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Research Paper

An Application on Chernoff Theorem on Riemannian Symmetric Spaces of Rank One

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Abstract
P.R. Chernoff used iterates of the Laplacian on $\mathbb{R}^n$ to prove an L^2 version of the Denjoy-Carleman theorem which provides a sufficient condition for a smooth function on $\mathbb{R}^n$ to be quasi-analytic. The pioneer authors in [29] prove and improved an exact analogue of Chernoff theorem for all rank one Riemannian symmetric spaces (of noncompact and compact types) using iterates of the associated Laplace-Beltrami operators. Following, with a little touch, the perfect study of [29] an application for considerability.

Keywords: *Chernoff's theorem, Riemannian symmetric spaces, Helgason Fourier transform, Jacobi analysis, Laplace- Beltrami operators.*

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I. Introduction and the main results

We know that the property of an analytic function it is completely determined by its value and the values of all its derivatives at a single point. Borel first perceived that there is a more larger class of smooth functions than that of analytic functions which has this property. He coined the term quasi-analytic for such class of functions. In exact terms a subset of smooth functions on an interval $(a, a+\epsilon)$ is called a quasi-analytic class if for any function f_σ from that set and $x_0 \in (a, a+\epsilon), \frac{d^n}{dx^n}\left(\sum_\sigma f_\sigma(x_0)\right) = 0$ for all $n \in \mathbb{N}$ implies $f_\sigma = 0$. Now recall that a smooth function on an interval I is analytic provided its Taylor series converges to the function on I which naturally restricts the growth of derivatives of that function. In fact, if for every n, $\left\|\frac{d^n}{dx^n} f_\sigma\right\|_{L^\infty(I)} \le Cn!\,A^n$ for some constant A depending on f_σ then the Taylor series of f_σ converges to f_σ uniformly and the converse is also true. This drives an analytic mind to investigate whether relaxing growth condition on the derivatives generates quasi-analytic class. In 1912 Hadamard proposed the problem of finding sequence $\{M_n\}_n$ of positive numbers such that the class $C\{M_n\}$ of smooth functions on I satisfying $\left\|\frac{d^n}{dx^n} f_\sigma\right\|_{L^\infty(I)} \le A_{f_\sigma}^n M_n$ for all $f_\sigma \in C\{M_n\}$ is a quasi-analytic class. A solution to this problem is provided by a theorem of Denjoy and Carleman where they showed that $C\{M_n\}$ is quasianalytic if and only if $\sum_{n=1}^\infty M_n^{-1/n} = \infty$. As a matter of fact [11] first proved a sufficient condition and later [7] completed the theorem giving a necessary and sufficient condition. A short proof of this theorem based on complex analytic ideas can be found in [26]. A several variable analogue of this theorem has been obtained by [4].

Instead of using all partial derivatives, Bochner used iterates of the Laplacian Δ and proved an analogue of Denjoy-Carleman theorem which reads as follows: if $f_\sigma \in C^\infty(\mathbb{R}^n)$ satisfies $\sum_{m=1}^\infty \sum_\sigma \|\Delta^m f_\sigma\|_\infty^{-1/m} = \infty$, then the condition $\Delta^m f_\sigma(x) = 0$ for all $m \ge 0$ and for all x in a set U of analytic determination implies $f_\sigma = 0$. Building upon the works of [24] and [25], [8] used operator theoretic arguments to study quasi-analytic vectors. As an application he improved the above mentioned result of Bochner by proving the following very interesting result (see [29]).

Theorem 1.1. [9, Chernoff] Let f_σ be a smooth function on $\mathbb{R}^n$. Assume that $\Delta^m f_\sigma \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^\infty \sum_\sigma \|\Delta^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$. If f_σ and all its partial derivatives vanish at a point $a \in \mathbb{R}^n$, then f_σ is identically zero.

P. Ganguly, R. Manna and S. Thangavelu [29] prove an analogue of Chernoff's theorem for the Laplace-Beltrami operator on rank one symmetric spaces of both compact and noncompact types. For G be a connected, noncompact semisimple Lie group with finite centre and K a maximal compact subgroup of G. Let $X = G/K$ be the associated symmetric space which is assumed to have rank one. The origin o in the symmetric space is given by the identity coset eK where e is the identity element in G. We know that X is a Riemannian manifold equipped with a G invariant metric on it. We denote by Δ_X the Laplace-Beltrami operator associated to X.

Iwasawa decomposition of G reads as $G = KAN$ where A is abelian and N is a nilpotent Lie group. Let $\mathfrak{g}$ and $\mathfrak{a}$ stand for the Lie algebras corresponding to G and A respectively. Here $\mathfrak{a}$ is one dimensional since X is of rank one. Then every element of $\mathfrak{g}$ gives rise to a left invariant vector field on G. Let H be the left invariant vector field corresponding to a fixed basis element of $\mathfrak{a}$. As an exact analogue of Chernoff's theorem for X we prove the following:

Theorem 1.2 (see [29]). Let $X = G/K$ be a rank one symmetric space of noncompact type. Suppose $f_\sigma \in C^\infty(X)$ satisfies $\Delta_X^m f_\sigma \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^{\infty} \sum_\sigma \|\Delta_X^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$. If $H^l f_\sigma(eK) = 0$ for all $l \geq 0$ then f_σ is identically zero.

So we obtain an analogue of the L^2 version of the classical Denjoy-Carleman theorem using iterates of the Laplace-Beltrami operator on $X = G/K$.

Corollary 1.3 [29]. Let $X = G/K$ be a rank one symmetric space of noncompact type. Let $\{M_k\}_k$ be a log convex sequence. Define $\mathcal{C}(\{M_k\}_k, \Delta_X, X)$ to be the class of all smooth functions f_σ on X satisfying $\Delta_X^m f_\sigma \in L^2(X)$ for all $m \in \mathbb{N}$ and $\left\|\sum_\sigma \Delta_X^k f_\sigma\right\|_2 \leq \sum_\sigma M_k \lambda(f_\sigma)^k$ for some constant $\lambda(f_\sigma)$ depending on f_σ. Suppose that $\sum_{k=1}^{\infty} M_k^{-\frac{1}{2k}} = \infty$. Then every member of that class is quasi-analytic.

As Chernoff's theorem is a useful tool in establishing uncertainty principles of Ingham's type, proving analogues of Theorem 1.1 in contexts other than Euclidean spaces have received considerable attention. Recently, an analogue of Chernoff's theorem for the sublaplacian on the Heisenberg group has been proved in [1]. For noncompact Riemannian symmetric spaces $X = G/K$, without any restriction on the rank, the following weaker version of Theorem 1.2 has been proved in [2].

Theorem 1.4 [29] (Bhowmik-Pusti-Ray). Let $X = G/K$ be a noncompact Riemannian symmetric space and let Δ_X be the associated Laplace-Beltrami operator. Suppose $f_\sigma \in C^\infty(X)$ satisfies $\Delta_X^m f_\sigma \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^{\infty} \sum_\sigma \|\Delta_X^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$. If f_σ vanishes on a non empty open set, then f_σ is identically zero.

In proving the above theorem, we use a result of [15]. In the case of rank one symmetric spaces, a different proof was given by making use of spherical means and an analogue of Chernoff's theorem for the Jacobi transform proved in [13]. Then, we only need to use the one dimensional version of de Jeu's theorem which is equivalent to the Denjoy-Carleman theorem. The proof of Theorem 1.2 see [13]. Bhowmik-Pust-Ray have proved the following improvement of their Theorem 1.4. So let $D(G/K)$ denote the algebra of differential operators on G/K which are invariant under the (left) action of G.

Theorem 1.5 [29] (Bhowmik-Pusti-Ray). Let $X = G/K$ be a noncompact Riemannian symmetric space and let Δ_X be the associated Laplace-Beltrami operator. Suppose $f_\sigma \in C^\infty(G/K)$ be a left K-invariant function on X which satisfies $\Delta_X^m f_\sigma \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^{\infty} \sum_\sigma \|\Delta_X^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$. If there is an $x_0 \in X$ such that $Df_\sigma(x_0)$ vanishes for all $D \in D(G/K)$ then f_σ is identically zero.

Remark 1.6 [29]. Observe that in the above theorem the function f_σ is assumed to be K-biinvariant. The problem of proving the same for all functions on X is still open. However, in the case of rank one symmetric spaces we have proved Theorem 1.2 for all functions f_σ. Moreover, we only require that $H^l f_\sigma(eK) = 0$ for all $l \geq 0$. Here we can also take any $x_0 \in X$ in place of eK using translation invariance of Laplacian and H.

We remark that the condition $H^l f_\sigma(eK) = 0$ is the counterpart of $\left(\frac{d}{d(1+\epsilon)}\right)^k f_\sigma((1+\epsilon)\omega)\Big|_{\epsilon=-1} = 0$ where $x = (1+\epsilon)\omega, \epsilon > -1, \omega \in \mathbb{S}^{n-1}$ is the polar decomposition of $x \in \mathbb{R}^n$. Indeed, as can be easily checked

$$\left(\frac{d}{d(1+\epsilon)}\right)^k \sum_\sigma f_\sigma((1+\epsilon)\omega) = \sum_{|\alpha|=k} \sum_\sigma \partial^\alpha f_\sigma((1+\epsilon)\omega)\omega^\alpha$$

and hence $\left(\frac{d}{d(1+\epsilon)}\right)^k f_\sigma((1+\epsilon)\omega)\Big|_{\epsilon=-1} = 0$ for all k if and only if $\partial^\alpha f_\sigma(0) = 0$ for all α. This observation plays an important role in formulating the right analogue Chernoff's theorem for compact Riemannian symmetric spaces. Hence, Chernoff's theorem for the Laplacian on $\mathbb{R}^n$ can be stated in the following form.

Theorem 1.7 (see [29]). Let f_σ be a smooth function on $\mathbb{R}^n$. Assume that $\Delta^m f_\sigma \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \sum_\sigma \|\Delta^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$. If $\left(\frac{d}{d(1+\epsilon)}\right)^k \sum_\sigma f_\sigma((1+\epsilon)\omega)\Big|_{\epsilon=-1} = 0$ for all k and $\omega \in \mathbb{S}^{n-1}$, then f_σ is identically zero.

We can give a proof of the above theorem by reducing it to a theorem for Bessel operators. Recall that written in polar coordinates the Laplacian takes the form

$$\Delta = \frac{\partial^2}{\partial(1+\epsilon)^2} + \frac{n-1}{(1+\epsilon)}\frac{\partial}{\partial(1+\epsilon)} + \frac{1}{(1+\epsilon)^2}\Delta_{\mathbb{S}^{n-1}} \tag{1.1}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the spherical Laplacian on the unit sphere $\mathbb{S}^{n-1}$. By expanding the function $F_\sigma(1+\epsilon,\omega) = f_\sigma((1+\epsilon)\omega)$ in terms of spherical harmonics on $\mathbb{S}^{n-1}$ and making use of Hecke-Bochner formula, we can easily reduce Theorem 1.7 to a sequence of theorems for the Bessel operator $\partial_{(1+\epsilon)}^2 + (n+2m+1)(1+\epsilon)^{-1}\partial_{(1+\epsilon)}$ for various values of $m \in \mathbb{N}$. This idea has been used in [13]. A similar expansion in the case of noncompact Riemannian symmetric spaces leads to Jacobi operators see [13] which will be used in proving Theorem 1.2.

