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Research Paper

Fixed point theorem for (ϕ, MF) – contraction on C^* −algebra valued b- metric space

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Abstract: *In this paper, we will introduce a new notion of (ϕ, MF) – contraction in C^* −algebra valued b-metric space and prove some fixed point theorem for the same. An example is also provided to prove the validity of our results.*

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I. Introduction:

Banach [1] contraction principle is a very useful simple and classical tool in modern analysis and plays an important role for solving existence problem in various field of sciences. It is the first principle to get a fixed point for a self mapping on a complete metric space. Many researchers had generalized the Banach contraction principle see [3,4,5,10].

In 2012, D. Wardowski [9] introduce the concept of (ϕ, F) – contraction in metric space and proved some fixed point theorems. In 2022, Mohamed Rossafi [7] proved the fixed point theorem of (ϕ, MF) – contraction in C^* −algebra valued metric space.

In 2015, Ma and Jiang [3] established the notion of C^* −algebra valued b- metric space and proved some fixed point theorem for contractive type mappings.

Throughout this paper, we suppose that $\mathbb{A}$ is a unital C^*- algebra with a unit I_A. Set $\mathbb{A}_h = \{x \in \mathbb{A}: x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succcurlyeq \theta$. Using positive elements, one can define a partial ordering $\preccurlyeq$ on $\mathbb{A}_h$ as follows: $x \preccurlyeq y$ if and only if $y - x \succcurlyeq \theta$, where θ means the zero element in $\mathbb{A}$. Now $\mathbb{A}_+ = \{x \in \mathbb{A} : x \succcurlyeq \theta\}$ and $|x| = (x^*, x)^{\frac{1}{2}}$.

II. Preliminaries:

In this section, we shall give some basic definition which will be used in sequel.

Definition 2.1.**[3]** Let X be a non-empty set $s \succcurlyeq I_A$. Suppose the mapping $d: X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) $d(x,y) \preccurlyeq s[d(x,z) + d(z,y)]$for all $x, y, z \in X$.

Then d is called C^* algebra valued b- metric on X and $(X, \mathbb{A}, d)$ is called C^*- algebra valued b- metric space.

Definition 2.2.[3] Let $(X, \mathbb{A}, d)$ be a C^*- algebra valued b- metric space. Let $\{x_n\}$ be a sequence in X then

(i) $\{x_n\}$ is said to be Cauchy if for all $\theta \preccurlyeq c$, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$

$d(x_n, x_m$

(ii) $\{x_n\}$ is said to be converges to x if for all$\theta \preccurlyeq c$ there is $N \in \mathbb{N}$ such that for all $n \geq N$

(iii)$(X, \mathbb{A}, d)$ is a complete C^*- algebra valued b- metric space if every Cauchy sequence is convergent in X.

Definition 2.3.[9] Let $\mathcal{F}$ be the family of all functions F: $\mathbb{R}_+ \to \mathbb{R}$ and Φ be the family of all the functions $\phi: [0,\infty) \to [0,\infty)$ satisfying:

(i) F is strictly increasing.

(ii) For each sequence $\{x_n\}$ of positive numbers

$\lim_{n\to 0} x_n = 0$ if and only if$\lim_{n\to\infty} F(x_n) = -\infty$

(iii) $\lim\inf_{s\to a^+}\phi(s) > 0$ for all $s > 0$.

(iv) There exists $k \in [0,1]$ such that $\lim_{x\to 0} x^k F(x) = 0$.

Definition 2.4[9] Let (X, d) be a complete metric space. A mapping T: $X \to X$ is called a(ϕ, F) – contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\phi \in \Phi$ such that

$$(d(Tx,Ty)) > 0 \Longrightarrow Fd(Tx,Ty)) + \phi(d(x,y)) \leq F(d(x,y))$$

For all $x, y \in X$ for which $Tx \neq Ty$

Definition 2.5[10] Let (X, d) be a complete metric space. A self-map T: $X \to X$ is called a(ϕ, MF) – contraction on (X, d) if there exists $\tau > 0$ such for $x, y \in X$

$M(Tx,Ty) > 0 \Longrightarrow \tau + F(M(Tx,Ty) \leq F(M(x,y))$, where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$$

Definition 2.6[11] Let the function ϕ: $[0, \infty] \to [0, \infty]$ be positive if having the following constraints :

(i) ϕ is continuous and non- decreasing

(ii) $\phi(a) = \theta$ if and only if $a = \theta$

(iii) $\lim_{n\to\infty} \phi^n(a) = \theta$

Definition 2.7[11] Suppose that A and B are C^* –algebras.

