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Review Paper

Various Unimodular Fourier Multipliers on (Weighted) Wiener Amalgam Spaces

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Abstract
We study, following the authors in [27], the estimates boundedness of unimodular Fourier multipliers on Wiener amalgam spaces. For a real-valued homogeneous function μ on $\mathbb{R}^{1+\epsilon}$ of degree $\epsilon \geq 0$, we also show the estimate boundedness of the operator $e^{i\mu(D)}$ between the weighted Wiener amalgam spaces$W_s^{1+\epsilon,1+2\epsilon}$ and $W^{1+\epsilon,1+2\epsilon}$ for all $0 \leq \epsilon \leq \infty$ and $s > (1+\epsilon)\left[\epsilon\left|\frac{1-\epsilon}{2(1+\epsilon)}\right| + \left|\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\right|\right]$. This inference is shown to be optimal for regions $max\left(\frac{1}{1+2\epsilon},\frac{1}{2}\right) \leq \frac{1}{1+\epsilon}$ and $min\left(\frac{1}{1+2\epsilon},\frac{1}{2}\right) \geq \frac{1}{1+\epsilon}$.Hence, we show sufficient conditions for the boundedness of $e^{i\mu(D)}$ on $W^{1+\epsilon,1+2\epsilon}$ for $0 < \epsilon < 2$.

***Keywords**: boundedness of unimodular Fourier; Wiener amalgam spaces; real-valued homogeneous function; estimate boundedness; regions max.*

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I. Introduction

A Fourier multiplier $\sigma(D)$ in $\mathbb{R}^{1+\epsilon}$ is an operator whose action on a sequence of test functionsf_r is formally defined by

$$\sigma(D)f_r(x) = \frac{1}{(2(1+\epsilon))^{1+\epsilon}}\int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{i\xi\cdot x}\sigma(\xi)\hat{f}_r(\xi)d\xi.$$

The function σ is called the symbol of the multiplier or simply the multiplier. One can rewrite this operator as a convolution operator

$$\sigma(D)f_r(x) = \check{\sigma} * f_r(x),$$

where $\check{\sigma}$ is the (distributional) inverse Fourier transform. These operators are closely related to bounded translation invariant operators(see [14, 15]) and have immense applications to PDEs (see [5, 17, 20]).

In particular, unimodular Fourier multipliers $\sigma(D) = e^{i|D|^{2+\epsilon}}$ arise naturally as formal solutions for Cauchy problem for dispersive equations. The cases $(2+\epsilon) = 1, 2, 3$ are of greatest interest because they correspond to theresearch in wave equation, Schrödinger equation and Airy equationin order. Estimates boundedness of these multipliers on a particular space S means that the S-properties of the initial condition are preserved by time evolution.

The study of Fourier multipliers (problems) is to relate the boundedness of $\sigma(D)$ on certain spaces to that of the properties of the symbol σ. In $L^{1+\epsilon}$ the full resolution of this study is known as the Hörmander-Mihlin multiplier theorem (see [15]).Unfortunately, unimodular Fourier multipliers excludes the use of Hörmander-Mihlin due to singularity of the derivatives at the origin and large derivatives at infinity. In fact, the operator $e^{i|D|^{2+\epsilon}}$ is bounded on $L^{1+\epsilon}$ if and only if $\epsilon = 1$ (see [16]). In view of this unboundedness in $L^{1+\epsilon}$,unimodular Fourier multipliers are studied in [3] and [18] in more suitable space, the modulation space $M^{1+\epsilon,1+2\epsilon}$, where they proved boundedness. Now we say thatmodulation spaces are defined by measuring the time-frequency concentration of functions or distributions in the time-frequency plane. Concrete definition of modulation spaces will be given in the nextsection.

In [3] they showed the boundedness of $e^{i|D|^{2+\epsilon}}$ on $M^{1+\epsilon,1+2\epsilon}$ for $0 \leq \epsilon \leq 2$(the result for $\epsilon = 0$ has been already known before from [25]). An extention of this result was given in [18] which is stated as follows: for $\epsilon > 0$ the Fourier multiplier operator $e^{i|D|^{2+\epsilon}}$ is bounded between the weighted modulation space $M_s^{1+\epsilon,1+2\epsilon}$ and $M^{1+\epsilon,1+2\epsilon}$, for $s \geq (\epsilon)(1+\epsilon)|\frac{1-\epsilon}{2(1+\epsilon)}|$. Here $M_s^{1+\epsilon,1+2\epsilon} = \{f_r \in S'(\mathbb{R}^{1+\epsilon}) : (1-\Delta)^{\frac{s}{2}} f_r \in M^{1+\epsilon,1+2\epsilon}\}$ and s represents loss of derivatives.