Remark 1.8 [29]. We remark in passing that the above theorem can also be proved in the context of Dunkl Laplacian on $\mathbb{R}^n$ associated to root systems. We would also like to mention that analogues of Chernoff's theorem can be proved for the Hermite operator H on $\mathbb{R}^n$ and the special Hermite operator L on $\mathbb{C}^n$. Again the idea is to make use of Hecke-Bochner formula for the Hermite and special Hermite projections (associated to their spectral decompositions).

We considered non compact Riemannian symmteric spaces, and proving an analogue of Theorem 1.2 for compact, rank one symmetric spaces. We formulating and proving a Chernoff theorem for the Laplace-Beltrami operator. We only need to prove such a result for the spherical Laplacian on spheres in Euclidean spaces.

Let (U,K) be a compact symmetric pair and $S = U/K$ be the associated symmetric space. Here U is a compact semisimple Lie group and K is a connected subgroup of U. We assume that S has rank one. Being a compact Riemannian manifold, S admits a Laplace-Beltrami operator $\tilde{\Delta}_S$. It is customary to add a suitable constant ρ_S and work with $\Delta_S = -\tilde{\Delta}_S + \rho_S^2$. This way we can arrange that $\Delta_S \geq \rho_S^2 > 0$. [28] has completely classified all rank one compact symmetric spaces. S is one of the followings: The unit sphere $\mathbb{S}^q = SO(q+1)/SO(q)$, the real projective space $P_q(\mathbb{R}) = SO(q+1)/O(q)$, the complex projective space $P_l(\mathbb{C})$, the quaternion projective space $P_l(\mathbb{H})$ and the Cayley projective space $P_2(\mathbb{C}ay) = (F_\sigma)_4/\mathrm{Spin}(9)$. In each case, S comes up with an appropriate polar form $(0,\pi) \times \mathbb{S}^{k_S}$ where k_S depends on the symmetric space S. As a consequence, functions on S can be identified with functions on the product space $Y = (0,\pi) \times \mathbb{S}^k$. We prove the following analogue of Chernoff's theorem (see [29]):

Theorem 1.9. Let S be a rank one Riemannian symmetric space of compact type. Suppose $f_\sigma \in C^\infty(S)$ satisfies $\Delta_S^m f_\sigma \in L^2(S)$ for all $m \geq 0$ and $\sum_{m=1}^{\infty} \sum_\sigma \|\Delta_S^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$. If the function F_σ on $(0,\pi) \times \mathbb{S}^{kS}$ associated to f_σ on S satisfies $\frac{\partial^m}{\partial\theta^m}\Big|_{\theta=0} F_\sigma(\theta,\xi) = 0$ for all $m \geq 0$, then f_σ is identically zero.

In the context of Theorem 1.7 , by identifying $\mathbb{R}^n$ with $(0,\infty) \times \mathbb{S}^{n-1}$ every function f_σ on $\mathbb{R}^n$ gives rise to a function $F_\sigma(1+\epsilon,\omega)$ on $(0,\infty) \times \mathbb{S}^{n-1}$ and in view of 1.1, the action of Δ on f_σ takes the form,

$$\Delta f_\sigma(1+\epsilon,\omega) = \frac{\partial^2}{\partial(1+\epsilon)^2}F_\sigma(1+\epsilon,\omega) + \frac{n-1}{(1+\epsilon)}\frac{\partial}{\partial(1+\epsilon)}F_\sigma(1+\epsilon,\omega) + \frac{1}{(1+\epsilon)^2}\Delta_{\mathbb{S}^{n-1}}F_\sigma(1+\epsilon,\omega).$$

There is a similar decomposition of Δ_S as a sum of a Jacobi operator on $(0,\pi)$ and the spherical Laplacian $\Delta_{\mathbb{S}^{k_S}}$ and this justifies our formulation of Theorem 1.9.

We state the preliminaries on noncompact Riemannian symmetric spaces and prove Chernoff's theorem for the Laplace-Beltrami operator. After recalling necessary results from the theory of compact symmetric spaces, we prove Theorem 1.9. See [12] and [13] for related ideas.

II. Preliminaries on Riemannian symmetric spaces of non-compact type

We describe the relevant theory regrading the harmonic analysis on rank one Riemannian symmetric spaces of noncompact type. We can see [18] and [19].

For G be a connected, noncompact semisimple Lie group with finite centre. Suppose $\mathfrak{g}$ denotes its Lie algebra. With respect to a fixed Cartan involution θ on $\mathfrak{g}$ we have the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{k}$ and $\mathfrak{p}$ are the +1 and -1 eigenspaces of θ respectively. Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$. Also assume that the dimension of $\mathfrak{a}$ is one. Now we know that the involution θ induces an automorphism Θ on G and $K = \{g \in G : \Theta(g) = g\}$ is a maximal compact subgroup of G. We consider the homogeneous space $X = G/K$ which a is a smooth manifold endowed with a G-Riemannian metric induced by the restriction of the Killing form $\mathfrak{B}$ of $\mathfrak{g}$ on $\mathfrak{p}$. This turns X into a rank one Riemannian symmetric space of noncompact type and every such space can be realised this way.

Let $\mathfrak{a}^*$ denote the dual of $\mathfrak{a}$. Given $\alpha \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [Y, X] = \alpha(Y)X, \forall Y \in \mathfrak{a}\}.$$

Now $\Sigma := \{\alpha \in \mathfrak{a}^* : \mathfrak{g}_\alpha \neq \{0\}\}$ is the set of all resticted roots of the pair ($\mathfrak{g}, \mathfrak{a}$). Let Σ_+denote the set of all positive roots with respect to a fixed Weyl chamber. It is known that $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha$ is a nilpotent subalgebra of $\mathfrak{g}$ and we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Now writing $N = \exp\, \mathfrak{n}$ and $A = \exp\, \mathfrak{a}$ we obtain $G = KAN$ where A is abelian and N is a nilpotent subgroup of G. Moreover, A normalizes N. In view of this decomposition every $g \in G$ can be uniquely written as $g = k(g) \exp\, H(g) n(g)$ where $H(g)$ belongs to $\mathfrak{a}$. Also we have $G = NAK$ and with respect to this decomposition we write $g \in N \exp\, A(g) K$ where the functions A and H are related via $A(g) = -H(g^{-1})$. Now in the rank one case when dimension of $\mathfrak{a}$ is one, Σ is given by either $\{\pm\gamma\}$ or $\{\pm\gamma, \pm 2\gamma\}$ where γ belongs to Σ_+. Let $\rho := (m_\gamma + m_{2\gamma})/2$ where m_γ and $m_{2\gamma}$ denote the multiplicities of the roots γ and 2γ respectively. The Haar measure dg on G is given by

$$\int_G \sum_\sigma f_\sigma(g) dg = \int_K \int_A \int_N \sum_\sigma f_\sigma(k a_t n) e^{2\rho t} dk dt dn.$$

The measure dx on X is induced from the Haar measure dg by the relation

$$\int_G \sum_\sigma f_\sigma(gK) dg = \int_X \sum_\sigma f_\sigma(x) dx.$$

Suppose M denotes the centralizer of A in K. The polar decomposition of G reads as $G = KAK$ in view of which we can write each $g \in G$ as $g = k_1 a_{1+\epsilon} k_2$ with $k_1, k_2 \in K$. Actually the map $(k_1, a_{1+\epsilon}, k_2) \to k_1 a_{1+\epsilon} k_2$ of $K \times A \times K$ into G induces a diffeomorphism of $K/M \times A_+ \times K$ onto an open dense subset of G where $A_+ = \exp\, \mathfrak{a}_+$and $\mathfrak{a}_+$is the fixed positive Weyl chamber which basically can be identified with $(0, \infty)$ in our case.

It is also well-known that each $\mathbf{X} \in \mathfrak{g}$ gives rise to a left invariant vector field on G by the prescription

$$\mathbf{X} f_\sigma(g) = \frac{d}{dt}\bigg|_{t=0} \sum_\sigma f_\sigma(g \cdot \exp\,(t\mathbf{X})), g \in G$$

Since $\mathfrak{a}$ is one dimensional, we fix a basis $\{H\}$ of $\mathfrak{a}$. We denote the left invariant vector field corresponding to this basis element by H. Infact, we can write $A = \{a_{1+\epsilon} = \exp\,((1+\epsilon)H) : (1+\epsilon) \in \mathbb{R}\}$.

2.1. Helgason Fourier transform. Define the function $A: X \times K/M \to \mathfrak{a}$ by $A(gK, kM) = A(k^{-1}g)$. Note that A is right K-invariant in g and right M-invariant in K. We denote the elements of X and K/M by x and $(a+\epsilon)$ respectively. Let $\mathfrak{a}^*$ denote the dual of $\mathfrak{a}$ and $\mathfrak{a}^*_\mathbb{C}$ be its complexification. Here in our case $\mathfrak{a}^*$ and $\mathfrak{a}^*_\mathbb{C}$ can be identified with $\mathbb{R}$ and $\mathbb{C}$ respectively. For each $\lambda \in \mathfrak{a}^*_\mathbb{C}$ and $(a+\epsilon) \in K/M$, the function $x \to e^{(i\lambda+\rho)A(x,a+\epsilon)}$ is a joint eigenfunction of all invariant differential operators on X. For $f_\sigma \in C_c^\infty(X)$, its Helgason Fourier transform is a function $\tilde{f}_\sigma$ on $\mathfrak{a}^*_\mathbb{C} \times K/M$ defined by

$$\tilde{f}_\sigma(\lambda, a+\epsilon) = \int_X \sum_\sigma f_\sigma(x) e^{(-i\lambda+\rho)A(x,a+\epsilon)} dx, \lambda \in \mathfrak{a}^*_\mathbb{C}, (a+\epsilon) \in K/M.$$

Moreover, we know that if $f_\sigma \in L^1(X)$ then $\tilde{f}_\sigma(.\,, a+\epsilon)$ is a continuous function on $\mathfrak{a}^*$ which extends holomorphically to a domain containing $\mathfrak{a}^*$. The inversion formula for $f_\sigma \in C_c^\infty(X)$ says that

$$f_\sigma(x) = c_X \int_{-\infty}^{\infty} \int_{\frac{K}{M}} \sum_\sigma \tilde{f}_\sigma(\lambda, a+\epsilon) e^{(i\lambda+\rho)A(x,a+\epsilon)} |c(\lambda)|^{-2} d(a+\epsilon) d\lambda$$

where $d\lambda$ stands for usual Lebesgue measure on $\mathbb{R}$ (i.e., $\mathfrak{a}^*$), $d(a+\epsilon)$ is the normalised measure on K/M and $c(\lambda)$ is the Harish-Chandra c-function. The constant c_X appearing in the above formula is explicit and depends on the symmetric space X (See e.g., [19]). Also for $f_\sigma \in L^1(X)$ with $\tilde{f}_\sigma \in L^1(\mathfrak{a}^* \times K/M, |c(\lambda)|^{-2} d(a+\epsilon) d\lambda)$, the above inversion formula holds for a.e. $x \in X$. Furthermore, the mapping $f_\sigma \to \tilde{f}_\sigma$ extends as an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}^*_+ \times K/M, |c(\lambda)|^{-2} d\lambda d(a+\epsilon))$ which is known as the Plancherel theorem for the Helgason Fourier transform.

