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Research Paper

On a thermodynamic limit for the Collatz sequence $T(n) = \left[\frac{(3\cdot n+1)}{2},\ \frac{n}{2}\right]$: odd and even integer numbers

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ABSTRACT: *In this paper, we consider the usual Collatz sequence i.e. the T sequence, involving the first two primes, i.e. the sequence T(n) = [(3n+1)/2, n/2], n a positive integer.*
We first study it in a finite domain and find the solutions "odd→ odd", i.e. the cascades at k=1,2,3...
of the even numbers given by the factor $1/2^k$. The numerical experiments suggest, based on the structure of the cascades, that a formula can be proposed.
Thus, starting at a level N big at our convenience, intended to be a number greater than one unity or
equal to the maximum first odd d_1 of the cascades ($(3\cdot d_1+1)/2^k=d_2$), where d_1 and d_2 are two odd numbers, we then define a "thermodynamic" limit for the sequence $[d_1]$ and the related even [e], where all starting odd numbers d_1 are also multiples 3 and we obtain the density of the odd to the even in such a limit for N→∞ that is: lim [$(n_o/n_e)(N)$]→1, as expected.
An analogy is conjectured to hold for some single closed trajectories and is tentatively applied to
investigate the problem of the absence of other cycles as N→∞, i.e. to conjecture that the only cycle of the length N in the problem with an equal number of odd and of even is the multicycle (1, 2, 1, 2, 1, 2...) (the known cycle (1, 2) repeated an arbitrary number of times N/2).

KEYWORDS: *Collatz Sequence, Inverse Sequence, Numerical Experiments, Thermodynamic limit for the "chalice", Multicycles.*

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I. INTRODUCTION

In this work, we are concerned with the analysis of the Equations defining the Collatz sequence. [(3n+1)/2, n/2] [1] companion of the similar sequence [3n+1, n/2] [2], and the main question here is to analyse a thermodynamic limit for the inverse orbits represented by a chalice [3] of the above reduced sequence [4]. Many important works on the two versions exist; the main result of our approach and analysis (both numerical and also theoretical) is that (independent of the open question if other cycles or trajectories are going to infinity) for a large value of the integer N (big at our convenience) the following equality emerges:

$$\lim\left(\frac{n(\text{odd})}{n(\text{even})}\right) = \lim\frac{\frac{N}{2}}{\frac{N}{2}} = 1$$

i.e. the total number of the odd is equal the total to the total number of the even, a limit different of that consisting in increasing the depth in vertical way.

The paper is organized as follows: In Section II, we review the relation between the sequence determined numerically and computations up to k=40 (the depth), in accordance with the early result of Lagarias and Weiss, i.e., c = 1.33333... = 4/3 is given. In Section III, we introduce another thermodynamic limit and a connected theoretical model. We then analyze some finite sequences numerically, laying the groundwork for the result that odd and even numbers are equal in this limit. In section 4, we briefly compute small cycles with an equal number of odd and even cycles, and then present our conclusion.

II. THE GROW CONSTANT C=4/3, ODD AND EVEN

2.1 Fibonacci sequence for the reduced sequence

Some time ago (for the 3n+1 formulation) [2], it was considered a Fibonacci Sequence describing in some way the 3n+1 sequence, i.e. of the form: F(n) = F(n-1) +c·F(n-2), where c=1/3, and where the characteristic Equation given by $x^2 = x + 1/3$ has the two solutions:

$$x_{1,2} = \frac{1 \pm \sqrt{\frac{7}{3}}}{2}$$

where we omit the negative solution and the first positive solution is given by c=1.26376... notice that c-1 =0.26376.. (more or less ¼). In the model under analysis the above Equation is the same, but where (c=4/9), and the two solutions of the characteristic equation are given by:

$$\mathrm{x}_{1,2} = \frac{1 \pm \frac{5}{3}}{2} \quad \text{where } \mathrm{x}_1 = \frac{4}{3} \quad \text{and } \mathrm{x}_2 = -\frac{1}{3}$$

we omit the negative solution.

2.2 Numerical experiments on c

Advanced calculations, up to k=40 (the depth), confirm the value of Lagarias and Weiss [1]: we obtain c= 1.3333….. i.e. c=4/3 and c-1 =1/3. We add that, in this limit c-1 =1/3 means that the number of the odd (total) is 1/3 of that of the even. In the 3n+1 formulation, the ratio is less i.e. the irrational number 0.26… near 1/4.
Still, for the 3n+1 sequence, the above approach using a Fibonacci sequence gives a satisfactory approximation for the Collatz sequence (we have constructed the chalice of inverse orbits, as in previous computations).

