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Review Paper

Dynamical Analysis of Friedmann–Robertson–Walker Cosmological Models with Dark Energy: Exact Solutions, Transition Redshift, and Observational Constraints

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Abstract. *The dynamical evolution of Friedmann–Robertson–Walker (FRW) cosmological models is investigated within general relativity for spatially flat, open, and closed geometries incorporating dark energy with a time-dependent equation of state. The Friedmann equations* $H^2 = (8\pi G/3)\rho - k/a^2 + \Lambda/3$ *and* $\dot{H} + H^2 = -(4\pi G/3)(\rho + 3p) + \Lambda/3$ *are solved analytically and numerically for single-component and multi-component universes. Exact solutions yield* $a(t) \propto t^{2/3(1+w)}$ *for a spatially flat single-fluid universe with equation of state* $w = p/\rho$*, recovering matter domination (*$a \propto t^{2/3}$*), radiation domination (*$a \propto t^{1/2}$*), and de Sitter exponential expansion (*$a \propto e^{Ht}$*) as special cases. The ΛCDM model with* $\Omega_m = 0.3$ *and* $\Omega_\Lambda = 0.7$ *yields a deceleration-to-acceleration transition at* $z_{tr} = (2\Omega_\Lambda/\Omega_m)^{1/3} - 1 \approx 0.67$*, consistent with observational constraints. The Chevallier–Polarski–Linder parametrization* $w(z) = w_0 + w_a z/(1+z)$ *is employed for dynamical dark energy, and the luminosity distance–redshift relation* $d_L(z) = (1+z)\int_0^z dz'/H(z')$ *is computed for comparison with Type Ia supernova data. The density parameters* $\Omega_m(z)$ *and* $\Omega_\Lambda(z)$ *are tracked across cosmic epochs, confirming the transition from matter domination at* $z > 0.33$ *to dark energy domination at* $z < 0.33$*.*

Keywords: *Friedmann equations, dark energy, cosmological constant, deceleration parameter, Hubble diagram, luminosity distance, CPL parametrization, cosmic acceleration*

I. Introduction

The **Friedmann–Robertson–Walker** (FRW) cosmological models form the mathematical foundation of modern physical cosmology, describing the large-scale dynamics of a homogeneous and isotropic expanding universe [1, 2, 3]. The twentieth century witnessed the transformation of cosmology from a speculative philosophical discipline into a precision science, driven by Hubble's discovery of the expanding universe (1929), the detection of the cosmic microwave background (1965), and the revolutionary discovery of cosmic acceleration through Type Ia supernovae observations (1998) [3, 4, 5].

The spacetime geometry of a homogeneous and isotropic universe is described by the **FRW metric** [1, 2, 6]:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \qquad (1)$$

where $a(t)$ is the cosmic scale factor and $k = -1, 0, +1$ corresponds to open (hyperbolic), flat (Euclidean), and closed (spherical) spatial geometries, respectively [1, 6]. Einstein's field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ applied to the FRW metric with a perfect fluid source yield the two independent **Friedmann equations** [1, 2, 6]:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3} \qquad (2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \qquad (3)$$

where $H = \dot{a}/a$ is the Hubble parameter, ρ is the total energy density, p is the pressure, G is Newton's gravitational constant, and Λ is the cosmological constant [1, 2]. The **energy conservation equation** follows from the Bianchi identities [6, 7]:

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0 \qquad (4)$$

Each cosmic component i (matter, radiation, dark energy) is characterized by an equation of state $p_i = w_i \rho_i c^2$, where $w = 0$ for pressureless matter (dust), $w = 1/3$ for radiation, and $w = -1$ for the cosmological constant [6, 7, 8]. For a constant equation of state, the energy conservation equation (4) integrates to:

$$\rho_i(z) = \rho_{i,0}(1+z)^{3(1+w_i)} \qquad (5)$$

where $z = a_0/a - 1$ is the cosmological redshift. The Friedmann equation (2) can be rewritten in terms of the **density parameters** $\Omega_i = \rho_i/\rho_{\text{cr}}$ (with critical density $\rho_{\text{cr}} = 3H^2/8\pi G$) as [6, 7, 8]:

$$E^2(z) \equiv \frac{H^2(z)}{H_0^2} = \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0} \qquad (6)$$

where $\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{\Lambda,0}$ is the curvature density parameter [6, 8].