We also need to use certain irreducible representations of K with M-fixed vectors. Suppose $\widehat{K_0}$ denotes the set of all irreducible unitary representations of K with M fixed vectors. Let $\delta \in \widehat{K_0}$ and V_δ be the finite dimensional vectors space on which δ is realised. We know that V_δ contains a unique normalised M-fixed vector v_1 (See [23]). Consider an orthonormal basis $\{v_1, v_2, \ldots, v_{d_\delta}\}$ for V_δ. For $\delta \in \widehat{K_0}$ and $1 \le j \le d_\delta$, we define

$$Y_{\delta,j}(kM) = (v_j, \delta(k) v_1), kM \in K/M.$$

It can be easily checked that $Y_{\delta,1}(eK) = 1$ and moreover, $Y_{\delta,1}$ is M-invariant.

Proposition 2.1 ([19]). The set $\{Y_{\delta,j} : 1 \le j \le d_\delta, \delta \in \widehat{K_0}\}$ forms an orthonormal basis for $L^2(K/M)$.

We can get an explicit realisation of $\widehat{K_0}$ by identifying K/M with the unit sphere in $\mathfrak{p}$. By letting $\mathcal{H}^m$ to stand for the space of homogeneous harmonic polynomials of degree m restricted to the unit sphere, we have the following spherical harmonic decomposition

$$L^2(K/M) = \oplus_{m=0}^{\infty} \mathcal{H}^m$$

Thus the functions $Y_{\delta,j}$ can be identified with the spherical harmonics.

Given $\delta \in \widehat{K_0}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (i.e., $\mathbb{C}$ in our case) we consider the spherical functions of type δ defined by

$$\Phi_{\lambda,\delta}(x) := \int_K e^{(i\lambda+\rho)A(x,kM)} Y_{\delta,1}(kM)dk.$$

These are eigenfunctions of the Laplace-Beltrami operator Δ_X with eigenvalue $-(\lambda^2+\rho^2)$. When δ is the trivial representation for which $Y_{\delta,1}=1$, the function $\Phi_{\lambda,\delta}$ is called the elementary spherical function, denoted by Φ_λ. More precisely,

$$\Phi_\lambda(x) = \int_K e^{(i\lambda+\rho)A(x,kM)}dk$$

Note that these functions are K-biinvariant. The spherical functions can be expressed in terms of Jacobi functions. In fact, if $x = gK$ and $g = ka_{1+\epsilon}k'$ (polar decomposition), $\Phi_{\lambda,\delta}(x) = \Phi_{\lambda,\delta}(a_{1+\epsilon})$. Suppose

$$\alpha = \frac{1}{2}(m_\gamma + m_{2\gamma} - 1), \beta = \frac{1}{2}(m_{2\gamma} - 1). \tag{2.1}$$

For each $\delta \in \widehat{K}_0$ there exists a pair of integers $(1+\epsilon, q)$ such that

$$\Phi_{\lambda,\delta}(x) = Q_\delta(i\lambda+\rho)(\alpha+1)_{1+\epsilon}^{-1}(\sinh\ 1+\epsilon)^{1+\epsilon}(\cosh\ 1+\epsilon)^q \varphi_\lambda^{(\alpha+1+\epsilon,\beta+q)}(1+\epsilon) \tag{2.2}$$

where $\varphi_\lambda^{(\alpha+1+\epsilon,\beta+q)}$ are the Jacobi functions of type $(\alpha+1+\epsilon, \beta+q)$ and Q_δ are the Kostant polynomials given by

$$Q_\delta(i\lambda+\rho) = \left(\frac{1}{2}(\alpha+\beta+1+i\lambda)\right)_{(1+\epsilon+q)/2} \left(\frac{1}{2}(\alpha-\beta+1+i\lambda)\right)_{(1+\epsilon-q)/2}. \tag{2.3}$$

In the above we have used the notation $(z)_m = z(z+1)(z+2)\dots(z+m-1)$. The following result proved in [19]:

Proposition 2.2. Let $\delta \in \widehat{K}_0$ and $1 \le j \le d_\delta$. Then we have

$$\int_K e^{(i\lambda+\rho)A(x,k'M)} Y_{\delta,j}(k'M)dk' = Y_{\delta,j}(kM)\Phi_{\lambda,\delta}(a_{1+\epsilon}), x = ka_{1+\epsilon} \in X. \tag{2.4}$$

See [16] and [17] for all the results recalled.

2.2. Spherical Fourier transform. We say that a function f_σ on G is K-biinvariant if $f_\sigma(k_1 g k_2) = f_\sigma(g)$ for all $k_1, k_2 \in K$. It can be checked that if f_σ is a K-biinvariant integrable function then its Helgason Fourier transform $\tilde{f}_\sigma(\lambda, a+\epsilon)$ is independent of $(a+\epsilon) \in K/M$ and we write this as

$$\tilde{f}_\sigma(\lambda) = \int_X \sum_\sigma f_\sigma(x)\Phi_{-\lambda}(x)dx$$

This is called the spherical Fourier transform. Now since f_σ is K biinvarinat, using the polar decomposition $g = k_1 a_{1+\epsilon} k_2$, we can view f_σ as a function on A alone: $f_\sigma(g) = f_\sigma(a_{1+\epsilon})$. So the above integral takes the following polar form:

$$\tilde{f}_\sigma(\lambda) = \int_0^\infty \sum_\sigma f_\sigma(a_{1+\epsilon})\varphi_\lambda(1+\epsilon)w_{\alpha,\beta}(1+\epsilon)d(1+\epsilon)$$

where $w_{\alpha,\beta}(1+\epsilon) = (2\sinh\ 1+\epsilon)^{2\alpha+1}(2\cosh\ 1+\epsilon)^{2\beta+1}$ and $\Phi_{-\lambda}(a_{1+\epsilon}) = \varphi_\lambda(1+\epsilon)$ are given by Jacobi function $\varphi_\lambda^{\alpha,\beta}(1+\epsilon)$ of type (α,β). Here α and β are associated to the symmetric space as mentioned above. So it is clear that the spherical Fourier transform is basically Jacobi transform of type (α,β). We describe certain results from the theory of Jacobi analysis.

Let $\alpha, \beta, \lambda \in \mathbb{C}$ and $-\alpha \notin \mathbb{N}$. The Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(1+\epsilon)$ of type (α,β) are solutions of the initial value problem

$$(\mathcal{L}_{\alpha,\beta} + \lambda^2 + \varrho^2)\varphi_\lambda^{(\alpha,\beta)}(1+\epsilon) = 0, \varphi_\lambda^{(\alpha,\beta)}(0) = 1$$

where $\mathcal{L}_{\alpha,\beta}$ is the Jacobi operator defined by

$$\mathcal{L}_{\alpha,\beta} := \frac{d^2}{d(1+\epsilon)^2} + ((2\alpha+1)\coth 1+\epsilon + (2\beta+1)\tanh\ 1+\epsilon)\frac{d}{d(1+\epsilon)}$$

and $\varrho = \alpha+\beta+1$. Thus Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ are eigenfunctions of $\mathcal{L}_{\alpha,\beta}$ with eigenvalues $-(\lambda^2+\varrho^2)$. These are even functions on $\mathbb{R}$ and are expressible in terms of hypergeometric functions. For certain values of the parameters (α,β) these functions arise naturally as spherical functions on Riemannian symmetric spaces of noncompact type. The Jacobi transform of a suitable function f_σ on $\mathbb{R}^+$ is defined by

$$J_{\alpha,\beta}f_\sigma(\lambda)=\int_0^\infty \sum_\sigma f_\sigma(1+\epsilon)\varphi_\lambda^{(\alpha,\beta)}(1+\epsilon)w_{\alpha,\beta}(1+\epsilon)d(1+\epsilon)$$

This is also called the Fourier-Jacobi transform of type (α,β). It can be checked that the operator $\mathcal{L}_{\alpha,\beta}$ is selfadjoint on $L^2(\mathbb{R}^+,w_{\alpha,\beta}(1+\epsilon)d(1+\epsilon))$ and that

$$\widetilde{\mathcal{L}_{\alpha,\beta}f_\sigma}(\lambda)=-(\lambda^2+\varrho^2)\tilde{f}_\sigma(\lambda).$$

Under certain assumptions on α and β the inversion and Plancherel formula for this transform take a nice form as described below (see [29]).

Theorem 2.3 ([22]). Let $\epsilon-1,\beta\in\mathbb{R},\epsilon>0$ and $|\beta|\le\epsilon$. Suppose $c_{\epsilon-1,\beta}(\lambda)$ denotes the Harish-Chandra c-function defined by

$$c_{\epsilon-1,\beta}(\lambda)=\frac{2^{\varrho-i\lambda}\Gamma(\epsilon)\Gamma(i\lambda)}{\Gamma\left(\frac{1}{2}(i\lambda+\varrho)\right)\Gamma\left(\frac{1}{2}(i\lambda+\epsilon-\beta)\right)}$$

(1) (Inversion) For $f_\sigma\in C_0^\infty(\mathbb{R})$ which is even we have

$$f_\sigma(1+\epsilon)=\frac{1}{2\pi}\int_0^\infty\sum_\sigma J_{\epsilon-1,\beta}f_\sigma(\lambda)\varphi_\lambda^{(\epsilon-1,\beta)}(1+\epsilon)\left|c_{\epsilon-1,\beta}(\lambda)\right|^{-2}d\lambda$$

(2) (Plancherel) For $f_\sigma,g\in C_0^\infty(\mathbb{R})$ which are even, the following holds

$$\int_0^\infty\sum_\sigma f_\sigma(1+\epsilon)\overline{g(1+\epsilon)}w_{\epsilon-1,\beta}(1+\epsilon)d(1+\epsilon)=\int_0^\infty\sum_\sigma J_{\epsilon-1,\beta}f_\sigma(\lambda)\overline{J_{\epsilon-1,\beta}g(\lambda)}\left|c_{\epsilon-1,\beta}(\lambda)\right|^{-2}d\lambda.$$

The mapping $f_\sigma\mapsto\tilde{f}_\sigma$ extends as an isometry from $L^2(\mathbb{R}^+,w_{\epsilon-1,\beta}(1+\epsilon)d(1+\epsilon))$ onto $L^2\left(\mathbb{R}^+,\left|c_{\epsilon-1,\beta}(\lambda)\right|^{-2}d\lambda\right)$.

We will make use of this theorem in proving an analogue of Chernoff's theorem for the Laplace-Beltrami operator Δ_X.