A mappingϕ: $A \to B$ is said to be C^* –homomorphism if :

(i) $\phi(ax + by) = a\phi(x) + b\phi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$

(ii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$

(iii) $\phi(x^*) = \phi(x)^*$ for all $x \in A$ (iv) ϕ maps the unit in A to the unit in B.

III. Main Results:

In this section, we shall prove some fixed- point results for the $(\phi, MF) -$ type contractions in

C^*- algebra valued b- metric space.

Definition 3.1. Let

$$F: \mathbb{A}_+ \to \mathbb{A}$$

be a function satisfying followings

(i) F is continuous and non decreasing.
(ii) $F(t) = \theta$ if and only if $t = \theta$.

A mapping $T : X \to X$ is said to be a (ϕ, MF)- C^* –algebra valued b- contraction if there exist ϕ: $\mathbb{A}_+ \longrightarrow \mathbb{A}$ a mapping such that

$$M(Tx, Ty) \succcurlyeq \theta \Rightarrow F(M(Tx, Ty)) + \phi(M(x, y) \preccurlyeq F(M(x, y)) \text{ for all } x, y \in X \quad (1)$$

Where $M(x, y)$ =max$\{d(x, y), d(x, Tx), d(y, Ty), \frac{\mathrm{d(x,Ty)+d(y,Tx)}}{2s}\}$.

Theorem 3.2. Let $(X, \mathbb{A}, d)$ be a complete C^*- algebra valued b-metric space and let $T : X \to X$ be a (ϕ, MF)- C^* –algebra valued b- contraction mapping.

Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be arbitrary and fixed. Define a sequence $\{x_n\}$ by $x_{n+1} = T\,x_n$ for all $n \in \mathbb{N}$.
Clearly, if $xn+1 = xn$, then x_0 is a fixed point of T and is unique.

Define $d_n = d(x_{n+1}, x_n)$; $n = 0,1,2,3, \ldots ..$

Suppose that $x_{n+1} \neq x_n$ for every $n \in X$ then $d_n \succ \theta$ for all $n \in \mathbb{N}$ and using (1)

$$F(M(Tx_n, Tx_{n+1})) + \phi(M(x_n, x_{n+1})) \preccurlyeq F(M(x_n, x_{n+1}))$$

Where $M(xn, xn+1)$ =max$\{d(xn, xn+1), d(xn, Txn), d(xn+1, Txn+1), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2s}\}$

. $\quad$ =max$\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2s}\}$

=max$\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}}{2s}\}$

Using the triangle inequality, we have

$$\frac{d(x_n, x_{n+2}}{2s} \preccurlyeq \frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s}$$

$$= \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}$$

$\preccurlyeq \max \{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$

Consequently, we drive that

$M(xn, xn+1) = \max \{d(xn, xn+1), d(xn+1, xn+2)\}$.

And

$M(Txn, Txn+1) = \max \{d(xn+1, xn+2), d(xn+2, xn+3)\}$.

If $d(x_n, x_{n+1}) \preccurlyeq d(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}$, then

$M(xn, xn+1) = d(xn+1, xn+2)$ and $M(Txn, Txn+1) = d(xn+2, xn+3)$.

Then

$$F(M(Tx_n, Tx_{n+1})) + \phi(M(x_n, x_{n+1}) \preccurlyeq F(M(x_n, x_{n+1}))$$

Implies

$F(d(x_{n+2}, x_{n+3})) \preccurlyeq F(d(x_{n+1}, x_{n+2})) - \phi(d(x_{n+1}, x_{n+2})) \preccurlyeq F(d(x_{n+1}, x_{n+2}))$which a contradiction.

