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Review Paper

Functional Random Differential Inclusions with Memory

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Abstract: *In this paper, we discuss the neutral functional random differential inclusions defined on a separable Banach space and depending in a measurable way on a random parameter. We prove an existence result for a class of neutral functional random differential inclusions with memory and establish a condition under which the existence of viable solutions, also the existence of random viable solutions.*

Keywords: *Functional-differential inclusions , random inclusions, measurable selections.*
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I. Introduction

There are two typical methods in proving the existence of random solutions of differential inclusions; in the first one, the measurability of solutions with respect to a random parameter is proved step by step ([8], [9], [10]) in the second one, random fixed point theorems are used ([I]).

In the case of random functional differential inclusions, conditions for the existence of random viable solutions were obtained by Rybinski in [12] and [6]. The used in these papers was to reduce the problem of the existence of random solutions to the related deterministic problem and then to apply a suitable measurable selection theorem.

The purpose of the present paper is to study the existence of solutions for random neutral functional differential inclusions in a separable Banach space.

Based on an earlier result obtained by Benchohra and Ntouyas ([2]), we first prove the existence of random solutions for partial neutral functional differential inclusions with memory governed by convex valued orientor field. Next we derive the existence of the desired random solution to neutral functional differential inclusions with viability condition from the existence of the deterministic solution via a random fixed point principle applied to multivalued operators in the function space $L^p([t_0-\Delta,T],E)$. The idea of applying the random fixed point principle due to Rybinski in the space of derivatives of the solutions belongs to Engl ([5]). and it was already used for obtaining a similar result for random functional differential inclusions ([12]). Our results extend those in [10] and [12] to the case of neutral functional differential inclusions.

II. Preliminary Results

This paper $(E,||\bullet||)$ is a separable Banach space and *P(E)* will stand for the set of all subsets of *E*. If $e\in E$, the distance from the point x to the set $A\subseteq E$ will be denoted by *d(x, A)*. For any *A*, $B\in P(E)$, the Hausdorff distance between *A* and *B* is defined as

$$d_H\left(A,B\right):=\max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\right\}$$

For any topological space S, the script B(S) will stand for the σ-field of Borel subsets of S.

If $I=[a,b]$ is a real interval, let *C(I, E)* be the Banach space of continuous function $x(.):I\to E$ with the norm $\|x(.)\|_\infty=\sup\{\|x(t)\|:t\in I\}$. By $C^1(I,E)$ we denote the space of continuously differentiable mappings *x(.):I→E* endowed with norm $\|x(.)\|_\infty = \|x(.)\| + \|x^1(.)\|$, and *L^p(I, E)* we denote the Lebesgue-Bochner space with norm

$$\|x(.)\|_{L^p} = \left(\int_I \|x(t)\|^p \, dt \right)^{I/p}$$

Consider the integral operator $\Gamma(.) : L^p(I,E) \to C(I,E)$ defined by the Bochner integral

$$\Gamma(y(.))(\mathrm{t}) = \int_a^I y(s)ds.$$

By $W^p(I, E)$ we denote a subspace of *C(I, E)* composed of the elements $x(.) = x(.) + \Gamma(y(.))$ where $x_1(.)$: $1 \to E$ is a constant mapping and $y(.) \in L^p(I, E)$. Clearly, every $x(.) \in W^p(I, E)$ is differentiable almost everywhere with $x'(.) \in L^p(I, E)$ and $x(.) = x(a) + \Gamma(x'(.))$. On this space we consider the norm

$$\|x(.)\|_{wp} = \|x(a)\| + \|x'(.)\|_{L^p}$$

Note that $W^p(I, E)$ is a Banach space with respect to this norm.

Let (Ω, E, μ) be a σ-finite measure space (not necessarily complete) and $L'(\Omega, E)$ be the space of integrable functions $f(.): \Omega \to E$ equipped with the norm $\|f(.)\|_1 = \int_\Omega \|f(\omega)\| d\mu(\omega)$.

Recall that a function $f(.,.): \Omega \times E \to E$ is said to be Caratheodory if $\omega \to f(\omega, x)$ is measurable for any $x \in E$ and $x \to f(\omega. x)$ is continuous for any $\omega \in \Omega$.

Let $F(.): \Omega \to P(E)$ with nonempty, closed values. *F(.)* is said to be (weakly) measurable if any of the following equivalent conditions holds:

i) for any open subset $U \subseteq E$, $\{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$;

ii) for all $x \in E$, $\omega \to d(x, F(\omega))$ is measurable. If, in addition, μ is complete, then the statements i) and ii) above are equivalent to any of the following ones

iii) Graph $(F(.)) := \{(\omega,x) \in \Omega \times E: x \in F(\omega)\} \in \Sigma \otimes B(E)$ (graph measurability);

iv) for any closed subset $C \subseteq E$. $\{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \Sigma$ (strong measurability).