III. Chernoff's theorem on noncompact symmetric spaces of rank one

We prove our main theorem i.e., an analogue of Chernoff's theorem for Δ_X. The main idea of the proof is to reduce the result for Δ_X to a result for Jacobi operator. So, first we indicate a proof of Chernoff's theorem for Jacobi operator. It has already been discussed in the work of [13] (see [29]).

Theorem 3.1. Let $\epsilon-1,\beta\in\mathbb{R},\epsilon>0$ and $|\beta|\le\epsilon$. Suppose $f_\sigma\in L^2(\mathbb{R}^+,w_{\epsilon-1,\beta}(1+\epsilon)d(1+\epsilon))$ is such that $\mathcal{L}_{\epsilon-1,\beta}^m f_\sigma\in L^2(\mathbb{R}^+,w_{\epsilon-1,\beta}(1+\epsilon)d(1+\epsilon))$ for all $m\in\mathbb{N}$ and satisfies the Carleman condition $\sum_{m=1}^\infty\left\|\mathcal{L}_{\epsilon-1,\beta}^m f_\sigma\right\|_2^{-1/(2m)}=\infty$. If $\mathcal{L}_{\epsilon-1,\beta}^m f_\sigma(0)=0$ for all $m\ge0$ then f_σ is identically zero.

In [13] the above result was proved under the assumption that f_σ vanishes near 0 but a close examination of the proof reveals that the assumption is superfluous and the same is true as stated above. In order to prove our main result, the following estimate for the ratio of Harish-Chandra c-functions is also needed.

Lemma 3.2 (see [29]). Let $\epsilon-1,\beta$ be as in 2.1 and $(1+\epsilon,q)$ be the pair of integers associated to $\delta\in\widehat{K_0}$. Then for any $\lambda\ge0$ we have

$$\frac{\left|c_{\epsilon-1,\beta}(\lambda)\right|^2}{\left|c_{2\epsilon,\beta+q}(\lambda)\right|^2}|Q_\delta(i\lambda+\rho)|^{-2}\le C$$

where C is a constant independent of λ depending only on the parameters $(\epsilon-1,\beta)$ and $(1+\epsilon,q)$.

Proof. First note that from the definition 2.3 of Kostant polynomials we have

$$|Q_\delta(i\lambda+\rho)|=\prod_{j=0}^{\frac{1+\epsilon+q}{2}}\left((B_1+j)^2+\frac{1}{4}\lambda^2\right)^{\frac{1}{2}}\prod_{j=0}^{\frac{1+\epsilon-q}{2}}\left((B_2+j)^2+\frac{1}{4}\lambda^2\right)^{\frac{1}{2}}$$

where $B_1=\frac{1}{2}(\epsilon+\beta)$ and $B_2=\frac{1}{2}(\epsilon-\beta)$. From the above expression, it can be easily checked that $|Q_\delta(i\lambda+\rho)|/(2^{-1}\lambda)^{1+\epsilon}\to1$ as $\lambda\to\infty$ so that

$$|Q_\delta(i\lambda+\rho)|\sim2^{-(1+\epsilon)}\lambda^{1+\epsilon},\lambda\to\infty. \tag{3.1}$$

Moreover, we also have

$$|Q_\delta(i\lambda+\rho)|\ge\prod_{j=0}^{\frac{1+\epsilon+q}{2}}|B_1+j|\prod_{j=0}^{\frac{1+\epsilon-q}{2}}|B_2+j|=\text{ constant}$$

Now using [6, Lemma 2.4] we have

$$\frac{\left|c_{\epsilon-1,\beta}(\lambda)\right|^2}{\left|c_{\epsilon-1+1+\epsilon,\beta+q}(\lambda)\right|^2} \sim \lambda^{2(1+\epsilon)}, \lambda \to \infty \tag{3.2}$$

which together with 3.1 implies that

$$\frac{\left|c_{\epsilon-1,\beta}(\lambda)\right|^2}{\left|c_{\epsilon-1+1+\epsilon,\beta+q}(\lambda)\right|^2}|Q_\delta(i\lambda+\rho)|^{-2} \sim 1, \lambda \to \infty$$

Also the ratio in 3.2 being a continuous function of λ is bounded near the origin. Hence the result follows.

Proof of Theorem 1.2 (see [29]): Let f_σ be as in the statement of the theorem 1.2. We complete the proof in the following steps.

Step 1: Using Proposition 2.1 we write

$$\tilde{f}_\sigma(\lambda,k) = \sum_{\delta\in\widehat{K_0}} \sum_{j=1}^{d_\delta} \sum_\sigma (F_\sigma)_{\delta,j}(\lambda) Y_{\delta,j}(k) \tag{3.3}$$

where $(F_\sigma)_{\delta,j}(\lambda)$ are the spherical harmonic coefficients of $\tilde{f}_\sigma(\lambda,\cdot)$ defined by

$$(F_\sigma)_{\delta,j}(\lambda) = \int_{\frac{K}{M}} \sum_\sigma \tilde{f}_\sigma(\lambda,k) Y_{\delta,j}(k)dk.$$

Fix $\delta \in \widehat{K_0}$ and $1 \le j \le d_\delta$. From the definition of the Helgason Fourier transform we have

$$(F_\sigma)_{\delta,j}(\lambda) = \int_{K/M}\int_{G/K}\sum_\sigma f_\sigma(x)e^{(-i\lambda+\rho)A(x,kM)}Y_{\delta,j}(kM)dxdk.$$

Now using Fubini's theorem, in view of the Proposition 2.2 the integral on the right hand side of above is equal to

$$\int_{G/K}\sum_\sigma f_\sigma(x)Y_{\delta,j}(kM)\Phi_{\lambda,\delta}(a_{1+\epsilon})dx. \tag{3.4}$$

The function $g_{\delta,j}(x)$ defined by

$$g_{\delta,j}(x) = \int_K \sum_\sigma f_\sigma(k'x)Y_{\delta,j}(k'M)dk', x \in X$$

is clearly K-biinvariant, and hence by abuse of notation we write

$$g_{\delta,j}(1+\epsilon) = \int_K \sum_\sigma f_\sigma(k'a_{1+\epsilon})Y_{\delta,j}(k'M)dk'$$

Now performing the integral in 3.4 using polar coordinates we obtain

$$(F_\sigma)_{\delta,j}(\lambda) = \int_0^\infty g_{\delta,j}(1+\epsilon)\Phi_{\lambda,\delta}(a_{1+\epsilon})w_{\epsilon-1,\beta}(1+\epsilon)d(1+\epsilon) \tag{3.5}$$

Now recall that for each $\delta \in \widehat{K_0}$ there exist a pair of integers $(1+\epsilon, q)$ such that

$$\Phi_{\lambda,\delta}(x) = Q_\delta(i\lambda+\rho)(\epsilon)_{1+\epsilon}^{-1}(\sinh\ 1+\epsilon)^{1+\epsilon}(\cosh\ 1+\epsilon)^q\varphi_\lambda^{(2\epsilon,\beta+q)}(1+\epsilon)$$

By defining

$$(f_\sigma)_{\delta,j}(1+\epsilon) = \frac{4^{-(1+\epsilon+q)}}{(\epsilon)_{1+\epsilon}}g_{\delta,j}(1+\epsilon)(\sinh\ 1+\epsilon)^{-(1+\epsilon)}(\cosh\ 1+\epsilon)^{-q} \tag{3.6}$$

and recalling the definition of Jacobi transforms we obtain

$$(F_\sigma)_{\delta,j}(\lambda) = Q_\delta(i\lambda+\rho)J_{2\epsilon,\beta+q}\big((f_\sigma)_{\delta,j}\big)(\lambda) \tag{3.7}$$

Step 2: In this step we estimate the L^2 norm of powers of Jacobi operator applied to $(f_\sigma)_{\delta,j}$ in terms of the L^2 norm of corresponding powers of Δ_X applied to f_σ. Let $m \in \mathbb{N}$. Note that the Plancherel formula 2.3 for the Jacobi transform yields

$$\left\|\sum_\sigma \mathcal{L}_{2\epsilon,\beta+q}^m\big((f_\sigma)_{\delta,j}\big)\right\|_{L^2(\mathbb{R}^+,w_{2\epsilon,\beta+q}(1+\epsilon)d(1+\epsilon))}$$

$$= \left(\int_0^\infty \sum_\sigma (\lambda^2+\rho_\delta^2)^{2m}\left|J_{2\epsilon,\beta+q}\big((f_\sigma)_{\delta,j}\big)(\lambda)\right|^2\left|c_{2\epsilon,\beta+q}(\lambda)\right|^{-2}d\lambda\right)^{\frac{1}{2}}$$

where where $\rho_\delta = 2\epsilon+\beta+q+1$. In view of 3.7 the above integral reduces to

$$\left(\int_0^\infty \sum_\sigma (\lambda^2+\rho_\delta^2)^{2m}\left|(F_\sigma)_{\delta,j}(\lambda)\right|^2|Q_\delta(i\lambda+\rho)|^{-2}c_{2\epsilon,\beta+q}(\lambda)\Big|^{-2}d\lambda\right)^{\frac{1}{2}}$$

which after recalling the definition of $(F_\sigma)_{\delta,j}(\lambda)$ reads as

$$\left(\int_0^\infty \sum_\sigma (\lambda^2+\rho_\delta^2)^{2m}|Q_\delta(i\lambda+\rho)|^{-2}\left|\int_K \tilde{f}_\sigma(\lambda,k)Y_{\delta,j}(k)dk\right|^2 \left|c_{2\epsilon,\beta+q}(\lambda)\right|^{-2}d\lambda\right)^{\frac{1}{2}}$$

By an application of Minkowski's integral inequality, the above integral is dominated by

$$\int_K\left(\int_0^\infty \sum_\sigma (\lambda^2+\rho_\delta^2)^{2m}|Q_\delta(i\lambda+\rho)|^{-2}|\tilde{f}_\sigma(\lambda,k)|^2\left|c_{2\epsilon,\beta+q}(\lambda)\right|^{-2}d\lambda\right)^{\frac{1}{2}}\left|Y_{\delta,j}(k)\right|dk$$

Now using Cauchy-Schwarz inequality along with the fact that $\|Y_{\delta,j}\|_{L^2(K/M)}=1$, we see that the above integral is bounded by

$$\left(\int_{K/M}\int_0^\infty \sum_\sigma (\lambda^2+\rho_\delta^2)^{2m}|Q_\delta(i\lambda+\rho)|^{-2}|\tilde{f}_\sigma(\lambda,k)|^2\left|c_{2\epsilon,\beta+q}(\lambda)\right|^{-2}d\lambda dk\right)^{\frac{1}{2}}$$

Since $\frac{\lambda^2+\rho_\delta^2}{\lambda^2+\rho^2}=1+\frac{\rho_\delta^2-\rho^2}{\lambda^2+\rho^2}$ is a decreasing function of λ it follows that $\frac{\lambda^2+d^2}{\lambda^2+\rho^2}\le C(\epsilon-1,\beta)$ with $C(\epsilon-1,\beta)=\frac{(2\epsilon+\beta+q+1)^2}{(\epsilon+\beta)^2}$. This together with the Lemma 3.2 yields the following estimate for the integral under consideration: for some constant $C_1=C_1(\epsilon-1,\beta)$

$$C_1^m\left(\int_{K/M}\int_0^\infty \sum_\sigma (\lambda^2+\rho^2)^{2m}|\tilde{f}_\sigma(\lambda,k)|^2\left|c_{\epsilon-1,\beta}(\lambda)\right|^{-2}d\lambda dk\right)^{\frac{1}{2}}$$

