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Research Paper

On Surjectivity of Bures Isometries between Density Spaces of $C*$-Algebras:

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Abstract

L. Molnár [17] describe the need structure of all surjective, nonlinear or, nonaffine, Bures isometries between density spaces of C^*-algebras equipped with faithful traces. Although the mentioned maps are closely related to (linear) Jordan *-isomorphisms between the underlying algebras. We consider density spaces and the problem of the positive definite (and positive semidefinite) cones of C^*-algebras. Following [17] and we show a survey on the considerable study.

Keywords: Bures Isometries, C^*-Algebras, Hilbert space, Operator Algebra, Thompson metric, Jordan $*$-isomorphism, density spaces.

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1. Introduction and statements of the results

The Bures metric method appears in several different areas of mathematics and physics. For example, when defined on the state space of a quantum system, it is a quantum generalization of the Fischer information metric and plays important roles in quantum information geometry (see, e.g., [1], Sections 9.4-9.6, or any related one). Moreover, when defined on the cone of positive definite matrices, it appears in various optimisation problems, in the theory of optimal transport, etc. We also mention that, [2], introduced and studied in details a new concept of matrix means based on this metric.

The Bures distance in quantum information theory is closely connected to the concept of fidelity which is a natural extension of the basic notion of transition probability (cf. Wigner's fundamental theorem on the structure of quantum mechanical symmetry transformations) from pure states to the case of mixed states. We follow the validity of the verification of [17], in the Hilbert space framework of quantum mechanics, considering (mixed) states as density operators, i.e., positive (and trace-class) operators on a complex Hilbert space having trace 1. The symmetries of the space of all such operators (with or without the normalizing condition of having unit trace) corresponding to the notion of fidelity (which maps are anyway natural analogues of Wigner's symmetry transformations) and the ones corresponding to the Bures distance were completely described (see [7], and [10]).

The authors in [3] and [4] extended the concepts of fidelity and Bures metric to the more general setting of C^*-algebras equipped with faithful traces. That context is mainly algebraic and the density operators in the classical Hilbert space is naturally investigated by the positive elements of the algebra having unit trace. [3], in Section II, presented several results relating to fidelity preserving maps where the linearity of the transformations in question was always occur, in [4], around some senses, continued their investigations and studied Bures contractions, again under the strong condition of linearity of the transformations.

L. Molnár [17] describe the structure of all surjective maps between density spaces of C^*-algebras which preserve either the fidelity or the Bures distance without assuming any sort of linearity of the maps. This means that, in certain senses, we obtain serious strengthenings of some of the results in [3], [4] and also strong extensions of [7], [10] to the general setting of C^*-algebras equipped with faithful traces. So the main tool of these descriptions is the structural results on the so-called (surjective) Thompson isometries of positive definite cones in operator algebras see [8] for the full operator algebra over a Hilbert space and see [6] for general C^*-algebras.

We fix the notation and give the necessary definitions. By a C^*-algebra we always mean a unital one. For a complex Hilbert space H, the C^*-algebra of all bounded linear operators on H is denoted by $B(H)$. The set of all positive elements (i.e., self-adjoint elements with nonnegative spectrum) of the C^*-algebra $\mathcal{A}$ is denoted by $\mathcal{A}_+$. It is a cone and called the positive semidefinite cone of $\mathcal{A}$. The set of all invertible elements of $\mathcal{A}_+$is denoted by $\mathcal{A}_+^{-1}$. It is also a cone and we call it the positive definite cone of $\mathcal{A}$. If $\mathcal{A}$ is a C^*-algebra and τ is a positive linear functional (a linear functional on $\mathcal{A}$ sending positive elements to nonnegative real numbers) such that $\tau(x(x+\epsilon)) = \tau((x+\epsilon)x)$ holds for all $x, (x+\epsilon) \in \mathcal{A}$, then τ is called a trace. We say that the trace τ on $\mathcal{A}$ is faithful if $\tau(a_r) = 0, a_r \in \mathcal{A}_+$, implies $a_r = 0$.

