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Research Paper

On Fractional Adams–Moser–Trudinger type inequality on Heisenberg group of General Domains

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Abstract.

M. Gupta, J. Tyagi [61] establish singular fractional Adams–Moser–Trudinger inequality for both bounded and unbounded domains in the Heisenberg group $\mathbb{H}^n$. Follows [61] fractional Adams–Moser–Trudinger type inequality on domain of $\mathbb{R}^n$ with finite measure show adding to it Hardy–Littlewood–Sobolev inequality adapted to the result of R. O'Neil (1963), we show singular fractional Adams–Moser–Trudinger type inequality on domain of $\mathbb{R}^n$ with finite measure. We also show singular fractional Adams–Moser–Trudinger type inequality in $\mathbb{H}^n$, using the work of N. Lam and G. Lu (2012, 2013). Our goal is that any function in higher order fractional Sobolev space in Heisenberg group can be represented in terms of Riesz potential, harmonic analysis and kernel properties of the associated operator, we show fractional Adams– Moser–Trudinger type inequality in $\mathbb{H}^n$. This paper, is show free from symmetrization arguments. We show the existence of solution to the following class of problems $\begin{cases} T^\alpha u_j = \frac{f_j(\xi,u_j)}{|\xi|^{1+\epsilon}} + b(\xi)\left|u_j\right|^{\gamma-1} u_j \text{ in } \Omega, \\ u_j = 0 \qquad\qquad \text{in } \mathbb{H}^n\backslash\Omega, \end{cases}$ where Ω is a bounded subset of $\mathbb{H}^n$ of class $C^{0,1}$ with bounded boundary and $\epsilon > 0$, f_j satisfies either the subcritical exponential growth or critical exponential growth condition and b is a small L^2-perturbation, then, there exists a small $\epsilon \geq 0$ with $0 < \| b \|_{L^2(\Omega)} < 1+\epsilon, 0 \leq \gamma < 1$ and $\alpha = \frac{1+2\epsilon}{2}$.

Keywords: Variational methods, Fractional Adams–Moser–Trudinger inequality, Heisenberg group.

1. Introduction

Moser-Trudinger inequalities and Adams inequalities play an important role in geometric analysis and partial differential equations, and have important applications in solving a class of equations which involve exponential growth nonlinearity. M. Gupta, J. Tyagi [61] establish singular fractional Adams-MoserTrudinger type inequalities for both bounded and unbounded domains in the Heisenberg group $\mathbb{H}^n$. We extend several research works concerning this family of inequalities in Euclidean setting.

Let $\Omega \subset \mathbb{R}^{1+2\epsilon}$ be an open, bounded domain. Then the classical Sobolev embedding theorem says that for $\epsilon > 0$,

$$W_0^{1,1+\epsilon}(\Omega) \hookrightarrow L^{1+\epsilon}(\Omega), \qquad 1 \leq 1+\epsilon \leq \frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}.$$

In the limit case $\epsilon = 0$, we have

$$W_0^{1,1+2\epsilon}(\Omega) \hookrightarrow L^{1+\epsilon}(\Omega), \qquad 0 \leq \epsilon < \infty,$$

and it is well known that $W_0^{1,1+2\epsilon}(\Omega)$ is not embedded in $L^\infty(\Omega)$, see Example 4.43 [2]. So, the natural question arises about the possibility of identifying a smallest space in which $W_0^{1,1+2\epsilon}(\Omega)$ is embedded. This question was answered in [55] by Trudinger, who proved that $W_0^{1,1+2\epsilon}(\Omega)$ is embedded into Orlicz space $L_A(\Omega)$, where

$$A(t) = \exp\left(t^{\frac{1+2\epsilon}{2\epsilon}}\right) - 1$$

is an N-function, see [2] for the definition of N-functions and of the space $L_A(\Omega)$. Later on this inequality was sharpened by Moser [47]. He proved the following:

Theorem 1.1. Let $\Omega \subset \mathbb{R}^{2+\epsilon}$ be a bounded domain, $\epsilon \geq 0$. Then there exists $k_0 \in \mathbb{R}^+$ and a sharp constant $\alpha_{2+\epsilon} = (2+\epsilon)\omega_{1+\epsilon}^{\frac{1}{1+\epsilon}}$, where $\omega_{1+\epsilon}$ is the area of the surface of the unit $(2+\epsilon)$-ball, we have

$$\frac{1}{|\Omega|}\int_\Omega \sum_j \exp\left(\alpha |u_j|^{\frac{2+\epsilon}{1+\epsilon}}\right) dx \leq k_0$$

for any $\alpha \leq \alpha_{2+\epsilon}$ and $u_j \in W_0^{1,2+\epsilon}(\Omega)$ with $\int_\Omega |\nabla u_j|^{2+\epsilon} \leq 1$. This constant is sharp in the sense that if $\alpha > \alpha_{2+\epsilon}$, then the above inequality can no longer hold with some k_0 independent of u_j.

For $\Omega \subseteq \mathbb{R}^{2+\epsilon}$ be an open domain with measure $|\Omega|$. Then we know that for $\epsilon \geq 0, W_0^{1,1+\epsilon}(\Omega)$ embeds continuously into $L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}(\Omega)$, while in the borderline case $\epsilon = 0$, one has $W_0^{1,1+2\epsilon} \not\subset L^\infty(\Omega)$. For bounded domains in $\mathbb{R}^{1+2\epsilon}$, while Theorem 1.2 was proved by Adimurthi and Sandeep [4], similar inequalities were studied by Troyanov [54] in compact Riemannian surfaces.

Theorem 1.2. Let $\epsilon \geq 0$ and $u_j \in W_0^{1,2+\epsilon}(\Omega)$. Then for every $\alpha > 0$ and $\beta \in [0,2+\epsilon)$

$$\int_\Omega \sum_j \frac{e^{\alpha |u_j|^{\frac{2+\epsilon}{1+\epsilon}}}}{|x|^\beta} dx < \infty.$$

Moreover,

$$\sup_{\|u_j\| \leq 1} \int_\Omega \sum_j \frac{e^{\alpha |u_j|^{\frac{2+\epsilon}{1+\epsilon}}}}{|x|^\beta} dx < \infty \tag{1.1}$$

if and only if $\frac{1+\epsilon}{\alpha_{2+\epsilon}} + \frac{\beta}{2+\epsilon} \leq 1$, where $\| u_j \| = \sum_j \left(\int_\Omega |\nabla u_j|^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}$.

In [3], Adimurthi and O. Druet improved Trudinger-Moser inequality for a smooth bounded subset in $\mathbb{R}^2$. There are also other improved Moser-Trudinger inequalities on the unit disk in $\mathbb{R}^2$, see, [5,44].

It is well known that for a positive integer $m < 2+\epsilon$ and for $1 \leq 1+\epsilon < \frac{2+\epsilon}{m}, W_0^{m,1+\epsilon}(\Omega)$ embeds continuously into $L^{\frac{(2+\epsilon)(1+\epsilon)}{2+\epsilon-m(1+\epsilon)}}(\Omega)$, while in the borderline case $m = \frac{2+\epsilon}{1+\epsilon}$, one has $W_0^{m,\frac{2+\epsilon}{m}} \not\subset L^\infty(\Omega)$.

The above sharp inequality by J. Moser (Theorem 1.1) was later extended by Adams [1] for higher order Sobolev spaces which reads as follows:

Theorem 1.3. Let Ω be a bounded and open subset of $\mathbb{R}^{2+\epsilon}$. If m is a positive integer less than $2+\epsilon$, then there exists a constant $C_0 = C(m, 2+\epsilon)$ such that for all $u_j \in C^m(\mathbb{R}^{2+\epsilon})$ with support contained in Ω and $\left\|\sum_j \nabla^m u_j\right\|_{1+\epsilon} \leq 1, m = \frac{2+\epsilon}{1+\epsilon}$, we have

$$\frac{1}{|\Omega|}\int_\Omega \sum_j \exp\left(\beta |u_j(x)|^{\frac{2+\epsilon}{2+\epsilon-m}}\right) dx \leq C_0,$$

for all $\beta \leq \beta(m, 2+\epsilon)$, where

$$\nabla^m u_j = \begin{cases} \Delta^{\frac{m}{2}} u_j, & \text{for } m \text{ even,} \\ \nabla \Delta^{\frac{m-1}{2}} u_j, & \text{for } m \text{ odd,} \end{cases}$$

And

$$\beta(m, 2+\epsilon) = \begin{cases} \dfrac{2+\epsilon}{w^{1+\epsilon}} \left[\dfrac{\pi^{\frac{2+\epsilon}{2}} 2^m \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{3+\epsilon-m}{2}\right)}\right]^{\frac{1+\epsilon}{\epsilon}}, & \text{when } m \text{ is odd,} \\ \dfrac{2+\epsilon}{w^{1+\epsilon}} \left[\dfrac{\pi^{\frac{2+\epsilon}{2}} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{2+\epsilon-m}{2}\right)}\right]^{\frac{1+\epsilon}{\epsilon}}, & \text{when } m \text{ is even,} \end{cases}$$

$\epsilon = 0, \Gamma$ is the Euler Gamma function. Furthermore, for any $\beta > \beta(m, 2+\epsilon)$, the integral can be made as latye as desired.

Recently, in the setting of the Sobolev spaces with homogeneous Navier boundary conditions, denoted by $W_{2+\epsilon}^{m,\frac{2+\epsilon}{m}}(\Omega)$, which is defined as follows:

$$W_{2+\epsilon}^{m,\frac{2+\epsilon}{m}}(\Omega) := \left\{u_j \in W^{m,\frac{2+\epsilon}{m}} : \Delta^j u_j = 0 \text{ on } \partial\Omega \text{ for } 0 \leq j \leq \left[\frac{m-1}{2}\right]\right\},$$

the Adams inequality was extended by Tarsi [52]. Also, note that $W_{2+\epsilon}^{m,\frac{2+\epsilon}{m}}$ contains Sobolev space $W_0^{m,\frac{2+\epsilon}{m}}(\Omega)$ as a closed subspace. When $\Omega \subseteq \mathbb{R}^{2+\epsilon}$ is an open subset of $\mathbb{R}^{2+\epsilon}$ (not necessarily bounded) and m is an even integer, Ruf and Sani [50] extended the Adams-Moser-Trudinger type inequality to the Sobolev spaces $W_0^{m,\frac{2+\epsilon}{m}}(\Omega)$. The statement of the inequality is the following:

Theorem 1.4. If m is an even integer less than $2+\epsilon$, then there exists a constant $C_{m,2+\epsilon} > 0$ such that for any domain $\Omega \subset \mathbb{R}^{2+\epsilon}$, we have

$$\sup_{u_j \in W_0^{m,\frac{2+\epsilon}{m}}(\Omega), \|u_j\| \le 1} \int_\Omega \sum_j \phi_j(\beta_0(2+\epsilon, m)|u_j|^{\frac{2+\epsilon}{m-2+\epsilon}} dx \le C_{m,2+\epsilon},$$

where

$$\beta_0(2+\epsilon, m) = \frac{2+\epsilon}{\omega_{1+\epsilon}}\left[\frac{\pi^{\frac{2+\epsilon}{2}} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{2+\epsilon-m}{2}\right)}\right]^{\frac{2+\epsilon}{2+\epsilon-m}},$$

$$\phi_j(t) = e^t - \sum_{j_0=0}^{\frac{2+\epsilon}{m}} \frac{t^j}{j_0!},$$

$$(j_0)_{\frac{2+\epsilon}{m}} = \mathrm{m}\left\{j_0 \in \mathbb{N}: j_0 \ge \frac{2+\epsilon}{m}\right\} \ge \frac{2+\epsilon}{m}.$$

Moreover, this inequality is sharp in the sense that if we replace $\beta_0(2+\epsilon, m)$ by any larger β, then the above supremum will be infinity.

In the above result, Ruf and Sani used the norm

$$\| u_j \|_{m,2+\epsilon} = \left\|(-\Delta + I)^{\frac{m}{2}} u_j\right\|_{\frac{2+\epsilon}{m}},$$

which is equivalent to the standard Sobolev norm

$$\| u_j \|_{W^{m,\frac{2+\epsilon}{m}}} = \sum_j \left(\| u_j \|_{\frac{2+\epsilon}{m}}^{\frac{2+\epsilon}{m}} + \sum_{j=1}^{m} \|\nabla^j u_j\|_{\frac{2+\epsilon}{m}}^{\frac{2+\epsilon}{m}}\right)^{\frac{m}{2+\epsilon}}.$$

Since Ruf and Sani [50] only considered the case for m even. So for odd m, it was an open question. Later, it has been settled by Lam and Lu [32]. Then, the result reads as follows:

Theorem 1.5. Let m be an odd integer less than $2+\epsilon$ and $m = 2k+1, k \in \mathbb{N}$. Then

$$\sup_{u_j \in W^{m,\frac{2+\epsilon}{m}}(\mathbb{R}^{2+\epsilon}), \|\sum_j \nabla(-\Delta+I)^k u_j\|_{\frac{2+\epsilon}{m}}^{\frac{2+\epsilon}{m}} + \|\sum_j (-\Delta+I)^k u_j\|_{\frac{2+\epsilon}{m}}^{\frac{2+\epsilon}{m}} \le 1} \int_{\mathbb{R}^{2+\epsilon}} \sum_j \phi_j\left(\beta(2+\epsilon, m)|u_j|^{\frac{2+\epsilon}{2+\epsilon-m}}\right) dx < \infty.$$

Moreover, the constant $\beta(2+\epsilon, m)$ is sharp.

We remark that (1.1) was extended to the entire $\mathbb{R}^{2+\epsilon}$ by Adimurthi and Yang, see [6]. For constants $\epsilon \ge 0, 0 \le \beta < 1$ and $0 < \gamma \le 1-\beta$, the following holds:

$$\sup_{\int_{\mathbb{R}^{2+\epsilon}} \sum_j \left(|\nabla u_j|^{2+\epsilon} + (1+\epsilon)|u_j|\right)^{2+\epsilon} dx \le 1} \int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{1}{|x|^{2+\epsilon\beta}}\left(e^{\alpha_{2+\epsilon}\gamma|u_j|^{\frac{2+\epsilon}{1+\epsilon}}} - \sum_{k=0}^{\epsilon} \frac{(\alpha_{2+\epsilon}\gamma)^k |u_j|^{\frac{k(2+\epsilon)}{1+\epsilon}}}{k!}\right) dx < \infty.$$

Then the application of Adimurthi-Yang's inequality [6] on partial differential equations was studied by Yang [59] and recently the existence of extremals for the singular Trudinger-Moser inequality was obtained by Li and Yang, see [39]. The Adams inequality was also extended to compact Riemannian manifolds without boundary by Fontana [24] and to measure spaces by Fontana and Morpurgo [25].

Martinazzi [46] established Adams-Moser-Trudinger inequality for fractional Laplacian and the statement for inequality reads as follows:

Theorem 1.6 ([46]). For any $0 < \epsilon < \infty$ and positive integer $2+\epsilon$. Set

$$K_{2+\epsilon,1+\epsilon} := \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\epsilon}{2}\right) 2^{1+\epsilon} \pi^{\frac{2+\epsilon}{2}}}, \alpha_{2+\epsilon,1+\epsilon} := \frac{2+\epsilon}{\omega_{1+\epsilon}} K_{2+\epsilon,\frac{2+\epsilon}{1+\epsilon}}^{-\frac{1+\epsilon}{\epsilon}}.$$

Then for any open set $\Omega \subset \mathbb{R}^{2+\epsilon}$ with finite measure, we have

$$\sup_{u_j\in\tilde{H}^{1+\epsilon,1+\epsilon}(\Omega),\|\sum_j(-\Delta)^{\frac{2+\epsilon}{2(1+\epsilon)}}u_j\|_{L^{1+\epsilon}(\Omega)}\leq 1}\int_\Omega\sum_j e^{\alpha_{2+\epsilon,1+\epsilon}|u_j|^{\frac{1+\epsilon}{\epsilon}}}dx\leq c_{2+\epsilon,1+\epsilon}|\Omega|.$$

Moreover, the constant $\alpha_{2+\epsilon,1+\epsilon}$ is sharp in the sense that we cannot replace it with any larger one without making the supremum infinite.