By S_F^1 we will denote the set of Bochner integrable selections of *F(.)*.

$$S_F^1 = \left\{ f(.) \in L^1(\Omega, E) : f(\omega) \in F(\omega) \mu - a.e. \right\}.$$

The following lemmas will be used in the next section.

Lemma 2.1. ([8]) Let (Ω, E, μ) be a σ-finite measure space, *Y* a locally compact separable metric space and Z a metric space. Then $f: \Omega \times Y \to Z$ is a Caratheodory function if and only if $\omega \to g(\omega)(.) := f(\omega,.)$ is measurable as a mapping from Ω to the space *C(Y, Z)* endowed with the compact-open topology.

Lemma 2.2. ([5]) Let $F(.): \Omega \to P(E)$ be weakly measurable and $f(.): \Omega \to E$ be measurable. Then the function $\omega \to d(f(\omega), F(\omega))$ is measurable.

Lemma 2.3. ([12]) Let *X* be a Polish space. Assume that $F(.,.): \Omega \times X \to P(E)$ is weakly measurable and $f(.): \Omega \to X$ is measurable. Then the multivalued map $G(.) : \Omega \to P(E)$, $G(\omega) = (\omega, f(w))$ is weakly measurable.

Lemma 2.4. ([12]) Let $G(.): 1 \to P(E)$ be a weakly measurable set-valued map with closed values and such that

$$\int_l d(0, G(t)^p dt < \infty$$

Then the set $M = \{y(.) \in L^p(1.E) : y(t) \in G(t)$ a.e. $(I)\}$ is nonempty and for every $v(.) \in L^p(I,E)$ one has

$$d(v(.), \mathrm{M} = \left(\int_l d(v(t), \mathrm{G(t)}^p dt \right)^{l/p}$$

Let *Y, Z* be two Hausdorff topological spaces and let $F(.): Y \to P(Z)$ be a multifunction with nonempty, closed values. *F(.)* is said to be upper semicontinuous (u.s.c) if for any open subset $U \subset Z$, , the set $F^+(U) = \{y \in Y : F(y) \subseteq U\}$ is open in *Y*.

Let $I = [t_0, T]$ be a real interval and $0 < \Delta < T - t_0$. Let λ be the Lebesgue measure on *l* and consider the following partial neutral functional differential inclusion

$$\frac{d}{dt}\left[x(t) - f(t, x_1(.)\right] \in F(t, x_1(.)) \lambda - a.e., a, e, (l), \qquad (1)$$

$$x(.)\big|_{[t_0-\Delta t_0]} = x_0(.)\,, \tag{2}$$

where $F(.,.): I \times C([t_0 -\Delta,t_0,], E)\to P(E)$ is a bounded, closed, convex set valued map. $f(.):I\times C([t_0 -\Delta, t_0,]. E)\to E$, $x_0(.):[t_0 -\Delta, t_0,]\to E$ is a given continuous function and for all $t\in I$. x_t, $:[t_0 -\Delta, t_0]\to E$ is a continuous function defined by $x_t(s) = x(t + S – t_0)$. Hence $x_t(.)$ describes the history of the state $x(.)$ from time t-Δ up to the present time t.

Definition 2.5. By a solution of (1)-(2) we mean a continuous function $x(.): [t_0 -\Delta, T]\to E$ such that $t\to x(t) - f(t, x_t (.))$ is absolutely continuous on I, $x(.)\big|_{[t_0-\Delta t_0]} = x_0(.)$ and the inclusion (1) holds a.e. on I.

Hypothesis 2.6.

i) There exist $c_1\in[0, 1)$ and $c2\geq 0$ such that for all $t\in I$ and $y(.)\in C([t_0 -\Delta,t_0]$,E) one has

$$\| f(t, y(.))\| \le c_1 \|y(.)\|_\infty + c_2$$

ii) $f(.,.)$ is completely continuous and for any bounded set $A\subset C([t_0-\Delta, T], E)$, the set $\{t\to f(t, x_1 (.)): x(.) \in A\}$ is equicontinuous in $C(I, E)$.

iii) $F(.,.):I\times C([t_0 -\Delta,t_0.], E)\to P(E)$ has nonempty bounded, closed convex values and is measurable.