For $\mathcal{A}$ be a C^*-algebra and τ be a faithful trace on $\mathcal{A}$ and for any $a_r, (a_r+\epsilon) \in \mathcal{A}_+$, we set

$$F_\tau(a_r, a_r+\epsilon) = \tau\left(\left((a_r+\epsilon)^{1/2} a_r (a_r+\epsilon)^{1/2}\right)^{1/2}\right)$$

and

$$d_B^\tau(a_r, a_r+\epsilon) = \sqrt{\tau(a_r) + \tau(a_r+\epsilon) - 2F_\tau(a_r, a_r+\epsilon)}. \qquad (1)$$

The first quantity is called the fidelity of a_r and $(a_r+\epsilon)$, the second one is said to be the Bures distance between a_r and $(a_r+\epsilon)$. It can be shown that d_B^τ is a true metric on $\mathcal{A}_+$. In fact, see [4] but there d_B^τ was defined only on the so-called τ-density space of $\mathcal{A}$, i.e., on the convex set

$$\mathcal{D}_\tau(\mathcal{A}) = \{a_r \in \mathcal{A}_+ : \tau(a_r) = 1\}.$$

However, the ideas applied in [4] to prove that d_B^τ is a metric on $\mathcal{D}_\tau(\mathcal{A})$ can also be used (with some necessary modifications) to verify that the same is true on the whole set $\mathcal{A}_+$. We have to admit that the Bures metric as we have defined it in (1) is, on the density space $\mathcal{D}_\tau(\mathcal{A})$, in fact a constant multiple of the Bures metric defined in [4] (however, apparently, this does not affect the structure of the corresponding isometries).

Hence we describe the surjective Bures isometries between density spaces of C^*-algebras. So all such maps are in fact restrictions of (linear) Jordan *-isomorphisms between the full algebras multiplied by some invertible positive central element (central element of an algebra being an element which commutes with all elements). Before dealing with the statement we recall that a Jordan *-isomorphism (or, a C^*-isomorphism) between C^*-algebras $\mathcal{A}, \mathcal{A}'$ is a bijective linear map $J: \mathcal{A} \to \mathcal{A}'$ which satisfies $J(x(x+\epsilon) + (x+\epsilon)x) = J(x)J(x+\epsilon) + J(x+\epsilon)J(x)$ and $J(x)^* = J(x^*)$ for all $x, (x+\epsilon) \in \mathcal{A}$. We mention that the following results can also be viewed as characterizations of Jordan *-isomorphisms between C^*-algebras in terms of some quantum information theoretical numerical quantities.

Theorem 1[17]: Let $\mathcal{A}, \mathcal{A}'$ be C^*-algebras with faithful traces τ, τ', respectively, and let $\phi: \mathcal{D}_\tau(\mathcal{A}) \to \mathcal{D}_{\tau'}(\mathcal{A}')$ be a surjective map. Then ϕ is a Bures isometry, i.e., it satisfies

$$d_B^{\tau'}\big(\phi(a_r), \phi(a_r+\epsilon)\big) = d_B^\tau(a_r, a_r+\epsilon), a_r, (a_r+\epsilon) \in \mathcal{D}_\tau(\mathcal{A}) \qquad (2)$$

if and only if there is a Jordan *-isomorphism $\phi: \mathcal{A} \to \mathcal{A}'$ and an element $c \in \mathcal{A}_+'^{-1}$ central in $\mathcal{A}'$ such that $\phi(a_r) = cJ(a_r)$ holds for all $a_r \in \mathcal{D}_\tau(\mathcal{A})$ and $\tau'(cJ(x)) = \tau(x)$ holds for all $x \in \mathcal{A}$.

We know as before the Bures distance is an important metric also on positive definite cones. Our next result concerns that case and looks quite the same as the previous theorem.

Theorem 2[17]: Let $\mathcal{A}, \mathcal{A}'$ be C^*-algebras with faithful traces τ, τ', respectively, and let $\phi: \mathcal{A}_+^{-1} \to \mathcal{A}_+'^{-1}$ be a surjective map. Then ϕ is a Bures isometry, i.e., it satisfies

$$d_B^{\tau'}(\phi(a_r), \phi(a_r + \epsilon)) = d_B^\tau(a_r, a_r + \epsilon), a_r, (a_r + \epsilon) \in \mathcal{A}_+^{-1}$$

if and only if there is a Jordan *-isomorphism $\phi: \mathcal{A} \to \mathcal{A}'$ and an invertible positive central element $c \in \mathcal{A}'$ such that $\phi(a_r) = cJ(a_r)$ holds for all $a_r \in \mathcal{A}_+^{-1}$ and $\tau'(cJ(x)) = \tau(x)$ holds for all $x \in \mathcal{A}$.

Some simple modifications in the proof of the above theorem will give us the following corollary.

Corollary 3[17]: The statement in Theorem 2 remains valid also in the case where the positive definite cones are replaced by the corresponding positive semidefinite cones.

From the above results we can deduce the following corollary concerning fidelity preserving maps.