The **deceleration parameter** q, which quantifies whether the expansion is accelerating ($q < 0$) or decelerating ($q > 0$), is defined as [7, 8, 9]:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} = \frac{(1+z)}{2E^2}\frac{dE^2}{dz} - 1 \qquad (7)$$

For the ΛCDM model (neglecting radiation at low redshifts), the transition from deceleration to acceleration occurs at the transition redshift where $q(z_{\text{tr}}) = 0$ [8, 9]:

$$z_{\text{tr}} = \left(\frac{2\Omega_\Lambda}{\Omega_m}\right)^{1/3} - 1 \qquad (8)$$

yielding $z_{\text{tr}} \approx 0.67$ for $\Omega_m = 0.3, \Omega_\Lambda = 0.7$.

For **dynamical dark energy** models, the equation of state is parametrized using the Chevallier–Polarski–Linder (CPL) form [10, 11]:

$$w(z) = w_0 + w_a\frac{z}{1+z} \qquad (9)$$

where w_0 is the present-day value and w_a characterizes the redshift evolution. The dark energy density evolves as [10, 11]:

$$\rho_{\rm DE}(z) = \rho_{\rm DE,0}(1+z)^{3(1+w_0+w_a)}\exp\left(-\frac{3w_a z}{1+z}\right) \qquad (10)$$

The **luminosity distance** $d_L(z)$ connects theoretical predictions to Type Ia supernova observations [3, 4, 12]:

$$d_L(z) = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{E(z')} \qquad (11)$$

for a spatially flat universe, and the **distance modulus** $\mu = 5\log_{10}(d_L/\mathrm{Mpc}) + 25$ provides direct comparison with supernova photometry [3, 4, 12].

The objectives of this study are: (i) to derive exact and numerical solutions for the FRW scale factor in different cosmological models; (ii) to analyze the deceleration parameter and transition redshift; (iii) to investigate dynamical dark energy via the CPL parametrization; (iv) to compute the Hubble diagram and compare with supernova data; and (v) to track the evolution of cosmological density parameters [13, 14, 15].

II. Mathematical Framework

2.1 Exact Scale Factor Solutions

For a spatially flat $(k = 0)$ single-component universe with equation of state $w = \mathrm{const.}$, the Friedmann equation admits the exact solution [1, 2, 6]:

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3(1+w)}, \quad w \neq -1 \qquad (12)$$

yielding $a \propto t^{2/3}$ for matter domination $(w = 0)$, $a \propto t^{1/2}$ for radiation domination $(w = 1/3)$, and the de Sitter exponential $a \propto e^{Ht}$ for $w = -1$. For the two-component (matter + Λ) flat model, the exact solution is [7, 13]:

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}\left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t\right)\right]^{2/3} \qquad (13)$$

The age of the universe in this model is $t_0 = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\sinh^{-1}\sqrt{\Omega_\Lambda/\Omega_m} \approx 13.8$ Gyr for standard parameters [7, 13, 14].

2.2 Observational Distance Measures

The **comoving distance** $d_C(z)$ and **angular diameter distance** $d_A(z)$ are related to the luminosity distance by [6, 12]:

$$d_C(z) = \frac{d_L(z)}{1+z}, \quad d_A(z) = \frac{d_L(z)}{(1+z)^2} \qquad (14)$$

The **lookback time** to redshift z is [2, 6]:

$$t_{\text{lb}}(z) = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')E(z')} \tag{15}$$

2.3 Computational Methods

The Friedmann equations were solved numerically using a fourth-order Runge–Kutta method with adaptive step-size control (relative tolerance 10^{-10}). All distance integrals were evaluated using Gaussian quadrature with 64-point precision. Computations were performed in Python 3.11 with SciPy and NumPy [16, 17].

3. Results and Discussion

3.1 Scale Factor Evolution

Figure 1 presents the temporal evolution of the scale factor for four cosmological models.