III. NUMERICAL EXPERIMENTS

3.1 N=24, 120, 255, 4096 (as a counting procedure)

Example1.
We start with the case N=24. Figure 1 illustrates the emergence of the odd and even numbers up to N=24 (cascades in different colors).

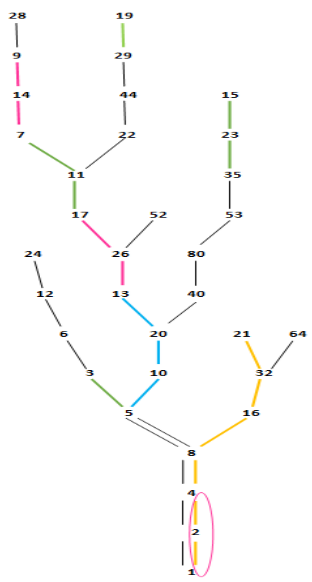

Figure 1. The odd and even numbers up to N=24 (odd: twelve, even: twelve).

Example 2.
We consider a chalice up to the number N=120. N/2=60 gives the number of odd numbers up to N=120. Without other assumptions, we now solve the following equations:

$$(3d_1+1)/2^k=d_2 \quad \text{with } d_1 \leq N \tag{1}$$

where d_1 and d_2 are two odd numbers and k=1,2,....is the order of the cascade of the even numbers.

$$k=1 : \quad (3d_1+1)/2 = d_2 \tag{2}$$

with the solution:

d_1= 3, 7, 11, 15,..,119 =3+4·p, p=0, 1,2,..29, n(d_1, k=1) =30=N/4.

We also count the numbers that are not multiples of 3 and given by+ $n_o(d_1, k=1)$=20 =(2/3)·30.
In general, the solutions of Eq. (1) up to k = 12, as given in [3], here up to k=12.

k	d_1	d_2
1	3+4·p	5+6·p
2	1+8·p	1+6·p
3	13+16·p	5+6·p
4	5+32·p	1+6·p
5	53+64·p	5+6·p
6	21+128·p	1+6·p
7	213+256·p	5+6·p
8	85+512·p	1+6·p
9	853+1024·p	5+6·p
10	341 +2048·p	1+6·p
11	3413+4096·p	5+6·p
12	1365+8192·p	1+6·p

Table 1

We notice that here for all k odd, d_2 = 5+6·p while for all k even, d_2=1+6·p.
For this numerical experiment, N=120 (notice that N is divisible here by 4 and by 3), the values are given on the Table 2; where $n(d_1,k)$ is the number of odd numbers d_1 with a cascade of order k and $n_o(d_1,k)$ those not multiple of 3.

k	$n(d_1,k)$	$n_o(d_1,k)$
1	30	20
2	15	10
3	7	5
4	4	3
5	2	1
6	1	0
7	0	0
8	1	1
9	0	0
10	0	0
	60	40

Table 2.

N/4=120/4 =30.
Notice, N/2=60 and N/3=40 =(2/3).(N/2).
We now count the total of the even numbers obtained in the cascades, which are smaller than N=120, starting at d_1 :

k=1: 6, 12, 24, 48, 96, 14, 28, 56, 112, 22, 44, 88, 30, 60, 120, 38, 76, 46, 92, 54, 108, 62, 70, 78, 86, 94, 102, 110, 118 (29 even numbers).
k=2: 2, 4, 8, 16, 32, 64, 18, 36, 72, 34, 68, 50, 100, 66, 82, 98, 114 (17 even numbers)
k=3: 13, 26, 52, 104, 58, 116, 90 (6 even numbers).
k=4: 10, 20, 40, 80,74 (5 even numbers)
k=5: 106 (1 even number).
k=6: 42, 84 (2 even numbers).

Total of the even different numbers smaller than or equal to N=120:

29+17+6+5+1+2 = 60=N/2.

Example3 .
Let now N=255. The numbers are given explicitly on the Table 3 below.

k	$n(d_1,k)$	$n_o(d_1,k)$	
1	64	42	(2/3).64=42.66
2	32	21	(2/3).32=21.33
3	16	11	(2/3).16=10.66
4	8	6	(2/3).8= 5.33
5	4	3	(2/3).4=2.66
6	2	2	(2/3).2 =1.33
7	1	1	(2/3).1=0.66
8	1	1	(2/3).1=0.66
	128	85	85.33≅ 85

Table 3.

