_{m=1}^N \mathcal{G}_n]$, from which the estimate follows by the linearity and the $L^{1+\epsilon}$-contractivity ofthe conditional expectation operator,

$$\|\textstyle\sum_{n=1}^N \sum_r \mathbb{E}[(f_r)_n|\mathcal{G}_n]\|_{L^{1+\epsilon}(X,\mathcal{F},\mu;E)} = \left\|\mathbb{E}[\sum_{n=1}^N \sum_r (f_r)_n \ |\times_{m=1}^n \mathcal{G}_n]\right\|_{L^{1+\epsilon}(X,\mathcal{F},\mu;E)} \le \sum_r \|\sum_M^N (f_r)_n\|_{L^{1+\epsilon}(X,\mathcal{F},\mu;E)}.$$

ByKolmogorov's definition of the conditional expectation, we have $\mathbb{E}[(f_r)_n|\mathcal{G}_n] = \mathbb{E}[(f_r)_n|\times_{m=1}^N \mathcal{G}_n]$ if and only if

$$\int_G \sum_r \mathbb{E}[(f_r)_n|\mathcal{G}_n]\, d\mu = \int_G \sum_r (f_r)_n\, d\mu \tag{2.1}$$

for all $G \in \times_{m=1}^N \mathcal{G}_n$. The collection of sets $G \in \times_{m=1}^N \mathcal{G}_n$satisfying the condition (2.1) is a λ-system (which means that the collection contains the empty set, is closed under taking complements and is closed under taking countable disjoint unions).Theσ-algebra $\times_{m=1}^N \mathcal{G}_n$is generated by the collection of sets $G_1 \times \ldots \times G_N$ with each, $G_n \in \mathcal{G}_n$, which is a$\pi-$ system (which means that the collection is closed under taking finite intersections). Dynkin's $\pi - \lambda$ theorem (for a proof, see, for example, the appendix of [4]) states that the λ-system and the σ-algebra both generated by the same π-system coalesce. Hence it suffices to check the condition (2.1) for the sets $G_1 \times \cdots \times G_N$with each $G_n \in \mathcal{G}_n$, which is done by using Fubini's theorem and Kolmogorov's definition of the conditional expectation,

$$\int_{G_1\times\ldots\times G_N} \textstyle\sum_r \mathbb{E}[(f_r)_n|\mathcal{G}_n] d\mu = \int_{G_n} \sum_r \mathbb{E}[(f_r)_n|\mathcal{G}_n] d\mu_n \prod_{m\ne n} d\mu_m\,(G_n) =$$
$$\int_{G_n} \textstyle\sum_r (f_r)_n \ d\mu_n \prod_{m\ne n} d\mu_m\,(G_m) = \int_{G_1\times\ldots\times G_N} \sum_r (f_r)_n d\mu.$$

2.2. Properties of R-bounds. We have collected some properties -bounds. For the proofs, see [16]. Let $(X, \mathcal{F}, \mu)$be a σ-finite measure space. Let Ebe a Banach space. Assume that $x \mapsto L_r(x)$ is an $\mathcal{L}(E)$ −valued function defined on X such that the function $x \mapsto L_r(x)e$ defined on X is strongly measurable for each $e \in E$. We define the operator$\int_X L_r(x)\, \lambda(x) d\mu(x) : E \to E\ by$

$$\left(\int_X \sum_r L_r(x)\,\lambda(x)d\mu(x)\right)e := \int_X \sum_r L_r(x)e\,\lambda(x)d\mu(x) \quad for\ all\ e \in E.$$

Proposition 2.2 (see [18])(Averaging preserves$R$-bounds)**.** Let$(X,\mathcal{F},\mu)$be a σ-finite measure space. LetSbean index set. Let$\{(L_r)_s\}_{s\in S}$be anindexed family of $\mathcal{L}(E)$ - valued functions defined on X such that the E- valued functions$x \mapsto (L_r)_s(x)e$defined on X is strongly μ-measurable for every $e \in E$ and every $s \in S$.Let$\{\lambda_s\}_{s\in S}$ be an indexed family of integrable real-valued functions. Then

$$\mathcal{R}\left(\left\{\int_X \sum_r (L_r)_s(x)e\,\lambda_s(x)d\mu(x)\right\}\right)$$
$$\leq \sup\left\{\int_X |\lambda_s(x)|\,d\mu(x)\right\}.\mathcal{R}(\{(L_r)_s(x): s \in S \text{ and } x \in X\}).$$

Proposition 2.3 (see [18])(Triangle inequality for R-bounds)**.** Let S and T be index sets.
Let$\{(L_r)_s\}_{s\in S}$and$\{M_t\}_{t\in T}$be indexed families of operators. Then

$$\mathcal{R}(\{M_s + (L_r)_t: s \in S, t \in T\}) \leq \mathcal{R}(\{M_s: s \in S\}) + \mathcal{R}(\{(L_r)_t: t \in T\}).$$

Proposition 2.4 (see [18])(Vector-valued Stein'sinequality)**.** Let E be a UMDspace. Let$(\Omega,\mathcal{F},\mathbb{P})$be a probability space. Let$(\mathcal{G}_n)_{n=1}^{\infty}$be a refining sequence of σ- algebras. Then the family

$$\{\mathbb{E}[.|\mathcal{G}_n]: L^{1+\epsilon}(\Omega;E) \to L^{1+\epsilon}(\Omega;E)\}_{n=1}^{\infty}$$

is R-bounded.Moreover,
$\mathcal{R}_{1+\epsilon}(\{\mathbb{E}[.|\mathcal{G}_n]: L^{1+\epsilon}(\Omega;E) \to L^{1+\epsilon}(\Omega;E)\}_{n=1}^{\infty}) \leq \beta_{1+\epsilon}(E)$.

2.3.Pythagoras' theorem for functions adapted to a sparse collection. Let μ be a Borel measure on $\mathbb{R}^d$. We use the notation$\langle f_r\rangle_Q^{\mu} := \frac{1}{\mu(Q)} f_r\,d\mu$. Let S be a collection of dyadic cubes. Foreach $S \in S$, let $ch_S(S)$ denote the collection of allmaximal$S' \in S$such that$S' \subseteq S$ and Let $E_S(S)$ denote the set $E_S(S) = S\setminus \cup_{S'\in ch_S(S)} S'$. For each $Q \in \mathcal{D}$, let $\pi_S(Q)$denote the minimal cube such that
$S \supseteq Q$. We say that the collection S is *sparse* if $\mu(E_S(S)) \geq \frac{1}{2}\mu(S)$ for every $S \in S$.

Lemma 2.5 (see [18])(Specialcase of the dyadic Carleson embedding theorem)**.** Let$E$be
a Banach space. Let $0 < \epsilon < \infty$. Assume that S is a sparse collection Then.

$$\left(\sum_{S\in S}\sum_r \left(\langle|f_r|_E\rangle_S^{\mu}\right)^{1+\epsilon}\mu(S)\right)^{1/2} \leq 2\frac{1+\epsilon}{\epsilon}\sum_r \|f_r\|_{L^{1+\epsilon}(\mu;E)}$$

Proof. For thedyadicHardy–Littlewoodmaximal function $M^{\mu}f_r = sup_{Q\in\mathcal{D}}1_Q\langle f_r\rangle_Q^{\mu}$, wehave $\langle|f_r|_E\rangle_S^{\mu} \leq inf_S M^{\mu}|f_r|_E$, and, moereover, $\|M|f_r|_E\|_{L^{1+\epsilon}(\mu)} \leq \frac{1+\epsilon}{\epsilon}\||f_r|_E\|_{L^{1+\epsilon}(\mu)}$. These factstogetherwiththeassumptions, Yield

$$\left(\sum_{S\in S}\sum_r \left(\langle|f_r|_E\rangle_S^{\mu}\right)^{1+\epsilon}\mu(S)\right)^{1/1+\epsilon} \leq 2^{1/1+\epsilon}\left(\sum_{S\in S}\int_{E_S(S)}\sum_r \left(\inf_S M^{\mu}|f_r|_E\right)d\mu\right)^{1/1+\epsilon}$$
$$\leq 2^{1/1+\epsilon}\sum_r \|M^{\mu}|f_r|_E\|_{L^{1+\epsilon}(\mu;E)} \leq 2^{1/1+\epsilon}\frac{1+\epsilon}{\epsilon}\sum_r \|f_r\|_{L^{1+\epsilon}(\mu;E)} = 2^{1/1+\epsilon}\frac{1+\epsilon}{\epsilon}\sum_r \|f_r\|_{L^{1+\epsilon}(\mu;E)}.$$

For each $S \in$ S, we define the operator $P_S$by setting

$$P_S f_r := \sum_{Q\in\mathcal{D}:\pi(Q)=S}\sum_r D_Q f_r$$

for every locally integrable $f_r : \mathbb{R}^d \to E$.

Lemma 2.6 (see [18])(Properties of the operators P_S)**.** *For each* $S \in$ S, the operator P_S hasthe following properties:

(i) $P_S f_r = \sum_{S'\in ch_S(S)}\sum_r \langle f_r\rangle_{S'}1_{S'} + \sum_r f_r 1_{E_S(S)} - \sum_r \langle f_r\rangle_S 1_S$.

(ii) $P_S f_r = f_r$ if andonlyif f_r is supportedon *S*, constant oneach $S' \in ch_S(S)$, and satisfies$\int_S \sum_r f_r\,d\mu = 0$.

(iii) $P_S^2 = P_S, P_S P_T = 0$whenever$T \in S$with$T \in S$.

(iv) $\int g_r\sum_r P_S\ f_r\,d\mu = \int\sum_r P_{S}g_r\,f_r\,d\mu$for every$f_r \in L^{1+\epsilon}(E)$ and $g_r \in L^{\frac{1+\epsilon}{\epsilon}}(E^*)$.

(v) $\sum_r \|P_S f_r\|_{L^{1+\epsilon}(E)} \leq 2\sum_r \|1_S f_r\|_{L^{1+\epsilon}(E)}$.

Proof. We prove the property (i), from which the other properties follow. On the one hand,

$$f_r 1_S = \sum_{Q:Q\subseteq S}\sum_r D_Q f_r + \sum_r \langle f_r\rangle_S 1_S,$$

on the other hand,

$$f_r 1_S = f_r 1_{E_S(S)} + \sum_{S'} \sum_r \quad f_r 1_{S'} = f_r 1_{E_S(S)} + \sum_{S' \in ch_S(S)} \sum_r \quad \left(\sum_{Q:Q \subseteq S'} D_Q f_r + \langle f_r \rangle_{S'} 1_{S'} \right).$$

Thus, by comparing,

$$\sum_{Q:Q \subseteq S} \sum_r \quad D_Q f_r - \sum_{S' \in ch_S(S)} \sum_{Q:Q \subseteq S'} \sum_r \quad D_Q f_r = f_r 1_{E_S(S)} + \sum_{S' \in ch_S(S)} \sum_r \quad + \langle f_r \rangle_{S'} 1_{S'} - \sum_r \quad \langle f_r \rangle_S 1_S.$$

Observing that

$$\sum_{Q:Q \subseteq S'} - \sum_{S' \in ch_S(S)} \sum_{Q:Q \subseteq S'} \quad = \sum_{Q:\pi_S(Q)=S}$$

completes the proof.

The following variant of Pythagoras' theorem in the case $E = \mathbb{R}$ was proven by [11, Lemma 7] by using a multilinear estimate. We next give a different proof of the theorem, which extends it to an arbitrary Banach space *E*.

Lemma 2.7 (see [18])(Pythagoras' theorem for sparsely supported, piecewise constant functions)**.** Let E be a Banach space. Let $0 \leq \epsilon < \infty$. Let S be a sparse collectionof dyadic cubes. For each $S \in S$, (assume) that $f_r s$ is a function that is supported on S and constant on each $S' \in ch_S(S)$. Then

$$\left\| \sum_S \sum_r \quad (f_r)_S \right\|_{L^{1+\epsilon}(E)} \leq 3(1+\epsilon) \sum_r \quad \left(\sum_S \| (f_r)_S \|_{L^{1+\epsilon}(E)}^{1+\epsilon} \right)^{1/1+\epsilon}$$

Moreover, the reverse estimate

$$\left(\sum_S \sum_r \quad \| f_r s \|_{L^{1+\epsilon}(E)}^{1+\epsilon} \right)^{1/1+\epsilon} \leq 6 \frac{1+\epsilon}{\epsilon} \sum_r \quad \left\| \sum_S f_r s \right\|_{L^{1+\epsilon}(E)}$$

Holds if, in addition, one of the following conditions is satisfied:

$$\text{(i)} \int_S \sum_r \quad (f_r)_S \, d\mu = 0, \text{ or (ii) } E = \mathbb{R} \text{ and } (f_r)_S \geq 0,$$

But may in general fail otherwise.