Now $x_n+1 \neq x_n$ for every $n \in X$then $d_n \succ \theta$ for all $n \in \mathbb{N}$ and using (1) the following holds for every $n \in \mathbb{N}$

$F(d_n) \preccurlyeq F(d_{n-1}) - \phi(d_{n-1}) \prec F(d_{n-1})$ (2)

Hence F is non decreasing and so the sequence (d_n) is monotonically decreasing in $\mathbb{A}$.So there exist $\theta \preccurlyeq t \in \mathbb{A}$ such that

$d(x_n, x_{n+1}) \to t$ as $n \to \infty$ From (2) we obtain $lim F(d_n) = \theta$ that together with (ii) gives

$$\lim_{n\to\infty} d_n = \theta$$

Now we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$.

Let $n, p \in \mathbb{N}$. Then

$$d(x_n, x_n+p) \preccurlyeq s\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})\}$$
$$\preccurlyeq sd(xn, xn+1) + s^2\{d(xn+1, xn+2) + d(xn+2, xp)\}$$

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$\preccurlyeq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \ldots + s^{p-1}d(x_{n+p-1}, x_{n+p})$. Taking the limit as $n \to \infty$ we get

$\lim d(x_n, x_{n+p}) = \theta$.

Thus $\{x_n\}$ is a Cauchy sequence. Since the space is complete,

There exist $x \in X$ such that $\lim x_n = x$.

Again T is continuous. Therefore $\lim Tx_n = Tx$ i.e. $\lim x_{n+1} = x = Tx$.

Thus x is a fixed point of T.

To show the uniqueness, let y be another fixed point of T.

Then by given condition

$$F(M(Tx, Ty)) + \phi(M(x, y)) \preccurlyeq F(M(x, y))$$

$\Rightarrow F(M(x, y)) \preccurlyeq F(M(x, y)) - \phi(M(x, y))$

Which is a contradiction.

Therefore T has a unique fixed point in X.

Example 3.3 Let $X = [0,1]$ and $\mathbb{A} = \mathbb{R}^2$ Then $\mathbb{A}$ is a C_*- algebra valued b-metric space with norm defined by $\|(x, y)\| = (x^2 + y^2)^{1/2}$

And let $d: X \times X \to \mathbb{A}$ on X be defined by

$$d(x, y) = (|x - y|, 0)$$

Then (X, d) is a C_*- algebra valued b-metric space with $s = 2$

A mapping $T: X \to X$ given by $Tx = \frac{x+1}{2}$ is continuous with respect to $\mathbb{A}$.

Let $F: \mathbb{A} \to \mathbb{A}$. Define by

$F(x, y) = (\left(\frac{x-y}{2}\right)^2, 0)$

It is clear that F satisfies (i) and (ii).

Now $M(x, y) = d(x, y)$ and

$(M(Tx, Ty)) = d(Tx, Ty) = d\left(\frac{x+1}{2}, \frac{y+1}{2}\right) = \frac{x+1}{2} - \frac{y+1}{2} = \frac{x-y}{2}$

We have $F(M(Tx, Ty)) = F(d(Tx, Ty)) = F\left(d\left(\frac{x+1}{2}, \frac{y+1}{2}\right)\right) = (\frac{x+1}{2} - \frac{y+1}{2})^2 = (\frac{x-y}{2})^2$

And $\frac{1}{4}\left(\frac{x-y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 \leq -\frac{1}{2}\left(\frac{x-y}{2}\right)^2$.

T satisfies all the condition of (ϕ, MF) contraction with

$$\Phi(M(x,y)) = \frac{1}{2}(\frac{x-y}{2})^2$$

Therefore T has a unique fixed point . Clearly 1 is the unique fixed point of T.