Corollary 4[17]: Let $\mathcal{A}, \mathcal{A}'$ be C^*-algebras with faithful traces τ, τ', respectively, and let $\phi: \mathcal{D}_\tau(\mathcal{A}) \to \mathcal{D}_{\tau'}(\mathcal{A}')$ be a surjective map. Then ϕ preserves fidelity, i.e., it satisfies

$$F_{\tau'}(\phi(a_r), \phi(a_r + \epsilon)) = F_\tau(a_r, a_r + \epsilon), a_r, (a_r + \epsilon) \in \mathcal{D}_\tau(\mathcal{A})$$

if and only if there is a Jordan *-isomorphism $\phi: \mathcal{A} \to \mathcal{A}'$ and an invertible positive central element $c \in \mathcal{A}'$ such that $\phi(a_r) = cJ(a_r)$ holds for all $a_r \in \mathcal{A}_+^{-1}$ and $\tau'(cJ(x)) = \tau(x)$ holds for all $x \in \mathcal{A}$.

The statement remains true when the density spaces are replaced by the corresponding positive definite cones or by the corresponding positive semidefinite cones.

2. Proofs

We devoted to the proofs of the results. So some auxiliary statements are needed. The following one gives a sort of characterization of the order by means of fidelities.

Lemma 5[17]: Let $\mathcal{A}$ be a C^*-algebra with a faithful trace τ and let $a_r, (a_r + \epsilon) \in \mathcal{A}_+$. The following assertions are equivalent:

(i) $a_r^2 \leq (a_r + \epsilon)^2$;

(ii) $\tau\left((x a_r^2 x)^{1/2}\right) \leq \tau\left((x(a_r + \epsilon)^2 x)^{1/2}\right)$ holds for all $x \in \mathcal{A}_+^{-1}$.

Consequently, for any $a_r, (a_r + \epsilon) \in \mathcal{A}_+$we have $a_r \leq a_r + \epsilon$ if and only if $F_\tau(a_r, x) \leq F_\tau(a_r + \epsilon, x)$ holds for all $x \in \mathcal{A}_+^{-1}$.

Proof. The implication $(i) \Rightarrow (ii)$ is easy. Assume (i) holds. We have $x a_r^2 x \leq x(a_r + \epsilon)^2 x$ for all $x \in \mathcal{A}_+^{-1}$. Since the square root function is operator monotone, we have $(x a_r^2 x)^{1/2} \leq (x(a_r + \epsilon)^2 x)^{1/2}$ and then, by the positivity of the linear functional τ, we obtain (ii).

Now assume that (i) holds. First observe that using the continuity of the linear functional τ (which is a consequence of its positivity) and the continuity of the multiplication and square root operation, we have that the inequality in (ii) holds for any $x \in \mathcal{A}_+$. Consider $\mathcal{A}$ as a norm closed *-subalgebra of $B(H)$ for some complex Hilbert space H containing the identity I. Clearly, $s_m = (a_r + \epsilon)^2 - a_r^2$ is a self-adjoint operator. Consider the spectral measure of s_m on the Borel subsets of the real line. Let p be the spectral measure of $]-\infty, 0[$. We have $p s_m p \leq 0$. For any nonnegative continuous real function $f \leq \chi_{]-\infty,0[}(\chi_M$

stands for the characteristic function of the set) we have $f(s_m)s_m f(s_m) = f(s_m)ps_m pf(s_m) \leq 0$. It follows that $f(s_m)(a_r+\epsilon)^2 f(s_m) \leq f(s_m)a_r^2 f(s_m)$ which implies

$$\tau(f(s_m)(a_r+\epsilon)^2 f(s_m)) \leq \tau(f(s_m)a_r^2 f(s_m)).$$

But, by our assumption, the reverse inequality also holds and hence, by the faithfulness of τ, we can infer $f(s_m)(a_r+\epsilon)^2 f(s_m) = f(s_m)a_r^2 f(s_m)$, i.e., $f(s_m)s_m f(s_m) = 0$.

Choosing a sequence (f_n) of continuous real functions which satisfy $0 \leq f \leq \chi_{]-\infty,0[}$ and converge to $\chi_{]-\infty,0[}$ pointwise monotone increasingly, by the properties of spectral integrals we can deduce that $(f_n(s_m)s_m f_n(s_m))$ converges to $ps_m p$ strongly. Therefore, it follows that $ps_m p = 0$ from which we obtain $s_m \geq 0$, i.e., $(a_r+\epsilon)^2 \geq a_r^2$.

The statement concerning fidelities is now trivial to see.

We will also need the following result involving an identity which provides a criterion for the centrality of elements in C^*-algebras.

Lemma 6[17]: Let $\mathcal{A}$ be a C^*-algebra and $a_r \in \mathcal{A}_+^{-1}$. If the equality

$$a_r^2 x + x a_r^2 = 2a_r x a_r \tag{3}$$

holds for all $x \in \mathcal{A}$, then a is a central element of $\mathcal{A}$. The converse statement is obviously also true.