Proof. First, we prove the direct estimate. By duality, it is equivalent to the esti-mate

$$\int \sum_S \sum_r \quad f_r s g_r \, d\mu \leq 3(1+\epsilon) \left(\sum_S \sum_r \quad \| f_r s \|_{L^{1+\epsilon}(E)}^{1+\epsilon} \right)^{1/1+\epsilon} \| g_r \|_{L^{\frac{1+\epsilon}{\epsilon}}(E)}.$$

Since $(f_r)_S$is supportedon S, sinse $f_r s$is constant on $S' \in ch_S(S)$, and since S is partitioned by ch_S, we have

$$\int \sum_S \sum_r \quad f_r s g_r \, d\mu$$

$$= \sum_S \int_S \sum_r \quad (f_r) \quad s g_r \, d\mu$$

$$= \sum_S \sum_{S' \in ch_S(S)} \sum_r \quad \langle f_r s \rangle_{S'} \int_{S'} g_r \, d\mu + \int \sum_S \sum_r \quad 1_{E_S(S)} (f_r)_r s g_r \, d\mu.$$

We can estimate the second term by Hölder'sinequality and the pairwise disjointness of the sets $E_S(S)$,

$$\left| \int \sum_S 1_{E_S(S)} \sum_r \quad (f_r) \quad s g_r \, d\mu \right| \leq \sum_r \quad \left\| \sum_S 1_{E_S(S)} (f_r) s \right\|_{L^{1+\epsilon}(E)} \quad \| g_r \|_{L^{1+\epsilon}(E^*)}$$

$$= \sum_r \quad \left(\sum_S \left\| 1_{E_S(S)} (f_r) s \right\|_{L^{1+\epsilon}(E)}^{1+\epsilon} \right)^{1/1+\epsilon} \quad \| g_r \|_{L^{\frac{1+\epsilon}{\epsilon}}(E^*)}.$$

We can estimate the first term as follows.

$$\left| \sum_S \sum_{S' \in ch_S(S)} \sum_r \quad \langle (f_r) s \rangle_{S'} \int_{S'} g_r \, d\mu \right|$$

$$\leq \sum_S \sum_{S' \in ch_S(S)} \sum_r \quad |\langle (f_r) s \rangle_{S'}|_E \, \mu(S')^{1/1+\epsilon} \frac{\left| \int_{S'} g_r d\mu \right|_{S'}}{\mu(S')} \mu(S')^{1/\frac{1+\epsilon}{\epsilon}}$$

$$\leq \left(\sum_S \sum_{S' \in ch_S(S)} \sum_r \quad |\langle (f_r) s \rangle_{S'}|_E^{1+\epsilon} \, \mu(S') \right)^{1/1+\epsilon} \left(\sum_S \sum_{S' \in ch_S(S)} \left(\frac{\int_{S'} |g_r|_{E^*} \, d\mu}{\mu(S')} \right)^{\frac{1+\epsilon}{\epsilon}} \mu(S') \right)^{1/\frac{1+\epsilon}{\epsilon}}$$

$$\leq \left(\sum_S \int_S \sum_r \langle (f_r)s\rangle_E^{1+\epsilon} \, d\mu\right)^{1/1+\epsilon} \left(\sum_S \sum_{S'\in ch_S(S)} \sum_r \langle |g_r|_{E^*}\rangle_{S'}^{\frac{1+\epsilon}{\epsilon}} \mu(S')\right)^{1/\frac{1+\epsilon}{\epsilon}}$$

The proof of the direct estimate is completed by the special case of the dyadic Carleson embedding theorem, Lemma 2.5.

Next, we prove the reverse estimate under the assumption that$\int_S (f_r)_s = 0$. By duality, this estimate is equivalent to the estimate

$$\sum_S \int \sum_r (f_r)sg_rs \, d\mu \leq 6\frac{1+\epsilon}{\epsilon}\sum_r \left\|\sum_S f_rs\right\|_{L^{1+\epsilon}(E)} \left(\sum_S \|g_rs\|_{L^{\frac{1+\epsilon}{\epsilon}}(E^*)}^{\frac{1+\epsilon}{\epsilon}}\right)^{1/\frac{1+\epsilon}{\epsilon}}$$

For arbitraryfunctions $g_rs \in L^{\frac{1+\epsilon}{\epsilon}}(E^*)$ .Bythepropertiesof the operators P_s , Lemma 2.6,we have that $\int \sum_r f_rsg_rs \, d\mu = \int \sum_r P_S f_rsg_rs \, d\mu = \int \sum_r P_S^2 f_rsg_rs \, d\mu = \int \sum_r P_S(f_r)_s P_{sg_rs} \, d\mu = \int \sum_r P_S(f_r)_S \sum_T P_T g_{r_T} \, d\mu = \int \sum_r f_rs \sum_T P_{Tg_rT} \, d\mu$.. Note that, although the functions$(g_r)_T$ are arbitrary, the Functions$P_T(g_r)_T$ satisfy the assumptions for the direct estimate: Each $P_T(g_r)_T$ is supported on$T$, and constant on each $T' \in ch_S(T)$. Thus, by Hölder's inequality and the direct estimate,

$$\begin{aligned}
\sum_S \int \sum_r f_rs(g_r)_rs \, d\mu &= \int \sum_S \sum_r f_rs \sum_{T\in S} P_T(g_r)_T \, d\mu \\
&\leq \sum_r \left\|\sum_S f_rs\right\|_{L^{1+\epsilon}(E)} \left\|\sum_{T\in S} P_T(g_r)_T\right\|_{L^{\frac{1+\epsilon}{\epsilon}}(E^*)} \\
&\leq 3\frac{1+\epsilon}{\epsilon}\sum_r \left\|\sum_S f_rs\right\|_{L^{1+\epsilon}(E)} \left(\sum_{T\in S} \|P_T(g_r)_T\|_{L^{\frac{1+\epsilon}{\epsilon}}(E^*)}^{\frac{1+\epsilon}{\epsilon}}\right)^{1/\frac{1+\epsilon}{\epsilon}} \\
&\leq 6\frac{1+\epsilon}{\epsilon}\sum_r \left\|\sum_S f_rs\right\|_{L^{1+\epsilon}(E)} \left(\sum_{T\in S} \|(g_r)_T\|_{L^{\frac{1+\epsilon}{\epsilon}}(E^*)}^{\frac{1+\epsilon}{\epsilon}}\right)^{1/\frac{1+\epsilon}{\epsilon}}
\end{aligned}$$

Next, we prove the reverse estimate under the assumption that $E = \mathbb{R}$ and $f_rs \geq 0$. Since f_rs issupported$S$ on S, sincef_rs is constanon $S' \in ch_S(S)$,
since $S$is partitioned by $ch_S(S)$and $E_S(S)$and since $\mu(E') \in 2\mu(E_S(S'))$we can write

$$\begin{aligned}
\|f_rs\|_{L^{1+\epsilon}(\mathbb{R})}^{1+\epsilon} &= \sum_{S'\in ch_S(S)} \sum_r |\langle f_rs\rangle_{S'}|^{1+\epsilon} \mu(S') + \int \sum_r 1_{E_S(S)} |f_rs|^{1+\epsilon} \, d\mu \\
&\leq 2\sum_r \sum_{S'\in ch_S(S)} \langle |f_rs|\rangle_{S'}^{1+\epsilon} \mu(E_S(S')) + \sum_r \int 1_{E_S(S)} \, |f_rs|^{1+\epsilon} d\mu \\
&= 2\int \sum_{S'\in ch_S(S)} \sum_r 1_{E_S(S')} \, |f_rs|^{1+\epsilon} d\,\mu + \int \sum_r 1_{E_S(S)} \, |f_rs|^{1+\epsilon} d\mu.
\end{aligned}$$

Summing over S and taking into account that $E_S(S)$are pairwise disjoint yields

$$\sum_S \sum_r \|f_rs\|_{L^{1+\epsilon}(\mathbb{R})}^{1+\epsilon} \leq 3\int \sum_r \left(\sum_S 1_{E_S(S')} \, |f_rs|\right)^{1+\epsilon} d\mu.$$

Using the assumption that $f_rs \geq 0$completes the proof.

We note that a simpleexample shows that the reverse estimate may in generalfail $\mathbb{R}$.Indeed, let $S \coloneqq [0,1), S_- \coloneqq [0, 1/2), f_rs \coloneqq 1_{S_-}$, and$f_rs_- = -1s_-$. Then $\|f_rs\|_{L^{1+\epsilon}(\mathbb{R};\mathbb{R})}^{1+\epsilon} + \|f_rs_-\|_{L^{1+\epsilon}(\mathbb{R};\mathbb{R})}^{1+\epsilon} = 2|S_-|$ but $\|f_rs + f_rs_-\|_{L^{1+\epsilon}(\mathbb{R};\mathbb{R})} = 0$

III. Decoupling of the sum of martingale differences

For$(X, \mathcal{F}, \mu)$be a σ-finite measure space. Let $(\mathcal{A}_n)_{n=-\infty}^{\infty}$be a refining
Sequence of countable partitions of Xinto measurable sets of finite positive
measure. Let $\mathcal{A} \coloneqq \bigcup_{n=-\infty}^{\infty} \mathcal{A}_n$for eachlet $\text{child}_{\mathcal{A}}(K) \coloneqq \{K' \in \mathcal{A}_{n+1}: K' \subseteq K\}$. For each $K \in \mathcal{A}$ Let $(f_r)_K$ be a function that is supported on K and constant on $K' \in \text{child}_{\mathcal{A}}(A)$ and such that$\int_K \sum_r (f_r)_k \, d\mu = 0$ Let $(Y_K, \mathcal{G}_K, v_K)$ Let be the probability space such that $Y_K \coloneqq K, \mathcal{G}_K$ is the σ-algebra generated by$\{K\}\cup\text{child}_{\mathcal{A}}(K)$, and $v_K = \mu(K)^{-1}\mu|K$. Let $(Y, \mathcal{G}, v)$be the product probability space of the spaces $(Y_K, \mathcal{G}_K, v_K)_{K\in\mathcal{A}}$,
We notice that the sequence$(d_K)_{K=-\infty}^{\infty}$ with $d_k(x, \mathcal{Y}) \coloneqq \sum_{K\in\epsilon_{K-1}} (f_r)_k(x)$is a

Martingale difference sequence adaptedtothe filtration$(\mathcal{F}_k)_{k=-\infty}^{\infty}$generated by the refining sequenceof partitions$(\mathcal{A}_k)_{k=-\infty}^{\infty}$. Conversely,each martingale difference sequence$(d_k)_{k=-\infty}^{\infty}$ adapted to thefiltrationcan be written as$d_k := \sum_{K\in\mathcal{A}_{n=1}}(f_r)_k$, where for each $K \in \mathcal{A}_{k-1}$the function $(f_r)_k$ is defined by $(f_r)_k := 1_K d_k = \sum_{K'\in\mathcal{A}_k:K'\subseteq K}\langle d_x\rangle_{k'} 1_{k'}$.

A variant of the following decoupling equality was proven by [10, Theorem 6.1] as a corollary of [13, Theorem 2.2] decoupling inequality for UMD-valued martingale difference sequences.

Theorem 3.1 (see [18])(Decoupling equality for piecewise constant, cancellative functions)**.**
Let $0 < \epsilon < \infty$. Let E be a UMD space. Then

$$\frac{1}{\beta_{1+\epsilon}(E)}\left(\mathbb{E}\left\|\sum_{K\in\mathcal{A}}\sum_{r} \in_K 1_K(x)(f_r)_K(\mathcal{Y}_K)\right\|_{L^{1+\epsilon}(d\mu(x)\times dv(\mathcal{Y});E)}^{1+\epsilon}\right)^{1/1+\epsilon}$$

$\leq \|\sum_{K\in\mathcal{A}}\sum_r \quad (f_r)_K(x)\|_{L^{1+\epsilon}(d\mu(x);E)}$.

$$\leq \beta_{1+\epsilon}(E)\left(\mathbb{E}\left\|\sum_{K\in\mathcal{A}}\sum_{r} \in_K 1_K(x)(f_r)_K(\mathcal{Y}_K)\right\|_{L^{1+\epsilon}(d\mu(x)\times dv(\mathcal{Y});E)}^{1+\epsilon}\right)^{1/1+\epsilon}$$

Here we give another proof of theequality: Roughly speaking, we construct auxiliary martingale differences $u_K(x,\mathcal{Y}_K)$ such that $(f_r)_K(x) = u_K(x,\mathcal{Y}_K) + v_A(x,\mathcal{Y}_A)$and $1_K(x)(f_r)_K(\mathcal{Y}_K) = u_K \quad (x,\mathcal{Y}_K) - v_K(x,\mathcal{Y}_K)$,from which the decoupling equality follows by the definition of the UMD property. Let $d_K$be a martingale difference sequence adapted to the filtration $\mathcal{F}_K$. We write

$$d_K(x,\mathcal{Y}) = \sum_{K\in\mathcal{A}_{k-1}} 1_K(x)\, d_K(x) 1_K(\mathcal{Y}_K)$$

and

$$\tilde{d}_K(x,\mathcal{Y}) := \sum_{K\in\mathcal{A}_{k-1}} 1_K(x)\, d_K(x) 1_K(\mathcal{Y}_K).$$

Proposition 3.2 (see [18])(Constructingauxiliary martingale differences)**.** There exists a martingale difference sequence$(u_K)_{K\in\frac{1}{2}\mathbb{Z}}$on the product measure space$(X\times Y,\mathcal{F}\times\mathcal{G},\mu\times\nu)$such that

$$d_K = u_K + u_{K+1/2}, \qquad \text{and} \qquad \tilde{d}_K = u_K - u_{K-1/2}.$$

Proof. Let d_K be a martingale difference sequence d_K adapted to the filtration $\mathcal{F}_K$ generated by a refining sequence of partitions A_K. The $\mathcal{F}_K$-measurability of $d_K$means that $d_K$equals to a constant $\langle d_K\rangle_K$ on $K \in \mathcal{A}_k$. Thus, we can write

$$d_K = \sum_{K\in\mathcal{A}_{k-1}} 1_K d_K = \sum_{K\in\mathcal{A}_{k-1}} 1_K \sum_{\substack{K'\in\mathcal{A}_k\\ K'\subseteq K}} 1_{\acute{K}} d_K, = \sum_{K\in\mathcal{A}_{k-1}} 1_K \sum_{\substack{K'\in\mathcal{A}_k\\ K'\subseteq K}} 1_{K'}\langle d_K\rangle_{K'}.$$