We notice that here, starting at 64 = (255+1)/4 = 64, we have a geometric progression with ratio q = 1/2, except for the last number at k = 8, which is 1. We have 128 odd numbers, 127 even numbers and 128+127=255=N.

Example 4
N=4096 and more.

k	$n_1(d_1,k)$	$n_{th}(k)$	$n_{even}(k)$
1	1024	1024	1023
2	512	512	514
3	256	256	255
4	128	128	129
5	64	64	63
6	32	32	33
7	16	16	15
8	8	8	9
9	4	4	3
10	2	2	3
11	1	1	0
12	1	1	1
	2048	2048	2048

Table 4.

N/2 =4096/2=2048

Here too, we have a geometric progression with a ratio of q=1/2, excluding the 1 for k=12. For the second column (th), we have taken the ceiling of ½, i.e., 1, which gives 2048 too. See below for the symbol th by means of our definition. For the third column for the even we have anticipated the use of the formula (4) below.

3.2 A thermodynamic limit for the chalice of the(3·n+1)/2 (as a model)

Let N be big at our convenience, and we define:
Our "thermodynamic" limit is defined by $N\to\infty$ where, from above we consider all the solutions of Eq.(1), i.e. up to $k\to\infty$ and N is greater than one unity or equal to the greater odd number d_l considered on the chalice (<N, N big at our convenience) (Figure 2 shows the chalice of the sequence (inverse orbits) up to the level 8 i.e. (k=0,..8)).

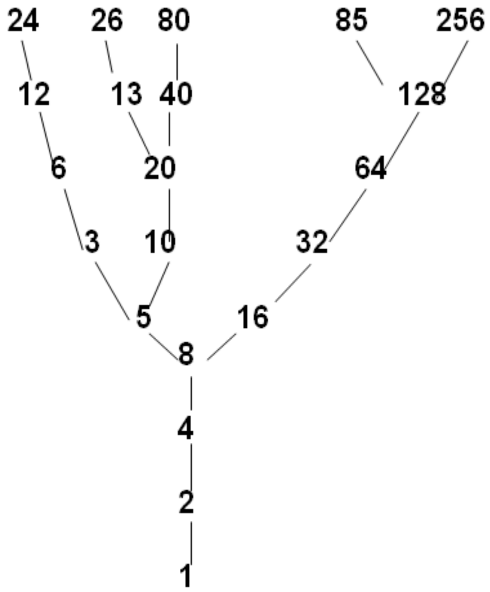

Figure 2. The chalice of the inverse orbits up to k=8 (k=0 is the odd, i.e. the number 1).

We now establish - as a model - a formula in the thermodynamic limit $N\to\infty$ suggested by the previous numerical computations (Tables). We want to specify the amount of the cascades of the even to control the ratio - odd to the even - in such a model as $N\to\infty$.
For this, we note that in a geometrical progression of first member a and of ratio q <1, given by(a, a·q, a·q²....), if the first term is equal to the sum of all the other infinite terms, then q=1/2.
In fact if k

$$a=\left[\sum_{k=0}^{\infty}(aq^k)\right]-a=\frac{a}{1-q}-a\to 2=\frac{1}{1-q} \tag{3}$$

and q=1/2 (See above : N=120; N/4=30 where approximatively we have q=1/2.
We now remark that in our thermodynamic limit $N\to\infty$ the cascades of the number 2, i.e. $(1/2^k)$ are allowed to extend up to $k=\infty$ (an example of a long cascade: take $n=(2^k-1)/3$, k large and even; for k=10, n=34; for k=20, n=349525).
From the above calculations, we have that $n_{odd}(k=1) = N/4$, i.e. $(3+4\cdot p) < N$, $p_{max} = (N-3)/4 = N/4 -0.75 >(N/4)-1$ and $n_{odd}(k=1) = N/4$; for the even we may also compute all inverse cascades of the even above all $d_l(k, N)$ for

$$(a+b\cdot p)\cdot 2^n = N,\ \ n=\sum_{p=0}^{p_{max}(a,b)}\text{floor}\left(\log_2\left(\frac{N}{a+b\cdot p}\right)\right) \tag{4}$$