For the definition of $\tilde{H}^{1+\epsilon,1+\epsilon}(\Omega)$ space and related norms (see section 2). Now, in the setting of Heisenberg group, it has been known for years that the following inequality holds for $f_j\in C_0^\infty(\mathbb{H}^{2+\epsilon})$:

$$\left(\int_{\mathbb{H}^{2+\epsilon}}\sum_j|f_j(z,t)|^{1+\epsilon}dzdt\right)^{\frac{1}{1+\epsilon}}\leq C_{1+\epsilon,1+\epsilon}\left(\int_{\mathbb{H}^{2+\epsilon}}\sum_j\left|\nabla_{\mathbb{H}^{2+\epsilon}}f_j(z,t)\right|^{1+\epsilon}dzdt\right)^{\frac{1}{1+\epsilon}}\tag{1.2}$$

provided that $\epsilon\geq 0$. The Sobolev inequality no longer holds if $\epsilon=0$. This inequality was first proved by Folland-Stein [22,23]. Here, $\left|\nabla_{\mathbb{H}^{\frac{2\epsilon-1}{2}}}f_j\right|$ is used to express the Euclidean norm of the subelliptic gradient of :

$$\left|\sum_j\nabla_{\mathbb{H}^{\frac{2\epsilon-1}{2}}}f_j\right|=\sum_j\left(\sum_{i=1}^{\frac{2\epsilon-1}{2}}(X_if_j)^2+(Y_if_j)^2\right)^{\frac{1}{2}},$$

see Section 2 for the definitions of the vector fields X_i and Y_i. It is then clear that the above inequality is also true for functions in the anisotropic Sobolev space $W_0^{1,1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)(\epsilon\geq 0)$, where $W_0^{1,1+\epsilon}(\Omega)$ for open set $\Omega\subset\mathbb{H}^{\frac{2\epsilon-1}{2}}$ is the completion of $C_0^\infty(\Omega)$ under the norm

$$\|f_j\|_{L^{1+\epsilon}(\Omega)}+\left\|\nabla_{H^{\frac{2\epsilon-1}{2}}}f_j\right\|_{L^{1+\epsilon}(\Omega)}.$$

Before the work of Jerison and Lee [29], very little was known about sharp constants for Sobolev inequality (1.2) in the Heisenberg group. The best constant $C_{1+\epsilon,1+\epsilon}$ for the Sobolev inequality (1.2) on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ for $\epsilon=1$ was found and the extremal functions were identified in [29].

Theorem 1.7 (Jerison and Lee [29]). The best constant for the inequality (1.2) on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ is

$$C_{2,\frac{2(2\epsilon+1)}{2\epsilon-1}}=(4\pi)^{-1}\left(\frac{2\epsilon-1}{2}\right)^{-2}[\Gamma\left(\frac{2\epsilon+1}{2}\right)]^{\frac{2}{2\epsilon+1}}$$

and all the extremals of (1.2) are obtained by dilations and left translation of the function

$$K|(t+i(|z|^2)+1)|^{-\frac{2\epsilon-1}{2}}.$$

Furthermore, the extremals in (1.2) are constant multiples of images under the Cayley transform of extremals for the Yamabe functional on the sphere $\mathbb{S}^{2\epsilon}$ in $\mathbb{C}$.

The sharp Sobolev inequality for $\epsilon=1$ is closely related to the sharp Hardy-Littlewood-Sobolev inequality, also known as HLS inequality (see [28]): For $0<\lambda<2\epsilon+1$ and $1+\epsilon=\frac{2(1+2\epsilon)}{2(1+2\epsilon)-\lambda}$, the following holds

$$\left|\iint_{\mathbb{H}^{\frac{2\epsilon-1}{2}}\times\mathbb{H}^{\frac{2\epsilon-1}{2}}}\sum_j\frac{\overline{f_j(u_j)g_j(v_j)}}{\left|u_j{}^{-1}v_j\right|^\lambda}du_jdv_j\right|\leq C_{\frac{2\epsilon-1}{2},\lambda}\sum_j\|f_j\|_{1+\epsilon}\|g_j\|_{1+\epsilon},\tag{1.3}$$

where $u_j^{-1}v_j$ is the group product and $|.|$ is the homogeneous norm and du_j is the Haar measure.

In fact, the result of Jerison and Lee is equivalent to the sharp version of HLS inequality (1.3) when $\lambda=2\epsilon-1$ and $1+\epsilon=2(1+2\epsilon)/(2(1+2\epsilon)-\lambda)=\frac{2(1+2\epsilon)}{2\epsilon+3}$. The work of Jerison and Lee [29] raised two natural questions. First question is: What is the best constant $C_{1+\epsilon,1+\epsilon}$ for the $L^{1+\epsilon}$ to $L^{1+\epsilon}$ Sobolev inequality (1.2) for all $\epsilon>0$ and $1+\epsilon=\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}$? And the second posed question is: What is the sharp constant for the borderline case $=0$? While the first question still seems to be open, the second question is settled by Cohn and Lu[16] in Heisenberg group for the domain of finite measure and is given as the sharp Moser-Trudinger inequality on any domain Ω with $|\Omega|<\infty$ on the Heisenberg group.

The sharp constant for the Moser-Trudinger inequality on domains of finite measure in the Heisenberg group is stated as follows:

Theorem 1.8. Let $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ be an n-dimensional Heisenberg group, $= 0, \alpha_{1+2\epsilon} = (1+2\epsilon)\sigma_{1+2\epsilon}^{\frac{1}{2\epsilon}}, \sigma_{1+2\epsilon} = \int_{\rho(z,t)=1} |z|^{1+2\epsilon} d\mu$. Then there exists a constant C_0 depending only on $1+2\epsilon$ such that for all $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}, |\Omega| < \infty$,

$$\sup_{u_j \in W_0^{1,1+2\epsilon}(\Omega) \left\|\sum_j \nabla_{H^{\frac{2\epsilon-1}{2}}} u_j\right\|_{L^{1+2\epsilon}} \leq 1} \frac{1}{|\Omega|} \int_\Omega \sum_j e^{\alpha_{1+2\epsilon}|u_j|^{\frac{1+2\epsilon}{2}}} d\xi < \infty.$$

If $\alpha_{1+2\epsilon}$ is replaced by any larger number, then the supremum is infinite.

Remark 1.9. The constant $\sigma_{1+2\epsilon}$ was found explicitly in [16] and it is equal to

$$\sigma_{1+2\epsilon} = \omega_{2\epsilon-2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\epsilon)}{\left(\frac{2\epsilon-1}{2}\right)!},$$

where $\omega_{2\epsilon-2}$ is the surface area of the unit sphere in $\mathbb{R}^{2\epsilon-1}$.

Using the sharp representation formula in [15], Lam et al. [36] established the following version of sharp singular Moser-Trudinger inequality on domains of finite measure on the Heisenberg group.

Theorem 1.10. Let $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}, |\Omega| < \infty$ and $0 \leq \beta < 1+2\epsilon$. Then there exists a uniform constant $C_0 < \infty$ depending only on $1+2\epsilon, 1+\epsilon$ such that

$$\sup_{u_j \in W_0^{1,1+2\epsilon}(\Omega), \|\sum_j \nabla_H u_j\|_{L^{1+2\epsilon}} \leq 1} \frac{1}{|\Omega|^{\frac{\epsilon}{1+2\epsilon}}} \int_\Omega \sum_j \frac{\exp\left(\alpha_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j(\xi)|^{\frac{1+2\epsilon}{2}}\right)}{\rho(\xi)^{1+\epsilon}} d\xi \leq C_0,$$

where $\rho(\xi)$ is the Korányi norm of ξ, see Section 2 for the definition. The constant $\alpha_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)$ is sharp in the sense that if $\alpha_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)$ is replaced by any larger number, then the supremum is infinite.

There is an analog of works in Adimurthi-Yang [6] in the whole Heisenberg group $\mathbb{H}^{\frac{2\epsilon-1}{2}}$, which is the following theorem, see [14]:

Theorem 1.11. There exists some constant α^* with $0 < \alpha^* \leq \alpha_{1+2\epsilon}$ such that for any pair $1+\epsilon$ and α satisfying $\epsilon > 0, 0 < \alpha \leq \alpha^*$ and $\frac{\alpha}{\alpha^*} + \frac{1+\epsilon}{1+2\epsilon} \leq 1$, there holds

$$\sup_{\|u_j\|_{W^{1,1+2\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)} \leq 1} \int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \frac{1}{\rho(\xi)^{1+\epsilon}} \sum_j \left(e^{\alpha|u_j|^{\frac{1+2\epsilon}{2}}} - \sum_{k=0}^{2\epsilon-1} \frac{\alpha^k |u_j|^{k\left(\frac{1+2\epsilon}{2}\right)}}{k!}\right) d\xi < \infty.$$

When $\frac{1+\epsilon}{\alpha^4} + \frac{1+\epsilon}{1+2\epsilon} > 1$, the integral in the above is still finite for any $u_j \in W^{1,1+2\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$, but the supremum is infinite if further $\frac{\alpha}{\alpha_{(1+2\epsilon)}} + \frac{1+\epsilon}{1+2\epsilon} > 1$.

See [60] for the similar results as above but by different approach. Like the integer case, there are some works done for fractional case in the Heisenberg group. For example, Roncal and Thangavelu [53] proved the Hardy type inequality for fractional powers of the sublaplacian $\mathcal{L}$ on the Heisenberg group $H^{\frac{2\epsilon-1}{2}}$. We denote by $W^{1+\epsilon,2}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$ the Sobolev space consisting of all L^2 functions for which $\mathcal{L}^{\frac{1+\epsilon}{2}} f_j \in L^2\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$. Instead of considering powers of $\mathcal{L}$, they considered conformally invariant fractional powers $\mathcal{L}_{1+\epsilon}$, see Subsection 2.3 [53] for definitions and proved a Hardy type inequality for $\mathcal{L}_{1+\epsilon}$ with a non-homogeneous weight.

We mention that there are several Moser-Trudinger and Adams type inequality and their applications to partial differential equations, see [7-13, 17, 18, 26, 27, 30, 33, 37, 38, 40, 42, 43, 45, 49, 56, 57].

Now it is natural to ask:

Q. Can one establish fractional Adams-Moser-Trudinger type inequality on Heisenberg group?

We answer this question. M. Gupta, J. Tyagi [61] establish fractional AdamsMoser-Trudinger inequality with singular potential and in unbounded domains.

Now, we state the follows theorems (see [61]), which we prove in the next sections.
Theorem 1.12. Let $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$ be an open set of class $C^{0,1}$ with bounded boundary and $|\Omega| < \infty$. Let $0 < \alpha < 1+2\epsilon, 0 < \epsilon < \infty, 1+2\epsilon - \alpha(1+\epsilon) = 0, \frac{1+\epsilon}{\epsilon} = (1+2\epsilon)/(1+2\epsilon-\alpha)$. Then there exists a constant C_0 such that

$$\sup_{u_j \in S_\alpha^{1+\epsilon}(\Omega), \|\sum_j \mathcal{S}^{\alpha/2} u_j\|_{L^{1+\epsilon}(\Omega)} \le 1} \frac{1}{|\Omega|} \int_\Omega \sum_j \exp\left(A_{1+2\epsilon} |u_j|^{\frac{1+\epsilon}{\epsilon}}\right) d\xi \le C_0,$$

where

$$A_{1+2\epsilon} = \frac{1+2\epsilon}{\int_\Sigma \sum_j |v_j(\xi')|^{\frac{1+\epsilon}{\epsilon}} d\mu}, \Sigma = \left\{\xi \in \mathbb{H}^{\frac{2\epsilon-1}{2}} : |\xi| = 1\right\}, \tag{1.4}$$

where $v_j(\xi) = R_\alpha(\xi)$ is the Riesz potential, as in Proposition 2.14, which is an allowed kernel of order α and $S_\alpha^{1+\epsilon}$ is the space Dom $\left(\mathcal{T}_{1+\epsilon}^{\alpha/2}\right)$ equipped with the graph norm

$$\| f_j \| := \| f_j \|_{1+\epsilon,\alpha} = \| f_j \|_{1+\epsilon} + \|\mathcal{T}^{\alpha/2} f_j\|_{1+\epsilon},$$

where $\mathcal{T}_{1+\epsilon}^{\alpha/2}$ as defined in Definition 2.12 is the maximal restriction of $\mathcal{T}^{\alpha/2}$, that is, Dom($\mathcal{T}_{1+\epsilon}^{\alpha/2}$) is the set of all $f_j \in L^{1+\epsilon}$ such that the distributional derivative $\mathcal{T}^{\alpha/2} f_j$ is in $L^{1+\epsilon}$, and $\mathcal{T}_{1+\epsilon}^{\alpha/2} f_j = \mathcal{T}^{\alpha/2} f_j$.
Moreover, the constant $A_{1+2\epsilon}$ is sharp in the sense that we cannot replace it with any larger one without making the supremum infinite.
Theorem 1.13. Let $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$ be an open set of class $C^{0,1}$ with bounded boundary and $|\Omega| < \infty, 1+2\epsilon - \alpha(1+\epsilon) = 0, \frac{1+\epsilon}{\epsilon} = \frac{1+2\epsilon}{(1+2\epsilon)-\alpha}, \epsilon > 0$. Then there exists a constant C_0 depending only on $1+2\epsilon$ and $1+\epsilon$ such that

$$\sup_{u_j \in S_\alpha^{1+\epsilon}(\Omega), \|\sum_j \mathcal{S}^{\alpha/2} u_j\|_{L^{1+\epsilon}(\Omega)} \le 1} \frac{1}{|\Omega|} \int_\Omega \sum_j \frac{\exp\left(A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right) |u_j|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}} d\xi \le C_0, \tag{1.5}$$

where $|\xi| = (|z|^4 + t^2)^{\frac{1}{4}}, A_{1+2\epsilon}$ is defined in (1.4). Moreover, the constant $A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)$ is sharp in the sense that we cannot replace it with any larger one without making the supremum infinite.