The martingale difference property$\mathbb{E}[d_k|\mathcal{F}_{k-1}] = 0$ means that for every $K \in \mathcal{A}_{k-1}$we have

$$\int_K d_k d_\mu = \sum_{\substack{K'\in\mathcal{A}_k\\ K'\subseteq K}} \langle d_k\rangle_{K'}\,\mu(K') = 0.$$

First, we consider a fixed $K \in \mathcal{A}_{k-1}$. Let v_K be the measure$v_K := \mu(K)^{-1}\mu|K$ resctrictedto the sub-σ-algebra $\mathcal{Y}_K$ that is generated by the collection $\{K\}\cup\{K' \in \mathcal{A}_k : K' \subseteq K\}$. Note thet the functions

$$d_k(x,\mathcal{Y}_K) := 1_K(x) d_k(x) 1_K(\mathcal{Y}_K) = \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K}} \langle d_k\rangle_A\, 1_A(x) 1_B(\mathcal{Y}_K)$$

And

$$\tilde{d}_K(x,\mathcal{Y}_K) := 1_K(x) d_K(\mathcal{Y}_K) 1_K(\mathcal{Y}_K) = \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K}} \langle d_k\rangle_B\, 1_A(x) 1_B(\mathcal{Y}_K)$$

Are equally distributed in the measure space $(\mathbb{R}^d \times K, \mu\times v_K, \mathcal{F}\times\mathcal{Y}_K)$,which is to say that the functions take the same values in sets of equal measure. We define the functions $u_K(x,\mathcal{Y}_K)$and $v_K(x,\mathcal{Y}_K)$by the pair of equations

$$d_K(x,\mathcal{Y}_K) =: u_K(x,\mathcal{Y}_K) + v_K(x,\mathcal{Y}_K),$$
$$\tilde{d}_K(x,\mathcal{Y}_K) =: u_K(x,\mathcal{Y}_K) - v_K(x,\mathcal{Y}_K).$$

Therefore, the function $u_K(x,\mathcal{Y}_K)$can be written out as

$$u_K(x,\mathcal{Y}_K) = \frac{1}{2}\left(d_K(x,\mathcal{Y}_K) + \tilde{d}_K(x,\mathcal{Y}_K)\right)$$
$$= \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K}} \frac{1}{2}(\langle d_k\rangle_A + \langle d_k\rangle_B) 1_A(x) 1_B(\mathcal{Y}_K)$$

$$= \sum_{\substack{A\in\mathcal{A}_k\\ A\subseteq K}} \langle d_k\rangle_A \, 1_{A\times A}(x,\mathcal{Y}_K)$$

$$+ \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K; A<B}} \frac{1}{2}(\langle d_k\rangle_A + \langle d_k\rangle_B)1_{A\times B\cup B\times A}(x,\mathcal{Y}_K)$$

where in the last step we introduced some order among the finite family

$$\{A\in\mathcal{A}_K : A\subseteq K\} = \{A_i\}_{i=1}^{I(K)}$$

and defined $A < B$ if and only if $A = A_i, B = A_j$. And $i < j$.The function $v_K(x,\mathcal{Y}_K)$can be written out as

$$v_K(x,\mathcal{Y}_K) = \frac{1}{2}\left(d_K(x,\mathcal{Y}_K) - \tilde{d}_K(x,\mathcal{Y}_K)\right)$$

$$= \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K}} \frac{1}{2}(\langle d_k\rangle_A - \langle d_k\rangle_B)1_A(x)1_B(\mathcal{Y}_K)$$

$$= \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K; A<B}} \frac{1}{2}(\langle d_k\rangle_A - \langle d_k\rangle_B)\left(1_{A\times B}(x,\mathcal{Y}_K) - 1_{B\times A}(x,\mathcal{Y}_K)\right)$$

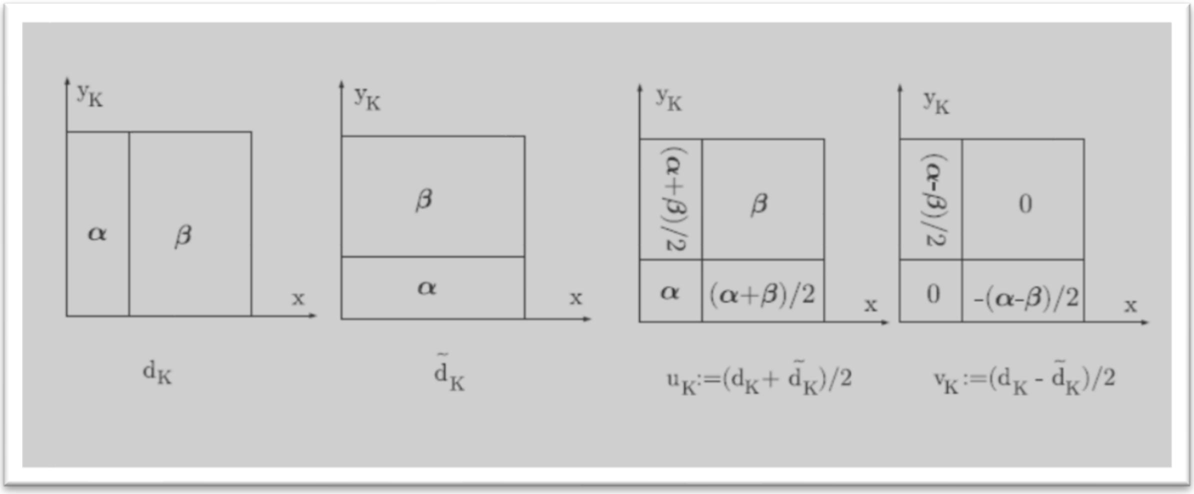

Figure 1[18]. Functions $d_K, \tilde{d}_K, u_K$, and v_K.

Next,we define the productmeasure space.For each $K\in\mathbb{Z}$and$K\in\mathcal{A}_{k-1}$let $(Y_K,\mathcal{G}_K,v_K)$be the probability space such that $Y_K := K, \mathcal{G}_K$is the σ-algebra
generated by$\{K\}\subseteq\{K'\in\mathcal{A}_k : K'\subseteq K\}$, and $v_K = \mu(K)^{-1}\mu|K$. Let $(Y,\mathcal{Y},v)$be the product probability space of the spaces$(Y_K,\mathcal{G}_K,v_K)$. We recall that the product space $Y$is the Cartesian product $Y = \prod_{K\in\mathcal{A}} Y_K$the product σ-algebra $\mathcal{G}$in the case of a countable index set) is the collection

$$\mathcal{G} := \left\{\prod_{K\in\mathcal{A}}^{*} G_K : G_K\in\mathcal{G}_K\right\}$$

and the product measure v is the unique measure on $(Y,\mathcal{G})$that satisfies

$$v\left(\prod_{K\in\mathcal{A}}^{*} G_K\right) = \prod_{K\in\mathcal{A}}^{*} v_K\,(G_K),$$

where $*$in the product indicates thatfor only finitely manyG_K we have $G_K \neq Y_K$.
Next, we prove that the sequence$(\ldots, u_k, u_{k+1/2}, u_{k+1}, u_{k+1+1/2}, \ldots)$defined by

$$u_K(x,\mathcal{Y}) := \textstyle\sum_{K\in\mathcal{A}_{k-1}} u_K(x,\mathcal{Y}_K),$$

and

$$v_k(x,\mathcal{Y}) := u_{k+1/2}(x,y) = \sum_{K\in\mathcal{A}_{k-1}} v_K(x,\mathcal{Y}_K)$$

is a martingale difference sequence in the measure space$(X\times Y,\mathcal{F}\times\mathcal{G},\mu\times v)$. Proving this is based on the following observations:

(a) For each $K\in\mathcal{A}_{k-1}$, the function $u_k(x,\mathcal{Y})$depends on $x$and $\mathcal{Y}_K$"in a symmetric way" (seeFigure 1);

(b) For each $K \in \mathcal{A}_{k-1}$, the function $u_k(x,\mathcal{Y})$depends on x and $\mathcal{Y}_K$"in an anti-symmetric way"− (see Figure 1);

(c) The function $u_K$averages to zero on the set $K \times K$because $d_k$itself is a martingale difference; Indeed,

$$\int_K \int_K u_K(x,\mathcal{Y})\, d\mu(x) dv_K(\mathcal{Y}_k)$$

$$= \frac{1}{\mu(K)}\left[\sum_{\substack{A\in\mathcal{A}_k\\ A\subseteq K}} \langle d_k\rangle_A\, \mu(A)\mu(A) + \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K, A<B}} \frac{1}{2}(\langle d_k\rangle_A + \langle d_k\rangle_B).2\mu(A)\mu(B)\right]$$

$$= \frac{1}{\mu(K)} \sum_{\substack{A\in\mathcal{A}_k\\ A\subseteq K}} \langle d_k\rangle_A\, \mu(A) \sum_{\substack{B\in\mathcal{A}_k\\ B\subseteq K}} \mu(B) = \sum_{\substack{A\in\mathcal{A}_k\\ A\subseteq K}} \langle d_k\rangle_A\, \mu(A) = \int_K d_k(x)\, d\mu(x) = 0$$

(d) The function v_K takes equal positive and negative values on two halves of the symmetric sets $A \times B \cup B \times A$with $A < B$, whereas the function u_k takes equal values on both the halves. Moreover, the function v_K takes zero value on the symmetric sets $A \times A$. Thus, for any function $\emptyset(u_K)$, we have

$$\int_{K\times K} u_K \emptyset(u_K)\, d\mu dv_K$$

$$= \sum_{\substack{A,B\in\mathcal{A}_k\\ A,B\subseteq K, A<B}} \left(\int_{A\times B} u_K d\mu dv_K + \int_{B\times A} u_K d\mu dv_K\right) \langle\emptyset(u_K)\rangle^{\mu\times v_K}_{A\times B\cup B\times A} = 0$$

Where theaverage$\langle\emptyset(u_K)\rangle^{\mu\times v_K}_{A\times B\cup B\times A}$ denotes the constant value of $\emptyset(u_K)$on the set$A \times B \cup B \times A$.

We define the filtration$(\mathcal{U}_k)_{k\in\frac{1}{2}\mathbb{Z}}$as follows. For each $k \in \mathbb{Z}$, we define the σ- algebra$\mathcal{U}_k$ as the σ-algebra generated by the functions $\{u_j\}_{j;j\leq k}$ and $\{1_K\}_{K\in\mathcal{A}_{k-1}}$ and, similarly, the σ-algebra $\mathcal{U}_{k+1/2}$ as the σ-algebra generated by the functions and $\{1_K\}_{K\in\mathcal{A}_{k-1}}$. We note that the functions $\{1_K\}_{K\in\mathcal{A}_{k-1}}$ are Included for technical reasons: They ensure that each $\mathcal{U}_k$ with $k \in \frac{1}{2}\mathbb{Z}$, is σ-finite so that taking the conditional expectation with respect to it makes sense. Now, by definition, each u_k is measurable with respect to $\mathcal{U}_k$, and, furthermore,$\{\mathcal{U}_k\}_{k\in\frac{1}{2}\mathbb{Z}}$is a filtration. Next, we check that $\mathbb{E}[u_k|\mathcal{U}_{k-1/2}] = 0$which is equivalent to checking that

$$\int_{X\times Y} u_k\, \emptyset\left(\{u_l\}_{l:l\leq k-1/2}, \{1_K\}_{K\in\mathcal{A}_{[k-1/2]-1}}\right) d\mu dv = 0$$

for all functions $\emptyset(\dots, u_l, \dots, u_{k-1/2}, 1_K, \dots) =: \left(\{u_l\}_{l:l\leq k-1/2}, \{1_K\}_{K\in\mathcal{A}_{[k-1/2]-1}}\right)$

First, we checkthat $\mathbb{E}[u_k|\mathcal{U}_{k-1/2}] = 0$for $k \in \mathbb{Z}$. We have

$$\int_{X\times Y} u_k\, \emptyset(\{u_l\}_{l:l\leq k-1/2}, \{1_K\}_{K\in\mathcal{A}_{k-2}}) d\mu dv$$

$$= \sum_{K\in\mathcal{A}_{k-1}} \int_{X\times Y} u_K\, \emptyset(\{u_l\}_{l:l\leq k-2}, \{1_K\}_{K\in\mathcal{A}_{k-2}}) d\mu dv$$

Each of the functions$\{u_l(x,\mathcal{Y})\}_{l:l\leq k-1/2}$, and $\{1_{K'}(x)\}_{K'\in\mathcal{A}_{k-2}}$is constant with respect to $x \in K \in \mathcal{A}_{k-1}$; We denote these constant values by their Averages. Moreover, $u_K(x,\mathcal{Y})$dependson $\mathcal{Y}$only by $\mathcal{Y}_K$. Therefore, by pulling out the constant, and integrating out the independent variables, we obtain

$$\int_{X\times Y} u_K\, \emptyset(\{u_l\}_{l:l\leq k-1/2}, \{1_K\}_{K\in\mathcal{A}_{k-2}}) d\mu \times dv$$

$$= \int_{X\times Y} u_K\, d\mu dv$$

$$\cdot \int_{\prod_{\substack{K'\in\mathcal{A}:\\ K'\neq K}} Y_{K'}} \emptyset(\{\langle u_l\rangle^{\mu}_K\}_{l:l\leq k-1/2}, \{\langle 1_{K'}\rangle^{\mu}_K\}_{K'\in\mathcal{A}_{k-2}}) \prod_{\substack{K'\in\mathcal{A}:\\ K'\neq K}} dvK'$$

The observation (c) states that$\int_{K\times Y_K} u_K\, d\mu dv_K = 0$.