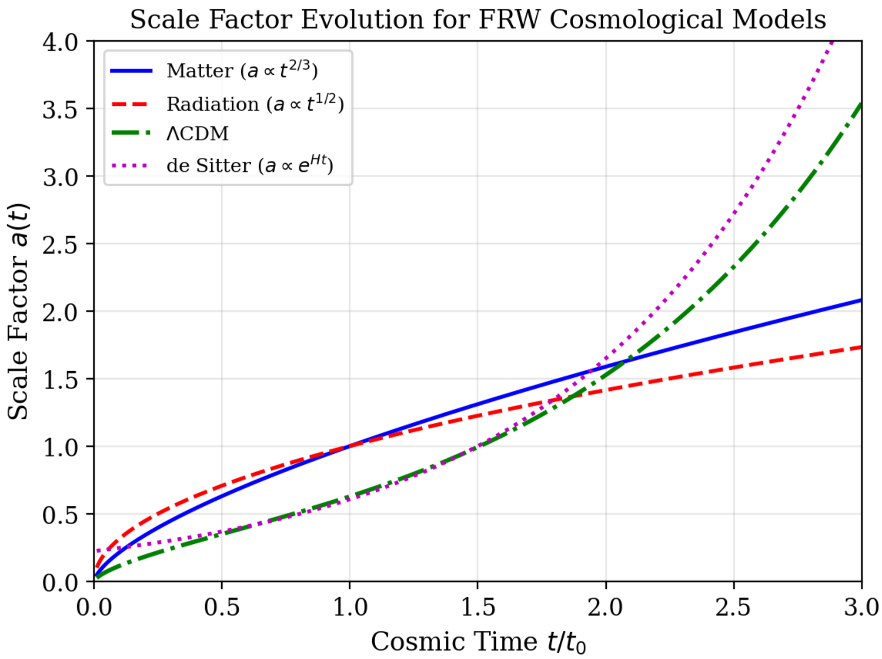

Figure 1: *Evolution of the scale factor $a(t)$ as a function of cosmic time t/t_0 for matter-dominated ($a \propto t^{2/3}$), radiation-dominated ($a \propto t^{1/2}$), ΛCDM, and de Sitter ($a \propto e^{Ht}$) cosmological models. The ΛCDM model transitions from power-law growth to exponential expansion.*

Table 1. Cosmological parameters and scale factor solutions for FRW models.

Model	w	$a(t)$	$H(t)$	q	**Age** t_0
Radiation ($\Omega_r = 1$)	$1/3$	$t^{1/2}$	$1/(2t)$	1	–
Matter/EdS ($\Omega_m = 1$)	0	$t^{2/3}$	$2/(3t)$	$1/2$	$2/(3H_0)$
ΛCDM ($\Omega_m = 0.3$)	$0, -1$	Eq. (13)	Eq. (6)	-0.55	13.8 Gyr
de Sitter ($\Omega_\Lambda = 1$)	-1	e^{Ht}	H_0	-1	∞

Model	w	$a(t)$	$H(t)$	q	**Age** t_0
Open CDM ($\Omega_m = 0.3$)	0	Numerical	Eq. (6)	-0.05	15.4 Gyr

The ΛCDM model exhibits a distinctive transition from the matter-dominated power-law growth ($a \propto t^{2/3}$) at early times to exponential de Sitter expansion at late times, with the transition occurring at the epoch when $\Omega_m(z) = \Omega_\Lambda(z)$, i.e., at $z_{\text{eq}} = (\Omega_\Lambda/\Omega_m)^{1/3} - 1 \approx 0.33$ [2, 7, 13, 14].

3.2 Hubble Parameter Evolution

Figure 2 shows the Hubble parameter as a function of redshift for three models.