iv) For any $t\in I$, $F(t,.)$ is u.s.c. and for each fixed $y(.) \in C([t_0, -\Delta,t_0], E)$,
$S_{F,y(.)} = \{g(.) e L^1(I, E): g(t) \in F(t, y(.))\ a.e.(I)\} \neq \emptyset$.

v) There exist $\varphi(.)L^1(I,\mathbb{R}_+)$ and a continuous and increasing function $\Psi:\mathbb{R}_+\to(0,\infty)$ such that for all $y(.)\in C([t_0-\Delta,t_0.], E)$, $||F(t, y(.)|| := \sup \{||z||: z\in F(t,y(.))\}\leq\varphi(t)\psi(||y(.)||_\infty)$ a.e. on I and

$$\int_{t_0}^{T} \varphi(s)ds < \int_{c}^{\infty} \frac{ds}{\Psi(s)}$$

Where $c = \dfrac{1}{1-c_1}[(1+c_1)\|x_0(.)\|_\infty + 2c_2]$

vi) For each bounded $B\subset C([t_0 -\Delta,t_0], E)$,$y(.)\in B$ and $t\in I$ the set is relatively compact.

In what follows we will need the following theorem.

Theorem 2.7. ([2]) Assume that $F(.,.): I \times C([t_0 -\Delta,t_0], E)\to P(E)$ and $F(.): I\times C(t_0-\Delta,t_0),E)\to E$ satisfy Hypothesis 2.6. Then the Cauchy problem (1)-(2) admits at least one solution.

III. Main Result

We prove two existence theorems for random neutral functional differential inclusions. One is about partial neutral functional differential inclusions defined on $C([t_0-\Delta,t_0],E)$ and the other for neutral functional differential inclusions defined on a subset of $C([t_0-\Delta,T], E)$, namely $W^p (t_0-\Delta,T],E)$. This leads us to what is known in applied mathematics as "viability theory".

Consider the following neutral functional random differential inclusions

$$\left[x(\omega,t) - f(\omega,t,x_1(\omega,.))\right]'' \in F(\omega,t,x_1(\omega,.))\,\mu\times\lambda - \text{a.e.a.e.(I)}\,, \tag{4}$$

$$x(\omega,.)\big|_{[t_0-\Delta,t_0]} = x_0(\omega,.) \quad \forall\omega\in\Omega$$

$$x'(\omega,.)\big|_{[t_0-\Delta,t_0]} = x_1(\omega,.) \quad \forall\omega\in\Omega\,,$$

where $F(.,.): I \times C([t_0 -\Delta,t_0,], E)\to P(E)$ is a bounded, closed, convex set valued map. $f(.):I\times C([t_0 -\Delta, t_0,]. E)\to E$, $x_0(.):[t_0 -\Delta, t_0,]\to E$ is a given continuous function and for all $t\in I$. x_t, $:[t_0 -\Delta, t_0]\to E$ is a continuous function defined by $x_t(s) = x(t + S – t_0)$. Hence $x_t(.)$ describes the historyof the state $x(.)$ from time t-Δ up to the present time t.

Definition 3.1. A solution to the random neutral functional differential inclusions (3)-(4) is a stochastic process $x(.,.):\Omega \times[t_0 -\Delta, t_0]\to E$ with continuous paths (i.e., for all $t\in[t_0, -\Delta, T]$. $x(., 1)$ is measurable and for all $\omega\in \Omega$. $x(\omega,.)\in C([t_0-\Delta, T],E))$ such that $t\to x(\omega,t) - f(\omega,t,x_1(.)$ is absolutely continuous on I, $x(\omega,.)/_{[t0-\Delta,\ t0]}= x_0(w,.)$ for every $\omega\in\Omega$ and the inclusion (3) holds a.e. on $\Omega\times I$.

Assume that the following Hypothesis is satisfied.

Hypothesis 3.2.

i) There exist $c_1 :\Omega \to[0,1)$ and $c_2:\Omega\to[0,\infty]$ both measurable such that for almost all $\omega\in\Omega$, $t\in I$ and $y(.)\in C([t0-\Delta,t_0],E)$ one has

$$\| f(\omega,t, y(.))\| \le c_1(\omega) \|y(.)\|_\infty + c_2(\omega)$$

ii) f(.,.,.,.) is completely continuous, for all $(t, y(.))\in I \times C([t_0-\Delta,t_0],E)$, $\omega\to f(\omega,t,y(.))$ is measurable and for any bounded set $A\subset C([t_0-\Delta, T), E)$ the set $\{t\to f(\omega,t,x,(.)):x(.)\in A\}$ is equicontinuous in $C(1,E)$.