Finally, we check that $\mathbb{E}[u_{k+1}|\mathcal{U}_k] = 0$ for $k \in \mathbb{Z}$. Again, we have

$$\int_{X\times Y} u_{k+1/2}\,\emptyset\big(\{u_l\}_{l:l\le k},\{1_K\}_{K\in\mathcal{A}_{k-1}}\big)d\mu dv$$
$$=\sum_{K\in\mathcal{A}_{k-1}}\int_{K\times Y} u_K\,\emptyset\big(\{u_l\}_{l:l\le k},\{1_K\}_{K\in\mathcal{A}_{k-1}}\big)\,d\mu dv.$$

Note that eachof thefunctions$\{u_l(x,\mathcal{Y})\}_{l:l\le k-1}$, and $\{1_K(x)\}_{K'\in\mathcal{A}_{k-1}}$is constant with respect to $x\in K\in\mathcal{A}_{k-1}$; We denote these constant values by Their averages. furthermore, $u_k(x,\mathcal{Y})=u_K(x,\mathcal{Y})+\sum_{\substack{K'\in\mathcal{A}_{k-1}:\\ K'\ne K}} u_{K'}(x,\mathcal{Y})=u_K(x,\mathcal{Y})$ for $x\in K\in\mathcal{A}_{k-1}$. Moreover,$u_K(x,\mathcal{Y})$depends on $\mathcal{Y}$ only by$\mathcal{Y}_K$.

Therefore, again by pulling out the constant, and integrating out the independent variables, we obtain

$$\int_{X\times Y} u_K\,\emptyset\big(\{u_l\}_{l:l\le k},\{1_K\}_{K\in\mathcal{A}_{k-1}}\big)d\mu dv$$
$$=\int_{K\times Y_K} u_K\,\Phi_K(u_K)d\mu dv_k$$

Where

$$\Phi_K(u_K):=\int_{\prod_{\substack{K'\in\mathcal{A}:\\ K'\ne K}} Y_{K'}}\Phi\big(\{\langle u_l\rangle_K^{\mu}\}_{l:l\le k-1},\{\langle 1_{K'}\rangle_K^{\mu}\}_{K'\in\mathcal{A}_{k-1}}\big)\prod_{\substack{K'\in\mathcal{A}:\\ K'\ne K}} dvK'$$

The observation (d) states that$\int_{X\times Y_K} u_K\,\Phi_K(u_K)d\mu dv_K=0$.

IV. Vector-valued dyadic shifts are bounded

Let$L:=\max\{i,j\}+1$. By picking every L:thlengthscale,we decompose the collection $\mathcal{D}$of dyadic cubes to subcollections $\mathcal{D}_{l \bmod L}$, with $l=0,\dots,L-1$,such that forevery $K\in\mathcal{D}_{l \bmod L}$ we have that both $D_K^i f_r$and $D_K^j g_r$are constant on $K'\in child_{\mathcal{D}_{l \bmod L}}(K)$ and have zero average on K. More specifically for each $l=1,\dots,L-1$ let $\mathcal{D}_{l \bmod L}=\bigcup_{K=-\infty}^{\infty}\{K\in\mathcal{D}:\ell(K)=2^{-KL+l}\}$.Then

$$S^{ji}f_r=\sum_{K\in\mathcal{D}}\sum_r D_K^j A^{ji} D_K^i f_r=\sum_{l=0}^{L-1}\sum_{K\in\mathcal{D}_{l \bmod L}}\sum_r D_K^j A^{ji} D_K^i f_r$$

This decomposition is done in orderto decouple by using Theorem3.1. We consider a fixed l. We write $e_K:=D_K^j A^{ji} D_K^i f_r$. Let $dv_K(x)=|K|^{-1}1_K(x)dx$be the Lebesgue measure restricted and normalized to the dyadic cube K. Let $v$denote the product measure $\times_{K\in\mathcal{A}} v_K$on the product space.$Y:=\prod_{K\in\mathcal{A}} K$ By Theorem 3.1,

$$\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} 1_K(x)\,e_K(x)\right\|_{L^{1+\epsilon}(dx,E)}^{1+\epsilon}\le\beta_{1+\epsilon}(E)^{1+\epsilon}\mathbb{E}\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} e_K 1_K(x)e_K(\mathcal{Y}K)\right\|_{L^{1+\epsilon}(dx\times dv(\mathcal{Y});E)}^{1+\epsilon}$$

We write$e_K(\mathcal{Y}_K)=\sum_r D_K^j A_K D_K^i f_r(\mathcal{Y}K)=:\sum_r D_K^j(g_r)_K(\mathcal{Y}K)$. By using Lemma 2.1 together with the fact that $D_K^j$is a difference of two conditional expectations, we obtain

$$\left\|e_K 1_K(x)\sum_{K\in\mathcal{D}_{l \bmod L}}\sum_r D_K^j(g_r)_K(\mathcal{Y}K)\right\|_{L^{1+\epsilon}(dv(\mathcal{Y});E)}^{1+\epsilon}$$
$$\le 2^{1+\epsilon}\left\|\sum_{K\in\mathcal{D}_{l \bmod L}}\sum_r e_k\;1_K(x)(g_r)_K(\mathcal{Y}K)\right\|_{L^{1+\epsilon}(dx\times dv(\mathcal{Y});E)}^{1+\epsilon}$$

We write$(g_r)_K(\mathcal{Y}_K)=\sum_r A_K D_K^j f_r(\mathcal{Y}K)=:\sum_r A_K(f_r)_K(\mathcal{Y}_K)$. By introducing an independent copy $(\tilde{Y},\tilde{v})$ of the probability space (Y,v),we write

$$A_K(f_r)_K(\mathcal{Y}K)\frac{1_K(\mathcal{Y}K)}{|K|}\int_K\sum_r a_K(\mathcal{Y}_K,x')\,(f_r)_K(x')dx'$$
$$=\int_{\tilde{Y}}\sum_r 1_K(\mathcal{Y}K)\,a_K\big(\mathcal{Y}K,\tilde{\mathcal{Y}}K\big)(f_r)_K(x')d\tilde{v}(\tilde{\mathcal{Y}}).$$

By Jensen's inequality,

$$\left|\int_{\tilde{Y}} \sum_{K\in\mathcal{D}_{l \bmod L}} \sum_{r} e_K 1_K(x) a_K(\mathcal{Y}K, \tilde{\mathcal{Y}}K)(f_r)_K(\tilde{\mathcal{Y}}K) d\tilde{v}(\tilde{\mathcal{Y}})\right|_E^{1+\epsilon}$$

$$\leq \int_{\tilde{Y}} \left|\sum_{K\in\mathcal{D}_{l \bmod L}} \sum_{r} e_K 1_K(x) a_K(\mathcal{Y}K, \tilde{\mathcal{Y}}K)(f_r)_K(\tilde{\mathcal{Y}}K)\right|_E^{1+\epsilon} d\tilde{v}(\tilde{\mathcal{Y}}).$$

Since the family of operators $\{a_K(x,x'): K \in \mathcal{D}, x \in K, x' \in K\}$is R-bounded, we have

$$\mathbb{E}\left|\left|\sum_{K\in\mathcal{D}_{l \bmod L}} \sum_r e_K 1_K a_K(\mathcal{Y}K, \tilde{\mathcal{Y}}K)(x)(f_r)_K(\tilde{\mathcal{Y}}K)\right|\right|_E^{1+\epsilon} \leq$$

$$\mathcal{R}_{1+\epsilon}(\{a\})^{1+\epsilon}\mathbb{E}\left|\sum_{K\in\mathcal{D}_{l \bmod L}} \sum_r e_K 1_K(x)(f_r)_K(\tilde{\mathcal{Y}}K)\right|_E^{1+\epsilon}.$$

Altogether, we have obtained that

$$\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} 1_K(x) e_K(x)\right\|_{L^{1+\epsilon}(dx;E)}^{1+\epsilon}$$

$$\leq \left(2\beta_{1+\epsilon}(E)\mathcal{R}_{1+\epsilon}(\{a\})\right)^{1+\epsilon}\int_Y \mathbb{E}\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} \sum_r e_K 1_K(x) D_K^i(f_r)(\tilde{\mathcal{Y}}K)\right\|_{L^{1+\epsilon}(dx\times d\tilde{v}(\tilde{\mathcal{Y}});E)}^{1+\epsilon} dv(\mathcal{Y})$$

$$= \left(2\beta_{1+\epsilon}(E)\mathcal{R}_{1+\epsilon}(\{a\})\right)^{1+\epsilon}\mathbb{E}\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} \sum_r e_K 1_K(x) D_K^i(f_r)(\tilde{\mathcal{Y}}K)\right\|_{L^{1+\epsilon}(dx\times d\tilde{v}(\tilde{\mathcal{Y}});E)}^{1+\epsilon}.$$

Since $D_K^i D_K^m = 0$ whenever $i \neq m$, we canwrite$D_K^i = D_K^i \sum_{m=0}^{L-1} D_K^m$. By using Lemma 2.1 together with the fact that $D_K^j$is a difference of two conditional expectations, we obtain

$$\mathbb{E}\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} \sum_r e_K 1_K(x) D_K^i \sum_{m=0}^{L-1} D_K^m(f_r)(\tilde{\mathcal{Y}}K)\right\|_{L^{1+\epsilon}(dx\times d\tilde{v}(\tilde{\mathcal{Y}});E)}^{1+\epsilon}$$

$$\leq 2^{1+\epsilon}\mathbb{E}\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} e_K 1_K(x) \sum_{m=0}^{L-1}\sum_r D_K^m(f_r)(\tilde{\mathcal{Y}}K)\right\|_{L^{1+\epsilon}(dx\times d\tilde{v}(\tilde{\mathcal{Y}});E)}^{1+\epsilon}$$

We have that$\sum_{m=0}^{L-1}\sum_r D_K^m f_r$ is constant on $K' \in child_{\mathcal{D}_{l \bmod L}}(K)$and has zero average on K. Therefore, by removing the decoupling using Theorem 3.1, we obtain

$$\mathbb{E}\left\|\sum_{K\in\mathcal{D}_{l \bmod L}} e_K 1_K(x) \sum_{m=0}^{L-1}\sum_r D_K^m f_r(\tilde{\mathcal{Y}}K)\right\|_{L^{1+\epsilon}(dx\times d\tilde{v}(\tilde{\mathcal{Y}});E)}^{1+\epsilon}$$

$$\leq \left\|\sum_{K\in\mathcal{D}_{l \bmod L}} \sum_{m=0}^{L-1}\sum_r D_K^i f_r(x)\right\|_{L^{1+\epsilon}(dx;E)}^{1+\epsilon} = \left\|\sum_{K\in\mathcal{D}}\sum_r D_K f_r\right\|_{L^{1+\epsilon}(\mathbb{R}^d;E)}^{1+\epsilon} = \sum_r \|f_r\|_{L^{1+\epsilon}(\mathbb{R}^d;E)}^{1+\epsilon}.$$

The proof is completed.

V. Sufficient condition for the boundedness of dyadic paraproducts

From the fact that $\left\|\langle f_r\rangle_{Q_0} 1_{Q_0}\right\|_{L^{1+\epsilon}(E)} \to 0$ as $\ell(Q_0) \to \infty$ it follows that the functions of the form $f_r \coloneqq \sum_{Q_0}(f_r)_{Q_0} \coloneqq \sum_{Q_0}\left(f_r - \langle f_r\rangle_{Q_0}\right) 1_{Q_0}$, where Q are disjoint dyadic cubes, are dense in$L^{1+\epsilon}(E)$. Hence it suffices to prove the estimate

$$\left\|\sum_{Q\in\mathcal{D}(Q_0)}\sum_r D_Q b\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(\mathbb{R}^d;E)}$$
$$\leq 6.2^d(1+\epsilon)(\frac{1+\epsilon}{\epsilon})\beta_{1+\epsilon}(E)^2\beta_{1+\epsilon}(T)\|b\|_{BMO_{1+\epsilon}(\mathbb{R}^d;T)}\sum_r \|f_r\|_{L^{1+\epsilon}(\mathbb{R}^d;E)}$$

Uniformlyforall $Q_0\in\mathcal{D}$.Now,we fixadyadic Q_0. Let $\mathcal{D}(Q_0):=\{Q\in\mathcal{D}: Q\subseteq Q_0\}$. Let $S:=S(Q_0)\subseteq\mathcal{D}(Q_0)$ be a sparse collection that contains the cube Q_0.For each $Q\in\mathcal{D}$, let $\pi_S(Q)$denote the minimal dyadic cube $S\in S$such that $S\supseteq Q$. We rearrange the summation as$\sum_{Q\in\mathcal{D}(Q_0)}=\sum_{S\in S}\sum_{\substack{Q\in\mathcal{D}(Q_0):\\ \pi(Q)=S}}$. By the variant ofPythagoras' theorem, Theorem 2.7, we obtain

$$\left\|\sum_{Q\in\mathcal{D}(Q_0)}\sum_r D_Q b\,\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(E)} = \left\|\sum_{S\in S}\sum_{\substack{Q\in\mathcal{D}(Q_0):\\ \pi(Q)=S}}\sum_r D_Q b\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(E)}.$$
$$\leq 3(1+\epsilon)\left(\sum_{S\in S}\left\|\sum_{\substack{Q\in\mathcal{D}(Q_0):\\ \pi(Q)=S}}\sum_r D_Q b\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}\right)^{1/1+\epsilon}$$

It remains to choose the sparse collection Sso that

(5.1) $\quad\left\|\sum_{\substack{Q\in\mathcal{D}(Q_0):\\ \pi(Q)=S}}\sum_r D_Q b\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(E)}\leq C_{b,E,1+\epsilon,d}\ \sum_r \langle |f_r|_E\rangle_S|S|^{1/1+\epsilon},$

Which, by the special case of the dyadic Carleson embedding theorem, Lemma 2.5, completes the proof by the estimate

$$\left(\sum_{S\in S}\left\|\sum_{\substack{Q\in\mathcal{D}(Q_0):\\ \pi(Q)=S}}\sum_r D_Q b\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}\right)^{1/1+\epsilon}\leq C_{b,E,1+\epsilon,d}\left(\sum_{S\in S}\sum_r \langle |f_r|_E\rangle_S^{1+\epsilon}|S|\right)^{1/1+\epsilon}$$
$$\leq C_{b,E,1+\epsilon,d}2\,(\frac{1+\epsilon}{\epsilon})\sum_r \|f_r\|_{L^{1+\epsilon}(E)}.$$

Next, we choose the collection S so that the estimate (5.1) is satisfied. For each $S\in\mathcal{D}$, let $ch_S(S)$be the collection of all the maximal dyadic subcubes $S'\nsubseteq S$such that

(5.2) $\quad\langle |f_r|_E\rangle_{S'}>2\langle |f_r|_E\rangle_S$.