and compute this for all a and b given before (as an example for N=120,$d_l=3+4\cdot p$, $p_{max}=N/4-1= 29$, a=3).
Computation with the formula (4) gives:
$n_{even}(d_l=3+4\cdot p) = 29$ and $n_{even}(d_l=1+8\cdot p) =17$ (here $p_{max}= 14$; finally: $n_{even}(N=120) = 29+17+6+5+ +1+2+0$ (k=7) =60=N/2.
Thus, the odd numbers are equal to the even numbers as expected. In fact, we may also compute (using our

model with a = q= ½) the cascades of the even that are 1/2 every time k, when is increasing by one, and the number of the even in such a cascade increases by one (as an example: $(1/2)\cdot 2 \rightarrow (1/2)^2\cdot 3$)

$$n_{even} = \frac{N}{8} + \frac{N}{16}\cdot 2 + \frac{N}{32}\cdot 3 + \frac{N}{64}\cdot 4 + \cdots$$

$$= \frac{N}{8}\cdot\left(1 + 1 + \frac{3}{4} + \frac{4}{8} + \cdots\right) =$$

$$= \frac{N}{8}\cdot\left(1 + \frac{\partial}{\partial a}\frac{a^2}{(1-a)}\bigg|_{a=\frac{1}{2}}\right) = \frac{N}{8}\cdot(1+2+1) = \frac{N}{2}$$

Since

$$\sum_{k=2} 2\cdot a^k\cdot k = 2\cdot a^2\cdot\frac{(2-1\cdot a)}{(1-a)^2} = 3 \ \text{ for } a = \frac{1}{2},$$

thus, for a =q=1/2, $n_{even} = (N/8)\cdot(1+3) = N/2$.

It should be remarked that our thermodynamic limit has been obtained (as a model of our numerical experiments!) in considering all cascades starting with the greatest odd d_1 (N or N+1) and counting the number of even in such cascades beginning at the maximum odd number allowed N.

But in this way we are counting to much (as an example 5→16→8→4→2 and 21→64→32→16→ →8→4→2. But from the calculations, we see that the even in the inverse orbits of the set of the d_1 behaves as the set of the odd d_1, i.e. with q=1/2 and n_{even} = N/2 (See the Table for N=4096). Thus, the correct formula (4) with the inverse orbits indicates N/4 even above $d_1(k=1)$, N/8 even above $n_1(k=2)$ and so on, asymptotically at N→∞, i.e. total n_{even} = N/ 2. In general, for the cardinalities of the odd, as well for the even numbers, we have the following. Here, we treat the case k=1 i.e. where $d_1 = 3+4\cdot p$, $p=0..p_m$. For the odd numbers i.e. $d_1(k=1) = 3+4\cdot p$, we consider the inequality:

$$x-1 \le \text{floor}(x) \le x+1 \qquad (5)$$

From above the maximum of $p = p_m = \text{floor}((N-3)/4)$ which gives the number $d_1(k=1) = 1 + p_m$, with:

$$\frac{(N-3)}{4} \le 1 + p_m \le \frac{(N-3)}{4} + 1 = \frac{(N+1)}{4} + 1$$

i.e for large N, $n_{odd}(k=1) = 1 + p_M \rightarrow N/4$.

For the even numbers:

$$(3+4\cdot p)\cdot 2^{n_p} \le N$$

where is an n_p integer. Then

$$n_{ev}(k=1) = \sum_{p=0}^{p_m} n_p = \text{floor}\left[\frac{1}{\log(2)}\cdot\sum_{p=0}^{p_m}\left(\log\left(\frac{N}{3+4\cdot p}\right)\right)\right]$$

For a lower bound the inequality (5) is too weak.

We then consider the approximation where above instead of 3+4·p we set 4+4·p and we verify that for "some N and p's", we have:

$$\text{floor}\left[\left(\frac{1}{\log(2)}\right)\cdot\log\left(\frac{N}{(3+4\cdot p)}\right)\right]\geq\log\left(\frac{N}{4\cdot(1+p)}\right)$$