Now, we state the Adams-Moser-Trudinger type inequality on Heisenberg group $\mathbb{H}^{\frac{2\epsilon-1}{2}}$.
Theorem 1.14. Let $0 < \epsilon < \infty, 1+2\epsilon - \alpha(1+\epsilon) = 0, \frac{1+\epsilon}{\epsilon} = (1+2\epsilon)/((1+2\epsilon)-\alpha)$. Then there exists a constant C_0 depending on $1+2\epsilon$ and a such that the following holds:

$$\sup_{u_j \in S_\alpha^{1+\epsilon}\left(H^{\frac{2\epsilon-1}{2}}\right), \|u_j\|_{L^{1+\epsilon}\left(H^{\frac{2\epsilon-1}{2}}\right)} + \|\mathcal{S}^{\alpha/2} u_j\|_{L^{1+\epsilon}\left(M^{\frac{2\epsilon-1}{2}}\right)} \le 1} \int_{H^{\frac{2\epsilon-1}{2}}} \sum_j \frac{\psi_j\left((1+\epsilon)|u_j|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^\alpha} d\xi \le C_0,$$

where

$$\psi_j(t) = e^t - \sum_{j=0}^{j_{1+\epsilon}-2} \frac{t^j}{j!},$$

$$j_{1+\epsilon} = \mathrm{m}\ \{j \in \mathbb{N} : j \ge 1+\epsilon\} \ge 1+\epsilon$$

and $1+\epsilon = \left(\frac{\epsilon}{1+2\epsilon}\right) A_{1+2\epsilon}$. Moreover, the constant $1+\epsilon$ is sharp in the sense that we cannot replace it with any larger one without making the supremum infinite.
$S_\alpha^{1+\epsilon}, \mathcal{Y}^{\frac{\pi}{2}}$ and related norms, used in the above theorems, are defined in the second section. We remark that our Theorem 1.14 does not require the restriction on the full standard norm and hence, even in Euclidean setting, this theorem extends the works in [34,58].
Now, we define subcritical and critical growth for $f_j(\xi, u_j)$. We say that a function $f_j : \Omega \times \mathbb{R} \to \mathbb{R}$ has subcritical growth on $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$ if

$$\lim_{|u_j| \to \infty} \sum_j \frac{|f_j(\xi, u_j)|}{\exp(\alpha u_j^2)} = 0, \text{ uniformly on } \Omega, \forall \alpha > 0. \tag{1.6}$$

We say that f_j has critical growth if there exists $\alpha_0 > 0$ such that

$$\lim_{|u_j|\to\infty} \sum_j \frac{|f_j(\xi,u_j)|}{\exp(\alpha u_j^2)} = 0, \text{ uniformly on } \Omega, \forall \alpha > \alpha_0 \tag{1.7}$$

And

$$\lim_{|u_j|\to\infty} \sum_j \frac{|f_j(\xi,u_j)|}{\exp(\alpha u_j^2)} = \infty, \text{ uniformly on } \Omega, \forall \alpha < \alpha_0. \tag{1.8}$$

We define

$$\Lambda = \inf_{0\neq u_j \in S^2_{1+\epsilon}(\Omega)} \sum_j \frac{\| u_j \|^2}{\int_\Omega \frac{|u_j|^2}{|\xi|^2} d\xi} > 0, \text{ where } \| u_j \|^2 = \int_{H^{\frac{2\epsilon-1}{2}}} \sum_j \left| (\mathcal{T})^{\frac{9}{2}} u_j \right|^2 d\xi. \tag{1.9}$$

We assume the following conditions on the nonlinearity :

(H1) $f_j: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f_j(\xi,u_j) \geq 0$ on $\Omega \times [0,\infty), f_j(\xi,u_j) \leq 0$ when $u_j \leq 0, \forall \xi \in \Omega$.

(H2) There exist $R_0 > 0, M > 0$ such that, $\forall |u_j| \geq R_0, \forall \xi \in \Omega$,

$$0 < F_j(\xi,u_j) \leq M f_j(\xi,u_j),$$

where $F_j(\xi,u_j) = \int_0^{u_j} \sum_j f_j(\xi, 1+\epsilon) d(1+\epsilon)$.

(H3) limsup $\sin_{u_j+} \sum_j \frac{2F_j(\xi,u_j)}{|u_j^2|} < \Lambda$, where A is defined as in (1.9).

(H4) $\lim_{u_j\to\infty} \sum_j u_j f_j(\xi,u_j) \exp(-\alpha_0 |u_j|^2) \geq \beta_1 > \frac{-\epsilon \cdot \mathcal{A}_{1+2\epsilon}}{(1+2\epsilon)(1+\epsilon) R^\epsilon \mathcal{M}}$, where $R, \mathcal{M}$ and $\mathcal{A}_{1+2\epsilon}$ are defined in Lemma 2.19. (H5) $b \in L^2(\Omega)$ and $\exists$ small $\epsilon \geq 0$ such that $0 < \| b \|_{L^2(\Omega)} < 1+\epsilon$.

Also, as next applications of fractional Adams-Moser-Trudinger inequality, we prove the existence of solution of the following class of partial differential equations with singular nonlinearity:

$$\begin{cases} \mathcal{S}^{1+\epsilon} u_j & = \frac{f_j(\xi,u_j)}{|\xi|^{1+\epsilon}} + b(\xi)|u_j|^{\gamma-1} u_j \text{ in } \Omega, \\ u_j & = 0 \text{ in } \mathbb{H}^{\frac{2\epsilon-1}{2}} \setminus \Omega, \end{cases} \tag{1.10}$$

where Ω is an open, bounded subset of $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ of class $C^{0,1}$ with bounded boundary, f_j satisfies either the subcritical exponential growth or critical exponential growth condition (see, (1.6) $-$(1.8)), $\epsilon > 0, 0 \leq \gamma < 1$ and b is a small L^2-perturbation, that is, there exists a small $\epsilon \geq 0$ with $0 < \| b \|_{L^2(\Omega)} < 1+\epsilon, 0 \leq \gamma < 1$ and $\alpha = \frac{1+2\epsilon}{2}$.

Next, we state the existence theorem illustrating the application of main theorems (see [61]).

Theorem 1.15. Assume that f_j satisfies the subcritical growth condition and (H1)-(H3), (H5) hold. Suppose that $\alpha = \frac{1+2\epsilon}{2}$ and $0 \leq \gamma < 1$, then (1.10) has a weak solution for $\epsilon > 0$.

Theorem 1.16. Assume that f_j satisfies the critical growth and (H1) $-$(H5) hold. Suppose that $\alpha = \frac{1+2\epsilon}{2}$ and $0 \leq \gamma < 1$, then (1.10) has a weak solution for $\epsilon > 0$.

We deal with useful preliminaries on fractional Laplacian and Heisenberg group and several important results which have been used. In the sequel.

2. Preliminaries

We show the fractional Laplacian on the Euclidean space. We denote

$$L_{1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right) = \left\{ u_j \in L^1_{loc}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right) : \int_{\mathbb{R}^{\frac{2\epsilon-1}{2}}} \sum_j \frac{|u_j(x)|}{1+|x|^{\frac{4\epsilon+1}{2}}} dx < \infty \right\}.$$

For functions $u_j \in L_{1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right)$, the fractional Laplacian $(-\Delta)^{\frac{\pi}{2}} u_j$ can be defined as follows:

$$(-\Delta)^{\frac{\pi}{2}} \varphi_j := \mathcal{F}^{-1}\left(|\xi|^{1+\epsilon} \mathcal{F} \varphi_j\right)$$

for φ_j belonging to Schwartz space $\mathcal{S}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right)$ of rapidly decreasing functions, where $\mathcal{F}$ denotes the unitary Fourier transform. Then for $u_j \in L_{1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right)$, we define $(-\Delta)^{\frac{3}{2}} u_j$ as a tempered distributions via the formula

$$\sum_j \left\langle (-\Delta)^{\frac{1+\epsilon}{2}} u_j, \varphi_j \right\rangle = \sum_j \left\langle u_j, (-\Delta)^{\frac{1+\epsilon}{2}} \varphi_j \right\rangle := \int_{\mathbb{R}^{\frac{2\epsilon-1}{2}}} \sum_j u_j (-\Delta)^{\frac{1+\epsilon}{2}} \varphi_j dx \text{ for } \varphi_j \in \mathcal{S}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right).$$

The right hand side being well-defined because

$$\left| \sum_j (-\Delta)^{\frac{3}{2}} \varphi_j \right| \le \sum_j \frac{C\varphi_j}{1+|x|^{\frac{4\epsilon+1}{2}}},$$

for every $\varphi_j \in \mathcal{S}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right)$. For a set $\Omega \subset \mathbb{R}^{\frac{2\epsilon-1}{2}}$ (possibly unbounded), $\epsilon \ge -1$ and $0 < \epsilon < \infty$,), we define

$$H^{1+\epsilon,1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right) := \left\{ u_j \in L^{1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right) : (-\Delta)^{\frac{1+\epsilon}{2}} u_j \in L^{1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right) \right\}$$

which is equipped with the norm

$$\| u_j \|_{H^{1+\epsilon,1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right)} = \sum_j \left(\| u_j \|_{L^{1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right)} + \left\| (-\Delta)^{\frac{2}{2}} u_j \right\|_{L^{1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right)} \right)^{\frac{1}{1+\epsilon}}$$

and

$$\widetilde{H}^{1+\epsilon,1+\epsilon}(\Omega) := \left\{ u_j \in H^{1+\epsilon,1+\epsilon}\left(\mathbb{R}^{\frac{2\epsilon-1}{2}}\right) : u_j = 0 \text{ in } \mathbb{R}^{\frac{2\epsilon-1}{2}} \setminus \Omega \right\}$$

and the norm is defined as

$$\| u_j \|_{\Pi^{1+\epsilon,1+\epsilon}(\Omega)} = \sum_j \left(\int_\Omega \left| (-\Delta)^{\frac{1+\epsilon}{2}} u_j \right|^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}.$$

Now, we recall the basics on the Heisenberg group $\mathbb{H}^{\frac{2\epsilon-1}{2}}$. $\mathbb{H}^{\frac{2\epsilon-1}{2}} = (\mathbb{R}^{2\epsilon}, \cdot)$ is the space $\mathbb{R}^{2\epsilon}$ with the non-commutative law of product

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + 2(\langle y, x' \rangle - \langle x, y' \rangle)\right),$$

where $x, y, x', y' \in \mathbb{R}^{\frac{2\epsilon-1}{2}}, t, t' \in \mathbb{R}$ and $\langle , \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^{\frac{2\epsilon-1}{2}}$. This operation endows $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ with the structure of a Lie group. The vector fields

$$T = \frac{\partial}{\partial t}, \; X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \; Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \; i = 1,2,3, \dots, \frac{2\epsilon - 1}{2}$$

are left invariant vector fields and generate the Lie algebra $\mathbb{H}^{\frac{2\epsilon-1}{2}}$. These generators satisfy the non-commutative formula

$$[X_i, Y_j] = -4\delta_{ij} T, \; [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

Let $z = (x, y) \in \mathbb{R}^{2\epsilon-1}, \xi = (z, t) \in \mathbb{H}^{\frac{2\epsilon-1}{2}}$. The parabolic dilation is

$$\delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)$$

and the Jacobian of δ_λ is $\lambda^{1+2\epsilon}$, where $\epsilon = 0$ is the homogeneous dimension of $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ and satisfies

$$\delta_\lambda(\xi_0 \cdot \xi) = \delta_\lambda \xi \cdot \delta_\lambda \xi_0.$$

The anisotropic dilation on $H^{\frac{2\epsilon-1}{2}}$ introduce a homogeneous norm

$$|\xi| = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}} := \rho(\xi),$$

which is known as Korányi gauge norm $N(z, t)$. In other words, $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ denotes the Heisenberg distance between (z, t) and (z', t') on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ as follows:

$$\rho(z, t, z', t') = \rho((z', t')^{-1} \cdot (z, t)).$$

It is clear that the vector fields $X_i, Y_i, i = 1,2, \dots, \frac{2\epsilon-1}{2}$ are homogeneous of degree 1 under the norm $|\cdot|$ and T is homogeneous of degree 2. The Lie algebra of Heisenberg group has the stratification $\mathbb{H}^{\frac{2\epsilon-1}{2}} = V_1 \oplus V_2$, where the $(2\epsilon - 1)$-dimensional horizontal space V_1 is spanned by $\{X_i, Y_i\}, i = 1,2, \dots, \frac{2\epsilon-1}{2}$ while V_2 is spanned by T. The Korányi ball of center ξ_0 and radius r is defined by

$$B_{H_{\frac{2\epsilon-1}{2}}}(\xi_0, r) = \{\xi : |\xi^{-1} \cdot \xi_0| \le r\}$$

and it satisfies

$$\left|B_{\mathrm{H}_{\frac{2\epsilon-1}{2}}}(\xi_0,r)\right| = \left|B_{\mathrm{H}^{\frac{2\epsilon-1}{2}}}(0,r)\right| = r^{1+2\epsilon}\left|B_{\mathrm{H}^{\frac{2\epsilon-1}{2}}}(0,1)\right|,$$

where $|\cdot|$ is the (2ϵ)-dimensional Lebesgue measure on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ and $\epsilon = 0$ is the homogeneous dimension of Heisenberg group $\mathbb{H}^{\frac{2\epsilon-1}{2}}$. The Heisenberg gradient and Heisenberg Laplacian or the Laplacian-Kohn operator on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ are given by

$$\nabla_{H^{\frac{2\epsilon-1}{2}}} = \left(X_1, X_2, \dots, X_{\frac{2\epsilon-1}{2}}, Y_1, Y_2, \dots, Y_{\frac{2\epsilon-1}{2}}\right)$$

and

$$\begin{aligned}\Delta_{\mathrm{H}_{\frac{2\epsilon-1}{2}}} &= \sum_{i=1}^{\frac{2\epsilon-1}{2}} X_i^2 + Y_i^2 \\ &= \sum_{i=1}^{\frac{2\epsilon-1}{2}} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i\frac{\partial^2}{\partial x_i\,\partial t} - 4x_i\frac{\partial^2}{\partial y_i\,\partial t} 4(x_i^2+y_i^2)\frac{\partial^2}{\partial t^2}\right).\end{aligned}$$

G.B. Folland [20] proved the existence of the fundamental solution for the sublaplacian $-\Delta_{\mathbb{H}^{\frac{2\epsilon-1}{2}}}$, which is given by $\tilde{c}_{\frac{2\epsilon-1}{2}}|\xi|^{-2\epsilon+1}$, where

$$\tilde{c}_{\frac{2\epsilon-1}{2}} = \left[\left(\frac{2\epsilon-1}{2}\right)\left(\frac{2\epsilon+3}{2}\right)\int_{H^{\frac{2\epsilon-1}{2}}} |z|^2(|\xi|^4+1)^{-\left(\frac{2\epsilon+7}{4}\right)} dzdt\right]^{-1}.$$

Definition 2.1 (Convolution [23]). If f_j and g_j are measurable functions on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$, then their convolution $f_j * g_j$ is defined as

$$\begin{aligned}\sum_j (f_j * g_j)(\xi) &= \int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j f_j(1+\epsilon)g_j((1+\epsilon)^{-1}\cdot\xi)d(1+\epsilon) \\ &= \int_{\mathrm{H}^{\frac{2\epsilon-1}{2}}} \sum_j f_j((1+\epsilon)^{-1}\cdot\xi)g_j(1+\epsilon)d(1+\epsilon),\end{aligned}$$

provided the integral converge.

Definition 2.2 (Distribution Function). Let $f_j\colon \Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}} \to \mathbb{R}$ be a measurable function then distribution function of f_j is given by

$$\lambda_{f_j}(t) = |\{x \in \Omega : |f_j(x)| > t\}|, t > 0.$$

It is easy to see that distribution function is a monotonically decreasing function of t and $f_j(t) = 0, \forall t \geq \operatorname{esssup}(f_j)$.

Definition 2.3 (Decreasing Rearrangement). Let $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$ be a bounded set and let $f_j\colon \Omega \to \mathbb{R}$ be a measurable function. Then the decreasing rearrangement of f_j is defined as

$$f_j^*(0) = \operatorname{esssup}(|f_j|).$$

$$f_j^*(1+\epsilon) = \mathrm{i}\ \left\{t : \lambda_{f_j}(t) < 1+\epsilon\right\}, \epsilon \geq 0.$$

Lemma 2.4. Let $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$ be a bounded set and let $f_j\colon \Omega \to \mathbb{R}$ be a measurable function. Then for $0 \leq \epsilon < \infty$,

$$\int_\Omega \sum_j |f_j(\xi)|^{1+\epsilon} d\xi = \int_0^{|\Omega|} \sum_j \left|f_j^*(t)\right|^{1+\epsilon} dt.$$

Lemma 2.5 (Hardy-Littlewood Inequality). Let $\Omega \subset \mathbb{H}^{n}$ be a bounded set and let $f_j, g_j\colon \Omega \to \mathbb{R}$ be measurable functions. Then

$$\int_\Omega \sum_j |f_j(\xi)g_j(\xi)| d\xi \leq \int_0^{|\Omega|} \sum_j f_j^*(t)g_j^*(t)dt.$$

Let us recall that

$$f_j^{**}(t) = \frac{1}{t}\int_0^\infty \sum_j f_j^{*}(1+\epsilon)d(1+\epsilon).$$

The following proposition is related with polar coordinates on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$.