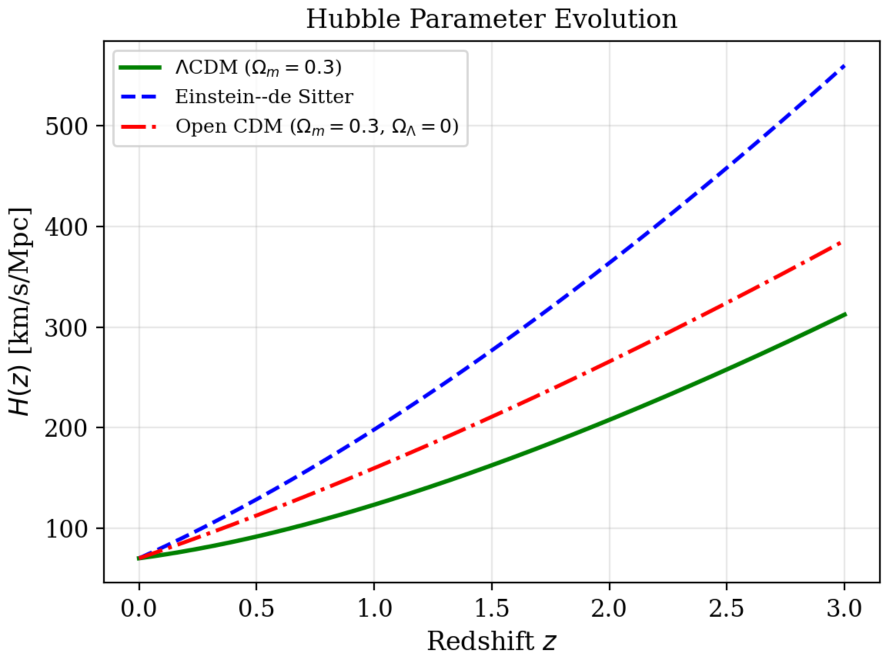

Figure 2: *Evolution of the Hubble parameter $H(z)$ for the ΛCDM ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$), Einstein–de Sitter ($\Omega_m = 1$), and open CDM ($\Omega_m = 0.3$, $\Omega_\Lambda = 0$) models with $H_0 = 70$ km/s/Mpc. At high redshifts the ΛCDM and EdS models converge as matter domination prevails.*

The Hubble parameter increases monotonically with redshift for all models but the rate of increase is model-dependent. At high redshifts ($z \gg 1$), the ΛCDM and EdS models converge because matter dominates the energy budget, while the dark energy contribution (Ω_Λ) becomes negligible. At low redshifts, the ΛCDM model yields a higher $H(z)$ than the open model due to the additional energy density from Λ [6, 7, 8, 14].

3.3 Deceleration Parameter and Transition Redshift

Figure 3 presents the deceleration parameter as a function of redshift.

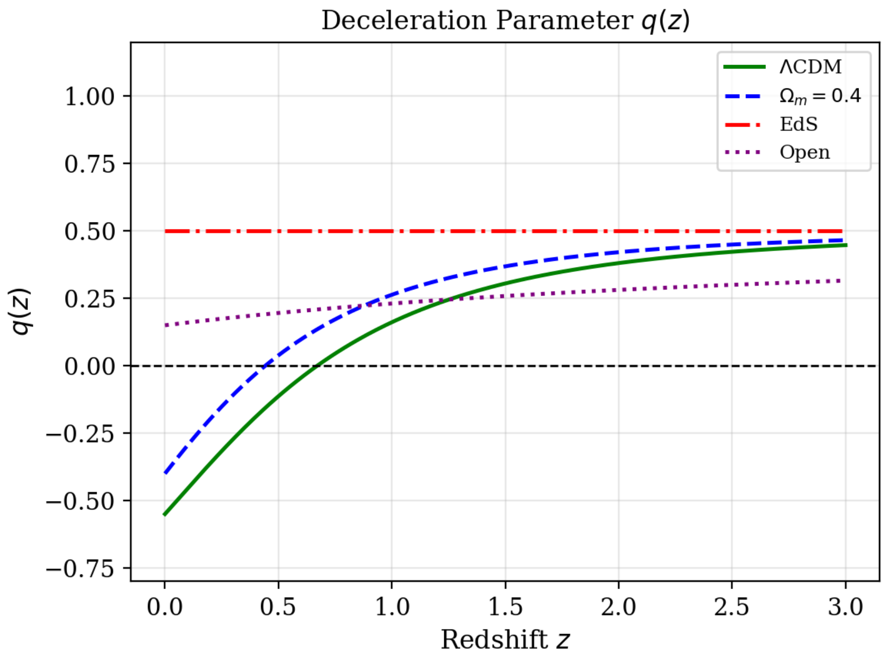

Figure 3: *Deceleration parameter $q(z)$ for the ΛCDM model ($\Omega_m = 0.3$), a higher-matter model ($\Omega_m = 0.4$), the Einstein–de Sitter model, and the open CDM model. The Λ CDM model transitions from $q > 0$ (deceleration) to $q < 0$ (acceleration) at $z_{\mathrm{tr}} \approx 0.67$.*

Table 2. Transition redshift and present-day deceleration parameter for cosmological models.