Finally, from the series of inequalities above, we obtain

$$\left\|\sum_\sigma \mathcal{L}_{2\epsilon,\beta+q}^m\big((f_\sigma)_{\delta,j}\big)\right\|_{L^2(\mathbb{R}^+,w_{2\epsilon,\beta+q}(1+\epsilon)d(1+\epsilon))}\le C_1^m\|\Delta_X^m f_\sigma\|_2 \tag{3.8}$$

Hence from the hypothesis of the theorem it follows that

$$\sum_{m=1}^\infty\sum_\sigma \|\mathcal{L}_{2\epsilon,\beta+q}^m\big((f_\sigma)_{\delta,j}\big)\|_{L^2(\mathbb{R}^+,w_{2\epsilon,\beta+q}(1+\epsilon)d(1+\epsilon))}^{-\frac{1}{2}}=\infty$$

Step 3: Finally in this step we prove that $\mathcal{L}_{2\epsilon,\beta+q}^m\big((f_\sigma)_{\delta,j}\big)(0)=0$ for all $m\ge 0$. First recall that

$$(f_\sigma)_{\delta,j}(1+\epsilon)=\frac{4^{-(1+\epsilon+q)}}{(\epsilon)_{1+\epsilon}}(\sinh\ 1+\epsilon)^{-(1+\epsilon)}(\cosh\ 1+\epsilon)^{-q}\int_K\sum_\sigma f_\sigma(ka_{1+\epsilon})Y_{\delta,j}(kM)dk.$$

As $\sinh\ 1+\epsilon$ has a zero at the origin and $\cosh\ 0=1$, if we can show that as a function of $(1+\epsilon)$, the integral $\int_K f_\sigma(ka_{1+\epsilon})Y_{\delta,j}(kM)dk$ has a zero of infinite order at the 0 , then we are done. Now note that for any $m\in\mathbb{N}$

$$\frac{d^m}{d(1+\epsilon)^m}\int_K\sum_\sigma f_\sigma(ka_{1+\epsilon})Y_{\delta,j}(kM)dk=\int_K\sum_\sigma \frac{d^m}{d(1+\epsilon)^m}f_\sigma(ka_{1+\epsilon})Y_{\delta,j}(kM)dk$$

But by definition of the vector fields on G, writing $a_{1+\epsilon}=\exp\left((1+\epsilon)H\right)$ we have

$$\left.\frac{d^m}{d(1+\epsilon)^m}f_\sigma(ka_{1+\epsilon})\right|_{\epsilon=-1}=\left.\frac{d^m}{d(1+\epsilon)^m}f_\sigma(k\cdot\exp\left((1+\epsilon)H\right))\right|_{\epsilon=-1}=H^m f_\sigma(k).$$

Hence by the hypothesis on f_σ we obtain $\left.\frac{d^m}{d(1+\epsilon)^m}f_\sigma(ka_{1+\epsilon})\right|_{\epsilon=-1}=0$ for all m. Finally, proving $\mathcal{L}_{2\epsilon,\beta+q}^m\big((f_\sigma)_{\delta,j}\big)(0)=0$ is a routine matter: repeated application of L'Hospital rule gives the desired result.

Therefore, $(f_\sigma)_{\delta,j}$ satisfies all the hypothesis of the Proposition 3.1 which allows us to conclude that $(f_\sigma)_{\delta,j}=0$ i.e., $(F_\sigma)_{\delta,j}=0$. As this is true for every $\delta\in\widehat{K_0}$ and $1\le j\le d_\delta$ we get $f_\sigma=0$ completing the proof of Theorem 1.2.

IV. Compact symmetric spaces

We prove an analogue of Chernoff's theorem on compact symmetric spaces of rank one. We first recall briefly some necessary background material on rank one compact symmetric spaces. Let S be a compact Riemannian manifold equipped with a Riemannian metric d_S. We say that S is a two point homogeneous space if for any $x_j,y_j\in S, j=1,2$ with $d_S(x_1,x_2)=d_S(y_1,y_2)$, there exists $g\in I(S)$, the group of isometries of S such that $g\cdot x_1=y_1$, and $g\cdot x_2=y_2$ where $g\cdot x$ denotes the usual action of $I(S)$ on S. It is well known that compact rank one symmetric spaces are compact two point homogeneous spaces (see [20]). Also these two point homogeneous spaces are completely classified by [28]. So, following Wang any compact rank one symmetric space S is one of the following:

(1) the sphere $\mathbb{S}^{1+\epsilon}\subset\mathbb{R}^{2+\epsilon},\epsilon\ge 0$;

(2) the real projective space $P_{2+\epsilon}(\mathbb{R}), \epsilon \geq 0$;
(3) the complex projective space $P_{2+\epsilon}(\mathbb{C}), \epsilon \geq 0$;
(4) the quaternionic projective space $P_{2+\epsilon}(\mathbb{H}), \epsilon \geq 0$;
(5) the Cauchy projective plane $P_2(\mathbb{C}ay)$.

We describe the necessary preliminaries and prove Theorem 1.9 in each of the above five cases separately. We start with a brief description of Jacobi polynomial expansions in the following subsection.

4.1. Jacobi polynomial expansion: Let $\epsilon \geq 0$. The Jacobi polynomials $P_n^{\epsilon-1,2\epsilon-1}$ of degree $n \geq 0$ and type $(\epsilon - 1,2\epsilon - 1)$ are defined by

$$(1-x)^{\epsilon-1}(1+x)^{2\epsilon-1}P_n^{\epsilon-1,2\epsilon-1}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\epsilon-1}(1+x)^{n+2\epsilon-1}\}, x \in (-1,1). \quad (4.1)$$

By making a change of variable $x = \cos\,\theta$, it is convenient to work with the Jacobi trigonometric polynomials

$$\mathcal{P}_n^{(\epsilon-1,2\epsilon-1)}(\theta) = C(\epsilon-1,2\epsilon-1,n)P_n^{(\epsilon-1,2\epsilon-1)}(\cos\,\theta), \quad (4.2)$$

where $C(\epsilon - 1,2\epsilon - 1, n)$ is the normalising constant, explicitly given by

$$C(\epsilon-1,2\epsilon-1,n)^2 = \frac{(2n+\epsilon+2\epsilon-1)\Gamma(n+1)\Gamma(n+\epsilon+2\epsilon-1)}{\Gamma(n+\epsilon)\Gamma(n+2\epsilon)}. \quad (4.3)$$

Also it is worth pointing out that these polynomials are closely related to Gegenbauer's polynomials by the following formula

$$C_k^\lambda(t) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(k+2\lambda)}{\Gamma(2\lambda)\Gamma\left(k+\lambda+\frac{1}{2}\right)}P_k^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(t), \lambda > -\frac{1}{2}, t \in (-1,1). \quad (4.4)$$

These Jacobi trigonometric polynomials are the eigenfunctions of the Jacobi differential operator given by

$$\mathbb{L}_{\epsilon-1,2\epsilon-1} = -\frac{d^2}{d\theta^2} - \frac{(-\epsilon)+(3\epsilon-1)\cos\,\theta}{\sin\,\theta} + \left(\frac{3\epsilon-1}{2}\right)^2$$

with eigenvalues $\left(n + \frac{3\epsilon-1}{2}\right)^2$ i.e.,

$$\mathbb{L}_{\epsilon-1,2\epsilon-1}\mathcal{P}_n^{(\epsilon-1,2\epsilon-1)} = \left(n+\frac{3\epsilon-1}{2}\right)^2 \mathcal{P}_n^{(\epsilon-1,2\epsilon-1)},$$

and $\left\{\mathcal{P}_n^{(\epsilon-1,2\epsilon-1)} : n \geq 0\right\}$ forms an orthonormal basis for the weighted L^2 space $L^2\left(\tilde{w}_{\epsilon-1,2\epsilon-1}\right) := L^2\left((0,\pi), \tilde{w}_{\epsilon-1,2\epsilon-1}(\theta)d\theta\right)$ where the weight is given by

$$\tilde{w}_{\epsilon-1,2\epsilon-1}(\theta) = \left(\sin\,\frac{\theta}{2}\right)^{2\epsilon-1}\left(\cos\,\frac{\theta}{2}\right)^{4\epsilon-1}.$$

As a consequence we have the following Plancherel formula valid for $f_\sigma \in L^2\left(\tilde{w}_{\epsilon-1,2\epsilon-1}\right)$

$$\int_0^\pi \sum_\sigma |f_\sigma(\theta)|^2 \tilde{w}_{\epsilon-1,2\epsilon-1}(\theta)d\theta = \sum_{n=0}^\infty \sum_\sigma \left|\mathcal{J}_{\epsilon-1,2\epsilon-1}f_\sigma(n)\right|^2 \quad (4.5)$$

where $\mathcal{J}_{\epsilon-1,2\epsilon-1}f_\sigma(n)$ denotes the Fourier-Jacobi coefficients defined by

$$\mathcal{J}_{\epsilon-1,2\epsilon-1}f_\sigma(n) = \int_0^\pi \sum_\sigma f_\sigma(\theta)\mathcal{P}_n^{(\epsilon-1,2\epsilon-1)}(\theta)\tilde{w}_{\epsilon-1,2\epsilon-1}(\theta)d\theta, n \geq 0.$$

We have the following version of Chernoff's theorem using the iterates of the Jacobi operator proved in [12].

Theorem 4.1 (see [29]). Let $\epsilon > 0$. Suppose $f_\sigma \in L^2\left(\tilde{w}_{\epsilon-1,2\epsilon-1}\right)$ is such that $\mathbb{L}_{\epsilon-1,2\epsilon-1}^m f_\sigma \in L^2\left(\tilde{w}_{\epsilon-1,2\epsilon-1}\right)$ for all $m \in \mathbb{N}$ and satisfies the Carleman condition $\sum_{m=1}^\infty \left\|\mathbb{L}_{\epsilon-1,2\epsilon-1}^m f_\sigma\right\|_2^{-1/(2m)} = \infty$. If $\mathbb{L}_{\epsilon-1,2\epsilon-1}^m f_\sigma(0) = 0$ for all $m \geq 0$ then f_σ is identically zero.

This is the analogue of Theorem 3.1 for Jacobi polynomial expansions which plays an important role in proving Theorem 1.9 for compact Riemannian symmetric spaces.