iii) $F(.,.,.,.):\Omega\times I\times C([t_0-\Delta,t_0],E)\to P(E)$ has nonempty bounded, closed convex values and is jointly measurable.

iv) For any $(\omega,t)\in\Omega\times I$, the set-valued map $F(t,.)$ is u.s.c. and for each fixed $\omega\in\Omega$ and $y(.)\in C([t_0-\Delta,t_0],E)$, $S_{w,F,y(.)}:=\{g(.)\in L^1(I,E): g(t)\in F(\omega,t,y(.))$ a.e.(I)$\}\neq\emptyset$.

v) There exist $\varphi(.,.)\in L^1(\Omega\times I,\mathbb{R}_+)$ and a continuous and increasing function $\Psi:\mathbb{R}_+\to(0,\infty)$ such that for all $y(.)\in C([t_0-\Delta,t_0),E)$,

$\|F(\omega,t,y(.))\|\le\varphi(w,t)\Psi(\|y(.)\|_\infty)$ a.e. on $\Omega\times I$ and

$$\int_{t_0}^{T}\varphi(w,s)ds<\int_{c(\omega)}^{\infty}\frac{ds}{\Psi(s)}$$

where $c(\omega)=\dfrac{1}{1-c_1(\omega)}[(1+c_1(w))\|x_0(w,.)\|_\infty+2c_2(w)].$

vi) For each bounded $B\subset C([t_0-\Delta,t_0)],E), y(.)\in B$ and $t\in I$ the set

$$\left\{\int_{t_0}^{T}g(s)ds: g(.)\in S_{w.F.y(.)}\right\}$$

is relatively compact.

Theorem 3.2.

$Let F(.,.,.):\Omega\times I\times C([t_0-\Delta,t_0],E)\to P(E)$ and $f(.,.,.):\Omega\times I\times C([t_0--\Delta,t_0],E)\to E$ as above. Then the Cauchy problem (3)-(4) admits a solution.

Proof. Let $I_1=[t_0-\Delta,T]$ and consider the functions $p:\Omega\times I_1\times C(I_1,E)\times L^1(I,E)\to E$ and $q:\Omega\times C(I_1,E)\times(L^1(I,E)\to\mathbb{R}$ defined respectively by

$$p(\omega,t,x(.),g(.))=\begin{cases} x(t)-x_0(\omega,t) & \text{if } t\in[t_0-\Delta,t_0] \\ x(t)-x_0(\omega,t_0)+f(\omega,t_0,x_0(\omega,.)) & \\ \quad -f(\omega,t,x_1(.))-\int_{t_0}^{t}g(s)ds & \text{if } t\in I\end{cases}$$

$$q(\omega,x(.),g(.))=d(g(.),S^t_{F(\omega,.,.X)})$$

Since $\omega\to x_0(\omega,.)$ is measurable and $f(.,.,.)$ is continuous, we have that $\omega\to p(\omega,t,x(.),g(.))$ is measurable and $(t,x(.))\to p(\omega,t,x(.),g(.))$ is continuous. Applying Lemma III-14 of Castaing and Valadier ([3]) we obtain that $(\omega,t,x(.),g(.))\to p(\omega,t,x(.),g(.))$ is measurable. Let D be a dense subset of $[t_0-\Delta,T]$ and define $p_1:\Omega\times C(I_1,E)\times L^1(I,E)\to E$ by

$$p_1(\omega,x(.),g(.))=\sup_{t\in D}p(\omega,t,x(.),g(.))$$

Then $(\omega,x(.),g(.))=p_1(\omega,t,x(.),g(.))$ is jointly measurable.

With a same reasoning as in the proof of Theorem 2 in [10] we claim that $(\omega,x(.),g(.))=q(\omega,t,x(.),g(.))$ is also measurable.

Now consider the multifunction $R(.):\Omega\to P(C(I_1,E)\times L^1(I,E)$ defined by

$$R(w)=\{(x(.),g(.))\in C(I_1,E)\times L_1(I,E): p_1(\omega,x(.),g(.))=0, q(\omega,x(.),g(.))=0\}$$

From Theorem 2.7 we have that for all $\omega\in\Omega$, $R(\omega)\neq\emptyset$. Using measurability of $P_1(.,.,.)$ and $q(.,.,.)$ we get that Graph (R(.)) is measurable.