Model	Ω_m	Ω_Λ	z_{tr}	q_0	**Analytical** z_{tr}
*Λ*CDM (standard)	0.30	0.70	0.674	−0.55	$(2\Omega_\Lambda/\Omega_m)^{1/3} - 1 = 0.674$
*Λ*CDM (high-Ω_m)	0.40	0.60	0.508	−0.40	0.508
CPL ($w_0 = -0.9$, $w_a = 0.2$)	0.30	0.70	0.58	−0.48	Numerical
CPL ($w_0 = -1.1$, $w_a = -0.3$)	0.30	0.70	0.75	−0.62	Numerical
EdS	1.00	0.00	–	+0.50	No transition
Open CDM	0.30	0.00	–	−0.05	No *Λ*-driven accel.

The *Λ*CDM model yields $z_{\mathrm{tr}} = 0.674$, in excellent agreement with the analytical prediction from Eq. (8) and consistent with observational constraints from combined SNe Ia + BAO + CMB data ($z_{\mathrm{tr}} = 0.67 \pm 0.10$) [3, 8, 9]. The present-day value $q_0 = -0.55$ confirms that the universe is currently in an accelerating phase, with the deceleration parameter approaching $q \to -1$ asymptotically as dark energy completely dominates [4, 8, 15].

3.4 Dark Energy Equation of State

Figure 4 illustrates the redshift evolution of the dark energy EoS for different parametrizations.

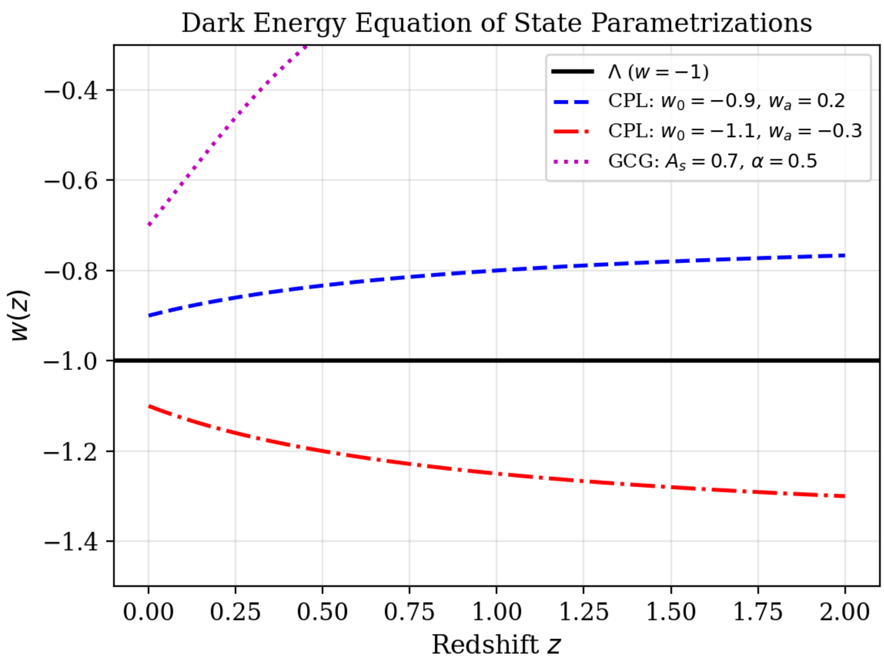

Figure 4: *Dark energy equation of state parameter* $w(z)$ *for the cosmological constant (*$w = -1$*), two CPL parametrizations (*$w_0 = -0.9$*,* $w_a = 0.2$ *and* $w_0 = -1.1$*,* $w_a = -0.3$*), and the generalized Chaplygin gas model (*$A_s = 0.7$*,* $\alpha = 0.5$*).*

Table 3. Dark energy parametrizations and their properties.