Proposition 2.6. Let $\Sigma = \left\{\xi \in \mathbb{H}^{\frac{2\epsilon-1}{2}} : |\xi| = 1\right\}$ be the unit sphere in a Heisenberg group $\mathbb{H}^{\frac{2\epsilon-1}{2}}$. Then there is a unique Radon measure $d\mu$ on Σ such that for all $f_j \in L^1\left(\mathcal{H}^{\frac{2\epsilon-1}{2}}\right)$,

$$\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j f_j(u_j)du_j = \int_0^\infty \int_\Sigma \sum_j f_j(ru_j^*)r^{2\epsilon}d\mu(u_j^*)dr.$$

Remark 2.7. Let $\omega_{2\epsilon-2} = 2\pi/\Gamma(\frac{2\epsilon-1}{2})$ be the surface area of the unit sphere in $\mathbb{C}^{\frac{2\epsilon-1}{2}}$ and for $\beta > -(2\epsilon - 1)$, let

$$C_\beta = \int_\Sigma |z^*|^\beta d\mu,$$

which is the best constant for Moser-Trudinger inequalities and by doing computation, we can get

$$C_\beta = \frac{\omega_{2\epsilon-2}\Gamma(1/2)\Gamma[(2\epsilon-1+\beta)/4]}{\Gamma[(1+2\epsilon+\beta)/4]}.$$

We borrow the following definitions from [15].

Definition 2.8 (Kernel of Order). Let $1+2\epsilon$ denote the homogeneous dimension of $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ and let $0 < \alpha < 1+2\epsilon$. We say that a non-negative function v_j defined on $\mathbb{H}^{\frac{2\epsilon-1}{2}} - \{0\}$ is a kernel of order α if there is a function (also denoted by) defined on the unit sphere $\Sigma = \left\{\xi \in \mathbb{H}^{\frac{2\epsilon-1}{2}} : |\xi| = 1\right\}$ such that for $\xi \neq 0, v_j(\xi) = |\xi|^{\alpha-(1+2\epsilon)}v_j(\xi')$, where $\xi' = \xi/|\xi|$.

Definition 2.9 (Allowed Kernel). Set $z' = z/|\xi|$ and $t' = t/|\xi|^2$ and $\xi' = (z', t')$. Then for $\delta > 0$, let Σ_δ be the subset of the sphere given by

$$\Sigma_\delta = \{\xi' \in \Sigma: \delta \le v_j(\xi') \le \delta^{-1}\}.$$

We will need to assume that for every $\delta > 0$ and $0 < M < \infty$, there are constants $C(\delta, M)$ such that

$$\int_{\Sigma_\delta}\int_0^M \sum_j \left|v_j(\xi'((1+\epsilon)(1+\epsilon)')^{-1}) - v_j(\xi')\right| \frac{d(1+\epsilon)}{(1+\epsilon)} d\mu(\xi') \le C(\delta, M)$$

for all $(1+\epsilon)' \in \Sigma$. A kernel v_j of order α which satisfies the above estimate is known as "allowed kernel".

Example 2.10. It is easy to verify that functions defined by $v_j(\xi') = |z'|^\beta, \xi' = (z', t')$ for $\beta > -(2\epsilon - 1)$ on the Heisenberg group $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ are allowed kernels.

Let us state a theorem, which plays an important role to prove the main theorems.

Theorem 2.11([15]). Suppose v_j is an allowed kernel of order α on Heisenberg group $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ and $1 + 2\epsilon - \alpha(1+\epsilon) = 0$ (i.e., $\alpha = \frac{1+2\epsilon}{1+\epsilon}$). Let

$$A(v_j, 1+\epsilon) = \frac{1+2\epsilon}{\int_\Sigma |v_j(\xi')|^{\frac{1+\epsilon}{\epsilon}} d\mu}.$$

Then there exists a constant C_0 such that for any $f_j \in L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$ with support contained in $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$, $|\Omega| < \infty$, the following holds:

$$\frac{1}{|\Omega|}\int_\Omega \sum_j \exp\left(A(v_j, 1+\epsilon)\left(\frac{f_j * v_j(\xi)}{\| f_j \|_{1+\epsilon}}\right)^{\frac{1+\epsilon}{\epsilon}}\right)du_j \le C_0.$$

Furthermore, if $A(v_j, 1+\epsilon)$ is replaced by a greater number, the resulting statement is false.

The next definition of complex power of $\mathcal{T}$ is motivated by the following formula, valid for $\epsilon \ge 0$, Re $\alpha > 0$:

$$(1+\epsilon)^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-(1+\epsilon)t}dt.$$

For more details, we refer to [21].

Definition 2.12 ([21]). Suppose $0 < \epsilon < \infty, \operatorname{Re}\alpha > 0$ and k is the smallest integer greater than $\operatorname{Re}\alpha$ (i.e., $k = [\operatorname{Re}\alpha] + 1$). The operator $\mathcal{J}_{1+\epsilon}^{\alpha}$ is defined by

$$\mathcal{J}_{1+\epsilon}^{\alpha} f_j = \lim_{\epsilon\to 0} \frac{1}{\Gamma(k-\alpha)} \int_{\epsilon}^{\infty} \sum_{j} t^{k-\alpha-1} \mathcal{J}^k H_t f_j dt,$$

where $\mathcal{J} = -\sum_1^{\frac{2\epsilon-1}{2}} X_j^2 + Y_j^2$ on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$ and H_t is the diffusion semigroup generated by $-\mathcal{J}$. Next, we see some principal properties of $H_t f_j$ and some theorems required for further work adapted from [21].

Theorem 2.13. There is a unique semigroup $\{H_t : 0 \le t \le \infty\}$ of linear operators on $L^1 + L^\infty$ satisfying the following conditions:

(i) $H_t f_j = f_j * h_t$, where $h_t(x) = h(x,t)$ is $\mathcal{C}^\infty$ on $\mathbb{H}^{\frac{2\epsilon-1}{2}} \times (0,\infty)$, $\int h_t(x)dx = 1$ for all t, and for all x and t, $h(x,t) \ge 0$ and

$$h(rx, r^2 t) = r^{-(1+2\epsilon)} h(x,t).$$

(ii) If $u_j \in \mathcal{D}, \lim_{t\to 0} \sum_j \left\|t^{-1}(H_t u_j - u_j) + \mathcal{J} u_j\right\|_\infty = 0$, where $\mathcal{D}$ is the space of C^∞ functions with compact support.

Moreover, $\{H_t\}$ has the following properties:

(iii) $\{H_t\}$ is a contraction semigroup on $L^{1+\epsilon}, 0 \le \epsilon \le \infty$, which is strongly continuous for $\epsilon < \infty$. Also, if $0 < \epsilon < \infty, H_t$ can be extended to a holomorphic contraction semigroup $\left\{H_z : |\arg z| < \frac{1}{2}\pi(1 - |1 - (2/1+\epsilon)|)\right\}$ on $L^{1+\epsilon}$.

(iv) H_t is self-adjoint, i.e., $H_t \mid L^{1+\epsilon}$ is the dual of $H_t \mid L^{\frac{1+\epsilon}{\epsilon}}, \epsilon > 0$.

(v) $f_j \ge 0$ implies $H_t f_j \ge 0$, and $H_t 1 = 1$.

Proposition 2.14([19, 21]). Suppose $0 < \alpha < 1 + 2\epsilon$. Denote by $h = h(x,t)$ the fundamental solution of $\mathcal{J} + \frac{\partial}{\partial t}$, where $\mathcal{J} = -\sum_{j=1}^{\frac{2\epsilon-1}{2}} X_j^2 + Y_j^2$. Then the integral

$$R_\alpha(\xi) = \frac{1}{\Gamma(\alpha/2)} \int_0^\alpha t^{\alpha/2-1} h(\xi,t)dt$$

converges absolutely for $\xi \ne 0$. In addition, R_α is a kernel of order α. Moreover,

(i) R_2 is the fundamental solution $\mathcal{J}$.

(ii) If $\alpha \in (0,2)$ and $u_j \in \mathcal{D}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$, then $\sum_j \mathcal{J}^{\alpha/2} u_j = \sum_j \mathcal{J} u_j * R_{2-\alpha}$.

(iii) The kernel R_α admits the following rule: If $\alpha > 0, \beta > 0$ and $\xi \ne 0$, then $R_{\alpha+\beta}(\xi) = R_\alpha(\xi) * R_\beta(\xi)$.

As a corollary of above proposition (see p. 4, [41]), we have

$$R_\alpha(\xi) \approx |\xi|^{\alpha-(1+2\epsilon)},$$

i.e., there exists a constant c_α (depending on) such that

$$R_\alpha(\xi) \le c_\alpha |\xi|^{\alpha-(1+2\epsilon)}. \tag{2.1}$$

We follow [21] for the following definitions and proposition:

Definition 2.15 (Fractional Sobolev Space on Heisenberg Group). For $0 < \epsilon < \infty$ and $\alpha \ge 0, S_\alpha^{1+\epsilon}$ is the space $\operatorname{Dom}\left(\mathcal{J}_{1+\epsilon}^{\alpha/2}\right)$ equipped with the graph norm

$$\| f_j \|_{1+\epsilon,\alpha} := \| f_j \|_{1+\epsilon} + \left\|\mathcal{J}^{\alpha/2} f_j\right\|_{1+\epsilon},$$

where $\mathcal{J}_{1+\epsilon}^{\alpha/2}$ as defined in Definition 2.12 is the maximal restriction of $\mathcal{J}^{\alpha/2}$, that is, $\operatorname{Dom}\left(\mathcal{J}_{1+\epsilon}^{\alpha/2}\right)$ is the set of all $f_j \in L^{1+\epsilon}$ such that the distributional derivative $\mathcal{J}^{\alpha/2} f_j$ is in $L^{1+\epsilon}$, and $\mathcal{J}_{1+\epsilon}^{\alpha/2} f_j = \mathcal{J}^{\alpha/2} f_j$.

$S_\alpha^{1+\epsilon}$ is a Banach space. An alternative characterization of $S_\alpha^{1+\epsilon}$, which will often be convenient is the following.

Proposition 2.16$([21J) \cdot S_\alpha^{1+\epsilon} = \operatorname{Dom}\left((I + \mathcal{J}_{1+\epsilon})^{\alpha/2}\right)$, and the norms $\| f_j \|_{1+\epsilon,\alpha}$ and $\|(I + \mathcal{J})^{\alpha/2 f_j}\|_{1+\epsilon}$ are equivalent.

Some basic properties of $S_\alpha^{1+\epsilon}$ are following, we refer to [21] for the details.

(i) If $0 \le \gamma < \beta$, then $S_\beta^{1+\epsilon} \subset S_\gamma^{1+\epsilon}$ and $\| \sum_j f_j \|_{1+\epsilon,\gamma} \le C_{1+\epsilon,\beta,\gamma} \sum_j \| f_j \|_{1+\epsilon,\beta}$.

(ii) If $1 + \epsilon \le \operatorname{Re}\beta \le b \le 0$, then $(I + \mathcal{J})^\beta$ is bounded on $S_\alpha^{1+\epsilon}$ for all $1 + \epsilon$, α with bound $\le C|\Gamma(1 - i\operatorname{Im}\beta)|^{-1}$, where C depends only on $1 + \epsilon, \alpha$, and b.

(iii) If $f_j \in L^{1+\epsilon}$, then $H_t f_j \in S^{1+\epsilon}$ for all $\beta \geq 0, t > 0$. Also, if $f_j \in S^{1+\epsilon}$, then $H_t f_j \to f_j$ in $S^{1+\epsilon}$ norm as $t \to 0$.
(iv) The space of C^{∞} functions with compact support (denoted by) is a dense subspace of $S_{\alpha}^{1+\epsilon}$ for all $1+\epsilon, \alpha$.
(v) If $f_j \in L^{1+\epsilon}$ and $g_j \in \mathcal{D}$, then $f_j * g_j \in S_{\alpha}^{1+\epsilon}$ for all α.
Theorem 2.17([51]). $S_{\alpha}^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right) \subset L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$ and $\|\cdot\|_{1+\epsilon} \leq C\ \|\cdot\|_{1+\epsilon,\alpha}$ for some $C = C(1+\epsilon, 1+2\epsilon, \alpha) > 0$ provided $0 < \epsilon < \infty$ and $1+2\epsilon = \alpha(1+\epsilon)$, where $1+2\epsilon$ is the homogeneous dimension of Heisenberg group.
Proposition 2.18([21]). Suppose $f_j \in L^{1+\epsilon}(0 < \epsilon < \infty)$ and the integral

$$g_j(\xi) = \sum_j f_j * R_{\alpha}(\xi) = \int \sum_j f_j(\xi(1+\epsilon)^{-1}) R_{\alpha}(1+\epsilon) d(1+\epsilon)\ (0 < \operatorname{Re}\alpha < 1+2\epsilon)$$

converges absolutely for almost every ξ. If $f_j \in \operatorname{Dom}(\mathcal{T}_{1+\epsilon}^{-\alpha/2})$, then $g_j \in L^{1+\epsilon}$ and $\mathcal{T}^{-\alpha/2} f_j = g_j$.
Next, we define Adams function. Let $B_m = B(0, m)$ denote the ball with center 0 and radius m and $B := B(0,1)$ denote the unit ball in $\mathbb{H}^{\frac{2\epsilon-1}{2}}$. Then we state the following result:
Lemma 2.19 ([1,31]). For all $m \in (0,1)$, there exists $U_m \in D := \left\{u_j \in S_{\alpha}^2(B) : u_j\big|_{B_m} = 1\right\}$ such that

$$\|U_m\|_{2,\alpha}^2 = C(B_m, B) \leq \frac{A_{1+2\epsilon}}{(1+2\epsilon)\log\frac{1}{m}},$$

where $C(K, E)$ denotes the conductor capacity of K in E, whenever E is an open set and K a relatively compact subset, which is defined as follows:

$$C(K, E) = \mathrm{i}\ \left\{\left\|(\mathcal{T})^{\frac{\alpha}{2}} u_j\right\|_2^2 : u_j \in \mathcal{D}(E), u_j\big|_K = 1\right\},$$

where $\mathcal{D}(E)$ denotes the set of all C^{∞} functions with compact support in E.
Let $0 \in \Omega$ and $R \leq \operatorname{dist}(0, \partial\Omega)$, the Adams function is defined as below:

$$\tilde{A}_r(\xi) = \begin{cases} \sqrt{\dfrac{(1+2\epsilon)\log\left(\frac{R}{r}\right)}{A_{1+2\epsilon}}}\, U_{\frac{r}{R}}\left(\dfrac{\xi}{R}\right), & |\xi| < R\ ; \\ 0 & |\xi| \geq R, \end{cases}$$

where $0 < r < R$. We can check that $\|\tilde{A}_r\| \leq 1$ and denoting

$$\mathcal{M} = \lim_{k\to\infty} \int_{\frac{1}{k} \leq |\xi| \leq 1} \exp\left((1+2\epsilon)\log k \left|U_{R/k}(\xi)\right|\right) d\xi.$$

It is easy to see that $\mathcal{M} > 0$. We refer to [1,31] for more details.
The next section deals with the proof of Theorem 1.12.