By the dyadic nestedness andmaximality, thecollection $ch_S(S)$is pairwisedisjoint. We define recursively $S_0:=\{Q_0\}$and $S_{n+1}:=\bigcup_{S\in S_n}ch_S(S)$. Let $S:=\bigcup_{n=0}^{\infty}S_n$. Wedefine the pairwise disjoint sets $E_S(S):=S\setminus\bigcup_{S'\in ch_S(S)}S'$.By construction,

$$\sum_{S'\in ch_S(S)}|S'|\leq\frac{1}{2}|S|,$$

which is to say that $|E_S(S)|\geq\frac{1}{2}|S|$. Hence the collection $S$is sparse.

Next, we chack that $\int_Q\sum_r f_r\,dx=\int_S\sum_r f_r\,dx$ for $(f_r)_S:=f_r1_{E_S(S)}+\sum_{S'\in ch_S(S)}\sum_r \langle f_r\rangle_{S'}1_{S'}$ whenever $\pi(Q)=S$. Firstly, the set $Q$is partioned by $E_S(S)\cap Q$and$\{S'\in ch_S(S): S'\cap Q\neq\emptyset\}$. Secondly, by the dyadic nestedness,$S'\cap Q\neq\emptyset$implies that either $Q\subseteq S'$or$S'\nsubseteq Q$. The alternative$Q\subseteq S'$ is excluded because $\pi_S(S)=Q$means that S is the minimal $S''\in S$such that $Q\subseteq S''$. Hence $S'\nsubseteq Q$ for all $S'\in Q$with $S'\cap Q\neq\emptyset$. Therefore

$$\int_Q\sum_r f_r\,dx=\int_Q\sum_r f_r1_{E_S(S)}+\sum_{S'\in ch_S(S):S'\nsubseteq Q}\int_{S'}\sum_r f_r\,dx$$
$$=\int_Q\sum_r f_r1_{E_S(S)}+\sum_{S'\in ch_S(S):S'\nsubseteq Q}\int_Q\sum_r \langle f_r\rangle_{S'}1_{S'}\,dx=\int_Q\sum_r (f_r)_S\,dx.$$

Next, we check that $|(f_r)_S|_E\leq 2.2^d\langle |f_r|_E\rangle_S$almost everywhere. First,let$x\in E_S(S)$. Then, by construction,for all $Q\in\mathcal{D}$ such that $Q\ni x$ we have $\langle |f_r|_E\rangle_Q\leq 2\langle |f_r|_E\rangle_S$. Therefore, by the Lebesgue differentiation theorem,$|f_r(x)|_E\leq 2\langle |f_r|_E\rangle_S$. for almost every such x. Let $S'\in$ child(S) By the maximalityof S',the dyadic parent S' of S'satisfies the opposite $\langle|f_r|_E\rangle_{S'} \leq 2\langle|f_r|_E\rangle_S$of the inequality (5.2). By doubling,$\langle|f_r|_E\rangle_{S'} \leq 2^d\langle|f_r|_E\rangle_{\hat{S}'}$Altogether$\langle|f_r|_E\rangle_{S'} \leq 2.2^d\langle|f_r|_E\rangle_{S'}$. Altogether, we have

$$\left\| \sum_{\substack{Q\in\mathcal{D}(Q_0):\\ \pi(Q)=S}} \sum_r D_Q b\langle f_r\rangle_Q \right\|_{L^{1+\epsilon}(E)} = \left\| \sum_{\substack{Q\in\mathcal{D}(Q_0):\\ \pi(Q)=S}} \sum_r D_Q b\langle (f_r)_S\rangle_Q \right\|_{L^{1+\epsilon}(E)}$$

With$\|(f_r)_S\|_{L^\infty(E)} \leq 2.\,2^d\langle|f_r|_E\rangle_S$. The proof is completed by Lemma 5.1.

Lemma 5.1 (see [18]). Let $0<\epsilon<\infty$. Let$E$be$a$UMDspace.Assume that$\mathcal{T}$is$a$UMD subspace of$\mathcal{L}(E)$. Let$S$ be a dyadic cube and let $Q(S)$be a collection of dyadic subcubes of$S$ Then

$$\left\| \sum_{Q\in\mathcal{Q}(S)} \sum_r D_Q b\langle f_r\rangle_Q \right\|_{L^{1+\epsilon}(E)} \leq \beta_{1+\epsilon}(E)^2\beta_{1+\epsilon}(\mathcal{T})\|b\|_{BMO_{1+\epsilon}(\mathcal{T})}\sum_r \|f_r\|_{L^\infty(S;E)}|S|^{1/1+\epsilon}$$

for any$f_r \in L^\infty(S;E)$and $b\in BMO_{1+\epsilon}(\mathbb{R}^d;\mathcal{T})$.

Proof without the decoupling equality. By the UMD property and the Kahane con-traction principle, we obtain $\left\|\sum_{Q\in\mathcal{Q}(S)}\sum_r D_Q b\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(e)}^{1+\epsilon} \leq \beta_{1+\epsilon}(E)^{1+\epsilon}\mathbb{E}\left\|\sum_{Q:Q\subseteq S}\sum_r \epsilon_Q D_Q b\langle f_r\rangle_Q 1_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}$.

We expand

$$D_Q b = \sum_{\eta\in\{0,1\}^d\backslash\{0\}} \langle b, h_Q^\eta\rangle h_Q^\eta.$$

where, for each $Q = I_1\times\ldots\times I_d$ and $\eta=(\eta_1,\ldots,\eta_d)\in\{0,1\}^d$, we have $h_Q^\eta = h_{I_1}^{\eta_1}\ldots h_{I_d}^{\eta_d}$ with $h_I := h_I^1 := \frac{1}{\sqrt{|I|}}\left(1_{I_{le(f_r)t}} - 1_{I_{rigrht}}\right)$ and $h_I^0 = \frac{1}{\sqrt{|I|}}1_I$. Therefore

$$\left(\mathbb{E}\left\|\sum_{Q\subseteq S}\sum_r \epsilon_Q D_Q b\langle f_r\rangle_Q 1_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}\right)^{1/1+\epsilon}$$

$\leq \sum_{\eta\in\{0,1\}^d\backslash\{0\}}\left(\mathbb{E}\left\|\sum_{Q\subseteq S}\sum_r \epsilon_Q h_Q^\eta\langle\langle b,h_Q^\eta\rangle f_r\rangle_Q 1_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}\right)^{1/1+\epsilon}$.

Next,weconsidera fixedη. We observe that, at each point $x\in\mathbb{R}^d$, we have $h_Q^\eta(x) = \pm|h_Q^\eta(x)|$and that $|h_Q^\eta|$is constant on Q. Hence

$$\mathbb{E}\left\|\sum_{Q\subseteq S}\sum_r \epsilon_Q h_Q^\eta\langle\langle b,h_Q^\eta\rangle f_r\rangle_Q 1_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon} = \mathbb{E}\left\|\sum_{Q\subseteq S}\sum_r \epsilon_Q\langle\langle b,h_Q^\eta\rangle|h_Q^\eta|f_r\rangle_Q 1_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}$$

By the vector-valuedSteininequality, and the observationthat, at each point$x\in\mathbb{R}^d$, we have $h_Q^\eta(x)=\pm|h_Q^\eta(x)|$, we obtain

$$\mathbb{E}\left\|\sum_{Q\subseteq S}\sum_r \epsilon_Q\langle\langle b,h_Q^\eta\rangle|h_Q^\eta|f_r\rangle 1_Q\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon} \leq \beta_{1+\epsilon}(E)^d\left\|\sum_{Q\subseteq S}\sum_r \epsilon_Q\langle b,h_Q^\eta\rangle|h_Q^\eta|f_r\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}$$

$$= \beta_{1+\epsilon}(E)^{1+\epsilon}\left\|\sum_{Q\subseteq S}\epsilon_Q\langle b,h_Q^\eta\rangle h_Q^\eta f_r\right\|_{L^{1+\epsilon}(E)}^{1+\epsilon}$$

By assumption, we have $b:\mathbb{R}^d\to\mathcal{T}$with $\mathcal{T}\subseteq\mathcal{L}(E)$. By the pointwise norm estimate,

$$\left\|\sum_{Q\subseteq S}\sum_r \epsilon_Q\langle b,h_Q^\eta\rangle h_Q^\eta f_r\right\|_{L^{1+\epsilon}(E)} \leq \sum_r \|f_r\|_{L^\infty(E)}\left\|\sum_{Q\subseteq S}\epsilon_Q\langle b,h_Q^\eta\rangle h_Q^\eta\right\|_{L^{1+\epsilon}(\mathcal{T})}$$

We can view $\langle b,h_Q^\eta\rangle h_Q^\eta$as a subsequence of a martingale difference sequence (thanks to Emil Vuorinen for pointing this out!).We spli Qinto two subsets Q_+^η andQ_-^η according to the value ofh_Q^η,

$Q_+^\eta := \bigcup_{\substack{Q'\in child(Q):\\ \langle h_Q^\eta\rangle_{Q'}=+|Q|^{-1/2}}} Q'andQ_-^\eta := \bigcup_{\substack{Q'\in child(Q):\\ \langle h_Q^\eta\rangle_{Q'}=-|Q|^{-1/2}}} Q'$

The corresponding martingale differences are

$$U_Q^\eta b := -\langle b\rangle_Q 1_Q + \left(\langle b\rangle_{Q_-^\eta}1_{Q_-^\eta} + \langle b\rangle_{Q_+^\eta}1_{Q_+^\eta}\right)$$

And

$$V_Q^\eta b := \left(\langle b\rangle_{Q_-^\eta}1_{Q_-^\eta} + \langle b\rangle_{Q_+^\eta}1_{Q_+^\eta}\right) + \sum_{Q'\epsilon child(Q)}\sum_r \langle f_r\rangle_{Q'}1_Q$$

By construction, $D_Q b = U_Q^\eta b + V_Q^\eta b$ and $U_Q^\eta b = \langle b, h_Q^\eta\rangle h_Q^\eta$. Hence, for any signs ϵ_Q, we have

$$\left\|\sum_{Q\subseteq S} \epsilon_Q \langle b, h_Q^\eta\rangle h_Q^\eta\right\|_{L^{1+\epsilon}(\mathcal{T})} = \left\|\sum_{Q\subseteq S}\left(\epsilon_Q U_Q^\eta b + 0.V_Q^\eta b\right)\right\|_{L^{1+\epsilon}(\mathcal{T})} \le \beta_{1+\epsilon}(\mathcal{T})\left\|\sum_{Q\subseteq S}\left(U_Q^\eta b + V_Q^\eta b\right)\right\|_{L^{1+\epsilon}(\mathcal{T})}$$

$$= \beta_{1+\epsilon}(\mathcal{T})\left\|\sum_{Q\subseteq S} D_Q b\right\|_{L^{1+\epsilon}(\mathcal{T})}$$

We can expand $\sum_{Q\subseteq S} D_Q b = 1(b - \langle b\rangle_S)$. By the definition of the BMO space,

$$\|1_S(b - \langle f_r\rangle_S)\|_{L^{1+\epsilon}(\mathcal{T})} \le \|b\|_{BMO_{1+\epsilon}(\mathcal{T})}|S|^{1/1+\epsilon}.$$

Proof with the decoupling equality. By the decoupling equality, Theorem 3.1,

$$\left\|\sum_{Q\in\mathcal{Q}(S)}\sum_r D_Q b\,\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(dxd\mu(\mathcal{Y}))}^{1+\epsilon}$$

$$\le \beta_{1+\epsilon}(E)^{1+\epsilon}\mathbb{E}\left\|\sum_{Q\in\mathcal{Q}(S)}\sum_r \epsilon_Q\, D_Q b(\mathcal{Y}Q)1_Q(x)\langle f_r\rangle_Q\right\|_{L^{1+\epsilon}(dxd\mu(\mathcal{Y}))}^{1+\epsilon}$$

Now, at each point $\mathcal{Y}Q \in Q$, we have $D_Q b(\mathcal{Y}Q)\langle f_r\rangle_Q = \langle D_Q b(\mathcal{Y}Q) f_r\rangle_Q$. By the vector-valued Stein inequality,

$$\mathbb{E}\left\|\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq S}}\sum_r \epsilon_Q\, 1_Q(x)\langle D_Q b(\mathcal{Y}Q) f_r\rangle_Q\right\|_{L^{1+\epsilon}(dx;E)}^{1+\epsilon}$$

$$\le \beta_{1+\epsilon}(E)^{1+\epsilon}\mathbb{E}\left\|\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq S}}\sum_r \epsilon_Q\, 1_Q(x) D_Q b(\mathcal{Y}Q) f_r(x)\right\|_{L^{1+\epsilon}(dxd\mu(\mathbb{E});E)}^{1+\epsilon}$$