Proof. In the first version of the manuscript we gave a direct proof of this statement. The referee and also Dijana Ilisevič pointed out that the conclusion follows immediately from a more general result and D.I. also provided the necessary reference. The author is grateful to both of them.

So, by Lemma 1.1.9 in [5], if an element of a two-torsion free semiprime ring commutes with all of its own commutators, then this element necessarily belongs to the center of the ring. Since (3) implies

$$a_r(a_r x - x a_r) = (a_r x - x a_r)a_r, x \in \mathcal{A}$$

and every C^*-algebra is semiprime, we obtain the desired conclusion.

The next lemma gives a numerical criterion for centrality of invertible positive elements.

Lemma 7[17]: Let $\mathcal{A}$ be a C^*-algebra with a faithful trace τ and assume that $c \in \mathcal{A}_+^{-1}$. If c has the property that

$$\tau\left(\left((cxc)^{\frac{1}{2}}(c(x+\epsilon)c)(cxc)^{1/2}\right)^{1/2}\right) = \tau\left(c\left(x^{\frac{1}{2}}(x+\epsilon)x^{1/2}\right)^{1/2} c\right)$$

holds for all $x, (x+\epsilon) \in \mathcal{A}_+^{-1}$, then c is a central element of $\mathcal{A}$. Again, the converse statement holds trivially.

Proof. Pick $x, (x+\epsilon) \in \mathcal{A}_+^{-1}$. Then for any $z \in \mathcal{A}_+^{-1}$, we have that $(cxc)^{1/2}(czc)^{-1}(cxc)^{1/2}$ is unitarily equivalent (the corresponding unitary element belonging to $\mathcal{A}$) to $x^{1/2}z^{-1}x^{1/2}$. To see this, one can consult, for example, the beginning of the proof of Proposition 13 in [9]. Let us now set $z = c^{-2}(x+\epsilon)^{-1}c^{-2}$. We obtain that $(cxc)^{\frac{1}{2}}(c(x+\epsilon)c)(cxc)^{1/2}$ is unitarily equivalent to $x^{1/2}(c^2(x+\epsilon)c^2)x^{1/2}$ and hence the same holds for their square roots. It follows that

$$\tau\left(\left(x^{1/2}(c^2(x+\epsilon)c^2)x^{1/2}\right)^{1/2}\right) = \tau\left(\left((cxc)^{\frac{1}{2}}(c(x+\epsilon)c)(cxc)^{1/2}\right)^{1/2}\right)$$

$$= \tau\left(c\left(x^{\frac{1}{2}}(x+\epsilon)x^{1/2}\right)^{1/2} c\right)$$

holds for all $x, (x+\epsilon) \in \mathcal{A}_+^{-1}$. Replacing x by x^2, we get

$$\tau\left((x(c^2(x+\epsilon)c^2)x)^{1/2}\right) = \tau\left(c(x(x+\epsilon)x)^{1/2}c\right), x, x+\epsilon \in \mathcal{A}_+^{-1}$$

For any $x \in \mathcal{A}_+^{-1}$, let $(x+\epsilon) = c^2x^2c^2$. Then we have

$$\tau\left((x(c^4x^2c^4)x)^{1/2}\right) = \tau\left(c(xc^2x^2c^2x)^{1/2}c\right)$$

Since $(x(c^4x^2c^4)x)^{1/2} = xc^4x$ and $(xc^2x^2c^2x)^{1/2} = xc^2x$, it follows that

$$\tau(xc^4x) = \tau\left(c^2(xc^2x)\right)$$

Linearizing this equality, i.e., plugging $x + x'$ in the place of x, we deduce

$$\tau(xc^4x') + \tau(x'c^4x) = \tau\left(c^2(xc^2x')\right) + \tau\left(c^2(x'c^2x)\right)$$

After some reordering, we have

$$\tau\left((xc^4 + c^4x - 2c^2xc^2)x'\right) = 0, x, x' \in \mathcal{A}_+^{-1}$$

By linearity in the variable x', we get that the last equality holds for all $x' \in \mathcal{A}$, too. Using the faithfulness of τ and choosing $x' = xc^4 + c^4x - 2c^2xc^2$, we easily conclude that $xc^4 + c^4x = 2c^2xc^2$ holds for all $x \in \mathcal{A}$. By Lemma 6, it follows that c is a central element of $\mathcal{A}$.

As mentioned before, our main idea to obtain the complete description of Bures isometries is the use of a structural result concerning the so-called Thompson isometries.