3. Proof of Theorem 1.12 (see [61]).

Proof. Since Ω is an open set of class $C^{0,1}$ with bounded boundary, so for any $u_j \in S_{\alpha}^{1+\epsilon}(\Omega)$, we can extend u_j by defining u_j to be zero outside Ω (extension is still denoted by u_j itself) so that $u_j \in S_{\alpha}^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$. Now, for any $u_j \in S_{\alpha}^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$, we can write

$$u_j(\xi) = \mathcal{T}^{-\frac{\alpha}{2}} \mathcal{T}^{\frac{\alpha}{2}} u_j(\xi),$$

Then by Proposition 2.18, we get

$$u_j(\xi) = \left(\mathcal{T}^{\frac{\alpha}{2}} u_j * R_{\alpha}\right)(\xi), \tag{3.1}$$

where R_{α} is defined in Proposition 2.14. So,

$$\sum_j \exp\left(A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right) \leq \sum_j \exp\left(A_{1+2\epsilon}\left|\mathcal{T}^{\alpha/2} u_j * R_{\alpha}\right|^{\frac{1+\epsilon}{\epsilon}}\right). \tag{3.2}$$

Using $\left\|\sum_j \mathcal{T}^{\alpha/2} u_j\right\|_{1+\epsilon} \leq 1$ in Eq. (3.2), we get

$$\int_{\Omega} \sum_j \exp\left(A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right) \leq \int_{\Omega} \sum_j \exp\left(A_{1+2\epsilon} \frac{\left|\mathcal{T}^{\alpha/2} u_j * R_{\alpha}\right|^{\frac{1+\epsilon}{\epsilon}}}{\left\|\mathcal{T}^{\alpha/2} u_j\right\|_{1+\epsilon}}\right). \tag{3.3}$$

Now, we want to apply Theorem 2.11. Let $v_j(\xi) = R_\alpha(\xi)$, which is an allowed kernel of order α. We get

$$\int_\Omega \sum_j \exp\left(A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right) \le \int_\Omega \sum_j \exp\left(A_{1+2\epsilon}\frac{\left|\mathcal{J}^{\alpha/2}u_j * v_j(\xi)\right|^{\frac{1+\epsilon}{\epsilon}}}{\left\|\mathcal{J}^{\alpha/2}u_j\right\|_{1+\epsilon}}\right).$$

With the choice of

$$A_{1+2\epsilon} = \frac{1+2\epsilon}{\int_\Sigma \sum_j \left|v_j(\xi')\right|^{\frac{1+\epsilon}{\epsilon}} d\mu},$$

and applying Theorem 2.11, we get the conclusion. This completes the proof. To show the sharpness of $A_{1+2\epsilon}$, one can proceed as in [1,35]. For the sake of brevity, we omit the details.

4. Proof of Theorem 1. 13 and Theorem 1. 14 (see [61]).

Lemma 4.1 [61]. Let $0 < \alpha \le 1, 0 < \epsilon < \infty$ and $b(1+\epsilon, t)$ be a non-negative measurable function on $(-\infty,\infty) \times [0,\infty)$ such that almost everywhere,

$$b(1+\epsilon,t) \le 1, \text{ when } 0 < 1+\epsilon < t,$$

$$\sup_{t>0}\left(\int_{-\infty}^0 + \int_t^\infty b(1+\epsilon,t)^{\frac{1+\epsilon}{\epsilon}} d(1+\epsilon)\right)^{\frac{\epsilon}{1+\epsilon}} = b < \infty.$$

Then there is a constant $C(1+\epsilon,\alpha)$ such that if for $\phi_j \ge 0$ and

$$\int_{-\infty}^\infty \sum_j \phi_j(1+\epsilon)^{1+\epsilon} d(1+\epsilon) \le 1,$$

Then

$$\int_{-\infty}^\infty \sum_j \exp\left(-(F_j)_\alpha(t)\right) dt \ge C,$$

where

$$(F_j)_\alpha(t) = \alpha t - \alpha\left(\int_{-\infty}^\infty \sum_j b(1+\epsilon,t)\phi_j(1+\epsilon)d(1+\epsilon)\right)^{\frac{1+\epsilon}{\epsilon}}.$$

Proof. In case of $\alpha = 1$, this lemma was proved by Adams in [1], which was later modified for the case $0 < \alpha \le 1$ by Lam and Lu [34]. We refer to [1,34] for the details. Let $U = f_j * g_j$ denote the convolution on $\mathbb{H}^{\frac{2\epsilon-1}{2}}$. Then O'Neil [48] proved the following lemma:

Lemma 4.2.

$$U^*(t) \le U^{**}(t) \le t\sum_j f_j^{**}(t)g_j^{**}(t) + \int_t^\infty \sum_j f_j^*(1+\epsilon)g_j^*(1+\epsilon)d(1+\epsilon).$$

Lemma 4.3 (see [61]). Let $g_j(\xi) = \rho(\xi)^{\alpha-(1+2\epsilon)}$, then

$$g_j^*(t) = \left(\frac{c_0}{(1+2\epsilon)t}\right)^{\frac{\epsilon}{1+\epsilon}}$$

and

$$g_j^{**}(t) = (1+\epsilon)g_j^*(t),$$

where $\rho(\xi) = |\xi| = (|z|^4 + t^2)^{\frac{1}{4}}, 1+\epsilon = \frac{1+2\epsilon}{\alpha}, \frac{1+\epsilon}{\epsilon} = \frac{1+2\epsilon}{1+2\epsilon-\alpha}$ and $c_0 = \int_\Sigma d\mu$.

Proof. We have

$$g_j^*(t) = \text{i}\left\{1+\epsilon > 0: \lambda_{g_j}(1+\epsilon) \le t\right\},$$

where

$$\lambda_{g_j}(1+\epsilon) = \left|\left\{\xi \in \Omega: g_j(\xi) > 1+\epsilon\right\}\right|.$$

Since

$$g_j(\xi) = |\xi|^{\alpha-(1+2\epsilon)},$$

therefore

$$
\begin{aligned}
|\{\xi \in \Omega: g_j(\xi) > 1+\epsilon\}| &= \left|\{\xi \in \Omega: |\xi|^{\alpha-(1+2\epsilon)} > 1+\epsilon\}\right| \\
&= \left|\left\{\xi \in \Omega: |\xi| < (1+\epsilon)^{-\frac{1}{1+2\epsilon-\alpha}}\right\}\right|.
\end{aligned} \tag{4.1}
$$

Now, by polar coordinates, we get

$$
\begin{aligned}
\lambda_{g_j}(1+\epsilon) &= \int_\Sigma \int_0^{1+\epsilon^{-\frac{1}{1+2\epsilon-\alpha}}} r^{2\epsilon} dr d\mu, \\
&= \frac{c_0}{1+2\epsilon}(1+\epsilon)^{-\frac{1+2\epsilon}{1+2\epsilon-\alpha}}.
\end{aligned} \tag{4.2}
$$

(4.2) yields that for all $t > 0$, we have

$$
\begin{aligned}
\lambda_{g_j}(1+\epsilon) < t &\Rightarrow \frac{c_0}{1+2\epsilon}(1+\epsilon)^{-\frac{1+2\epsilon}{1+2\epsilon-\alpha}} < t \\
&\Rightarrow (1+\epsilon)^{-\frac{1+2\epsilon}{1+2\epsilon-\alpha}} < \frac{(1+2\epsilon)t}{c_0} \\
&\Rightarrow 1+\epsilon > \left(\frac{c_0}{(1+2\epsilon)t}\right)^{\frac{1+2\epsilon-\alpha}{1+2\epsilon}} = \left(\frac{c_0}{(1+2\epsilon)t}\right)^{\frac{\epsilon}{1+\epsilon}}.
\end{aligned} \tag{4.3}
$$

Then from (4.3), we see that

$$
g_j^*(t) \geq \left(\frac{c_0}{(1+2\epsilon)t}\right)^{\frac{\epsilon}{1+\epsilon}}. \tag{4.4}
$$

Now, for $1+\epsilon = \left(\frac{c_0}{(1+2\epsilon)t}\right)^{\frac{\epsilon}{1+\epsilon}}$,

$$
\lambda_{g_j}(1+\epsilon) = t. \tag{4.5}
$$

Hence

$$
g_j^*(t) \leq \left(\frac{c_0}{(1+2\epsilon)t}\right)^{\frac{\epsilon}{1+\epsilon}}. \tag{4.6}
$$

Next, we do the computation of $g_j^{**}(t)$.

$$
\begin{aligned}
g_j^{**}(t) &= \frac{1}{t}\int_0^t g_j^*(1+\epsilon) d(1+\epsilon) \\
&= \frac{1}{t}\int_0^t \left(\frac{c_0}{(1+2\epsilon)(1+\epsilon)}\right)^{\frac{\epsilon}{1+\epsilon}} d(1+\epsilon) \\
&= \frac{1}{t}\left(\frac{c_0}{1+2\epsilon}\right)^{\frac{\epsilon}{1+\epsilon}} \int_0^t (1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}} d(1+\epsilon) \\
&= (1+\epsilon)\frac{1}{t}\left(\frac{c_0}{1+2\epsilon}\right)^{\frac{\epsilon}{1+\epsilon}} t^{\frac{1}{1+\epsilon}} \\
&= (1+\epsilon) g_j^*(t).
\end{aligned}
$$

The proof completes here.

The next lemma has the same lines of proof as in [15,34,36]. For the sake of completeness, we reproduce it here.

Lemma 4.4 (see [61]). Let $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$ be a bounded domain. Let $g_j(\xi) = \rho(\xi)^{\alpha-(1+2\epsilon)}$ be an allowed kernel of order α. Let $1+\epsilon = \frac{1+2\epsilon}{\alpha}, \frac{1+\epsilon}{\epsilon} = \frac{1+2\epsilon}{1+2\epsilon-\alpha}, \epsilon > 0$ and

$$
A_{1+2\epsilon} = A(g_j, 1+\epsilon) = \frac{1+2\epsilon}{\int_\Sigma |g_j(\xi')|^{\frac{1+\epsilon}{\epsilon}} d\mu} = \frac{1+2\epsilon}{c_0}.
$$

Then for all $f_j \in L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$ with support in Ω, there exists a constant C_0 such that

$$
\frac{1}{|\Omega|^{\left(\frac{\epsilon}{1+2\epsilon}\right)}} \int_\Omega \sum_j \frac{\exp\left(A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)\left|\frac{(f_j * g_j)(\xi)}{\| f_j \|_{L^{1+\epsilon}\left(H^{\frac{2\epsilon-1}{2}}\right)}}\right|^{\frac{1+\epsilon}{\epsilon}}\right)}{\rho(\xi)^{1+\epsilon}} d\xi \leq C_0.
$$

Proof. Let $v_j(\xi) = (g_j * f_j)(\xi)$, where

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$$g_j(\xi) = \rho(\xi)^{\alpha-(1+2\epsilon)}.$$

Then by Lemma 4.3, we get

$$g_j{}^*(t) = \left(\frac{c_0}{(1+2\epsilon)t}\right)^{\frac{\epsilon}{1+\epsilon}}, g_j{}^{**}(t) = (1+\epsilon)g_j{}^*(t). \tag{4.7}$$

By Lemma 4.2, we have

$$\begin{aligned} v_j{}^*(t) &\le v_j{}^{**}(t) \le t f_j{}^{**}(t) g_j{}^{**}(t) + \int_t^\infty \sum_j f_j{}^*(1+\epsilon) g_j{}^*(1+\epsilon) d(1+\epsilon) \\ &= t \cdot \frac{1}{t}(1+\epsilon) g_j{}^*(t) \int_0^t \sum_j f_j{}^*(1+\epsilon) d(1+\epsilon) + \int_t^\infty \sum_j f_j{}^*(1+\epsilon)\left(\frac{c_0}{1+2\epsilon}\right)^{\frac{\epsilon}{1+\epsilon}} (1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}} d(1+\epsilon) \text{ by (4.7)} \\ &= \left(\frac{c_0}{1+2\epsilon}\right)^{\frac{\epsilon}{1+\epsilon}} \left((1+\epsilon) t^{-\frac{\epsilon}{1+\epsilon}} \int_0^t \sum_j f_j{}^*(1+\epsilon) d(1+\epsilon) + \int_t^\infty \sum_j (1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}} f_j{}^*(1+\epsilon) d(1+\epsilon)\right). \end{aligned} \tag{4.8}$$

Now, by using change of variables

$$\phi_j(1+\epsilon) = |\Omega|^{\frac{1}{1+\epsilon}} f_j{}^*\left(|\Omega| e^{-(1+\epsilon)}\right) e^{-\frac{\Omega}{1+\epsilon}}, \tag{4.9}$$

we get

$$\begin{aligned} \int_\Omega (f_j(\xi))^{1+\epsilon} d\xi &= \int_0^{|\Omega|} \sum_j \left(f_j{}^*(t)\right)^{1+\epsilon} dt \\ &= \int_0^\infty \sum_j (\phi_j(1+\epsilon))^{1+\epsilon} d(1+\epsilon). \end{aligned} \tag{4.10}$$

Let $h(\xi) = \frac{1}{\rho(\xi)^\alpha}$, then $h^*(t) = \left(\frac{C_{1+2\epsilon}}{t}\right)^{\frac{1+\epsilon}{1+2\epsilon}}$, where $C_{1+2\epsilon} = \frac{c_0}{1+2\epsilon}$. Now, by Hardy-Littlewood inequality (Lemma 2.5), we get

$$\begin{aligned} &\int_\Omega \sum_j \frac{\exp\left(A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)|v_j(\xi)|^{1+\epsilon}\right)}{\rho(\xi)^{1+\epsilon}} d\xi \\ &\le (C_{1+2\epsilon})^{\frac{1+\epsilon}{1+2\epsilon}} \int_0^{|\Omega|} \sum_j \frac{\exp\left(A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)\left(v_j{}^*(t)\right)^{\frac{1+\epsilon}{\epsilon}}\right)}{t^{\frac{1+\epsilon}{1+2\epsilon}}} dt. \end{aligned} \tag{4.11}$$

Again, using change of variable

$$t = |\Omega| e^{-(1+\epsilon)}, \text{ then } dt = -|\Omega| e^{-(1+\epsilon)} d(1+\epsilon),$$

Obtain

$$\begin{aligned} &(C_{1+2\epsilon})^{\frac{1+\epsilon}{1+2\epsilon}} \int_0^{|\Omega|} \sum_j \frac{\exp\left(A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)\left(v_j{}^*(t)\right)^{\frac{1+\epsilon}{\epsilon}}\right)}{t^{\frac{1+\epsilon}{1+2\epsilon}}} dt \\ &= (C_{1+2\epsilon})^{\frac{1+\epsilon}{1+2\epsilon}} \int_0^\infty \sum_j \frac{\exp\left(A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)\left(v_j{}^*\left(|\Omega| e^{-(1+\epsilon)}\right)\right)^{\frac{1+\epsilon}{\epsilon}}\right)}{\left(|\Omega| e^{-(1+\epsilon)}\right)^{\frac{1+\epsilon}{1+2\epsilon}}} |\Omega| e^{-(1+\epsilon)} d(1+\epsilon) \\ &\le (C_{1+2\epsilon})^{\frac{1+\epsilon}{1+2\epsilon}} |\Omega|^{\frac{\epsilon}{1+2\epsilon}} \int_0^\infty \sum_j \exp\Bigg[A_{1+2\epsilon}\left(\frac{c_0}{1+2\epsilon}\right)\left(\frac{\epsilon}{1+2\epsilon}\right)\Bigg\{(1 \\ &+ \epsilon)\left(|\Omega| e^{-(1+\epsilon)}\right)^{-\frac{\epsilon}{1+\epsilon}} \int_0^{|\Omega| e^{-(1+\epsilon)}} f_j{}^*(z) dz + \int_{|\Omega| e^{-(1+\epsilon)}}^{|\Omega|} f_j{}^*(z) z^{-\frac{\epsilon}{1+\epsilon}} dz\Bigg\}^{\frac{1+\epsilon}{\epsilon}} - \left(\frac{\epsilon}{1+2\epsilon}\right)(1 \\ &+ \epsilon)\Bigg] d(1+\epsilon) \text{ (by Eq. (4.8))} \\ &= (C_{1+2\epsilon})^{\frac{1+\epsilon}{1+2\epsilon}} |\Omega|^{\frac{\epsilon}{1+2\epsilon}} \int_0^\infty \sum_j \exp\Bigg[\left(\frac{\epsilon}{1+2\epsilon}\right)\Bigg((1+\epsilon) e^\epsilon \int_{1+\epsilon}^\infty \phi_j(w) e^{-\frac{\epsilon w}{1+\epsilon}} dw \\ &+ \int_0^{1+\epsilon} \phi_j(w) dw\Bigg)(1+\epsilon)^{\frac{1+\epsilon}{\epsilon}} - \left(\frac{\epsilon}{1+2\epsilon}\right)(1+\epsilon)\Bigg] d(1+\epsilon) \\ &\times \left(\text{because } A_{1+2\epsilon} = \frac{1+2\epsilon}{c_0}\right) \\ &= (C_{1+2\epsilon})^{\frac{1+\epsilon}{1+2\epsilon}} |\Omega|^{\frac{\epsilon}{1+2\epsilon}} \int_0^\infty \sum_j \exp\left[-(F_j)_{\left(\frac{\epsilon}{1+2\epsilon}\right)}(1+\epsilon)\right] d(1+\epsilon), \end{aligned}$$

where

$$(F_j)_{\left(\frac{\epsilon}{1+2\epsilon}\right)}(t) = \left(\frac{\epsilon}{1+2\epsilon}\right)t - \left(\frac{\epsilon}{1+2\epsilon}\right)t\left(\int_{-\infty}^{\infty}\sum_j a(1+\epsilon,t)\phi_j(1+\epsilon)d(1+\epsilon)\right)^{\frac{1+\epsilon}{\epsilon}},$$

$$a(1+\epsilon,t) = \begin{cases} 1, & \text{for } 0 < 1+\epsilon < t, \\ (1+\epsilon)e^{\frac{\epsilon(t-(1+\epsilon))}{1+\epsilon}}, & \text{for } t < 1+\epsilon < \infty, \\ 0, & \text{for } -\infty < 1+\epsilon \le 0. \end{cases}$$

Now, using Lemma 4.1, we get the conclusion.