By the pointwise norm estimate,

$$\mathbb{E}\left\|\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq S}}\sum_r \epsilon_Q\, 1_Q(x) D_Q b(\mathcal{Y}Q) f_r(x)\right\|_{L^{1+\epsilon}(dxd\mu(\mathcal{Y});\mathcal{T})}^{1+\epsilon}$$

$$\le \sum_r \|f_r\|_{L^\infty(E)}^{1+\epsilon}\mathbb{E}\left\|\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq S}} \epsilon_Q\, 1_Q(x) D_Q b(\mathcal{Y}Q)\right\|_{L^{1+\epsilon}(dxd\mu(\mathcal{Y});\mathcal{T})}^{1+\epsilon}.$$

By the decoupling equality, Theorem 3.1,

$$\left\|\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq S}} \epsilon_Q 1_Q(x) D_Q b(\mathcal{Y}Q)\right\|_{L^{1+\epsilon}(dxd\mu(\mathcal{Y});\mathcal{T})}^{1+\epsilon} \le \beta_{1+\epsilon}(\mathcal{T})^{1+\epsilon}\left\|\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq S}} D_Q b(x)\right\|_{L^{1+\epsilon}(dx;\mathcal{T})}^{1+\epsilon}.$$

Remark. In the scalar-valued setting, we obtain the following proof of the boundedness of the dyadic paraproduct: Let S be the collection of dyadic cubes that is iteratively chosen by the condition $\langle|f_r|\rangle_{S'} > 2\langle|f_r|\rangle_S$. Hence $\left|\langle f_r\rangle_Q\right| \le 2\langle|f_r|\rangle_S$ whenever $\pi_S(Q) = S$. From the variant of Pythagoras' theorem (Lemma 2.7), Burkholder's inequality, and the special case of the dyadic Carleson embedding theorem (Lemma 2.5) it follows that

$$\left\|\sum_Q\sum_r \langle f_r\rangle_Q D_Q b\right\|_{L^{1+\epsilon}(\mathbb{R}^d;\mathbb{R})} \le 3(1+\epsilon)\left(\sum_S\left\|\sum_{\pi_S(Q)=S}\sum_r \langle f_r\rangle_Q\, D_Q b\right\|_{L^{1+\epsilon}(\mathbb{R}^d;\mathbb{R})}^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$\le 3(1+\epsilon)2\beta_{1+\epsilon}(\mathbb{R})\left(\sum_S\sum_r \langle|f_r|\rangle_S^{1+\epsilon}\left\|\sum_{Q:Q\subseteq S} D_Q b\right\|_{L^{1+\epsilon}(\mathbb{R}^d;\mathbb{R})}^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$\le 6(1+\epsilon)\beta_{1+\epsilon}(\mathbb{R})\|b\|_{BMO_{1+\epsilon}(\mathbb{R}^d;\mathbb{R})}\left(\sum_r\sum_S \langle|f_r|\rangle^{1+\epsilon}|S|\right)^{1/1+\epsilon}$$

$$\le 6(1+\epsilon)\beta_{1+\epsilon}(\mathbb{R})\|b\|_{BMO_{1+\epsilon}(\mathbb{R}^d;\mathbb{R})}2\left(\frac{1+\epsilon}{\epsilon}\right)\sum_r \|f_r\|_{L^{1+\epsilon}(\mathbb{R}^d;\mathbb{R})}$$

Note that $\beta_{1+\epsilon}(\mathbb{R}) = \max\left\{1+\epsilon, \frac{1+\epsilon}{\epsilon}\right\} - 1$, which was proven by Burkholder [3].

VI. Vector-valued dyadic representation theorem

The proof of the vector-valued dyadic representation theorem follows verbatim the proof of the scalar-valued one that is given in Hytönen's lecture notes on the A_2 theorem [8], except for the estimation of matrix elements: In the scalar-valued case, the absolute value of the matrix elements (which are real numbers) is estimated, whereas in the vector-valued case, the R-bound of the matrix elements (which are operators) needs to be estimated. For readability, we have sketched the whole proof here.

6.1.Expanding the dual pairing by means of dyadic shifts. By expanding$g_r \in L^{\frac{1+\epsilon}{\epsilon}}(\mathbb{R}^d; E^*)$ as

$$g_r = \sum_{J\in\mathcal{D}}\sum_r \quad D_J g_r = \sum_{J\in\mathcal{D}}\sum_{\eta=1}^{2^d-1}\sum_r \quad \langle g_r, h_J^\eta\rangle h_J^\eta\,,$$

where$h_J^\eta$with $\eta = 1, \dots, 2^d - 1$and $J \in \mathcal{D}$are L^2-normalized Haar functions, and similarly, $f_r \in L^{1+\epsilon}(\mathbb{R}^d; E)$the dual pairing is written as

$$\langle g_r, Tf_r\rangle = \textstyle\sum_{I\in\mathcal{D},J\in\mathcal{D}}\sum_r \quad \langle g_r, h_J\rangle\,\langle h_I, Th_I\rangle\langle h_I, f_r\rangle.$$

The index ηwill be suppressed from now on. To control the relative arrangement of I and J and whence the size of matrix elements, the notion of a good dyadic cube is introduced.

Definition 6.1 (see [18])(Gooddyadic cube)**.** Fixa boundary exponent$\gamma \in (0,1)$and an ancestor threshold $r \in \mathbb{N}$. A dyadic cube $I \in \mathcal{D}$is goodif we have

$$\operatorname{dist}(I, K^c) > \left(\frac{\ell(I)}{\ell(K)}\right)^\gamma \ell(K)$$

for every dyadic ancestor $K \in \mathcal{D}$ of the dyadic cube I such that $\ell(K) \geq 2^r\ell(I)$.

To restrict to the good cubes in the dual pairing, the randomized dyadic systems are introduced. Let $\mathcal{D}^0$designate the standard dyadic system. For every parameter $(\omega_i)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} =: \Omega$and every $I \in \mathcal{D}^0$, the translated dyadic cube $I + \omega$ is defined by

$$I + \omega := I + \textstyle\sum_{j:2^{-j}<l(I)} 2^{-j}\omega_i.$$

For each$\omega \in \Omega$, the translated dyadic system $\mathcal{D}^\omega$is defined by$\mathcal{D}^\omega := \{I + \omega : I \in \mathcal{D}^0\}$. The parametersetis equipped with the naturalprobability

Measure: Each component $\omega_j \in \{0,1\}^d$ has an equal probability $2^{-d}$of taking any of the 2^d values and all components are stochastically independent. By construction, the position and the goodness of a dyadic cube $I \in \omega$ are stochastically independent. Also by construction, the probability $P_\omega(\{I + \omega \in \mathcal{D}^\omega \text{ is good }\}) =: \pi_{good}$does not depend on $I \in \mathcal{D}^0$, and, as calculated in [8, Lemma 2.3] ,

$$\pi_{good} \geq 1 - \frac{8d}{\gamma}2^{-r\gamma}$$

Inparticular, for any boundary exponent $\gamma \in (0,1)$ we canmake the probability π_{good} strictly positive by choosing the ancestor threshold $r \in \mathbb{N}$sufficiently large.

The following proposition was proven by [8, Proposition 3.5]. (For an earlier version of the proposition, see [9, Theorem 3.1].)

Proposition 6.2 (see [18])(Discarding the bad cubes)**.** Assume that$T: L^{1+\epsilon}(\mathbb{R}^d; \mathbb{R}) \to L^{1+\epsilon}(\mathbb{R}^d; \mathbb{R})$is bounded. Then

$$\langle \beta, T(1+2\epsilon)\rangle = \frac{1}{\pi_{good}}\mathbb{E}_\omega \sum_{\substack{I\in\mathcal{D}^\omega, J\in\mathcal{D}^\omega:\\ smaller\{I,J\} is good}} \langle \beta, h_J\rangle\,\langle h_J, Th_I\rangle\langle h_I, 1+2\epsilon\rangle$$

For all. $\beta \in C_0^1(\mathbb{R}^d; \mathbb{R})$ and $\alpha \in C_0^1(\mathbb{R}^d; \mathbb{R})$.

Let $C_0^1(\mathbb{R}^d; \mathbb{R}) \otimes E$ denote the set of all finite linear combinations of the form

$$f_r = \sum_{\pi_{good}}^{I} \alpha_i e_i \text{ with } \alpha_i \in C_0^1(\mathbb{R}^d; \mathbb{R}) \text{and } e_i \in E,$$

set which is dense in$L^{1+\epsilon}(\mathbb{R}^d; E)$. By linearity, Theorem 6.2 extends to vector-valued functions.

Corollary 6.3 (see [18]). Let E be a Banach space. Assume that $T: L^{1+\epsilon}(\mathbb{R}^d; E) \to L^{1+\epsilon}(\mathbb{R}^d; E)$is bounded. Then

$$\langle g_r, Tf_r\rangle = \frac{1}{\pi_{good}}\mathbb{E}_\omega \sum_{\substack{I\in\mathcal{D}^\omega, J\in\mathcal{D}^\omega:\\ \text{smaller}\{I,J\}\text{is good}}} \sum_r \quad \langle g_r, h_J\rangle\,\langle h_J, Th_I\rangle\langle h_I, f_r\rangle$$

for all$g_r \in C_0^1(\mathbb{R}^d; \mathbb{R}) \otimes E^*andf_r \in C_0^1(\mathbb{R}^d; \mathbb{R}) \otimes E$.

Next, the paraproducts are extracted. The dyadic system $\mathcal{D}^\omega$is suppressed in the notation from now on. Consider the summation

$$\sum_{\substack{I\in\mathcal{D}^{\omega},J\in\mathcal{D}^{\omega}:\\ \text{smaller}\{I,J\}\text{is good}}}\sum_{r}\langle g_r,h_J\rangle\langle h_J,Th_I\rangle\langle h_I,f_r\rangle$$

In the case '$I\nsubseteq J$', the paraproduct $\prod_{T^*1}^{*}$ is extracted as follows: Let J_I denote the dyadic childof J that contains I. Then

$$\langle h_J,Th_I\rangle=\langle 1_{J_I^c}h_J,Th_I\rangle+\langle h_I\rangle_I\langle 1_{J_I},Th_I\rangle=\langle 1_{J_I^c}\big(h_J-\langle h_I\rangle_I\big),Th_I\rangle+\langle h_J\rangle_I\langle 1_{J_I}+1_{J_I^c},Th_I\rangle$$

Summing the last term yields

(6.1)

$$\sum_{I,J:I\nsubseteq J}\sum_{r}\langle g_r,h_J\rangle\langle h_J\rangle_I\langle 1,Th_I\rangle\langle h_I,f_r\rangle=\sum_{I}\langle\sum_{J:J\supsetneq I}\sum_{r}\langle g_r,h_J\rangle h_J\rangle_I\langle 1,Th_I\rangle\langle f_r,h_I\rangle$$

$$=\langle\sum_{I}\sum_{r}\langle g_r\rangle_I\langle 1,Th_I\rangle h_I,f_r\rangle=:\sum_{r}\langle\prod_{T^*1}g_r,f_r\rangle.$$

Similarly, in the case"$J\subsetneq I$' the paraproduct Π_{T1} is extracted. For the remaining, it is supposed that the paraproducts are extracted, and hence the convention

$$\langle h_J,Th_I\rangle\coloneqq\langle 1_{J_I^c}\big(h_J-\langle h_I\rangle_I\big),Th_I\rangle\text{whenever } I\nsubseteq J,$$

is used together with the similar convention whenever $J\subsetneq I$.

Next, the summation is rearranged according to the minimal common dyadic ancestor of I and J, which is denoted by $I\vee.J$. (if $I\subseteq J$, then $I\vee J=J$. if $I\cap J=\phi$, then a common dyadic ancestor exists because one of the cubes is good.)