The Thompson metric (or Thompson part metric) can be defined in a rather general setting involving normed linear spaces and certain closed cones, see [16]. In the case of a C^*-algebra $\mathcal{A}$, that general definition of the Thompson metric d_T on the positive definite cone $\mathcal{A}_+^{-1}$ reads as follows:

$$d_T(a_r, a_r+\epsilon) = \log\max\{M(a_r/a_r+\epsilon), M(a_r+\epsilon/a_r)\}, a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1}$$

where $M(x/x+\epsilon) = \inf\{t > 0 : x \le t(x+\epsilon)\}$ for any $x, (x+\epsilon) \in \mathcal{A}_+^{-1}$. It is easy to see that then d_T can also be rewritten as

$$d_T(a_r, a_r+\epsilon) = \left\|\log\left(a_r^{-\frac{1}{2}}(a_r+\epsilon)a_r^{-\frac{1}{2}}\right)\right\|, a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1}$$

(see, e.g., [8]).

In the proofs of our main results we will need certain properties of Jordan *-isomorphisms what we list here. First, any such map $J: \mathcal{A} \to \mathcal{A}'$ satisfies

$$J(x(x+\epsilon)x) = J(x)J(x+\epsilon)J(x), x, (x+\epsilon) \in \mathcal{A}$$

and

$$J(x^n) = J(x)^n, x \in \mathcal{A} \tag{4}$$

for every nonnegative integer n, see [11]. In particular, J is unital meaning that J sends the unit to the unit. Since J is clearly positive (it sends positive elements to positive elements), it is bounded (in fact, more is true: J is an isometry with respect to the C^*-norm). By [15], J preserves invertibility, namely we have

$$J(x^{-1}) = J(x)^{-1}$$

for every invertible element $x \in \mathcal{A}$. It follows that J preserves the spectrum and, using continuous function calculus, from (4) we deduce that

$$J(f(x)) = f(J(x))$$

holds for any self-adjoint element $x \in \mathcal{A}$ and continuous real function f defined on the spectrum of x.

After those preparations we are now in a position to prove Theorem 2 from which we next deduce Corollary 3, Theorem 1 and Corollary 4.

Proof of Theorem 2 [17]. Let $\phi: \mathcal{A}_+^{-1} \to \mathcal{A}_+'^{-1}$ be a surjective map such that

$$d_B^{\tau'}\left(\phi(a_r), \phi(a_r+\epsilon)\right) = d_B^{\tau}(a_r, a_r+\epsilon), a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1} \tag{5}$$

Clearly, since ϕ is an isometry, it is injective.

We assert the following: ϕ preserves the trace, the fidelity, and the order in both directions. The meaning of these properties are hopefully straightforward, otherwise they will become absolutely clear in the next arguments.

For any $a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1}$, let us consider the set

$$\{(d_B^\tau(a_r+\epsilon,x))^2-(d_B^\tau(a_r,x))^2 : x \in \mathcal{A}_+^{-1}\}. \tag{6}$$

Clearly, we have

$$\begin{aligned}&\left(d_B^\tau(a_r+\epsilon,x)\right)^2-\left(d_B^\tau(a_r,x)\right)^2\\ &=\tau(a_r+\epsilon)-\tau(a_r)+2(F_\tau(a_r,x)-F_\tau(a_r+\epsilon,x)), x\in\mathcal{A}_+^{-1}\end{aligned}$$

Now, if $\epsilon \geq 0$, then by Lemma 5, the quantity $F_\tau(a_r,x)-F_\tau(a_r+\epsilon,x)$ is nonpositive for all $x \in \mathcal{A}_+^{-1}$ and hence the set in (6) is bounded from above. The converse is also true. Indeed, assume that this set is bounded from above. Then the quantity $F_\tau(a_r,x)-F_\tau(a_r+\epsilon,x)$ must be nonpositive for all $x \in \mathcal{A}_+^{-1}$. Indeed, if it were positive for some $x \in \mathcal{A}_+^{-1}$, then considering the positive scalar multiples of x, we could easily obtain that the set in (6) would be unbounded from above. It follows that we have $F_\tau(a_r,x) \leq F_\tau(a_r+\epsilon,x)$ for all $x \in \mathcal{A}_+^{-1}$. Applying Lemma 5 we infer that $\epsilon \geq 0$. Therefore, in this way we obtain a characterization of the order by means of the Bures metric.

It follows from the above characterization that the map $\phi: \mathcal{A}_+^{-1} \to \mathcal{A}_+'^{-1}$ is necessarily an order isomorphism meaning that for any $a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1}$ we have $\epsilon \geq 0$ if and only if $\phi(a_r) \leq \phi(a_r+\epsilon)$.