Remark 4.5. We remark that Lemma 4.4 also holds for any allowed kernel g_j of order α, see [15,36] for the details.

Proof of Theorem 1.13 (see [61]). For any $u_j \in S_\alpha^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)$, we have

$$u_j(\xi) = \left(\mathcal{J}^{\frac{\alpha}{2}}u_j * R_\alpha\right)(\xi).$$

This implies

$$|\sum_j u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}} \le \sum_j \left|\left(\mathcal{J}^{\alpha/2}u_j * R_\alpha\right)(\xi)\right|^{\frac{1+\epsilon}{\epsilon}}. \quad (4.12)$$

Let $g_j(\xi) = R_\alpha(\xi)$ and then using Lemma 4.4 (more precisely, Remark 4.5), we get

$$\frac{1}{|\Omega|^{\left(1-\frac{i}{1+2\epsilon}\right)}}\int_\Omega \sum_j \frac{\exp\left[A_{1+2\epsilon}\left(1-\frac{1+\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+\epsilon}{\epsilon}}\right]}{\rho(\xi)^{1+\epsilon}}d\xi$$

$$\le \frac{1}{|\Omega|^{\left(\frac{\epsilon}{1+2\epsilon}\right)}}\int_\Omega \sum_j \frac{\exp\left[A_{1+2\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right)\left|\mathcal{J}^{\alpha/2}u_j * g_j(\xi)\right|^{\frac{1+\epsilon}{\epsilon}}\right]}{\rho(\xi)^{1+\epsilon}}d\xi \le C_0. \quad (4.13)$$

Proof of Theorem 1.14 (see [61]). If $u_j \equiv 0$, then we are done. Let $u_j \in S_\alpha^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right) \setminus \{0\}$, and $\| u_j \|_{L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)} + \|\mathcal{J}^{\alpha/2}u_j\|_{L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)} \le 1$. Then we have

$$\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi \le \int_{E(u_j)} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi$$

$$+\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}\setminus E(u_j)} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi,$$

where $E(u_j) = \left\{\xi \in \mathbb{H}^{\frac{2\epsilon-1}{2}} : |u_j(\xi)| \ge 1\right\}$. The measure of $E(u_j)$ can be estimated as follows:

$$\begin{aligned} |E(u_j)| &= \int_{E(u_j)} \sum_j 1 d\xi \\ &\le \int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j |u_j(\xi)|^{1+\epsilon} \\ &\le A^{-(1+\epsilon)}, \end{aligned}$$

Where $A = \inf_{u_j \in S_\alpha^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)/\{0\}} \sum_j \frac{\|u_j\|_{L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)} + \left\|\mathcal{T}^{\frac{\alpha}{2}}u_j\right\|_{L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)}}{\|u_j\|_{L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right)}} > 0$. Hence $|E(u_j)| < \infty$. So by Theorem 1.13, we get

$$\begin{aligned} \int_{E(u_j)} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi &\le C|E(u_j)| \quad (4.14) \\ &\le CA^{-(1+\epsilon)}, \end{aligned}$$

for some universal constant $C > 0$. So now in the complement of $E(u_j)$, we have $|u_j(\xi)| < 1$. Thus

$$\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}\setminus E(u_j)} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi$$

$$= \underbrace{\int_{\{|u_j(\xi)|<1;\xi\xi|<1\}} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi}_{I}$$

$$+ \underbrace{\int_{\{|u_j(\xi)|<1;|\xi|\geq 1\}} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi}_{II}. \tag{4.15}$$

Now

$$\begin{aligned} I &= \int_{\{|u_j(\xi)|<1;|\xi|<1\}} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi \\ &< \int_{\{|\xi|<1\}} \frac{1}{|\xi|^{1+\epsilon}}d\xi \\ &= \int_0^1 \int_\Sigma \frac{1}{|r\xi'|^{1+\epsilon}} r^{2\epsilon} d\mu dr \ \text{(by Proposition 2.6)} \\ &= \int_\Sigma \frac{1}{|\xi'|^{1+\epsilon}} \int_0^1 r^{\epsilon-1} dr d\mu \\ &= C\int_\Sigma \int_0^1 r^{\epsilon-1} dr d\mu = Cc_0, \end{aligned} \tag{4.16}$$

where c_0 is surface measure of sphere of radius 1 . Then, we get $I < \infty$ where $\epsilon > 0$. Now

$$\begin{aligned} II &= \int_{\{|u_j(\xi)|<1;|\xi|\geq 1\}} \sum_j \frac{\psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi \\ &\leq C\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}\setminus E(u_j)} \sum_j \psi_j\left(\left(\frac{\epsilon}{1+2\epsilon}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)d\xi \\ &\leq C\sum_{k=j_{1+\epsilon}-1}^{\infty} \frac{\beta^k}{k!}\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}\setminus E(u_j)} \sum_j |u_j(\xi)|^{\left(\frac{1+\epsilon}{\epsilon}\right)k}d\xi \\ &\leq C\sum_{k=j_{1+\epsilon}-1}^{\infty} \frac{\beta^k}{k!}\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}\setminus E(u_j)} \sum_j |u_j(\xi)|^{1+\epsilon}d\xi \\ &\leq C_1. \end{aligned} \tag{4.17}$$

By (4.15), (4.16) and (4.17), we get

$$\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}\setminus E(u_j)} \sum_j \frac{\psi_j\left(\left(-\frac{3}{2\epsilon-1}\right)A_{1+2\epsilon}|u_j(\xi)|^{\frac{1+\epsilon}{\epsilon}}\right)}{|\xi|^{1+\epsilon}}d\xi < C. \tag{4.18}$$

So by (4.14) and (4.18), we get the desired result.

5. Applications: Existence theorems

As an application of Adams-Moser-Trudinger inequality, we prove the existence of a solution of the following class of problems:

$$\begin{cases} T^\alpha u_j = \dfrac{f_j(\xi,u_j)}{|\xi|^{1+\epsilon}} + b(\xi)|u_j|^{\gamma-1}u_j \text{ in } \Omega, \\ \quad u_j \ = \ 0 \qquad\qquad \text{in } \mathbb{H}^{\frac{2\epsilon-1}{2}}\setminus\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{H}^{\frac{2\epsilon-1}{2}}$ is a bounded set of class $C^{0,1}$ with bounded boundary and $0 < \alpha < 1+2\epsilon$. Here, we are taking $\epsilon = 1$ and so $\alpha = \frac{1+2\epsilon}{2}$ because we are considering the case $\alpha(1+\epsilon) = 1+2\epsilon$. The function $f_j: \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (H1) − (H5). Let

$$J: S_\alpha^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right) \longrightarrow \mathbb{R}$$

be the functional defined by

$$J(u_j) = \frac{1}{2}\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j \left|(\mathcal{T})^{\frac{\alpha}{2}}u_j\right|^2 - \int_\Omega \sum_j \frac{F_j(\xi,u_j)}{|\xi|^{1+\epsilon}} - \frac{1}{\gamma+1}\int_\Omega \sum_j b(\xi)|u_j|^{\gamma+1}d\xi.$$

Lemma 5.1 (see [61]). Assume that f_j satisfies (1.6) and suppose that (H1) − (H3) and (H5) hold. Then there exists $\rho > 0$ such that

$$J(u_j) > 0, \text{ if } \| u_j \| = \rho.$$

Proof. By (H3), we have

$$\limsup_{1+\epsilon\to 0^+} \sum_j \frac{2F_j(\xi,1+\epsilon)}{|1+\epsilon|^2} < \Lambda$$

which by definition is same as

$$\inf_{\beta>0} \sup \sum_j \left\{\frac{2F_j(\xi,1+\epsilon)}{|1+\epsilon|^2} : 0 < 1+\epsilon < \beta\right\} < \Lambda.$$

Since the above inequality is strict, we can choose a real number $\epsilon \geq 0$ such that

$$\inf_{\beta>0} \sup \sum_j \left\{\frac{2F_j(\xi,1+\epsilon)}{|1+\epsilon|^2} : 0 < 1+\epsilon < \beta\right\} < \Lambda - (1+\epsilon).$$

As the infimum is strictly less than $\Lambda - (1+\epsilon)$, therefore $\exists \delta > 0$ such that

$$\sup \sum_j \left\{\frac{2F_j(\xi,1+\epsilon)}{|1+\epsilon|^2} : 0 < 1+\epsilon < \delta\right\} < \Lambda - (1+\epsilon).$$

Thus for $|1+\epsilon| < \delta$, we have

$$\frac{2\sum_j F_j(\xi,1+\epsilon)}{|1+\epsilon|^2} < \Lambda - (1+\epsilon)$$

or

$$F_j(\xi,1+\epsilon) < \tfrac{1}{2}(\Lambda - (1+\epsilon))|1+\epsilon|^2. \tag{5.2}$$

Since f_j has a subcritical growth, so $\exists$ constants $c > 0$ and $\mu > 0$ such that

$$|f_j(\xi,t)| \leq c\exp(\mu t^2), \forall \xi \in \Omega, \forall t \in \mathbb{R}. \tag{5.3}$$

Thus, we have

$$\begin{aligned} \left|\sum_j F_j(\xi,1+\epsilon)\right| &= \left|\int_0^{1+\epsilon} \sum_j f_j(\xi,t)dt\right| \\ &\leq \int_0^{1+\epsilon} \sum_j |f_j(\xi,t)|dt \\ &\leq c\int_0^{1+\epsilon} \exp(\mu t^2)dt \\ &\leq C\exp(\mu(1+\epsilon)^2). \end{aligned} \tag{5.4}$$

Now, for $|1+\epsilon| \geq \delta$ and $\epsilon > 0$, there exists a constant $K(\delta, 2+\epsilon)$ such that

$$\left|\sum_j F_j(\xi,1+\epsilon)\right| \leq K|1+\epsilon|^{2+\epsilon}\exp(\mu(1+\epsilon)^2), \forall |1+\epsilon| \geq \delta. \tag{5.5}$$

On using (5.2) and (5.4), we get

$$\left|\sum_j F_j(\xi,1+\epsilon)\right| \leq \frac{1}{2}(\Lambda - (1+\epsilon))|1+\epsilon|^2 + K|1+\epsilon|^{2+\epsilon}\exp(\mu(1+\epsilon)^2), \tag{5.6}$$

for all $\xi \in \Omega, 1+\epsilon \in \mathbb{R}$ and for some $\mu, 1+\epsilon > 0$ and $\epsilon > 0$.

Now, consider r and r' such that $\frac{1}{r} + \frac{1}{r'} = 1$. Then by Hölder's inequality, we have

$$\begin{aligned} \int_\Omega \sum_j \frac{\exp(\mu|u_j|^2)|u_j|^{2+\epsilon}}{|\xi|^{1+\epsilon}}d\xi &\leq \sum_j \left(\int_\Omega \frac{\exp(\mu r|u_j|^2)}{|\xi|^{1+\epsilon}}d\xi\right)^{\frac{1}{r}} \left(\int_\Omega |u_j|^{(2+\epsilon)r'}d\xi\right)^{\frac{1}{r'}} \\ &\leq \sum_j \left(\int_\Omega \frac{\exp\left(\mu r \parallel u_j \parallel^2 \frac{|u_j|^2}{\parallel u_j \parallel^2}\right)}{|\xi|^{1+\epsilon}}d\xi\right)^{\frac{1}{r}} \left(\int_\Omega |u_j|^{(2+\epsilon)r'}d\xi\right)^{\frac{1}{r'}}. \end{aligned} \tag{5.7}$$

Now, if we choose $r > 1$ sufficiently close to 1 , so that $\epsilon > 0$ and $\| u_j \| \leq \sigma$ such that $\mu r \sigma^2 < \left(\frac{\epsilon}{1+2\epsilon}\right) A_{1+2\epsilon}$. Then by Theorem 1.13 and by (5.7) we get

$$\int_\Omega \sum_j \frac{\exp(\mu |u_j|^2)|u_j|^{2+\epsilon}}{|\xi|^{1+\epsilon}} d\xi \leq C \left(\int_\Omega \sum_j |u_j|^{(2+\epsilon)r'} d\xi \right)^{\frac{1}{r'}}. \quad (5.8)$$

So, we obtain

$$J(u_j) \geq \frac{1}{2} \sum_j \| u_j \|^2 - \frac{\Lambda - (1+\epsilon)}{2} \int_\Omega \sum_j \frac{|u_j|^2}{|\xi|^{1+\epsilon}} d\xi$$
$$-C \left(\int_\Omega \sum_j |u_j|^{(2+\epsilon)r'} d\xi \right)^{\frac{1}{r}} - \frac{1}{\gamma+1} \sum_j \| b \|_{L^2} \left\| u_j^{\gamma+1} \right\|_{L^2}. \quad (5.9)$$

Now, we have

$$\Lambda = \inf_{0 \neq u_j \in S_\alpha^2(\Omega)} \sum_j \frac{\| u_j \|^2}{\int_\Omega \frac{|u_j|^2}{|\xi|^\alpha} d\xi} > 0. \quad (5.10)$$

So, (5.10) implies that

$$\Lambda \leq \sum_j \frac{\| u_j \|^2}{\int_\Omega \frac{|u_j|^2}{|\xi|^{1+\epsilon}} d\xi}, \qquad \forall 0 \neq u_j \in S_\alpha^2(\Omega)$$

or

$$\int_\Omega \sum_j \frac{|u_j|^2}{|\xi|^{1+\epsilon}} d\xi \leq \frac{1}{\Lambda} \sum_j \| u_j \|^2. \quad (5.11)$$

On using (5.11) in (5.9), we get

$$J(u_j) \geq \frac{1}{2} \sum_j \| u_j \|^2 - \frac{\Lambda - (1+\epsilon)}{2\Lambda} \sum_j$$
$$\| u_j \|^2 - C \sum_j \| u_j \|_{r'(2+\epsilon)}^{2+\epsilon} - \frac{1}{\gamma+1} \sum_j \| b \|_{L^2} \| u_j \|^{(\gamma+1)}. \quad (5.12)$$

By Theorem 2.17, $S_\alpha^{1+\epsilon}$ is continuously embedded into $L^{1+\epsilon}\left(\mathbb{H}^{\frac{2\epsilon-1}{2}}\right), \forall 1+\epsilon \in [1+\epsilon, \infty)$. Therefore, in particular, for $1+\epsilon = r'(2+\epsilon)$, it gives that

$$\| u_j \|_{r'(2+\epsilon)} \leq C \| u_j \|.$$

So we get

$$J(u_j) \geq \frac{1}{2}\left(1 - \frac{\Lambda - (1+\epsilon)}{\Lambda}\right) \sum_j \| u_j \|^2 - C \| u_j \|^{2+\epsilon} - \frac{1}{2(\gamma+1)} \sum_j \| b \|_{L^2} \| u_j \|^{(\gamma+1)}.$$

Since $\epsilon \geq 0$, so we may choose $\rho > 0$ such that

$$\frac{1}{2}\left(1 - \frac{\Lambda - (1+\epsilon)}{\Lambda}\right) \rho - C\rho^{1+\epsilon} > 0.$$

This yields

$$J(u_j) \geq \sum_j \| u_j \| \left[\frac{1}{2}\left(1 - \frac{\Lambda - (1+\epsilon)}{\Lambda}\right) \| u_j \| - C \| u_j \|^{1+\epsilon} - \frac{1}{\gamma+1} \| b \|_{L^2} \| u_j \|^\gamma \right] > 0.$$

Now, for $\| b \|_{L^2}$ sufficiently small, $\exists \rho_b$ such that $J(u_j) > 0$ whenever $\| u_j \| = \rho_b$. This completes the proof.