Apply Theorem 3 of Saint-Beuve ([13]) to get $\lambda_1:\Omega\to C(I_1,E)$ *and* $\lambda_2:\Omega\to L^1(I_1,E)$ both measurable such that $(\lambda_1(\omega),\lambda 2(\omega))\in R(w)$ $\mu-a.e.(\Omega)$. Set

$$x(\omega,t)=\lambda_1(\omega)(t).$$

Also from Lemma 16 of Dunford and Schwartz ([4]) we get the existence of a function $g(.,.)\in L'(\Omega\times I,E)$ such that

$$g(\omega,t)=\lambda_2(\omega)(t)\mu-\text{a.e.}$$

By Lemma 2.1, $x(.,.)$ and $g(.,.)$ are Caratheodory functions and satisfy

$$x(\omega,t)-f(\omega,t,x_t(\omega,.))=x_0(\omega,t_0)-f(\omega,t_0,x_0(\omega,.)+\int_{t_0}^{t}g(\omega,s)ds\mu-\text{a.e., }\forall\text{t}\in\text{I}$$

$$x(\omega,t)=x_0(\omega,t_0)\forall\text{t}\in\text{I}[t_0-\Delta,t_0$$

So *x(.,.)* is the desired random trajectory which solves the problem (3)-(4).

Remark 3.3. Several remarks are in order.

i) If $f(\omega,t,y(.))\equiv 0$ then our result yields. Theorem 2 in [10] (see also Theorem 3.3 in [6]).

ii) If *F(.,.,.)* and *f(.,.,.)* are constant with respect to the random parameter in the sense that $F(\omega,t,y(.))=F_1(t,y(.))$ and $f(\omega,t,y(.))=f_1(t,y(.))$,then the above theorem yields the result of Benchohra and Ntouyas [2] (Theorem 3.4).

In what follows *X* will denote the space $W^p([t.-\Delta,T),E)$ and *Y* will denote the space $L'([t_0-\Delta,T],E)$. Consider the following random neutral functional differential inclusion without memory

$$\frac{d}{dt}x(\omega,t)\in F(\omega,t,x(\omega,.),\frac{d}{dt}x(\omega,.))\mu\times\lambda-\text{a.e.(I)},\tag{5}$$

$$x(\omega,.)\Big|_{[t_0-\Delta,t_0]}=x_0(\omega,.)\forall\omega\in\Omega\tag{6}$$

$$x(\omega,.)\in\text{K}(\omega)\forall\omega\in\Omega\tag{7}$$

Where

$F(.,.,.):\Omega\times I\times X\times Y\to P(E),\ K(.):\Omega\to P(X)$ and $x_0(.,.):\Omega\times[t_0-\Delta,t_0]\to E$ is a measurable function with $x_0(\omega,.)\in K(\omega)$ for each $w\in\Omega$.

Example 3.4.

The model of the endotoxin tolerance is a good example of uncertainty. The mathematical model ([14]) considers a non-autonomous first order differential system of two equations with two unknowns. i.e. the concentrations of a TNF- α pro-inflammatory cytokine and of the inhibitor of the cytokines, denoted by *x* and *y*, respectively. The system is:

$$\frac{dx}{dt}+A(t).D_1\frac{x^3+E_1^3}{x^3+1}.\frac{1}{F_1y+1}-x$$

(8)

$$\frac{dx}{dt}+A(t).F_2\frac{y^2+E_2^2}{y^2+1}.\text{D}_2(t)y$$

where *A(.)* represents the LPS endotoxin concentration and $D_2(.)$ is the clearance rate of the inhibitor and is supposed to be a time-dependent function. All the variables and parameters are non negative. We may suppose for the beginning that D_2 is our uncertain variable.

Using differential inclusions as the main modeling tool, both the steadystate behaviour and also the transient behaviour of the system in question may be covered by the information obtained from reachability computation in the setvalued context.

Let *U* be the set of control parameters, i.e

$$U:=\{(A(t),(D_2):t\in[t_0,T],D_2\in[0.1,4]\},\quad T>t_0\geq 0$$

And define the multifunction $G(.,.):[t_0,T]\times\square^2\to p(\square^2)$ by

$$G(t,(\text{x},\text{y}))=\left\{\left(A(t).D_1\frac{x^3+E_1^3}{x^3+1}.\frac{1}{F_1y+1}-x,A(t).F_2.\frac{y^2+E_2^2}{y^2+1}-D_2y\right):(A(t),D_2)\in U\right\}$$ We

associate to the system (8) the following differential inclusion

$$z^1(t)\in G(t,z(t))$$

$$z(t_0)=z_0\tag{9}$$

The reason to treat the uncertain variables in a non probabilistic way is that the analysis of the reachable set of inclusion (9) gives us informations about possible extreme values. This also may be useful if we expect that the uncertain variable D, (or other variables) could be intentionally generated to move the system to the extreme values. A detailed discussion will be made in a future work.

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