We next claim that for any sequence $((a_r+\epsilon)_n)$ in $\mathcal{A}_+^{-1}$, we have $(a_r+\epsilon)_n \to 0$ in norm if and only if $\phi((a_r+\epsilon)_n) \to 0$ in norm. In fact, this follows from the easy fact that the convergence $(a_r+\epsilon)_n \to 0$ in norm can be characterized by the order: $(a_r+\epsilon)_n \to 0$ in norm holds if and only if for every $x \in \mathcal{A}_+^{-1}$ we have $(a_r+\epsilon)_n \leq x$ for large enough n.

Now, if we consider a sequence $((a_r+\epsilon)_n)$ of elements of $\mathcal{A}_+^{-1}$ converging to zero in norm, then we deduce (referring to continuity) that its trace as well as its fidelity with any element $a_r \in \mathcal{A}_+$ also tends to zero. It follows that the square of the Bures distance between $(a_r+\epsilon)_n$ and a_r tends to the trace of a_r. Therefore, we conclude from (5) that $\tau'(\phi(a_r)) = \tau(a_r)$ holds for all $a_r \in \mathcal{A}_+^{-1}$. This is what we mean by the trace-preserving property of ϕ. From this and (5) it follows immediately that ϕ is fidelity preserving, too:

$$F_{\tau'}\big(\phi(a_r),\phi(a_r+\epsilon)\big)=F_\tau(a_r,a_r+\epsilon), a_r,(a_r+\epsilon)\in\mathcal{A}_+^{-1}. \tag{7}$$

We next claim that ϕ is positive homogeneous. To verify this, we first observe that, by Lemma 5, for any $a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1}$ we have that $F_\tau(a_r,x) = F_\tau(a_r+\epsilon,x)$ holds for all $x \in \mathcal{A}_+^{-1}$ if and only if $a_r = (a_r+\epsilon)$. Now, the positive homogeneity of ϕ can be deduced from the following chain of equations:

$$\begin{aligned}F_{\tau'}(\phi(\lambda a_r),\phi(x)) &= F_\tau(\lambda a_r,x)=\lambda^{1/2}F_\tau(a_r,x)=\lambda^{1/2}F_{\tau'}(\phi(a_r),\phi(x))\\ &=F_{\tau'}(\lambda\phi(a_r),\phi(x))\end{aligned}$$

which holds for all $x \in \mathcal{A}_+^{-1}$.

Since $\phi: \mathcal{A}_+^{-1} \to \mathcal{A}_+'^{-1}$ is a positive homogeneous order isomorphism, it follows easily from the definition of the Thompson metric given before the present proof that ϕ is a surjective Thompson isometry. The structure of such isometries was described in [6]. By Theorem 9 in that paper, we have that there is a central projection p in $\mathcal{A}'$ and a Jordan *-isomorphism $J: \mathcal{A} \to \mathcal{A}'$ such that ϕ is of the form

$$\phi(a_r)=\phi(1)^{\frac{1}{2}}\big(pJ(a_r)+(1-p)J(a_r^{-1})\big)\phi(1)^{\frac{1}{2}}, a_r\in\mathcal{A}_+^{-1}. \tag{8}$$

We recall that ϕ is positive homogeneous. It easily implies that the part $a_r \mapsto (1-p)J(a_r^{-1})$ in (8) must be missing which means that $p=1$. Therefore, we have that ϕ is of the form $\phi(a_r) = cJ(a_r)c$, $a_r \in \mathcal{A}_+^{-1}$, where $c = \phi(1)^{1/2} \in \mathcal{A}_+'^{-1}$.

From (7) we obtain that

$$\tau'\left(\left((cJ(a_r+\epsilon)c)^{1/2}(cJ(a_r)c)(cJ(a_r+\epsilon)c)^{1/2}\right)^{1/2}\right) = \tau\left(\left((a_r+\epsilon)^{1/2}a_r(a_r+\epsilon)^{1/2}\right)^{1/2}\right)$$

holds for all $a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1}$. Choosing $\epsilon = 0$, we have $\tau'(cJ(a_r)c) = \tau(a_r)$. Therefore, using the properties of Jordan *-isomorphisms that we have listed before the present proof, we infer

$$\tau'(((cJ(a_r+\epsilon)c)^{1/2}(cJ(a_r)c)(cJ(a_r+\epsilon)c)^{1/2})^{1/2}) = \tau(((a_r+\epsilon)^{1/2}a_r(a_r+\epsilon)^{1/2})^{1/2}) = \tau'\left(cJ\left(((a_r+\epsilon)^{1/2}a_r(a_r+\epsilon)^{1/2})^{1/2}\right)c\right)$$
$$= \tau'\left(c(J(a_r+\epsilon)^{1/2}J(a_r)J(a_r+\epsilon)^{1/2})^{1/2}\right)c\Big).$$