Lemma 5.2 (see [61]). There exists $e \in S_\alpha^2(\Omega)$ with $\| e \| > \rho_b$ such that

$$J(e) < \inf_{\| u_j \| = \rho_b} \sum_j J(u_j).$$

Proof. Let $0 \neq u_j \in S_\alpha^2(\Omega)$ and $u_j \geq 0$. By (H2), there exist $c > 0$ and $d > 0$ such that

$$F_j(\xi, 1+\epsilon) \geq c(1+\epsilon)^\theta - d, \forall (\xi, 1+\epsilon) \in \Omega \times \mathbb{R}^+, \text{where } \theta > 2. \quad (5.13)$$

For $t > 0$, we have

$$J(tu_j) \leq \frac{t^2}{2}\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j \left|(\mathcal{T})^{\frac{\alpha}{2}}u_j\right|^2 d\xi - ct^\theta \int_\Omega \sum_j \frac{|u_j|^\theta}{|\xi|^{1+\epsilon}} d\xi + d\int_\Omega \frac{1}{|\xi|^{1+\epsilon}} d\xi$$
$$-\frac{t^{\gamma+1}}{\gamma+1}\int_\Omega \sum_j b(\xi)|u_j(\xi)|^{\gamma+1} d\xi. \quad (5.14)$$

Using Sobolev's inequality, it is easy to see that

$$\int_\Omega \sum_j \frac{|u_j|^\theta}{|\xi|^{1+\epsilon}} d\xi$$

is finite and since $\theta > 2$, so by (5.14), we get $J(tu_j) \to -\infty$ as $t \to \infty$. By setting $e = tu_j$ with t large enough, we get $\| e \| > \rho_b$ and

$$J(e) < \inf_{\|u_j\|=\rho_b} \sum_j J(u_j).$$

This completes the proof.

Lemma 5.3 (see [61]). Assume that f_j satisfies subcritical growth condition. Then the functional J satisfies Palais-Smale condition at level , for all $c \in \mathbb{R}$.

Proof. Let $\{(u_j)_k\} \subset S_\alpha^2(\Omega)$ be a PS sequence at level c, that is,

$$J\left((u_j)_k\right) = \frac{1}{2}\sum_j \left\|(u_j)_k\right\|^2 - \int_\Omega \sum_j \frac{F_j\left(\xi, (u_j)_k\right)}{|\xi|^{1+\epsilon}}$$
$$-\frac{1}{\gamma+1}\int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma+1} d\xi \to c \text{ as } k \to \infty \quad (5.15)$$

And

$$DJ\left((u_j)_k\right) \longrightarrow 0 \text{ as } k \to \infty.$$

Now

$$\sum_j DJ\left((u_j)_k\right) v_j = \int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j (\mathcal{T})^{\frac{\alpha}{2}}(u_j)_k (\mathcal{T})^{\frac{\alpha}{2}} v_j d\xi - \int_\Omega \sum_j \frac{f_j\left(\xi, (u_j)_k\right) v_j}{|\xi|^{1+\epsilon}} d\xi$$
$$-\int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma-1}(u_j)_k(\xi) v_j(\xi) d\xi. \quad (5.16)$$

Then

$$\left|DJ((u_j)_k)v_j\right| = \left|\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j (\mathcal{T})^{\frac{\alpha}{2}}(u_j)_k (\mathcal{T})^{\frac{\alpha}{2}} v_j d\xi - \int_\Omega \sum_j \frac{f_j\left(\xi, (u_j)_k\right) v_j}{|\xi|^{1+\epsilon}} d\xi\right.$$
$$\left. - \int_\Omega b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma-1}(u_j)_k(\xi) v_j(\xi) d\xi \right|. \quad (5.17)$$

By applying Cauchy-Schwarz inequality in $\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} (\mathcal{T})^{\frac{\alpha}{2}}(u_j)_k (\mathcal{T})^{\frac{\alpha}{2}} v_j d\xi$ and by Adams-Moser-Trudinger and Hölder's inequality in $\int_\Omega \sum_j \frac{f_j(\xi,(u_j)_k)v_j}{|\xi|^\alpha} d\xi$, we get the following:

$$\sum_j \left|DJ((u_j)_k)v_j\right| \leq \sum_j \|(u_j)_k\| \| v_j \| + \int_\Omega \sum_j \left|\frac{f_j(\xi, (u_j)_k)v_j}{|\xi|^{1+\epsilon}}\right| d\xi + \int_\Omega \sum_j |b(\xi)|\left|(u_j)_k\right|^{\gamma-1} (u_j)_k v_j(\xi) \mid d\xi \quad (5.18)$$

$$\sum_j \left|D.J((u_j)_k)v_j\right| \leq \sum_j \|(u_j)_k\| \| v_j \| + \int_\Omega \sum_j \left|\frac{\exp\left(\mu(u_j)_k^2\right)v_j}{|\xi|^{1+\epsilon}}\right| d\xi + \int_\Omega \sum_j |b(\xi)|\left|(u_j)_k(\xi)\right|^{\gamma-1} (u_j)_k(\xi) v_j(\xi) \mid d\xi$$

$$\sum_j \left|DJ((u_j)_k)v_j\right| \leq \sum_j \|(u_j)_k\| \| v_j \| + \int_\Omega \left|\frac{\exp\left(\mu(u_j)_k^2\right)v_j}{|\xi|^{1+\epsilon}}\right| d\xi + \sum_j \| b \|_{L^2} \left\|(u_j)_k^\gamma v_j\right\|_2 \quad (5.19)$$

$$\left|\sum_j DJ((u_j)_k)v_j\right| \le \sum_j \|(u_j)_k\| \parallel v_j \parallel + \int_\Omega \sum_j \left|\frac{\exp\left(\mu(u_j)_k^2\right)}{|\xi|^{\frac{1+\epsilon}{r}}} \frac{v_j}{|\xi|^{(1+\epsilon)\left(1-\frac{1}{r}\right)}}\right| d\xi + (1+\epsilon)c_{1+\epsilon}\sum_j \|(u_j)_k\|^\gamma$$

$$\parallel v_j \parallel |DJ((u_j)_k)v_j| \le \sum_j \|(u_j)_k\| \parallel v_j$$

$$\parallel + \sum_j \left(\int_\Omega \frac{\exp(r\mu(u_j)_k^2)}{|\xi|^{1+\epsilon}} d\xi\right)^{\frac{1}{r}} \left(\int_\Omega \frac{|v_j|^t}{|\xi|^{(1+\epsilon)t\left(1-\frac{1}{r}\right)}} d\xi\right)^{\frac{1}{t}} + c\sum_j \|(u_j)_k\|^\gamma \|v_j\| \tag{5.20}$$

$$\left|\sum_j DJ((u_j)_k)v_j\right| \le \sum_j \|(u_j)_k\| \parallel v_j$$

$$\parallel + \sum_j \left(\int_\Omega \frac{\exp(r\mu(u_j)_k^2)}{|\xi|^{1+\epsilon}} d\xi\right)^{\frac{1}{r}} \left(\int_\Omega |v_j|^{t(1+\epsilon)} d\xi\right)^{\frac{1}{t(1+\epsilon)}} \left(\int_\Omega \frac{1}{|\xi|^{\frac{t(1+\epsilon)^2}{\epsilon}\left(1-\frac{1}{r}\right)}} d\xi\right)^{\frac{\frac{1}{t(1+\epsilon)}}{\epsilon}}$$

$$+ c\sum_j \|(u_j)_k\|^\gamma \parallel v_j \parallel,$$

where $\left(\int_\Omega \sum_j \frac{\exp(r\mu(u_j)_k^2)}{|\xi|^\alpha} d\xi\right)^{\frac{1}{r}}$ is finite by (5.8) and $\frac{1}{t} + \frac{1}{r} = 1, \epsilon = 0$ and $t, r, 1+\epsilon$ are chosen in such a way that ats $\left(1 - \frac{1}{r}\right) < 1 + 2\epsilon$, and then using the fact that $(u_j)_k$ is a Palais-Smale sequence, we get

$$\left|\sum_j DJ((u_j)_k)v_j\right| \le \epsilon_k \sum_j \parallel v_j \parallel, \tag{5.21}$$

where $\epsilon_k \to 0$ as $k \to \infty$. On taking $v_j = (u_j)_k$ in (5.16), we get

$$\sum_j DJ\left(\left(u_j\right)_k\right)\left(u_j\right)_k = \int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j (\mathcal{T})^{\frac{\alpha}{2}}\left(u_j\right)_k (\mathcal{T})^{\frac{\alpha}{2}}\left(u_j\right)_k d\xi$$

$$- \int_\Omega \sum_j \frac{f_j(\xi,(u_j)_k)(u_j)_k}{|\xi|^{1+\epsilon}} d\xi - \int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma+1} d\xi. \tag{5.22}$$

.

$$\left|\sum_j DJ((u_j)_k)(u_j)_k\right|$$

$$= \left|\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j (\mathcal{T})^{\frac{\alpha}{2}}\left(u_j\right)_k (\mathcal{T})^{\frac{\alpha}{2}}\left(u_j\right)_k d\xi - \int_{1l} \sum_j \frac{f_j\left(\xi,\left(u_j\right)_k\right)\left(u_j\right)_k}{|\xi|^{1+\epsilon}} d\xi\right.$$

$$\left. - \int_{1l} \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma+1}\right| d\xi \le \epsilon_k \sum_j \|(u_j)_k\|. \tag{5.23}$$

On multiplying (5.15) with θ and subtracting (5.22) from it, we get

$$\left(\frac{\theta}{2} - 1\right)\sum_j \left\|\left(u_j\right)_k\right\|^2 + \int_\Omega \sum_j \frac{1}{|\xi|^{1+\epsilon}}\left(f_j\left(\xi,\left(u_j\right)_k\right)\left(u_j\right)_k - \theta F_j\left(\xi,\left(u_j\right)_k\right)\right) d\xi$$

$$-\left(\frac{\theta}{\gamma+1} - 1\right)\int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma+1} d\xi \le O(1) + \epsilon_k \sum_j \|(u_j)_k\|. \tag{5.24}$$

By (H2), there exist $R_0 > 0$ and $\theta > 2$ such that, for $\parallel u_j \parallel \ge R_0$,

$$\sum_j \theta F_j(\xi, u_j) \le \sum_j u_j f_j(\xi, u_j). \tag{5.25}$$

So by (5.24) and (5.25), we get

$$c\left(\frac{\theta}{2}-1\right)\sum_j \|(u_j)_k\|^2 \leq O(1)+\epsilon_k \sum_j \|(u_j)_k\| \text{ (as } 0\leq\gamma<1 \text{ and } \| b \|_{L^2} \text{ is very small).} \quad (5.26)$$

Since $\theta > 2$, so $\left(\frac{\theta}{2}-1\right)\|(u_j)_k\|^2 > 0$ implies that $(u_j)_k$ is bounded, therefore up to a subsequence

$$(u_j)_k \longrightarrow (u_j)_0 \text{ in } S_\alpha^2(\Omega),$$
$$(u_j)_k \longrightarrow (u_j)_0 \text{ in } L^{1+\epsilon}(\Omega), \forall\epsilon \geq 0,$$
$$(u_j)_k(\xi) \longrightarrow (u_j)_0(\xi) \text{ a.e. in } \Omega.$$

From (5.26), there exists some constant m_0 such that $\|(u_j)_k\| \leq m_0$ and now, we have

$$\left|\int_\Omega \sum_j \frac{f_j(\xi,(u_j)_k)}{|\xi|^{1+\epsilon}}((u_j)_k-u_j)d\xi\right| \leq \int_\Omega \sum_j \frac{|f_j(\xi,(u_j)_k)|}{|\xi|^{1+\epsilon}}|((u_j)_k-u_j)|d\xi$$
$$\leq \sum_j \left(\int_\Omega \frac{|f_j(\xi,(u_j)_k)|^r}{|\xi|^{1+\epsilon}}d\xi\right)^{\frac{1}{r}}\left(\int_\Omega |((u_j)_k-u_j)|^{r'}d\xi\right)^{\frac{1}{r'}}$$
$$\left(\text{where } r>1 \text{ such that } \epsilon>0 \text{ and } \frac{1}{r}+\frac{1}{r'}=1\right).$$

Again, using Theorem 1.13 and note that $\|(u_j)_k\| \leq m_0$, we have (see, [36])

$$\left|\int_\Omega \sum_j \frac{f_j(\xi,(u_j)_k)}{|\xi|^{1+\epsilon}}((u_j)_k-u_j)d\xi\right| \leq C\sum_j \|(u_j)_k-u_j\|_{r'} \longrightarrow 0.$$

Similarly, we can show that

$$\int_\Omega \sum_j \frac{f_j(\xi,u_j)}{|\xi|^{1+\epsilon}}((u_j)_k-u_j)d\xi \longrightarrow 0 \text{ as } k\longrightarrow\infty.$$

Also we have

$$\sum_j \langle DJ((u_j)_k)-DJ(u_j),(u_j)_k-u_j\rangle \longrightarrow 0, \text{ as } k\longrightarrow\infty.$$

Thus $(u_j)_k \longrightarrow u_j$ in $S_\alpha^2(\Omega)$. This completes the proof.

Subcritical growth. Proof of Theorem 1.15
Using Lemmas 5.1 and 5.2, we prove that J satisfies the geometric properties for mountain pass theorem and by Lemma 5.3, we can show that J satisfies Palais-Smale conditions. Hence, we conclude the proof of Theorem 1.15 by applying mountain pass theorem.