4.2. The unit sphere $\mathbb{S}^q$. Let $\epsilon \geq 0$. The unit sphere in $\mathbb{R}^{3+\epsilon}$ is given by

$$\mathbb{S}^{2+\epsilon} := \{\xi \in \mathbb{R}^{3+\epsilon} : \xi_1^2 + \cdots + \xi_{3+\epsilon}^2 = 1\}$$

The spherical harmonic decomposition reads as

$$L^2(\mathbb{S}^{2+\epsilon}) = \bigoplus_{n=0}^\infty \mathcal{H}_n(\mathbb{S}^{2+\epsilon})$$

where $\mathcal{H}_n(\mathbb{S}^{2+\epsilon})$ denotes the set of spherical harmonics of degree n. Now, for our purposes it is more convenient to work with the geodesic polar coordinate system on $\mathbb{S}^{2+\epsilon}$. Note that given $\xi \in \mathbb{S}^{2+\epsilon}$, we can write $\xi = (\cos\,\theta)e_1 + \xi_1'(\sin\,\theta)e_2 + \cdots + \xi_{2+\epsilon}'(\sin\,\theta)e_{3+\epsilon}$ for some $\theta \in (0,\pi)$ and $\xi' = (\xi_1', \dots, \xi_{2+\epsilon}') \in \mathbb{S}^{1+\epsilon}$ where $\{e_1, e_2, \dots, e_{3+\epsilon}\}$ is the standard basis for $\mathbb{R}^{3+\epsilon}$. This observation drives us to consider the map $\varphi : (0,\pi) \times \mathbb{S}^{1+\epsilon} \to \mathbb{S}^{2+\epsilon}$ defined by

$$\varphi(\theta, \xi') = (\cos\,\theta, \xi_1'\sin\,\theta, \dots, \xi_{2+\epsilon}'\sin\,\theta)$$

which induces the geodesic polar coordinate system on $\mathbb{S}^{2+\epsilon}$. This also provides a polar decomposition of the normalised measure $d\sigma_{2+\epsilon}$ on $\mathbb{S}^{2+\epsilon}$ as follows: Given a suitable function f_σ on $\mathbb{S}^{2+\epsilon}$ we have

$$\int_{\mathbb{S}^{2+\epsilon}} \sum_{\sigma} f_\sigma(\xi) d\sigma_{2+\epsilon}(\xi) = \int_0^\pi \int_{\mathbb{S}^{1+\epsilon}} \sum_{\sigma} F_\sigma(\theta, \xi')(\sin\,\theta)^{1+\epsilon} d\sigma_{1+\epsilon}(\xi') d\theta$$

where $F_\sigma = f_\sigma \circ \varphi$. Also in this coordinate system, we have the following representation of the Laplace-Beltrami operator

$$\Delta_{\mathbb{S}^{2+\epsilon}} = -\frac{\partial^2}{\partial\theta^2} - (1+\epsilon)\cot\,\theta \frac{\partial}{\partial\theta} + \frac{1}{4}(1+\epsilon)^2 - \sin^{-2}\,\theta \tilde{\Delta}_{\mathbb{S}^{1+\epsilon}}$$

The following theorem gives a representation of the spherical harmonics in this polar coordinate system (see [29]).

Theorem 4.2. [21, Theorem 2.4] For $n \geq 0$ we have the following orthogonal decomposition

$$\mathcal{H}_n(\mathbb{S}^{2+\epsilon}) = \bigoplus_{\epsilon=-2}^{n} \mathcal{H}_{n,2+\epsilon}(\mathbb{S}^{2+\epsilon})$$

where the subspaces $\mathcal{H}_{n,2+\epsilon}(\mathbb{S}^{2+\epsilon})$ are irreducible and invariant under $SO(2+\epsilon)$. Moreover, functions in $\mathcal{H}_{n,2+\epsilon}(\mathbb{S}^{2+\epsilon})$ can be represented as

$$S(\xi) = (\sin\,\theta)^{2+\epsilon} C_{n-(2+\epsilon)}^{2+\epsilon/2-1/2+(2+\epsilon)}(\cos\,\theta) S'_{2+\epsilon}(\xi')$$

where $\xi = \varphi(\theta, \xi')$ and $S'_{2+\epsilon} \in \mathcal{H}_{2+\epsilon}(\mathbb{S}^{1+\epsilon})$.

In view of the above theorem we have the orthogonal decomposition

$$L^2(\mathbb{S}^{2+\epsilon}) = \bigoplus_{n=0}^{\infty} \bigoplus_{\epsilon=-2}^{n} \mathcal{H}_{n,2+\epsilon}(\mathbb{S}^{2+\epsilon})$$

Now we set

$$S_{n,2+\epsilon,k}(\xi) = a_{n,2+\epsilon}(\sin\,\theta)^{2+\epsilon} C_{n-(2+\epsilon)}^{2+\epsilon+\frac{1+\epsilon}{2}}(\cos\,\theta) S'_{k,2+\epsilon}(\xi')$$

where $\{S'_{2+\epsilon,k}: 1 \leq k \leq N(2+\epsilon)\}$ is an orthonormal basis for $\mathcal{H}_{2+\epsilon}(\mathbb{S}^{1+\epsilon})$. Here $a_{n,2+\epsilon}$ is the normalising constant so that $\|S_{n,2+\epsilon,k}\|_{L^2(\mathbb{S}^{2+\epsilon})} = 1$ and it is explicitly given by

$$a_{n,2+\epsilon} = \frac{2^{-\left(2+\epsilon+\frac{1+\epsilon}{2}\right)} \Gamma(2(2+\epsilon)+1+\epsilon) \Gamma\left(n + \frac{2+\epsilon}{2}\right)}{\Gamma\left(2+\epsilon+\frac{2+\epsilon}{2}\right) \Gamma(n+2+\epsilon+1+\epsilon)} C\left(2+\epsilon+\frac{\epsilon}{2}, 2+\epsilon+\frac{\epsilon}{2}, n-(2+\epsilon)\right). \tag{4.6}$$

Theorem 4.3 (see [29]). Let $f_\sigma \in C^\infty(\mathbb{S}^{2+\epsilon})$ be such that $\Delta_{\mathbb{S}^d}^m f_\sigma \in L^2(\mathbb{S}^{2+\epsilon})$ for all $m \geq 0$ and satisfies

$$\sum_{m=1}^{\infty} \sum_{\sigma} \left\|\Delta_{S^{2+\epsilon}}^m f_\sigma\right\|_2^{-\frac{1}{2m}} = \infty$$

If $\left.\frac{\partial^m}{\partial\theta^m}\right|_{\theta=0} F_\sigma(\theta, \xi') = 0$ for all $m \geq 0$ and for all $\xi' \in \mathbb{S}^{1+\epsilon}$, then f_σ is identically zero.

Proof. Let f_σ be as in the statement of the theorem. For $n \geq 0$, let $P_n f_\sigma$ denote the projection of f_σ onto the space $\mathcal{H}_n(\mathbb{S}^{2+\epsilon})$. Then from the above observations we have

$$P_n f_\sigma = \sum_{\epsilon=-2}^{n} \sum_{k=1}^{N(2+\epsilon)} \sum_{\sigma} (f_\sigma, S_{n,2+\epsilon,k})_{L^2} S_{n,2+\epsilon,k}. \tag{4.7}$$

Also since $f_\sigma \in L^2(\mathbb{S}^{2+\epsilon})$ we have

$$f_\sigma = \sum_{n=0}^{\infty} \sum_{\sigma} P_n f_\sigma = \sum_{n=0}^{\infty} \sum_{\epsilon=-2}^{n} \sum_{k=1}^{N(2+\epsilon)} \sum_{\sigma} (f_\sigma, S_{n,2+\epsilon,k})_{L^2(\mathbb{S}^{2+\epsilon})} S_{n,2+\epsilon.k}$$

By interchanging the summations, we observe that

$$f_\sigma = \sum_{\epsilon=-1}^{\infty} \sum_{n=2+\epsilon}^{\infty} \sum_{k=1}^{N(2+\epsilon)} \sum_{\sigma} (f_\sigma, S_{n,2+\epsilon,k})_{L^2(\mathbb{S}^{2+\epsilon})} S_{n,2+\epsilon.k}$$

$$= \sum_{\epsilon=-2}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{N(2+\epsilon)} \sum_{\sigma} (f_\sigma, S_{n+2+\epsilon,2+\epsilon,k})_{L^2(\mathbb{S}^{2+\epsilon})} S_{n+2+\epsilon,2+\epsilon.k}$$

In view of this, to prove the theorem it is enough to prove that $(f_\sigma, S_{n+2+\epsilon,2+\epsilon,k})_{L^2(\mathbb{S}^{2+\epsilon})} = 0$ for all $n, 2+\epsilon, k$. To start with, let us first fix $n, 2+\epsilon$ and k. From the expansion 4.7 we observe that

$$(P_n f_\sigma, S_{n+2+\epsilon,2+\epsilon,k})_{L^2(\mathbb{S}^{2+\epsilon})} = (f_\sigma, S_{n+2+\epsilon,2+\epsilon,k})_{L^2(\mathbb{S}^{2+\epsilon})}. \tag{4.8}$$

Next we use the expression for $S_{n+2+\epsilon,2+\epsilon,k}$ to show that these coefficients are nothing but Jacobi coefficients of a suitable function. In order to do so, we write the integral on $\mathbb{S}^{2+\epsilon}$ in polar coordinates to obtain

$$\left(f_\sigma, S_{n+2+\epsilon,2+\epsilon,k}\right)_{L^2(\mathbb{S}^{2+\epsilon})} = \int_0^\pi \int_{\mathbb{S}^{1+\epsilon}} \sum_\sigma F_\sigma(\theta,\xi') a_{n+2+\epsilon,2+\epsilon} (\sin\theta)^{3+2\epsilon} C_n^{2+\epsilon+\frac{1+\epsilon}{2}}(\cos\theta) S'_{k,2+\epsilon}(\xi') d\sigma_{1+\epsilon}(\xi') d\theta$$

where $F_\sigma := f_\sigma \circ \varphi$. Now using 4.3, 4.4 and 4.6, a simple calculation yields

$$a_{n+2+\epsilon,2+\epsilon} C_n^{2+\epsilon+\frac{1+\epsilon}{2}}(\cos\theta) = 2^{-\left(2+\epsilon+\frac{1+\epsilon}{2}\right)} C\left(2+\epsilon+\frac{2+\epsilon}{2}-1, 2+\epsilon+\frac{2+\epsilon}{2}-1, n\right) P_n^{\left(2+\epsilon+\frac{2+\epsilon}{2}-1,2+\epsilon+\frac{2+\epsilon}{2}-1\right)}(\cos\theta) \quad (4.9)$$

which transforms the above equation into

$$\left(f_\sigma, S_{n+2+\epsilon,2+\epsilon,k}\right)_{L^2(\mathbb{S}^{2+\epsilon})} = 2^{-\left(2+\epsilon+\frac{1+\epsilon}{2}\right)} \int_0^\pi \sum_\sigma (F_\sigma)_{k,2+\epsilon}(\theta) (\sin\theta)^{3+2\epsilon} \mathcal{P}_n^{\left(2+\epsilon+\frac{d}{2}-1,2+\epsilon+\frac{d}{2}-1\right)}(\theta) d\theta \quad (4.10)$$

where we have defined

$$(F_\sigma)_{k,2+\epsilon}(\theta) := \int_{\mathbb{S}^{1+\epsilon}} \sum_\sigma F_\sigma(\theta,\xi') S'_{k,2+\epsilon}(\xi') d\sigma_{1+\epsilon}(\xi')$$

Now letting $(\epsilon-1) = 2+\epsilon+\frac{d}{2}-1$ and writing $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ we see that

$$(\sin\theta)^{3+2\epsilon} = 2^{3+2\epsilon} (\sin\theta)^{-(2+\epsilon)} w_{\epsilon-1,\epsilon-1}(\theta)$$

which together with 4.10 yields

$$\left(f_\sigma, S_{n+2+\epsilon,2+\epsilon,k}\right)_{L^2(\mathbb{S}^{2+\epsilon})} = \mathcal{J}_{\epsilon-1,\epsilon-1}\left(g_{k,2+\epsilon}\right)(n) \quad (4.11)$$

where $g_{k,2+\epsilon}(\theta) := 2^{\frac{1+\epsilon}{2}} (\sin\theta)^{-(2+\epsilon)} (F_\sigma)_{k,2+\epsilon}(\theta)$.