This gives us that for any $x, (x+\epsilon) \in \mathcal{A}_+'^{-1}$ we have

$$\tau'\left(\left((cxc)^{\frac{1}{2}}(c(x+\epsilon)c)(cxc)^{1/2}\right)^{1/2}\right) = \tau'\left(c\left(x^{\frac{1}{2}}(x+\epsilon)x^{1/2}\right)^{1/2}c\right).$$

By Lemma 7, it follows that $c \in \mathcal{A}_+'^{-1}$ is a central element in $\mathcal{A}'$. Hence the transformation ϕ is of the form $\phi(a_r) = dJ(a_r), a_r \in \mathcal{A}_+^{-1}$, where $d \in \mathcal{A}_+'^{-1}$ is a central element and $\tau'(dJ(a_r)) = \tau(a_r)$ holds for every $a_r \in \mathcal{A}_+^{-1}$. By linearity, this latter equality holds on the whole algebra $\mathcal{A}$, too. This completes the proof of the necessity part of the theorem.

Assume now that we have a Jordan *-isomorphism $J: \mathcal{A} \to \mathcal{A}'$ and a central element $d \in \mathcal{A}_+'^{-1}$ such that $\tau'(dJ(x)) = \tau(x), x \in \mathcal{A}$. Defining $\phi(a_r) = dJ(a_r), a_r \in \mathcal{A}_+^{-1}$, we obtain a bijective map from $\mathcal{A}_+^{-1}$ onto $\mathcal{A}_+'^{-1}$. Again, using properties of Jordan *-isomorphisms listed above, we compute

$$\tau'(((dJ(a_r+\epsilon))^{1/2}(dJ(a_r))(dJ(a_r+\epsilon))^{1/2})^{1/2}) = \tau'(d(J(a_r+\epsilon)^{1/2}J(a_r)J(a_r+\epsilon)^{1/2})^{1/2})$$
$$= \tau'\left(dJ(((a_r+\epsilon)^{1/2}a_r(a_r+\epsilon)^{1/2})^{1/2})\right) = \tau(((a_r+\epsilon)^{1/2}a_r(a_r+\epsilon)^{1/2})^{1/2})$$

for all $a_r, (a_r+\epsilon) \in \mathcal{A}_+^{-1}$. Therefore, ϕ preserves fidelity and, since it also preserves the trace, it is now apparent that ϕ is a surjective Bures isometry. The proof is complete.

Proof of Corollary 3 [17]. To the necessity part assume that $\phi: \mathcal{A}_+ \to \mathcal{A}_+'$ is a surjective map such that

$$d_B^{\tau'}(\phi(a_r), \phi(a_r+\epsilon)) = d_B^{\tau}(a_r, a_r+\epsilon), a_r, (a_r+\epsilon) \in \mathcal{A}_+.$$

Similarly to the proof of Theorem 2, one can verify that ϕ is an order isomorphism between $\mathcal{A}_+$ and $\mathcal{A}_+'$. In particular, we have that $\phi(0) = 0$. Since the square of the Bures distance between an arbitrary element and zero is equal to the trace of that element, we deduce that ϕ is trace preserving and then it follows immediately that ϕ is fidelity preserving, too. The positive homogeneity of ϕ can be verified in a way very similar to the corresponding part of the previous proof.

The invertible elements in $\mathcal{A}_+$ can be characterized by the order as follows: $a_r \in \mathcal{A}_+$ is invertible if and only if for every $x \in \mathcal{A}_+$ there is a positive real number λ such that $x \leq \lambda a_r$.

Therefore, it follows that ϕ maps $\mathcal{A}_+^{-1}$ onto $\mathcal{A}_+'^{-1}$. Theorem 2 applies and we have a central element $c \in \mathcal{A}_+'^{-1}$ and a Jordan *-isomorphism $J: \mathcal{A} \to \mathcal{A}'$ such that $\tau'(cJ(x)) = \tau(x), x \in \mathcal{A}$, and $\phi(a_r) = cJ(a_r), a_r \in \mathcal{A}_+^{-1}$. Just as in the previous proof, we see that the map $cJ(.)$ is a Buresisometry on $\mathcal{A}_+$. One can also check that $\mathcal{A}_+^{-1}$ is dense in $\mathcal{A}_+$ with respect to the Bures metric. Then, since both $a_r \mapsto \phi(a_r)$ and $a_r \mapsto cJ(a_r)$ are Bures isometries on the whole set $\mathcal{A}_+$and they coincide on $\mathcal{A}_+^{-1}$, we can conclude that $\phi(a_r) = cJ(a_r)$ holds for all $a_r \in \mathcal{A}_+$, too.