Theorem 3.4 Let $(X, \mathbb{A}, d)$ be a complete C^* - algebra valued b-metric space and let $T: X \to X$ be a (ϕ, MF)- C^* –algebra valued b- contraction of Hardy Rogers type where

$M(x, y) = \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx)$ and $\alpha_i \geq 0, i \in \{1,2,3,4,5\}$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2s\alpha_4 + \alpha_5 < 1.$

Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary and fixed we define a sequence $\{x_n\}$,$x_{n+1} = T\ x_n$ for all $n \in \mathbb{N}$ .Clearly, if $x_{n+1} = x_n$, then x_0 is a fixed point of T and is unique. Now we show that $lim_{n\to\infty} d(x_n, x_{n+1}) = \theta$. We have

$$M(xn, xn+1) = \alpha1d(xn, xn+1) + \alpha2d(xn, Txn) + \alpha3d(xn+1, Txn+1) + \alpha4d(xn, Txn+1) + \alpha5d(xn+1, Txn)$$

$= \alpha1d(xn, xn+1) + \alpha2d(xn, xn+1) + \alpha3d(xn+1, xn+2) + \alpha4d(xn, xn+2) + \alpha_5 d(x_{n+1}, x_{n+1})$

$\preccurlyeq (\alpha_1 + \alpha_2 + s\alpha_4)d(xn, xn+1) + (\alpha3 + s\alpha4)d(xn+1, xn+2),$

and $M(Txn, Txn+1)) = \alpha1d(Txn, Txn+1) + \alpha2d(Txn, T2xn) + \alpha3d(Txn+1, T2xn+1) + \alpha_4 d(Tx_n, T^2x_{n+1}) + \alpha_5(Tx_{n+1}, T^2x_n)$

$= \alpha1d(xn+1, xn+2) + \alpha2d(xn+1, xn+2) + \alpha3d(xn+2, xn+3) + \alpha_4 d(x_{n+1}, x_{n+3}) + d(x_{n+2}, x_{n+2})$

$\preccurlyeq (\alpha_1 + \alpha_2 + s\alpha_4)d(x_{n+1}, x_{n+2}) + (\alpha_3 + s\alpha_4)d(x_{n+2}, x_{n+3}),$

If $d(xn, xn+1) \preccurlyeq d(xn+1, xn+2)$

Then

$M(x_n, x_{n+1}) \preccurlyeq (\alpha_1 + \alpha_2 + \alpha_3 + 2s\alpha_4)d(x_{n+1}, x_{n+2})$

And

$M(Tx_n, Tx_{n+1})) \preccurlyeq (\alpha_1 + \alpha_2 + \alpha_3 + 2s\alpha_4)d(x_{n+2}, x_{n+3})$

Then

$$F((\alpha_1 + \alpha_2 + \alpha_3 + 2s\alpha_4)d(x_{n+2}, x_{n+3})) \preccurlyeq F((\alpha_1 + \alpha_2 + \alpha_3 + 2s\alpha_4)d(x_{n+1}, x_{n+2})) - \phi(\alpha_1 + \alpha_2 + \alpha_3 + 2s\alpha_4)d(x_{n+1}, x_{n+2})$$

Using the property of F and ϕ we have

$F(d(x_{n+1}, x_{n+2})) \preccurlyeq F(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})).$

Since $(\alpha_1 + \alpha_2 + \alpha_3 + 2s\alpha_4) \leq 1.$

There exists $x \in \mathbb{A}$ such that $lim_{n\to\infty} d(x_n, x_{n+1}) = x.$

Taking $n \to \infty$ in $F(d(x_{n+1}, x_{n+2})) \preccurlyeq F(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1}))$

We have $F(x) \preccurlyeq F(x) - \phi(x)$ which is a contradiction unless $x = \theta$.

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta$.

Now we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$.
Let $n, p \in \mathbb{N}$. Then

$$d(x_n, x_{n+p}) \preccurlyeq s\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})\}$$

$$\preccurlyeq sd(x_n, x_{n+1}) + s^2\{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_p)\}$$

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$\preccurlyeq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{p-1} d(x_{n+p-1}, x_{n+p})$. Taking the limit as $n \to \infty$ we get $\lim d(x_n, x_{n+p})$

$= \theta$.

Thus $\{x_n\}$ is a Cauchy sequence. Since the space is complete. Thereexist $x \in X$

such that $\lim x_n = x$.