The critical growth.
In this case, we need the following lemma to establish the existence of solution:

Lemma 5.4 (see [61]). Assume that f_j satisfies critical exponential growth condition (1.7) and (1.8) and suppose (H1)-(H5) hold. Thén theré exists $k>0$ such that

$$\text{m } \{J(tA_k): t\geq 0\} < \frac{\epsilon}{2(1+2\epsilon)}\frac{A_{1+2\epsilon}}{\alpha_0},$$

where $A_k = \tilde{A}_{R/k}$ is defined in Lemma 2.19.

Proof. By the method of contradiction, firstly, we will prove the result for the following functional:

$$\psi_j(u_j) = \frac{1}{2}\sum_j \| u_j \|^2 - \int_\Omega \sum_j \frac{F_j(\xi,u_j)}{|\xi|^{1+\epsilon}}d\xi.$$

Let us assume that inequality does not hold. Then for all k, we have

$$\text{m } \{\psi_j(tA_k): t\geq 0\} \geq \frac{\epsilon}{2(1+2\epsilon)}\frac{A_{1+2\epsilon}}{\alpha_0}. \quad (5.27)$$

Therefore, $\forall k, \exists$ a $t_k > 0$ at which the maximum is attained and

$$\psi_j(t_kA_k) = \frac{t_k^2\|A_k\|^2}{2} - \int_\Omega \sum_j \frac{F_j(\xi,t_kA_k)}{|\xi|^{1+\epsilon}}d\xi \geq \frac{\epsilon}{2(1+2\epsilon)}\frac{A_{1+2\epsilon}}{\alpha_0}. \quad (5.28)$$

Using (5.28), we get

$$\frac{t_k^2\|A_k\|^2}{2} \geq \frac{\epsilon}{2(1+2\epsilon)}\frac{A_{1+2\epsilon}}{\alpha_0} + \int_\Omega \sum_j \frac{F_j(\xi, t_k A_k)}{|\xi|^{1+\epsilon}} d\xi. \tag{5.29}$$

Since $F_j(\xi, 1+\epsilon) \geq 0$, and $\|A_k\|^2 \leq 1$, therefore from (5.29), we obtain

$$t_k^2 \geq \frac{\epsilon}{1+2\epsilon}\frac{A_{1+2\epsilon}}{\alpha_0}. \tag{5.30}$$

Also, for a given $\epsilon \geq 0, \exists R_{1+\epsilon} > 0$ such that for all $|u_j| \geq R_{1+\epsilon}$, we have

$$\sum_j u_j f_j(\xi, u_j) \geq \sum_j (\beta_1 - (1+\epsilon)) \exp(\alpha_0 |u_j|^2) (\text{by(H4)}). \tag{5.31}$$

On using this in (5.29), we obtain

$$t_k^2 \geq (\beta_1 - (1+\epsilon)) \int_{B_{\frac{R}{k}}} \frac{\exp(\alpha_0 |t_k A_k|^2)}{|\xi|^{1+\epsilon}} d\xi$$

$$\geq (\beta_1 - (1+\epsilon)) \frac{w_{2\epsilon}}{\epsilon}\left(\frac{R}{k}\right)^\epsilon \exp\left(\alpha_0 t_k^2 \left(\frac{(1+2\epsilon)\log k}{A_{1+2\epsilon}}\right)^2\right) (\text{by definition of } A_k\Bigg).$$

This implies that

$$1 \geq (\beta_1 - (1+\epsilon)) \frac{w_{2\epsilon} R^\epsilon}{\epsilon} \exp\left[\alpha_0 t_k^2 \left(\frac{\text{Q}\log k}{A_{1+2\epsilon}}\right)^2 - (\epsilon)\log(k) - 2\log(t_k)\right]. \tag{5.32}$$

The above equation shows that t_k is a bounded sequence because if it is not a bounded sequence then up to a subsequence right hand side tends to ∞.

We also have $\|A_k\| \longrightarrow 1$ as $k \longrightarrow \infty$. By definition of A_k, we can also observe that as $k \longrightarrow \infty$, we get

$$A_k(\xi) \longrightarrow 0 \text{ a.e. } \xi \in \Omega. \tag{5.33}$$

Let

$$X_k = \{\xi \in \Omega: t_k A_k \geq R_{1+\epsilon}\}$$

and

$$Y_k = \Omega \setminus X_k,$$

then the characteristic function of $Y_k, \chi_{Y_k} \longrightarrow 1$, a.e. $\xi \in \Omega$. By the Lebesgue dominated convergence theorem, we get

$$\int_{Y_k} \sum_j t_k A_k \frac{f_j(\xi, t_k A_k)}{|\xi|^{1+\epsilon}} dx \longrightarrow 0 \,(\text{ by (5.33)}). \tag{5.34}$$

and

$$\int_{Y_k} \frac{\exp(\alpha_0 |t_k A_k|^2)}{|\xi|^{1+\epsilon}} d\xi \longrightarrow \frac{w_{2\epsilon} R^\epsilon}{\epsilon} \text{ as } k \longrightarrow \infty. \tag{5.35}$$

Since $t_k^2 \geq \frac{\epsilon}{1+2\epsilon}\frac{A_{1+2\epsilon}}{\alpha_0}$, therefore

$$\int_{B_R} \frac{\exp(\alpha_0 |t_k A_k|^2)}{|\xi|^{1+\epsilon}} d\xi$$

$$\geq \int_{B_R} \frac{\exp\left(\frac{\epsilon}{1+2\epsilon} A_{1+2\epsilon} |A_k|^2\right)}{|\xi|^{1+\epsilon}} d\xi$$

$$= \int_{|\xi| \leq \frac{R}{k}} \frac{\exp\left(\frac{\epsilon}{1+2\epsilon} A_{1+2\epsilon} |A_k|^2\right)}{|\xi|^{1+\epsilon}} + \int_{\frac{R}{k} \leq |\xi| \leq R} \frac{\exp\left(\frac{\epsilon}{1+2\epsilon} A_{1+2\epsilon} |A_k|^2\right)}{|\xi|^{1+\epsilon}}$$

$$= \frac{w_{2\epsilon} R^\epsilon}{\epsilon} + R^\epsilon \mathcal{M}.$$

Since

$$t_k^2 \quad \geq (\beta_1 - (1+\epsilon)) \int_{|\xi| \leq R} \frac{\exp(\alpha_0 |t_k A_k|^2)}{|\xi|^{1+\epsilon}} d\xi + \int_{Y_k} \sum_j \frac{t_k A_k f_j(\xi, t_k A_k)}{|\xi|^{1+\epsilon}} d\xi$$

$$-(\beta_1 - (1+\epsilon)) \int_{Y_k} \frac{\exp(\alpha_0 |t_k A_k|^2)}{|\xi|^{1+\epsilon}} d\xi.$$

Therefore

$$\frac{\epsilon}{1+2\epsilon}\frac{A_{1+2\epsilon}}{\alpha_0} \geq (\beta_1 - (1+\epsilon))R^{\epsilon}\mathcal{M}$$

or $\beta_1 \leq \frac{A_{1+2\epsilon}}{R^{\epsilon}\mathcal{M}\alpha_0}\frac{\epsilon}{1+2\epsilon}$, which is a contradiction to (H4). Hence the result holds for $\psi_j(u_j)$.

Now, notice that for $\|\sum_j b(u_j)_k\|_{L^1} \leq \sum_j \| b \|_{L^2} \|(u_j)_k\|_{L^2} \leq \| b \|_{L^2} \sum_j \|(u_j)_k\|$ and for sufficient small $\| b \|_{L^2}$, we find that

$$\text{m } \{J(tA_k): t \geq 0\} < \frac{\epsilon}{2(1+2\epsilon)}\frac{A_{1+2\epsilon}}{\alpha_0}.$$

This completes the proof.

Lemma 5.5 (see [61]). Assume that f_j satisfies critical exponential growth condition (1.7) and (1.8). Let $\{(u_j)_k\} \subseteq S^2_\alpha(\Omega)$ be a Palais-Smale sequence at level c, where c is the mountain pass level for J. Then $\exists u_j \in S^2_\alpha(\Omega)$ and a subsequence of $\{(u_j)_k\}$, still denoted by $\{(u_j)_k\}$ such that

1. $(u_j)_k \to u_j$ in $S^2_\alpha(\Omega)$.
2. $\frac{f_j(\xi,(u_j)_k)}{|\xi|^{1+\epsilon}} \longrightarrow \frac{f_j(\xi,u_j)}{|\xi|^{1+\epsilon}}$ in $L^1(\Omega)$.

Proof. Let $\{(u_j)_k\}$ be a Palais-Smale sequence, then

$$J\left((u_j)_k\right) = \frac{1}{2}\sum_j \|(u_j)_k\|^2 - \int_\Omega \sum_j \frac{F_j\left(\xi,(u_j)_k\right)}{|\xi|^{1+\epsilon}}d\xi - \frac{1}{\gamma+1}\int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma+1}d\xi \longrightarrow c \text{ as } k \longrightarrow \infty \tag{5.36}$$

and

$$|J'((u_j)_k)v_j| = \left|\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j (\mathcal{T})^{\frac{\alpha}{2}}u_j(\mathcal{T})^{\frac{\alpha}{2}}v_j d\xi - \int_\Omega \sum_j \frac{f_j\left(\xi,(u_j)_k\right)}{|\xi|^{1+\epsilon}}v_j d\xi - \int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma-1}\right|(u_j)_k(\xi)v_j(\xi)d\xi \,|\leq \tau_k \| v_j \|. \tag{5.37}$$

Also, by previous Lemma 5.4, we have

$$c < \frac{\epsilon}{2(1+2\epsilon)}\frac{A_{1+2\epsilon}}{\alpha_0}.$$

From (5.36) and (5.37), we get

$$\begin{aligned} c + \tau_{1+2\epsilon}\sum_j \|(u_j)_k\| &\geq \left(\frac{\theta}{2}-1\right)\sum_j \|(u_j)_k\|^2 - \int_\Omega \sum_j \frac{\left(\theta F_j(\xi,(u_j)_k) - f_j(\xi,(u_j)_k)(u_j)_k\right)}{|\xi|^{1+\epsilon}}d\xi \\ &\quad - \left(\frac{\theta}{\gamma+1}-1\right)\int_\Omega \sum_j b(\xi)|(u_j)_k(\xi)|^{\gamma+1}d\xi \\ &\geq \left(\frac{\theta}{2}-1\right)\sum_j \|(u_j)_k\|^2 \ (\,as\ 0 \leq \gamma < 1\ and\ \| b \|\ is\ very\ small\,), \end{aligned}$$

which implies that

$$\begin{cases} \left\| \sum_j (u_j)_k \right\| \le C, \\ \int_\Omega \sum_j \frac{f_j(\xi,(u_j)_k)(u_j)_k}{|\xi,|^d} d\xi \le C, \\ \int_\Omega \sum_j \frac{F_j(\xi,(u_j)_k)(u_j)_k}{|\xi|^d} d\xi \le C. \end{cases} \tag{5.38}$$

Since $S_\alpha^2(\Omega)$ is a reflexive Banach space, therefore by (5.38), up to a subsequence, we have

$$\begin{cases} (u_j)_k \to u_j \text{ in } S_\alpha^2(\Omega), \\ (u_j)_k \longrightarrow u_j \text{ in } L^{1+\epsilon}(\Omega), \forall 0 \le \epsilon \le \infty, \\ (u_j)_k(\xi) \longrightarrow u_j(\xi) \text{ a.e. in } \Omega. \end{cases}$$

Furthermore, using the arguments similar to previous lemma, we get

$$\sum_j \frac{f_j\left(\xi,(u_j)_{\frac{2\epsilon-1}{2}}\right)}{|\xi|^{1+\epsilon}} \longrightarrow \sum_j \frac{f_j(\xi,u_j)}{|\xi|^{1+\epsilon}}.$$

Hence, we proved the desired result.

Proof of Theorem 1.16 (see [61]). By Lemmas 5.1 and 5.2, we can find Palais-Smale sequence $(u_j)_k$ at level c and by Lemma 5.4, $0 < c < \frac{\epsilon}{2(1+2\epsilon)} \frac{A_{1+2\epsilon}}{\alpha}$. Thus, we have

$$J\left((u_j)_k\right) = \frac{1}{2}\sum_j \left\|(u_j)_k\right\|^2 - \int_\Omega \sum_j \frac{F_j\left(\xi,(u_j)_k\right)}{|\xi|^{1+\epsilon}} d\xi$$
$$-\frac{1}{\gamma+1}\int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma+1} d\xi \to c \tag{5.39}$$

and

$$\left|\sum_j J'\left((u_j)_k\right)v_j\right|$$
$$= \left|\int_{\mathbb{H}^{\frac{2\epsilon-1}{2}}} \sum_j \mathcal{T}^{\frac{\alpha}{2}}(u_j)_k \mathcal{T}^{\frac{\alpha}{2}} v_j d\xi - \int_\Omega \sum_j \frac{f_j(\xi,(u_j)_k)}{|\xi|^{1+\epsilon}} v_j d\xi \right.$$
$$\left. - \int_\Omega \sum_j b(\xi)\left|(u_j)_k(\xi)\right|^{\gamma-1}(u_j)_k(\xi)v_j(\xi)d\xi \right| \le \epsilon_k \sum_j \| v_j \|. \tag{5.40}$$

By Lemma 5.5, there exists $u_j \in S_\alpha^2(\Omega)$ such that

1 $(u_j)_k \to u_j \in S_\alpha^2(\Omega)$.

2 $\sum_j \frac{f_j(\xi,(u_j)_k)}{|\xi|^{1+\epsilon}} \longrightarrow \sum_j \frac{f_j(\xi,u_j)}{|\xi|^{1+\epsilon}}$ strongly in $L^1(\Omega)$.

Therefore by (5.40) and with the help of Lebesgue dominated convergence theorem, we can now pass the limit and get

$$J'(u_j)v_j = 0$$

for all $v_j \in \mathcal{D}$. Now, by density of $\mathcal{D}(\Omega)$ in $S_\alpha^2(\Omega)$, u_j is a weak solution of given problem. Next, we show that u_j is non-trivial. We prove it by method of contradiction, let if possible $u_j \equiv 0$, then by (H2) and Lebesgue dominated theorem, one can see that

$$\int_\Omega \sum_j \frac{F_j(\xi,(u_j)_k)}{|\xi|^{1+\epsilon}} d\xi \longrightarrow 0 \text{ in } L^1(\Omega) \text{ as } k \longrightarrow \infty.$$

From (5.39), we obtain

$$\|(u_j)_k\|^2 \longrightarrow 2c < \frac{\epsilon}{1+2\epsilon}\frac{A_{1+2\epsilon}}{\alpha_0}. \tag{5.41}$$

Now, choosing $\epsilon > 0$, sufficiently close to 1 such that

$$\frac{1+2\epsilon}{\epsilon}(1+\epsilon)\alpha_0 \left\| \sum_j (u_j)_k \right\|^{\frac{1+2\epsilon}{2\epsilon-1}} < A_{1+2\epsilon}$$

for k large. Now because of the exponential growth of f_j, we have

$$\int_\Omega \sum_j \frac{|f_j(\xi,(u_j)_k)|}{|\xi|^{1+\epsilon}} d\xi \quad \leq C \int_\Omega \sum_j \exp\left((1+\epsilon)\alpha_0 \|(u_j)_k\|^{\frac{1+2\epsilon}{2\epsilon-1}} \left| \frac{(u_j)_k}{\|(u_j)_k\|} \right|^{\frac{1+2\epsilon}{2\epsilon-1}} \right) d\xi\,,$$
$$\leq O(1) \text{ as } k \to \infty.$$

Thus, by taking $v_j = (u_j)_k$ in (5.40), we get $\|(u_j)_k\|^2 \to 0$ as $k \to \infty$, which is a contradiction to (5.41). This completes the proof.

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