By splitting the summation according to which one of the cubes I and J has smaller side length (and hence is good), and by rearranging the summation according to which cube $K$is the minimal common dyadic ancestor $I\vee J$and what is the size of I and J relative to $I\vee J$, one obtains

$$\sum_{\substack{I,J:\\ \text{smaller}\{I,J\}\text{ is good}}}=\sum_{i,j:i\geq j}\sum_{K}\sum_{\substack{I,J:I\vee J=K\\ I\text{ is good}\\ \ell(I)=2^{-i}\ell(K),\\ \ell(J)=2^{-j}\ell(K)}}+\sum_{i,j:i\geq j}\sum_{K}\sum_{\substack{I,J:I\vee J=K\\ I\text{ is good}\\ \ell(I)=2^{-i}\ell(K),\\ \ell(J)=2^{-j}\ell(K)}}$$

Note that, for $K=I\vee J$, one can write

$$\sum_{\substack{I,J:I\vee J=K,\\ I\text{ is good},\\ \ell(I)=2^{-i}\ell(K),\\ \ell(J)=2^{-j}\ell(K)}}\langle g_r,h_I\rangle\sum_{r}\langle h_I,Th_I\rangle\langle h_I,f_r\rangle=\sum_{r}\langle g_r,D_K^jA_K^{ij}D_K^if_r\rangle$$

by defining

$$A_K^{ij}f_r(x')=\frac{1_K(x')}{|K|}\int_K\sum_{r}a_K^{ij}(x',x)f_r(x)dx$$

with.

$$a_K^{ij}(x',x)\coloneqq|K|\sum_{\substack{I,J:I\vee J=K,\\ \text{smaller}\{I,J\}\text{is good},\\ \ell(I)=2^{-i}\ell(K),\\ \ell(J)=2^{-j}\ell(K)}}h_I(x')h_I(x)\langle h_I,Th_I\rangle$$

Altogether, it is obtained that

$$\langle g_r,Tf_r\rangle=\frac{1}{\pi_{\text{good}}}\mathbb{E}_\omega\sum_{i,j}\sum_{r}\langle g_r,\sum_{K\in\mathcal{D}^\omega}D_K^jA_K^{ij}D_K^if_r\rangle+\frac{1}{\pi_{\text{good}}}\mathbb{E}_\omega\sum_{r}\langle g_r,\Big(\prod\nolimits_{TI}^{\mathcal{D}^\omega}+\Big(\prod\nolimits_{T^*1}^{\mathcal{D}^\omega}\Big)^*\Big)f_r\rangle.$$

6.2. Estimating the R-bounds of the matrix elements. We may consider the case $i\geq j$(which means $\ell(I)\leq\ell(J)$), since, by duality, the case $i>j$can be treated similarly. It remains to estimate the R-bound of the family$\{a_K^{ij}(x,x'):K\in\mathcal{D},x\in K,x'\in K\}$of the operator-valued kernels defined by

$$a_K^{ij}(x',x)\coloneqq|K|\sum_{\substack{I,J:I\vee J=K,\\ I\text{ is good},\\ \ell(I)=2^{-i}\ell(K),\\ \ell(J)=2^{-j}\ell(K)}}h_I(x')h_I(x)\langle h_I,Th_I\rangle$$

with $i\geq j$(and hence $\ell(I)\leq\ell(J)$). We divide this into cases according to two criteria. The first criterion is whether K is much bigger than I. The second criterionis how thecubes$I$and J intersect:Whether $I\subsetneq J$ (in which case $K=J$), $I=J$ (in which case $K=I=J$), or $I\cap J=\emptyset$. In total, we have five cases:

- $\ell(K)>2^r\ell(I)$and $I\cap J=\emptyset$,

- $\ell(J) > 2^r \ell(I)$ and $I \nsubseteq J$ (in this case $K = J$),
- $\ell(K) \leq 2^r \ell(I)$ and $I \cap J = \emptyset$,
- $\ell(J) \leq 2^r \ell(I)$ and $I \nsubseteq J$ (in this case $K = J$), and
- $I = J$ (in this case $K = J$),

These cases are tackled in Lemmas 6.4 through 6.8, which complete the proof of the representation theorem by assuring that

$$\mathcal{R}\big(\{a_K^{ij}(x', x): K \in \mathcal{D}, x \in K, x' \in K\}\big) \lesssim_{r,\gamma,d} \left(\mathcal{R}_{cz_0} + \mathcal{R}_{cz_0} + \mathcal{R}_{WBP}\right) 2^{-(1-\epsilon)(1+2\epsilon)\max\{i,j\}},$$

under the choice $\gamma := \frac{\epsilon(1+2\epsilon)}{(1+2\epsilon)+d}$ of the boundary exponent $\gamma \in (0,1)$.

Lemma 6.4 (see [18]) ($Case$ '$\ell(I) \leq \ell(J), \ell(K) > 2^r \ell(I)$. and $I \cap J = \emptyset$') **.** Suppose that i and j are nonnegative integers such that $i \geq r$ and $\geq j$. Let

$$a_K^{ij}(x', x) := |K| \sum\nolimits_{I,J}' \langle h_J, T h_I \rangle\, h_I(x) h_J(x'),$$

wherethe summation is over all the dyadic cubes I and J such that $I \cap J = \emptyset$,
$I \vee J = K, \ell(I) = 2^{-i}\ell(K), \ell(J) = 2^{-j}\ell(K)$, and I is good with threshold r and exponent γ Then

$$\mathcal{R}\big(\{a_K^{ij}(x', x): K \in \mathcal{D}, x \in K \text{ and } x' \in K\}\big) \lesssim \mathcal{R}_{cz_0} 2^{-i((1+2\epsilon)(1-\gamma)-\gamma d)}$$

Proof. We observethat for eachtriplet (K, x, x') either the sum is empty or there is a unique $I_{K,x}$ and a unique $J_{K,x'}$ satisfying the summation condition. Let $\mathcal{Y}I_{K,x}$ denote the center of the dyadic interval $I_{K,x}$ By using the integral representation of the Calderón − Zygmund operator T, and by using the cancellation of the Haar functions, we write

$$a_K^{ij}(x', x) \int_{\mathbb{R}^d \times \mathbb{R}^d} \big(k(\mathcal{Y}', \mathcal{Y}) - k(\mathcal{Y}', \mathcal{Y}I_{K,x})\big) \left(\frac{|\mathcal{Y} - \mathcal{Y}'|}{|\mathcal{Y} - \mathcal{Y}I_{K,x}|}\right)^{1+2\epsilon} |\mathcal{Y} - \mathcal{Y}'|^d 1_{I_{K,x}}(\mathcal{Y}) 1_{J_{K,x'}}(\mathcal{Y}')$$

$$\times |K| \left(\frac{|\mathcal{Y} - \mathcal{Y}I_{K,x}|}{|\mathcal{Y} - \mathcal{Y}'|}\right)^{1+2\epsilon} \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^d} h_{I_{K,x}}(\mathcal{Y}) h_{J_{K,x'}}(\mathcal{Y}') h_{I_{K,x}}(x) h_{J_{K,x'}}(x') d\mathcal{Y} d\mathcal{Y}'$$

$$=: \int_{\mathbb{R}^d \times \mathbb{R}^d} L_{K,x,x'}(\mathcal{Y}\ \ , \mathcal{Y}') \times \lambda_{K,x,x'}(\mathcal{Y}\ \ , \mathcal{Y}') d\mathcal{Y} d\mathcal{Y}'$$

Under the assumptions, we have $|\mathcal{Y} - \mathcal{Y}I_{K,x}| < \frac{1}{2}|\mathcal{Y}\ \ , \mathcal{Y}'|$, which is checked in the following paragraph. Hence, by the Rademacher standard estimates,

$$\mathcal{R}\big(\{L_{K,x,x'}(\mathcal{Y}\ \ , \mathcal{Y}'): x \in K, x' \in K \text{ and } \mathcal{Y} \in \mathbb{R}^d, \mathcal{Y}' \in \mathbb{R}\}\big) \leq \mathcal{R}_{cz_0}.$$

Next, we show that

$$\sup\left\{\int_{\mathbb{R}^d \times \mathbb{R}^d} |\lambda_{K,x,x'}(\mathcal{Y}\ \ , \mathcal{Y}')|\, d\mathcal{Y}' d\mathcal{Y} : x \in K, x' \in K\right\} \lesssim_r 2^{-i((1+2\epsilon)(1-\gamma)-\gamma d)},$$

which, by Theorem 2.2, completes the proof. For the remaining, we suppress the dependence on thetriplet (K, x, x') in the notation. Since $\mathcal{Y} \in I$ and $\mathcal{Y}_I \in I$, we have $|\mathcal{Y} - \mathcal{Y}_I| \leq \frac{1}{2}\ell(I)$, and since $\mathcal{Y} \in I$ and $\mathcal{Y}' \in J$ we have $|\mathcal{Y}\ \ , \mathcal{Y}'| \geq \operatorname{dist}(I, J)$; hence

$$\left(\frac{|\mathcal{Y} - \mathcal{Y}_I|}{|\mathcal{Y}\ \ - \mathcal{Y}'|}\right)^{1+2\epsilon} \frac{1}{|\mathcal{Y}\ \ - \mathcal{Y}'|^d} \leq \left(\frac{|\ell(I)|}{|\operatorname{dist}(I, J)|}\right)^{1+2\epsilon} \frac{1}{\operatorname{dist}(I, J)^d}.$$

Therefore

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |\lambda(\mathcal{Y}\ \ , \mathcal{Y}')|\, d\mathcal{Y}' d\mathcal{Y} \leq \|h_I\|_\infty \|h_J\|_\infty \|h_I\|_I \|h_J\|_I |K| \left(\frac{\ell(I)}{\ell(K)}\right)^{1+2\epsilon} \frac{1}{\operatorname{dist}(I, J)^d}.$$

It remains to check that

$$\operatorname{dist}(I, J) \geq 2^\gamma \left(\frac{\ell(I)}{\ell(K)}\right)^\gamma \ell(K).$$

In particular, this implies that $|\mathcal{Y} - \mathcal{Y}I| \leq \frac{1}{2}|\mathcal{Y} - \mathcal{Y}'|$ denote the dyadic child of
K that contains I. Since $\ell(K) > 2^\gamma \ell(I)$, we have $\ell(K_I) > 2^\gamma \ell(I)$. Therefore, since I is good, we have that

$$\operatorname{dist}(I, K_I^c) > \left(\frac{\ell(I)}{\ell(K_I)}\right)^\gamma \ell(K_I) = 2^\gamma \left(\frac{\ell(I)}{\ell(K)}\right)^\gamma \ell(K).$$

If K_I intersected J, then either $K_I \nsubseteq I$ (which is not true because we assume that I and J are disjoint) or $K_I \supseteq I$ (which is not true because we assume that K is the minimal dyadic ancestor of I that contains J).Therefore K_I does not intersect J, and hence

$$\operatorname{dist}(I, J) > \operatorname{dist}(I, K_I^c)$$

The proof is completed.

Lemma 6.5 (see [18])(Case '$\ell(I) \leq \ell(K) \leq 2^{\gamma}\ell(I)$, and$I \cap J = \emptyset$'**)**. Suppose that $i$and j are nonnegative integers such that $i \leq r$ and $\geq j$. Let

$$a_K^{ij}(x',x) := |K| \sum_{I,J} \langle h_J, Th_I \rangle\, h_I(x) h_I(x'),$$

where the summationis over all the dyadic cubes I and J suchthat $I \cap J = \emptyset, I \vee J = K, \ell(I) = 2^{\gamma}\ell(K)$and I is good with threshold r and exponent γ. Then

$$\mathcal{R}\left(\left\{a_K^{ij}(x',x): K \in \mathcal{D}, x \in K \text{ and } x' \in K\right\}\right) \lesssim_{r,d} \mathcal{R}_{cz_0}$$

Proof. Wenote that for eachtriplet(K,x,x')either the sum is empty or there is a unique $I_{K,x}$and a unique $J_{K,x'}$satisfying the summation condition. By using the integral representation of the Calder$ón$ − Zygmund operator *T*, we write

$$a_K^{ij}(x',x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{I_{K,x}}(\mathcal{Y}) 1_{J_{K,x'}}(\mathcal{Y}') k(\mathcal{Y},\mathcal{Y}') |\mathcal{Y} - \mathcal{Y}'|^d$$

$$\times |K| \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^d} h_{I_{K,x}}(\mathcal{Y}) h_{J_{K,x'}}(\mathcal{Y}') h_{I_{K,x}}(x) h_{J_{K,x'}}(x') d\mathcal{Y} d\mathcal{Y}'$$

$$=: \int_{\mathbb{R}^d \times \mathbb{R}^d} L_{K,x,x'}(\mathcal{Y},\mathcal{Y}') \times \lambda_{K,x,x'}(\mathcal{Y},\mathcal{Y}') d\mathcal{Y} d\mathcal{Y}'$$

By the Rademacher standard estimates,

$$\mathcal{R}\left(\left\{L_{K,x,x'}(\mathcal{Y},\mathcal{Y}'): x \in K, x' \in K \text{ and } \mathcal{Y} \in \mathbb{R}^d, \mathcal{Y}' \in \mathbb{R}\right\}\right) \leq \mathcal{R}_{cz_0}.$$

We next check that

$$\sup\left\{\int_{\mathbb{R}^d \times \mathbb{R}^d} \left|\lambda_{K,x,x'}(\mathcal{Y},\mathcal{Y}')\right| d\mathcal{Y}' d\mathcal{Y} : x \in K. x' \in K\right\} \lesssim_{r,d} 1,$$

which, by Theorem 2.2, completes the proof.