Again T is continuous. Therefore $\lim_{n\to\infty} Tx_n = Tx$ i.e. $\lim_{n\to\infty} x_{n+1} = x = Tx$.

Thus x is a fixed point of T.

To show the uniqueness, let y be another fixed point of T. Then by given

condition

$$F(M(Tx, Ty)) + \phi(M(x, y)) \preccurlyeq F(M(x, y))$$

$\Rightarrow F(M(x, y)) \preccurlyeq F(M(x, y)) - \phi(M(x, y) \prec F(M(x, y))$

Which is a contradiction.

Therefore T has a unique fixed point in X.

Corollary 3.5. Let $(X, \mathbb{A}, d)$ be a complete C^*- algebra valued b-metric space and let $T : X \to X$ be a (ϕ, MF)- C^* −algebra valued b- contraction of Banach-type, where

$M(x, y) = \alpha d(x, y)$ and $0 < \alpha < 1$.

Then T has a unique fixed point in X.

Corollary 3.6. Let $(X, \mathbb{A}, d)$ be a complete C^*- algebra valued b-metric space and let $T : X \to X$ be a (ϕ, MF)- C^* −algebra valued b- contraction of Kannan-type, where $M(x, y) = s\{\alpha d(x, Tx) + \beta d(y, Ty)\}$ and $0 \leq \alpha + \beta < 1$.

Then T has a unique fixed point in X.

Corollary 3.7. Let $(X, \mathbb{A}, d)$ be a complete C^*- algebra valued b-metric space and let $T : X \to X$ be a (ϕ, MF)-Chatterjea-type C^* –algebra valued b- contraction, where $M(x, y) = s\{\alpha d(x, Ty) + \beta d(y, Tx)\}$and $\forall \alpha, \beta \geq 0, \alpha + \beta < 1$.

Then T has a unique fixed point in X.

Corollary 3.8 Let $(X, \mathbb{A}, d)$ be a complete C^*- algebra valued b-metric space and let $T : X \to X$ be a (ϕ, MF)-Reich-type C^* –algebra valued b- contraction, where

$M(x, y) = s\{\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)\}$ and $\forall \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma < 1$.

Then T has a unique fixed point in X.

References:

[1]. Banach S., Sur les operations dans les ensembles abstraits et leur application aux équations integrals,*Fundam. Math*., **3**(1)(1922), 133–181.

[2]. Barman D., Tiwary K.,Fixed point theorems for generalized F-contraction on metric space, *Sarajevo J. Math.* **17**(2021),119-128.

[3]. Ma Z. and Jiang L. "C^*- Algebra Valued -b Metric Spaces and related fixed point theorems", *Fixed Point Theory and Applications,***2015**(1)(2015): 222.

[4]. Piri H.,Kumam P., Some fixed point theorems concerning F-contraction in complete metric spaces ,*Fixed Point Theory Appl.,***2014** (2014) 210.

[5]. Piri H., RahroviS.,ZarghamiR.,somefixed point theorems on generalized asymmetric metric spaces, *Asian-European J.Math.* **14** (2021) 2150109.

[6]. Piri H., RahroviS.,MarasiH.,Kumam P., F-contraction on asymmetric metric spaces, *J.Math. Computer Sci.* **17** (2017) 32-40.

[7]. Rossafi M., Massit H., Kabbaj S., Fixed point theorem for $((\phi, MF)$-contraction onC^*- algebra valued metric spaces, *Assian J. Math Appl.* **(2022)** 2022 7

[8]. Samet B., Vetro C. and Vetro P., Fixed point theorems for α-ψ- contractive type mappings, *Nonlinear Analysis Theory Methods and Applications,* **75**(2012),21542165.

[9]. WardowskiD.,Fixed points of a new type of contractive mappings in complete metric space, *Fixed Point Theory Appl.,***2012** (2012) 94.

[10]. Wardowski D., Solving existence problems via F-contractions.Proc. Amer. Math. Soc.**146** (2018) 1585-1598.

[11]. Zhu K.,An introduction to operator Algebras, CRC Press, Boca Raton, FL, USA, 1961.