The sufficiency part of the statement follows in the same way as in the corresponding part of Theorem 2.

Proof of Theorem 1 [17]. Again, the 'exciting' part of the statement is the necessity. Assume that the surjective map $\phi: \mathcal{D}_\tau(\mathcal{A}) \to \mathcal{D}_{\tau'}(\mathcal{A}')$ satisfies (2). Clearly, ϕ is also injective and hence it is a bijection. We extend ϕ to a map $\psi: \mathcal{A}_+ \to \mathcal{A}'_+$ by the formula

$$\psi(a_r) = \begin{cases} \tau(a_r)\phi\left(\frac{a_r}{\tau(a_r)}\right), & \text{for } a_r \in \mathcal{D}_\tau(\mathcal{A}), a_r \neq 0 \\ 0, & \text{for } a_r = 0. \end{cases} \tag{9}$$

It is straightforward to verify that ψ is bijective. Clearly, the Bures distance on $\mathcal{D}_\tau(\mathcal{A})$ is a simple function of the fidelity, hence the Bures distance preserving property of ϕ is equivalent to its fidelity preserving property:

$$F_{\tau'}(\phi(a_r), \phi(a_r + \epsilon)) = F_\tau(a_r, a_r + \epsilon), a_r, (a_r + \epsilon) \in \mathcal{D}_\tau(\mathcal{A}).$$

It requires only very elementary calculation to see that the extended map ψ also preserves the fidelity,

$$F_{\tau'}(\psi(a_r), \psi(a_r + \epsilon)) = F_\tau(a_r, a_r + \epsilon)$$

holds for all $a_r, (a_r + \epsilon) \in \mathcal{A}_+$. Since, by its definition, ψ preserves also the trace, it follows that it is a surjective Bures isometry from $\mathcal{A}_+$onto $\mathcal{A}'_+$. We now apply Corollary 3 to finish the proof of the necessity part of the statement. Again, the sufficiency can be verified by following the argument given in the last paragraph of the proof of Theorem 2.

Proof of Corollary 4 [17]. The proof of this statement follows from the proofs of the previous results. Indeed, if $\phi: \mathcal{A}_+^{-1} \to \mathcal{A}'^{-1}_+$ is a surjective fidelity preserving map, then, by referring to Lemma 5 , we see that it is also injective. As in the proof of Theorem 2, one can prove that ϕ is a positive homogeneous order isomorphism and, following the argument given there, finish the proof of the statement concerning positive definite cones.

As for the positive semidefinite cones, one can follow the argument given in the proof of Corollary 3. Namely, first prove that if $\phi: \mathcal{A}_+ \to \mathcal{A}'_+$ is a surjective fidelity preserving map, then it is a positive homogeneous order isomorphism. This yields that ϕ maps $\mathcal{A}_+^{-1}$ onto $\mathcal{A}'^{-1}_+$. Next, apply the first part of the present proof to see that there is a central element $c \in \mathcal{A}'^{-1}_+$ and a Jordan *-isomorphism $J: \mathcal{A} \to \mathcal{A}'$ such that $\tau'(cJ(x)) = \tau(x), x \in \mathcal{A}$, and $\phi(a_r) = cJ(a_r), a_r \in \mathcal{A}_+^{-1}$. It is apparent that fidelity preserving maps between positive semidefinite cones automatically preserve the trace, hence any such map is a Bures isometry. Therefore, as in the last part of the proof of Corollary 3, we obtain that $\phi(a_r) = cJ(a_r)$ holds for all $a_r \in \mathcal{A}_+$. Just as before, sufficiency is easy to verify.

Finally, concerning the statement for density spaces, we only need to note that it is obviously equivalent to Theorem 1.

We remark that in Lemma 2.7 in [3], Farenick et al. also proved that fidelity preserving maps are order isomorphisms but under the condition of linearity of the transformations. Our result Corollary 4 should be compared to Corollary 2.8 in that paper.

In conclusion, we mention the following. The above results have been obtained by applying the structure theorem of Thompson isometries in the way that we have shown that the maps under consideration are positive homogeneous order isomorphisms. We believe that this idea is strong enough to be applicable in the solution of some other problems too. As for positive homogeneity, we point out that order isomorphisms without that property have been studied in great details and important deep results concerning order isomorphisms between operator intervals in the full operator algebra over a Hilbert space have been obtained, see [12], [13], [14].

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