For the remaining,we suppress the dependence on the triplet(K,x,x')in the notation. Since $\ell(K) \leq 2^{\gamma}\ell(I)$ and K ⊇ I. we have $2^{r+1} I \supseteq K$. and since $K \supseteq J$ and $I \cap J = \emptyset$, w have $(K \backslash I) \supseteq J$;hence$\left((2^{r+1} I) \backslash I\right) \supseteq J$.Since$\ell(I) \leq \ell(J), J \supseteq K$(and hence $\ell(J) \leq \ell(K)$) and$\ell(K) \leq 2^r \ell(I)$, we have$|I| \simeq_r |J| \simeq_r |K|$ Therefore

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |\lambda(\mathcal{Y},\mathcal{Y}')| d\mathcal{Y}' d\mathcal{Y} \leq |K| \|h_I\|_\infty^2 \|h_J\|_\infty^2 \int_I \int_J \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^d} d\mathcal{Y}' d\mathcal{Y}$$

$$\leq |K| \|h_I\|_\infty^2 \|h_J\|_\infty^2 \int_I \int_{(2^{r+1}I)\backslash I} \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^d} d\mathcal{Y}' d\mathcal{Y}$$

$$\lesssim_{r,d} |K| \|h_I\|_\infty^2 \|h_J\|_\infty^2 |I| \simeq_r 1.$$

Lemma 6.6 (see [18])(Case '*I* = *J* = *K*'**)**. Let

$$a_I^{00}(x',x) := |I| \langle h_I, Th_I \rangle h_I(x) h_I(x').$$

Then

$$\mathcal{R}(\{a_I(x',x): I \in \mathcal{D}, x \in I, and\ x' \in I\}) \lesssim_d \mathcal{R}_{CZ_0} + \mathcal{R}_{WBP}.$$

Proof. Let I_i(where $i = 1, \ldots, 2^d$) denote the dyadic children of *I*. By decomposing$1_I = \sum_{I_i} 1_i$,and using the integral representation of the Calder$ón$ − Zygmund operator T , we write

$$a_I^{00}(x',x) = \sum_{I_i, I_j} |I|\, h_I(x) h_I(x') \langle h_I \rangle_{I_i} \langle h_I \rangle_{I_j} \langle 1_{I_i}, T1_{I_j} \rangle$$

$$= \sum_{I_i} |I||I_i|\, h_I(x) h_I(x') \langle h_I \rangle_{I_i} \langle h_I \rangle_{I_i} \frac{\langle 1_{I_i}, T1_{I_i} \rangle}{|I_i|}$$

$$+ \sum_{I_i \neq I_j} |I||I_i|\, h_I(x) h_I(x') \langle h_I \rangle_{I_i} \langle h_I \rangle_{I_j} \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{I_i}(\mathcal{Y}) 1_{I_j}(\mathcal{Y}') k(\mathcal{Y},\mathcal{Y}') |\mathcal{Y} - \mathcal{Y}'|^d$$

$$\times \frac{1}{|I_i|} \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^d} 1_{I_j}(\mathcal{Y}) 1_{I_j}(\mathcal{Y}') d\mathcal{Y} d\mathcal{Y}'$$

$$= \sum_{I_i} \pm \frac{\langle 1_{I_i}, T1_{I_i} \rangle}{|I_i|} + \sum_{I_i \neq I_j} \pm \int_{\mathbb{R}^d \times \mathbb{R}^d} L_{I_i, I_j}(\mathcal{Y},\mathcal{Y}') \times \lambda_{I_i, I_j}(\mathcal{Y},\mathcal{Y}') d\mathcal{Y} d\mathcal{Y}'.$$

By the Rademacher standard estimates, we have

$$\mathcal{R}\left(\left\{L_{I_i, I_j}(\mathcal{Y},\mathcal{Y}'): I \in \mathcal{D}, I_i \neq I_j, \mathcal{Y} \in I_i, \mathcal{Y}' \in I_j, \right\}\right) \leq \mathcal{R}_{CZ_0}$$

Moreover, we have

$$\sup\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\lambda_{I_i,I_j}(\mathcal{Y},\mathcal{Y}')\right|d\mathcal{Y}'d\mathcal{Y}: I\in\mathcal{D}, I_i\neq I_j\right\}\leq\frac{1}{|I|}\int_I\int_{(3I)\setminus I}\frac{1}{|\mathcal{Y}-\mathcal{Y}'|^d}\lesssim_d 1.$$

By the Rademacher weak boundedness property, we have

$$\mathcal{R}\left(\left\{\frac{\langle 1_I, T1_I\rangle}{|I|}: I\in\mathcal{D}\right\}\right)\leq\mathcal{R}_{WBP}.$$

The proof is completed by using Theorem 2.2 and Proposition 2.3.

Lemma 6.7 (see [18]). (Case $'\ell(I)<2^{-r}\ell(J), I\nsubseteq J$).Suppose that i is a nonnegative integersuch that$i>r$. Let

$$a_J^{i0}(x',x):=|J|\sum_I{}'\langle 1_{J_I^c}\big(h_J-\langle h_J\rangle_{J_I}\big)Th_I\rangle\, h_I(x)h_J(x').$$

where J_I is the dyadic child of J that contains I and the summation is over allthe dyadic cubes I such that$I\nsubseteq J, \ell(I)=2^{-i}\ell(J)$and I is good with threshold r and

Exponent γ. Then

$$\mathcal{R}\big(\{a_J^{ij}(x',x): J\in\mathcal{D}, x\in J \text{ and } x'\in J\}\big)\lesssim_\gamma \mathcal{R}_{CZ_0}2^{-i(1+2\epsilon)(1-\gamma)}.$$

Proof. We observe that for each triplet (J,x,x')either the sum is empty or there is a unique $I_{J,x}$satisfying the summation condition. Let $\mathcal{Y}1_{J,x}$denote the center of the dyadic interval $I_{J,x}$. By using the integral representation of the Calder$\acute{o}$n $-$ Zygmund operator T and by using the cancellation of the Haar functions, we have

$$a_J^{i0}(x',x)=\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(k(\mathcal{Y}',\mathcal{Y})-k\big(\mathcal{Y}',\mathcal{Y}I_{J,x}\big)\Big)\left(\frac{|\mathcal{Y}-\mathcal{Y}'|}{|\mathcal{Y}-\mathcal{Y}I_{J,x}|}\right)^{1+2\epsilon}|\mathcal{Y}-\mathcal{Y}'|^d$$

$$\times|J|\left(\frac{|\mathcal{Y}-\mathcal{Y}I_{J,x}|}{|\mathcal{Y}-\mathcal{Y}'|}\right)^{1+2\epsilon}\frac{1}{|\mathcal{Y}-\mathcal{Y}'|^d}h_{I_{J,x}}(\mathcal{Y})1_{J_I^c}(\mathcal{Y}')\big(h_J(\mathcal{Y}')-\langle h_J\rangle_{J_I}\big)$$

$$.\,h_{I_{K,x}}(x)h_J(x')d\mathcal{Y}d\mathcal{Y}'$$

$$=:\int_{\mathbb{R}^d\times\mathbb{R}^d}L_{J,x,x'}(\mathcal{Y},\mathcal{Y}')\times\lambda_{J,x,x'}(\mathcal{Y},\mathcal{Y}')d\mathcal{Y}d\mathcal{Y}'.$$

Under the assumptions, we have, which is checked in next paragraph. Hence, by the Rademacher standard estimates, we have

$$\mathcal{R}\big(\{L_{J,x,x'}(\mathcal{Y},\mathcal{Y}'): x\in J, x'J \text{ and } \mathcal{Y}\in\mathbb{R}^d, \mathcal{Y}'\in\mathbb{R}\}\big)\leq\mathcal{R}_{CZ_0}.$$

Next, we show that

$$\sup\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\lambda_{J,x,x'}(\mathcal{Y},\mathcal{Y}')\right|d\mathcal{Y}'d\mathcal{Y}: x\in J, x'\in J\right\}\lesssim_\gamma 2^{-i(1+2\epsilon)(1-\gamma)},$$

which, by Theorem 2.2, completes the proof.

For the remaining,wesuppressthe dependence on the triplet (J,x,x')in the notation. Since $\operatorname{dist}(I,J_I^c)>l(I)$which follows from thefacts that $I$is good and $\ell(JK_I)\geq 2^r\ell(I)$, since $\mathcal{Y}\in I$, and since$\mathcal{Y}'\in J_I^c$, we have that$|\mathcal{Y}I-\mathcal{Y}|\leq\frac{1}{2}|\mathcal{Y},\mathcal{Y}'|$. Hence,by thetriangle inequality,$|\mathcal{Y}-\mathcal{Y}'|\geq\frac{2}{3}|\mathcal{Y}'-\mathcal{Y}I|$.Therefore

$$\left(\frac{|\mathcal{Y}-\mathcal{Y}I|}{|\mathcal{Y}-\mathcal{Y}'|}\right)^{1+2\epsilon}\frac{1}{|\mathcal{Y}-\mathcal{Y}'|^d}\lesssim_{d,1+2\epsilon}\ell(I)^{1+2\epsilon}\frac{1}{|\mathcal{Y}'-\mathcal{Y}I|^{(1+2\epsilon)+d}}.$$

Therefore

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|\lambda(\mathcal{Y},\mathcal{Y}')|\,d\mathcal{Y}'d\mathcal{Y}$$

$$(6.2)\lesssim_{d,1+2\epsilon}|J|\big\|h_J-\langle h_J\rangle_{J_i}\big\|_\infty\big\|h_J\big\|_\infty\|h_I\|_\infty\|h_I\|_I\ell(I)^{1+2\epsilon}\int_{J_I^c}\frac{1}{|\mathcal{Y}'-\mathcal{Y}I|^{(1+2\epsilon)+d}}d\mathcal{Y}'$$

$$\lesssim\left(\frac{\ell(I)}{\operatorname{dist}(I,J_I^c)}\right)^{1+2\epsilon}=\left(\frac{\ell(J_I)}{\operatorname{dist}(I,J_I^c)}\right)^{1+2\epsilon}\left(\frac{\ell(I)}{\ell(J_I)}\right)^{1+2\epsilon}$$

Since I is good and$\ell(J_I)\geq 2^r\ell(I)$ we have that

$$\operatorname{dist}(I,J_I^c)<l(J_I)\left(\frac{\ell(I)}{\ell(J_I)}\right)^\gamma$$

which concludes the proof.

Lemma 6.8 (see [18])(Case $'\ell(I)\geq 2^{-r}\ell(J), I\nsubseteq J'$). Suppose that i is a nonnegativeinteger such that$1\leq i\leq r$. Let

$$a_J^{i0}(x',x):=|J|\sum_I{}'\langle 1_{J_I^c}\big(h_J-\langle h_J\rangle_{J_I}\big), Th_I\rangle\, h_I(x)h_J(x').$$

where J_I is the dyadic child of J that contains I and the summation is over all the dyadic cubes I such that $I \subseteq J, \ell(I) = 2^{-i}\ell(J)$ and I is good with threshold r and exponent γ. Then

$$\mathcal{R}(\{a_J(x',x): J \in \mathcal{D} \text{ and} x' \in J\}) \lesssim_{d,1+2\epsilon} \mathcal{R}_{CZ_{1+2\epsilon}} + \mathcal{R}_{CZ_0}$$

Proof. We observe that for each triplet (J,x,x') either the sum is empty or there is a unique $I_{J,x}$ satisfying the summationcondition. Let $\mathcal{Y}I_{J,x}$ denote the center of the dyadic interval $I_{J,x}$. We split $1_{J_I^c} = 1_{J_I^c \cap (3I)} + 1_{J_I^c \cap (3I)^c}$. By using the integral representation of the Calderón − Zygmund operator , and by using the cancellation of the Haar functions, we have

$$a_J^{i0}(x',x)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} (k(\mathcal{Y},\mathcal{Y}') - k(\mathcal{Y}',\mathcal{Y}I)) \left(\frac{|\mathcal{Y} - \mathcal{Y}I_{K,x}|}{|\mathcal{Y} - \mathcal{Y}'|}\right)^{-(1+2\epsilon)} \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^{-d}} \times |J| \left(\frac{|\mathcal{Y} - \mathcal{Y}I_{I,x}|}{|\mathcal{Y} - \mathcal{Y}'|}\right)^{1+2\epsilon}$$

$$\cdot \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^d} h_{I_{J,x}}(\mathcal{Y}) 1_{J_I^c \cap (3I)^c}(\mathcal{Y}') (h_J(\mathcal{Y}') - \langle h_J \rangle_{J_I}) h_{I_{K,x}}(x) h_J(x') d\mathcal{Y} d\mathcal{Y}'$$

$$+ \int_{\mathbb{R}^d \times \mathbb{R}^d} k(\mathcal{Y}',\mathcal{Y}) |\mathcal{Y} - \mathcal{Y}'|^d$$

$$\times |J||\mathcal{Y} - \mathcal{Y}'|^d h_{I_{J,x}}(\mathcal{Y}) 1_{J_I^c \cap (3I)}(\mathcal{Y}') (h_J(\mathcal{Y}') - \langle h_J \rangle_{J_I}) h_{I_{K,x}}(x) h_J(x') d\mathcal{Y} d\mathcal{Y}'$$

$$=: \int_{\mathbb{R}^d \times \mathbb{R}^d} L_{J,x,x'}(\mathcal{Y},\mathcal{Y}') \times \lambda_{J,x,x'}(\mathcal{Y},\mathcal{Y}')\, d\mathcal{Y} d\mathcal{Y}'$$

$$+ \int_{\mathbb{R}^d \times \mathbb{R}^d} M_{J,x,x'}(\mathcal{Y},\mathcal{Y}') \times \lambda_{J,x,x'}(\mathcal{Y},\mathcal{Y}')\, d\mathcal{Y} d\mathcal{Y}'$$

By the Rademacher standard estimates,

$$\mathcal{R}(\{L_{J,x,x'}(\mathcal{Y},\mathcal{Y}'): x \in J, x' \in J \text{ and } \mathcal{Y} \in \mathbb{R}^d, \mathcal{Y}' \in \mathbb{R}\}) \le \mathcal{R}_{CZ_0}$$

And

$$\mathcal{R}(\{M_{J,x,x'}(\mathcal{Y},\mathcal{Y}'): x \in J, x' \in J \text{ and } \mathcal{Y} \in \mathbb{R}^d, \mathcal{Y}' \in \mathbb{R}\}) \le \mathcal{R}_{CZ_0}$$

The same calculation as in (6.2) yields

$$\sup\left\{\int_{\mathbb{R}^d \times \mathbb{R}^d} |\lambda_{J,x,x'}(\mathcal{Y},\mathcal{Y}')|\, d\mathcal{Y}' \quad d\mathcal{Y} := x \in J, x' \in J\right\} \lesssim_d 1.$$

Moreover, we have

$$\sup\left\{\int_{\mathbb{R}^d \times \mathbb{R}^d} |\mu_{J,x,x'}(\mathcal{Y},\mathcal{Y}')| := x \in J, x' \in J\right\}$$

$$\le |J| \|h_I\|_\infty \|h_J\|_\infty \|h_J - \langle h_J \rangle\|_\infty \|h_I\|_\infty \int_I \int_{(3I)\setminus I} \frac{1}{|\mathcal{Y} - \mathcal{Y}'|^d} \lesssim_d 1.$$

By Theorem 2.2 and Proposition 2.3, the proof is completed.

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