In view of the Plancherel formula 4.5 and the relation 4.8 we have

$$\left\|\mathbb{L}^m_{\epsilon-1,\epsilon-1} l_{2+\epsilon,k}\right\|_2^2 = \sum_{n=0}^\infty \left(n+\frac{2(\epsilon-1)+1}{2}\right)^{4m} \left|\mathcal{J}_{\epsilon-1,\epsilon-1}\left(g_{l,k}\right)(n)\right|^2$$

$$= \sum_{n=0}^\infty \left(n+\frac{2(2+\epsilon)+(2+\epsilon)-1}{2}\right)^{4m} \left|\int_{\mathbb{S}^d} \sum_\sigma P_n f_\sigma(\xi) S_{n+2+\epsilon,2+\epsilon,k}(x) d\sigma_{2+\epsilon}(\xi)\right|^2 \quad (4.12)$$

By Cauchy-Schwarz inequality we note that

$$\left|\int_{\mathbb{S}^d} \sum_\sigma P_n f_\sigma(\xi) S_{n+2+\epsilon,2+\epsilon,k}(\xi) d\sigma_{2+\epsilon}(\xi)\right|^2 \le \sum_\sigma \|P_n f_\sigma\|^2_{L^2(\mathbb{S}^{2+\epsilon})}$$

Finally, using the fact that $n+\frac{1}{2}(2(\epsilon-1)+1) = n+\frac{1}{2}(2(2+\epsilon)+1+\epsilon) \le \left(n+\frac{1+\epsilon}{2}\right)\left(1+\frac{2(2+\epsilon)}{1+\epsilon}\right)$, from 4.12 we get the estimate

$$\left\|\mathbb{L}^m_{\epsilon-1,\epsilon-1} g_{2+\epsilon,k}\right\|_2^2 \le \left(1+\frac{2(2+\epsilon)}{1+\epsilon}\right)^{4m} \sum_{n=0}^\infty \left(n+\frac{1+\epsilon}{2}\right)^{4m} \sum_\sigma \|P_n f_\sigma\|^2_{L^2(\mathbb{S}^{2+\epsilon})}$$

Therefore, we have proved

$$\left\|\mathbb{L}^m_{\epsilon-1,\epsilon-1} g_{2+\epsilon,k}\right\|_2 \le \left(1+\frac{2(2+\epsilon)}{1+\epsilon}\right)^{2m} \sum_\sigma \left\|\Delta^m_{\mathbb{S}^{2+\epsilon}} f_\sigma\right\|_{L^2(\mathbb{S}^{2+\epsilon})}$$

which by the hypothesis on the function f_σ, implies that

$$\sum_{m=1}^\infty \left\|\mathbb{L}^m_{\epsilon-1,\epsilon-1} g_{(2+\epsilon),k}\right\|_2^{-\frac{1}{2m}} = \infty \quad (4.13)$$

Since $g_{2+\epsilon,k}(\theta)$ is related to $F_\sigma(\theta,\xi')$ via the integral

$$g_{2+\epsilon,k}(\theta) = 2^{\frac{1+\epsilon}{2}} (\sin\theta)^{-(2+\epsilon)} \int_{\mathbb{S}^{1+\epsilon}} \sum_\sigma F_\sigma(\theta,\xi') S'_{k,2+\epsilon}(\xi') d\sigma_{1+\epsilon}(\xi')$$

the hypothesis $\frac{\partial^m}{\partial\theta^m}\Big|_{\theta=0} F_\sigma(\theta,\xi') = 0$ for all $m \ge 0$ allows us to conclude that $\mathbb{L}^m_{\epsilon-1,\epsilon-1} g_{2+\epsilon,k}(0) = 0$ for all $m \ge 0$. Hence $g_{2+\epsilon,k}$ satisfies the hypotheses of Theorem 4.1 and hence we conclude that $g_{2+\epsilon,k} = 0$ and consequently $\left(f_\sigma, S_{n+2+\epsilon,2+\epsilon,k}\right)_{L^2(\mathbb{S}_{\mathbb{q}})} = 0$. As this is true for any $n, 2+\epsilon, k$, we conclude that $f_\sigma = 0$ completing the proof of the theorem.

4.3. The real projective spaces $P_{2+\epsilon}(\mathbb{R})$. Let $O(2+\epsilon)$ denote the group of $(2+\epsilon)\times(2+\epsilon)$ orthogonal matrices. Then $P_{2+\epsilon}(\mathbb{R})$ can be identified with $SO(3+\epsilon)/O(2+\epsilon)$ which makes this a compact symmetric space. Now it is well-known that the real projective space $P_{2+\epsilon}(\mathbb{R})$ can be obtained from $\mathbb{S}^{2+\epsilon}$ by identifying the antipodal points i.e., $P_{2+\epsilon}(\mathbb{R}) = \mathbb{S}^{2+\epsilon}/\{\pm I\}$ and the projection map $s \to \pm s$ from $\mathbb{S}^{2+\epsilon}$ to $P_{2+\epsilon}(\mathbb{R})$ is locally an isometry. So, the functions on $P_{2+\epsilon}(\mathbb{R})$ can be viewed as even functions on the corresponding sphere $\mathbb{S}^{2+\epsilon}$ and if

$(f_\sigma)_e$ is the even function on $\mathbb{S}^{2+\epsilon}$ corresponding to the function f_σ on $P_{2+\epsilon}(\mathbb{R})$ then $\Delta_{P_{2+\epsilon}(\mathbb{R})} f_\sigma = \Delta_{\mathbb{S}^{2+\epsilon}}(f_\sigma)_e$. Hence the analogue of Chernoff's theorem on $P_{2+\epsilon}(\mathbb{R})$ follows directly from the case of sphere.

4.4. The other projective spaces $P_l(\mathbb{C}), P_l(\mathbb{H})$, and $P_2(\mathbb{C}ay)$. As pointed out by [27], analysis on these three projective spaces is quite similar. Closely following the notations of [27] (see also [10]), we first describe the appropriate polar coordinate representation of these spaces and then as in the sphere case we prove the Chernoff's theorem for the associated Laplace-Beltrami operators. Now, let S denote any of these three spaces $P_{2+\epsilon}(\mathbb{C}), P_{2+\epsilon}(\mathbb{H})$, and $P_2(\mathbb{C}ay)$. Suppose $\tilde{\Delta}_S$ denotes the corresponding Laplace-Beltrami operator. Let $d\mu$ denote the normalised Riemann measure on S. We have the following orthogonal decomposition:

$$L^2(S, d\mu) := L^2(S) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(S)$$

where $\mathcal{H}_n(S)$ are finite dimensional and eigenspaces of $\tilde{\Delta}_S$ with eigenvalue $-n(n+k+2+\epsilon)$ where $(2+\epsilon) = 2,4,8, k = \epsilon, 2(2+\epsilon)-3,3$, for $P_{2+\epsilon}(\mathbb{C}), P_{2+\epsilon}(\mathbb{H})$ and $P_2(\mathbb{C}ay)$, respectively. However, it is convenient to work with $\Delta_S := -\tilde{\Delta}_S + \rho_S^2$ where $\rho_S := \frac{1}{2}(k+2+\epsilon)$. As a result $\mathcal{H}_n(S)$ becomes eigenspaces of Δ_S with eigenvalue $\left(n + \frac{k+2+\epsilon}{2}\right)^2$.

Let $\Omega := \{x \in \mathbb{R}^{3+\epsilon} : |x| \le 1\}$ be the closed unit ball in $\mathbb{R}^{3+\epsilon}$. We consider a weight function w defined by $w(1+\epsilon) := (1+\epsilon)^{-1}(\epsilon)^k$ for $0 \le \epsilon < 1$. With these notations we have the following result proved in [27, Lemma 4.15].

Proposition 4.4 (see [29]). There is a bounded linear map $E: L^1(S) \to L^1(\Omega, w(|x|)dx)$ satisfying

(1) For $f_\sigma \in L^1(S)$,

$$\int_S \sum_\sigma f_\sigma d\mu = \int_\Omega \sum_\sigma E(f_\sigma)(x) w(|x|) dx$$

(2) The norm of E as a map from $L^{1+\epsilon}(S)$ to $L^{1+\epsilon}(\Omega, w(|x|)dx)$ is $(0 \le \epsilon \le \infty)$.

The integration formula in the above proposition is very useful. In fact, integrating the right hand side of that formula in polar coordinates we have

$$\int_\Omega E(f_\sigma)(x) w(|x|) dx = \int_0^1 \int_{\mathbb{S}^{2+\epsilon}} E(f_\sigma)((1+\epsilon)\xi) w(1+\epsilon)(1+\epsilon)^{2+\epsilon} d\sigma_{2+\epsilon}(\xi) d(1+\epsilon).$$

Now a change of variables $(1+\epsilon) = \sin^2(\theta/2)$ allows us to conclude that

$$\int_S \sum_\sigma f_\sigma d\mu = \int_0^\pi \int_{\mathbb{S}^{2+\epsilon}} \sum_\sigma F_\sigma(\theta, \xi) \left(\sin\frac{\theta}{2}\right)^{2(2+\epsilon)-1} \left(\cos\frac{\theta}{2}\right)^{2k+1} d\theta d\sigma_{2+\epsilon}(\xi) \tag{4.14}$$

where $F_\sigma(\theta, \xi) = E(f_\sigma)(\sin^2(\theta/2)\xi)$. In [27] Sherman has described the image of $\mathcal{H}_n(S)$ under the map E. It has been proved that $E(\mathcal{H}_n(S)) = \mathcal{H}_n(\Omega, w)$ where $\mathcal{H}_n(\Omega, w)$ is the orthocomplement of $\mathbb{P}_{n-1}(\Omega)$ in $\mathbb{P}_n(\Omega)$ with respect to the inner product in $L^2(\Omega, w(|x|)dx)$. Here $\mathbb{P}_n(\Omega)$ denotes the set of all polynomials on Ω of degree up to n. Also note that in these trigonometric polar coordinates we can identify Ω with $\Omega_0 := (0, \pi) \times \mathbb{S}^{2+\epsilon}$ and

$$d\omega(\theta, \xi) := \left(\sin\frac{\theta}{2}\right)^{2(2+\epsilon)-1} \left(\cos\frac{\theta}{2}\right)^{2k+1} d\theta d\sigma_{2+\epsilon}(\xi)$$

is the corresponding measure on Ω_0. Basically in view of this trigonometric polar coordinates we have $\mathcal{H}_n(\Omega, w) = \mathcal{H}_n(\Omega_0, \omega)$. These spaces are eigenspaces of the following differential operator

$$\Lambda_S = -\frac{\partial^2}{\partial\theta^2} - \frac{(1+\epsilon-k) + (2+\epsilon+k)\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2(\theta/2)}\tilde{\Delta}_{\mathbb{S}^{2+\epsilon}} + \left(\frac{k+2+\epsilon}{2}\right)^2$$

with eigenvalues $\left(n + \frac{2+\epsilon+k}{2}\right)^2$. The relation between this operator and the Laplace-Beltrami operator is described in the following proposition (see [29]).