Then we obtain:

$$n_{ev}\geq\left(\log\left(\frac{N}{4}\right)^{p_m+1}\cdot\left(\frac{1}{(1+p_m)!}\right)\right)\geq$$

$$\geq\left(\log\left(\frac{N}{4}\right)^{p_m+1}\cdot\left(\frac{1}{(1+p_m+1)!}\right)\right)$$

which for large N, with the Stirling formula, gives:

$$n_{ev}(k=1)\geq\log\left[\left(\frac{N}{4}\right)\cdot\left(\frac{1}{(1+p_m)}\right)\right]^{p_m+1}\cdot e^{(1+p_m+1)}=$$

$$=\log\left[\left(\frac{N}{4}\right)\cdot\left(\frac{1}{(1+p_m)}\right)\right]^{p_m+1}\cdot e^{(1+p_m+1)}\cdot e^{\left(-\left(\frac{1}{2}\right)\cdot\log(1+p_m+1)\right)}\geq$$

$$\geq\left(1+p_m-\left(\frac{1}{2}\cdot\log(1+p_m+1)\right)\right)\rightarrow\frac{N}{4}\cdot\left[1-\frac{1}{2}\cdot\log\left(\frac{N}{4}\right)/\left(\frac{N}{4}\right)\right]\rightarrow$$

$\rightarrow\sim N/4$ as N is large.

We now look at a possible upper bound still for the even numbers (k=1).

Here too, we set the following inequality:

$$\text{floor}\left[\left(\frac{1}{\log(2)}\right)\cdot\log\left(\frac{N}{(3+4\cdot p)}\right)\right]<\mathit{log}\left(\frac{N}{(4+4\cdot p)}\right)\cdot N^{\varepsilon(N)}$$

with $\varepsilon(N)\leq N^c/N$, $c<1$. As above we have verified that for some N and some p's the above inequality holds (with $c<1$) .
Then with the Stirling formula, we obtain :

$$n_{ev}(k=1)<(1+p_m+1)+\left(\frac{1}{2}\right)\cdot\log\left(\frac{N}{4}\right)+N^c\cdot\log(N)\rightarrow$$

$$\rightarrow\left(\frac{N}{4}\right)\cdot\left(1+\frac{1}{2}\cdot\frac{\log\left(\frac{N}{4}\right)}{\left(\frac{N}{4}\right)}+4\cdot\log(N)\cdot N^{c-1}\right)\rightarrow\frac{N}{4}$$

for N large at our convenience. (It should be added that for the even (k=1), the upper and the lower approximations are not rigorous i.e. may not be verified for each N).
We then apply our model with q=1/2 of Section 3.2 and we suggest that as N is large (thermodynamic limit), $(n_{ev}(N)/n_{od}(N))\rightarrow(N/2)/(N/2)=1$.
Our finding may appear as a triviality: i.e. there is no raison that the cardinality of the even number will be different from that of the odd numbers in our new thermodynamic limit, but we was unable to find such an

equality odd versus even without the above experiments and without a counting procedure by means of our model i.e. that with q=1/2 and also with the floor function.

IV. SMALL CYCLES WITH THE NUMBER OF ODD EQUAL TO THE NUMBER OF EVEN

In this Section we analyse small cycles where the cardinality of the odd is equal to that of the even, in connection with the above finding.
n=1 (1 odd and 1 even)

$$\frac{3 \cdot n + 1}{2^k} = n$$

with the solution; (1, 2, 1)

n=2 (2 odd and 2 even)

$$\frac{\left(3 \cdot \left(\frac{3 \cdot n + 1}{2^k}\right) + 1\right)}{2^p} = n$$

$$n \cdot \left(2^{k+p} - 9\right) = 3 + 2^k$$

with the solution k= p = 2 and n=1: the "2" cycle (1, 2, 1, 2, 1) and in general the conjecture is that we have the circuit (1, 2, 1), i.e. the cycle (1, 2, 1) of the (3n+1)/2 repeated an arbitrary number of times.

V. CONCLUSION

For decades, many scholars have been dedicated to solving the Collatz conjecture and exploring interesting connections with other fields [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

In this work, we have first computed the constant c =4/3 by means of a numerical program up to k=40 (vertical depth) to have at least 4 decimal for c i.e. c=1,3333...

We then considered the approach using the Fibonacci sequence, as was done some time ago for the 3·n+1 sequence, and specified the characteristic function. We then considered the solutions to the fundamental set of Equations relating two odd numbers through a cascade of even numbers, with odd numbers no larger than N, where N is arbitrarily large, i.e. our thermodynamic limit. The finding is that in such a limit different from the usual one, considered in Section II, the cardinality of the odd numbers should be equal to the cardinality of the even numbers. We also add that it will be very difficult to disprove such an equality for a finite value of N. In Section IV, we have considered small cycle**s** with an equal number of odd and of even and conjectured based on some solution that for the (3n+1)/2 sequence there is the unique circuit (1, 2, 1) i.e. the cycle (1, 2, 1), arbitrarily n time.

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