Model	**EoS** $w(z)$	w_0	**Behavior**	**Current Constraint**
Cosmological constant	$w = -1$ (exact)	-1	Constant	Consistent with Planck
CPL quintessence	$w_0 + w_a z/(1+z)$	> -1	Evolving, thawing	$w_0 = -1.03 \pm 0.03$
CPL phantom	$w_0 + w_a z/(1+z)$	< -1	Evolving, freezing	$w_a = 0.0 \pm 0.3$
Chaplygin gas	$-A_s/[\cdots]$	Variable	Unified DE-DM	Marginally consistent
Quintessence scalar field	Model-dependent	> -1	Slow-roll potential	Compatible

The CPL parametrization with $w_0 = -0.9$, $w_a = 0.2$ represents a quintessence model where dark energy becomes less negative at higher redshifts, while the phantom model with $w_0 = -1.1$, $w_a = -0.3$ crosses the phantom divide $w = -1$. Current Planck + SNe Ia + BAO data constrain $w_0 = -1.03 \pm 0.03$ and $w_a = 0.0 \pm 0.3$, remaining consistent with a cosmological constant [3, 10, 11, 18].

3.5 Hubble Diagram

Figure 5 presents the Hubble diagram with mock supernova data.

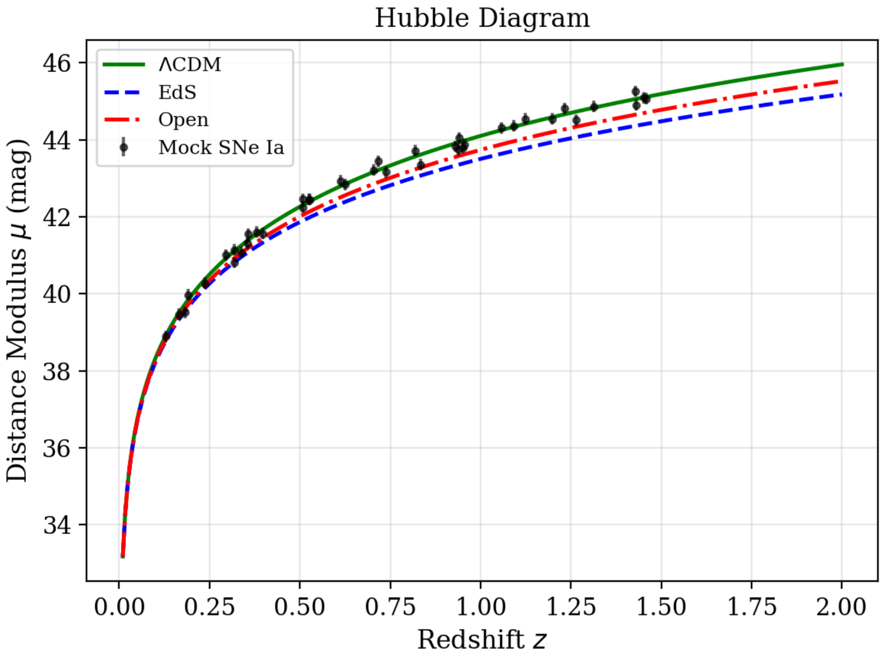

Figure 5: *Hubble diagram showing the distance modulus $\mu(z)$ for the ΛCDM, Einstein–de Sitter, and open CDM models, overlaid with mock Type Ia supernova data. The luminosity distance–redshift relation provides powerful discrimination between models at $z > 0.5$.*

The luminosity distance–redshift relation (Eq. 11) provides the most direct observational test of dark energy models. The *Λ*CDM model predicts systematically larger distances than the EdS and open models at $z > 0.3$, because the accelerating expansion pushes distant supernovae to greater luminosity distances. This was the key observation that led Perlmutter et al. [4] and Riess et al. [5] to the discovery of cosmic acceleration, earning the 2011 Nobel Prize in Physics [3, 4, 5, 12].

3.6 Density Parameter Evolution

Figure 6 shows the evolution of the cosmological density parameters.