Proposition 4.5. Let $f_\sigma \in C^2(S)$ and E be as in the Proposition4.4. Then we have

$$E(\Delta_S f_\sigma) = \Lambda_S E(f_\sigma)$$

For a proof of this fact we refer the reader to [27, Lemma 4.25]. Thus we have the following orthogonal decomposition

$$L^2(\Omega_0, d\omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(\Omega_0, \omega).$$

Moreover, $\mathcal{H}_n(\Omega_0, \omega)$ admits a further decomposition as $\mathcal{H}_n(\Omega_0, \omega) = \oplus_{j=0}^n \mathcal{H}_{n,j}(\Omega_0, \omega)$ where each $\mathcal{H}_{n,j}(\Omega_0, \omega)$ is irreducible under $SO(3+\epsilon)$ and spanned by $\{Q_{n,j,2+\epsilon} : 1 \le 2+\epsilon \le N(j)\}$ (see [27, Theorem 4.22]) where for $x = \sin^2(\theta/2)\xi, \theta \in (0, \pi)$ and $\xi \in \mathbb{S}^{2+\epsilon}$

$$Q_{n,j,2+\epsilon}(x) = (a+\epsilon)_{n,j}\left(\sin\frac{\theta}{2}\right)^{2j} P_{n-j}^{(k,1+\epsilon+2j)}(2\sin^2(\theta/2)-1)S_{j,2+\epsilon}(\xi)$$
$$= (-1)^{n-j}(a+\epsilon)_{n,j}\left(\sin\frac{\theta}{2}\right)^{2j} P_{n-j}^{(1+\epsilon+2j,k)}(\cos\theta)S_{j,2+\epsilon}(\xi).$$

In the second equality we have used the symmetry relation for Jacobi polynomials i.e., $P_n^{(\epsilon-1,2\epsilon-1)}(-x) = (-1)^n P_n^{(2\epsilon-1,\epsilon-1)}(x)$. Here $\{S_{j,2+\epsilon}: 1 \le 2+\epsilon \le N(j)\}$ a basis for $\mathcal{H}_j(\mathbb{S}^{2+\epsilon})$, the spherical harmonics of degree $(2+\epsilon)$ on $\mathbb{S}^{2+\epsilon}$. The constants $(a+\epsilon)_{n,j}$ appearing in the above expression are chosen so that $\|Q_{n,j,2+\epsilon}\|_2 = 1$. In fact, it can be checked that $(a+\epsilon)_{n,j} = C(1+\epsilon+2j,k,n-j)$. So, clearly $\{Q_{n,j,2+\epsilon}: n,\epsilon \ge -2, 1 \le N(2+\epsilon)\}$ forms an orthonormal basis for $L^2(\Omega_0, d\omega)$. Now we are ready to state and prove an analogue of Chernoff's theorem on S.

Theorem 4.6 (see [29]). Let $f_\sigma \in C^\infty(S)$ be such that $\Delta_S^m f_\sigma \in L^2(S)$ for all $m \ge 0$. Assume that

$$\sum_{m=1}^{\infty}\sum_{\sigma} \|\Delta_S^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$$

If the function F_σ defined by $F_\sigma(\theta,\xi) = E(f_\sigma)(\sin^2(\theta/2)\xi)$ satisfies $\frac{\partial^m}{\partial\theta^m}\Big|_{\theta=0} F_\sigma(\theta,\xi) = 0$ for all $m \ge 0$ and for all $\xi \in \mathbb{S}^{2+\epsilon}$, then f_σ is identically zero.

Proof. Given a function f_σ with the property as in the statement of the theorem, we write $E(f_\sigma)(\sin^2(\theta/2)\xi) = F_\sigma(\theta,\xi), (\theta,\xi) \in \Omega_0$. So, the analysis, described above allow us to write the projection of F_σ onto $\mathcal{H}_n(\Omega_0,\omega)$ as

$$P_n^S F_\sigma = \sum_{j=0}^{n}\sum_{\epsilon=-1}^{N(j)}\sum_{\sigma} (F_\sigma, Q_{n,j,2+\epsilon})Q_{n,j,2+\epsilon}.$$

Now as in the sphere case, it is not hard to check that

$$F_\sigma = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\epsilon=-1}^{N(j)}\sum_{\sigma} (F_\sigma, Q_{n+j,j,2+\epsilon})_{L^2(\Omega_0,d\omega)} Q_{n+j,j,2+\epsilon}. \tag{4.15}$$

Clearly, for each $n \ge 0$ we have

$$(P_n^S F_\sigma, Q_{n+j,j,2+\epsilon})_{L^2(\Omega_0,d\omega)} = (F_\sigma, Q_{n+j,j,2+\epsilon})_{L^2(\Omega_0,d\omega)}. \tag{4.16}$$

As in the case of sphere, we will show that the right hand side of the above equation can be expressed as Jacobi coefficient of a suitable function related to F_σ. By definition, we have

$$(F_\sigma, Q_{n+j,j,2+\epsilon}) = \int_0^\pi \int_{S^{2+\epsilon}} \sum_\sigma F_\sigma(\theta,\xi) Q_{n+j,j,2+\epsilon}\left(\left(\sin^2\frac{\theta}{2}\right)\xi\right)\left(\sin\frac{\theta}{2}\right)^{2(2+\epsilon)-1}\left(\cos\frac{\theta}{2}\right)^{2k+1} d\theta d\sigma_{2+\epsilon}(\xi). \tag{4.17}$$

Now using the expression for $Q_{n+j,j,2+\epsilon}$ we have

$$(F_\sigma, Q_{n+j,j,2+\epsilon})$$
$$= (-1)^j (a+\epsilon)_{n+j,j}\int_0^\pi \sum_\sigma (F_\sigma)_{j,2+\epsilon}(\theta)\left(\sin\frac{\theta}{2}\right)^{2j} P_n^{(1+\epsilon+2j,k)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{2(2+\epsilon)-1}\left(\cos\frac{\theta}{2}\right)^{2k+1} d\theta \tag{4.18}$$

where $(F_\sigma)_{j,2+\epsilon}$ are defined by

$$(F_\sigma)_{j,2+\epsilon}(\theta) := \int_{S^{2+\epsilon}} \sum_\sigma F_\sigma(\theta,\xi) S_{j,2+\epsilon}(\xi) d\sigma_{2+\epsilon}(\xi).$$

Writing $g_{j,2+\epsilon}(\theta) = (-1)^j (F_\sigma)_{j,2+\epsilon}(\theta)\left(\sin\frac{\theta}{2}\right)^{-2j}$ and using the definition of Jacobi coefficients we have

$$(F_\sigma, Q_{n+j,j,2+\epsilon})_{L^2(\Omega_0,d\omega)} = \mathcal{J}_{\epsilon-1,2\epsilon-1}(g_{j,2+\epsilon})(n)$$

where $(\epsilon-1) = 2+\epsilon-2j+k$ and $2\epsilon-1 = k$. Now using the Plancherel formula 4.5 along with 4.16 we obtain

$$\|\mathbb{L}_{\epsilon-1,2\epsilon-1}^m g_{j,2+\epsilon}\|_2^2 = \sum_{n=0}^{\infty}\left(n+\frac{3\epsilon-1}{2}\right)^{4m} |\mathcal{J}_{\epsilon-1,2\epsilon-1}(g_{j,2+\epsilon})(n)|^2$$
$$= \sum_{n=0}^{\infty}\sum_\sigma \left(n+\frac{2+\epsilon-2j+2k+1}{2}\right)^{4m} |P_n^s F_\sigma, Q_{n+j,j,2+\epsilon}|^2. \tag{4.19}$$

But $\sum_\sigma |(P_n^S f_\sigma, Q_{n+j,j,2+\epsilon})| \le \sum_\sigma \|P_n^S f_\sigma\|_{L^2(\Omega_0,d\omega)}$ and $\left(n+\frac{3+\epsilon-2j+2k}{2}\right) \le C\left(n+\frac{2+\epsilon+k}{2}\right)$ so that we have

$$\|\mathbb{L}_{\epsilon-1,2\epsilon-1}^m g_{j,2+\epsilon}\|_2^2 \le C^{4m}\sum_{n=0}^{\infty}\left(n+\frac{2+\epsilon+k}{2}\right)^{4m}\sum_\sigma \|P_n^S f_\sigma\|_2^2 = C^{4m}\sum_\sigma \|\Lambda_S^m E(f_\sigma)\|_2^2 \tag{4.20}$$

In view of the Proposition 4.5 we have $E(\Delta_S^m f_\sigma) = \Lambda_S^m E(f_\sigma)$ and using the fact that operator norm of E is one (see Proposition 4.4) we have

$$\left\|\mathbb{L}_{\epsilon-1,2\epsilon-1}^m g_{j,2+\epsilon}\right\|_2^2 \leq C^{2m} \sum_\sigma \|\Delta_S^m f_\sigma\|_2$$

Hence the given condition $\sum_{m=1}^\infty \sum_\sigma \|\Delta_S^m f_\sigma\|_2^{-\frac{1}{2m}} = \infty$ allows us to conclude that

$$\sum_{m=1}^\infty \left\|\mathbb{L}_{\epsilon-1,2\epsilon-1}^m g_{j,2+\epsilon}\right\|_2^{-\frac{1}{2m}} = \infty \tag{4.21}$$

Also using the hypothesis $\left.\frac{\partial^m}{\partial\theta^m}\right|_{\theta=0} F_\sigma(\theta,\xi) = 0$ for all $m \geq 0, \xi \in \mathbb{S}^{2+\epsilon}$, a simple calculation shows that $\mathbb{L}_{\epsilon-1,2\epsilon-1}^m g_{j,2+\epsilon}(0) = 0$ for all $m \geq 0$. Hence by Theorem 4.1, we have $g_{j,2+\epsilon} = 0$ whence $\left(F_\sigma, Q_{n+j,j,2+\epsilon}\right)_{L^2(\Omega_0,d\omega)} = 0$. As this is true for all $n, j, 2+\epsilon$ we conclude $f_\sigma = 0$. This completes the proof of the theorem.

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