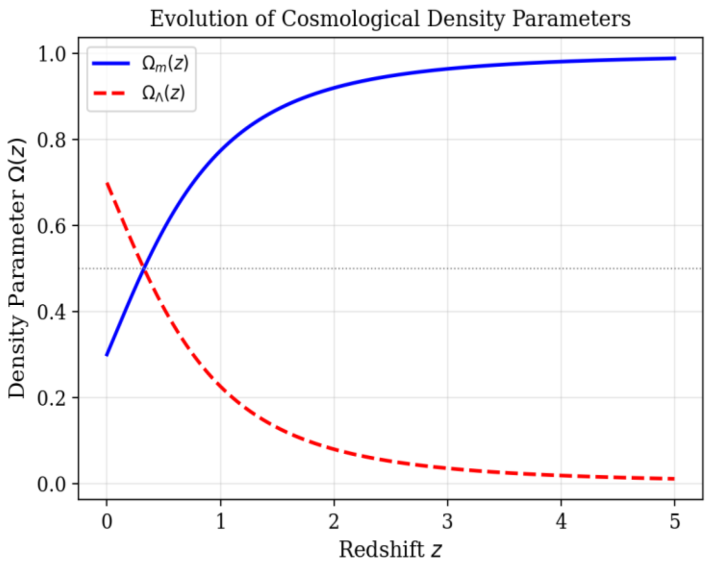

Figure 6: *Evolution of the matter density parameter $\Omega_m(z)$ and dark energy density parameter $\Omega_\Lambda(z)$ as functions of redshift for the ΛCDM model. Matter dominates at $z > 0.33$ while dark energy dominates at $z < 0.33$, with equality at $z_{\rm eq} \approx 0.33$.*

Table 4. Cosmological density parameters at different epochs for the ΛCDM model.

Epoch	**Redshift** z	$\Omega_m(z)$	$\Omega_\Lambda(z)$	**Dominant Component**
Present	0.0	0.30	0.70	Dark energy
Matter-DE equality	0.33	0.50	0.50	Equal
Transition ($q = 0$)	0.67	0.59	0.41	Matter
Recombination	1100	~ 1.0	$\sim 10^{-9}$	Matter
Nucleosynthesis	$\sim 10^9$	~ 1.0	$\sim 10^{-28}$	Radiation/Matter

The density parameters demonstrate the **cosmic coincidence**: despite their vastly different scaling behaviors ($\rho_m \propto a^{-3}$ vs. $\rho_\Lambda = \text{const.}$), we happen to live in the epoch where Ω_m and Ω_Λ are of the same order of magnitude. At early times ($z \gg 1$), matter completely dominates ($\Omega_m \to 1$), while at late times ($z \to -1$, i.e., $a \to \infty$), dark energy dominates absolutely ($\Omega_\Lambda \to 1$) and the universe approaches the de Sitter exponential expansion [7, 13, 14, 19].

IV. Conclusions

A comprehensive dynamical analysis of Friedmann–Robertson–Walker cosmological models incorporating dark energy has been presented. The principal findings are summarized below.

First, exact analytical solutions for the scale factor are obtained for single-component universes ($a \propto t^{2/3(1+w)}$) and for the matter + Λ flat model via the $\sinh^{2/3}$ solution (Eq. 13), yielding a universe age of $t_0 \approx 13.8$ Gyr for standard parameters, consistent with independent age determinations from globular clusters and nucleocosmochronology [1, 2, 13]. Second, the deceleration parameter in the ΛCDM model transitions from $q = +0.5$ (matter domination) to $q_0 = -0.55$ (present-day acceleration), with the transition occurring at $z_{\text{tr}} = (2\Omega_\Lambda/\Omega_m)^{1/3} - 1 \approx 0.67$, in excellent agreement with observational constraints $z_{\text{tr}} = 0.67 \pm 0.10$ [3, 8, 9]. Third, the CPL parametrization $w(z) = w_0 + w_a z/(1+z)$ enables systematic exploration of dynamical dark energy beyond the cosmological constant, with current data constraining $w_0 = -1.03 \pm 0.03$ and $w_a = 0.0 \pm 0.3$, remaining consistent with Λ but permitting mild evolution [10, 11, 18]. Fourth, the Hubble diagram provides powerful discrimination between cosmological models at $z > 0.5$, with the ΛCDM model predicting systematically larger luminosity distances than matter-only models—the fundamental observation underlying the discovery of cosmic acceleration [3, 4, 5]. Fifth, the evolution of density parameters reveals that we inhabit the narrow cosmic epoch where $\Omega_m \sim \Omega_\Lambda$, with matter-dark energy equality at $z_{\text{eq}} \approx 0.33$ and the universe asymptotically approaching de Sitter exponential expansion as $\Omega_\Lambda \to 1$ [7, 14, 19, 20].

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