We study following, J. Cunaran M. Sugimoto [27], the boundedness of the unimodular Fourier multipliers $e^{i|D|^{2+\epsilon}}$ with $\epsilon > 0$ on Wiener amalgam spaces $W^{1+\epsilon,1+2\epsilon}$. Similar to modulation spaces, Wiener amalgam spaces have been recognized to be appropriate in studying PDE problems since they treat local and global beheaviour of functions separately. There exist an important relationship between modulation spaces and Wiener amalagam spaces, namely, $W^{1+\epsilon,1+2\epsilon} = \mathcal{F}M^{1+2\epsilon,1+\epsilon}$. It is good to remark that boundedness of Fourier multipliers in $W^{1+\epsilon,1+2\epsilon}$ is equivalent to boundedness of pointwise multipliers in $M^{1+2\epsilon,1+\epsilon}$ and vice versa.

The main result is stated as follows.

Theorem 1.1 (see [27]). Let $\epsilon \geq 0$ and μ be a real-valued homogeneous function on $\mathbb{R}^{1+\epsilon}$ of degree $(2+\epsilon)$ which belongs to $C^\infty(\mathbb{R}^{1+\epsilon}\backslash\{0\})$. Let $0 \leq \epsilon \leq \infty$ and $s \in \mathbb{R}$. Then Fourier multiplier operator $e^{i\mu(D)}$ is bounded from $W_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$ to $W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$ whenever

$$s > (1+\epsilon)\left[\epsilon\left|\frac{1-\epsilon}{2(1+\epsilon)}\right| + \left|\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\right|\right].$$

We note the analogy of this theorem with the result in [18] for the case on Wiener amalgam spaces.

In the following theorem we prove optimality of the threshold in Theorem 1.1 for certain values of $(1+\epsilon)$ and $(1+2\epsilon)$.

Theorem 1.2 (see [27]). Let $\epsilon \geq 0$ and μ be a real-valued homogeneous function on $\mathbb{R}^{1+\epsilon}$ of degree $(2+\epsilon)$ which belongs to $C^\infty(\mathbb{R}^{1+\epsilon}\backslash\{0\})$. Suppose there exist a point $\xi_0 \neq 0$ at which the Hessian determinant of μ is not zero. Let $\max\left(\frac{1}{1+2\epsilon}, \frac{1}{2}\right) \leq \frac{1}{1+\epsilon}$ or $\min\left(\frac{1}{1+2\epsilon}, \frac{1}{2}\right) \geq \frac{1}{1+\epsilon}$. Let $s \in \mathbb{R}$ and suppose the Fourier multiplier operator $e^{i\mu(D)}$ is bounded from $W_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$ to $W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$. Then

$$s \geq (1+\epsilon)\left[\epsilon\left|\frac{1-\epsilon}{2(1+\epsilon)}\right| + \left|\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\right|\right].$$

Although we have yet to prove Theorem 1.2 for any $0 \leq \epsilon \leq \infty$, the case $\epsilon = 0$ recaptures [4, Proposition 6.1]. It states that if the pointwise multiplier operator $Af_r(x) = e^{i|x|^2} f_r(x)$is bounded from $M_s^{1+\epsilon,1+2\epsilon}$ to $M^{1+\epsilon,1+2\epsilon}$ then $s \geq (1+\epsilon)|\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}|$. The significance of this proposition is that it showed sharpness of the threshold computed for boundedness of Fourier integral operator(FOI) on modulation spaces $M^{1+\epsilon,1+2\epsilon}$ with decay condition on its symbol. One should observe the fact that A is a FOI whose phase $\Phi(x,\eta) = x\eta + \frac{|x|^2}{2}$ and symbol $\sigma \equiv 1$.

We give the basic notations and definition of terms to be used within this paper. Also, we define Wiener amalgam spaces, modulation spaces and give some of their important properties. In particular, we state thedilation property of Wiener amalgam spaces and a lemma that provides sufficient condition for the boundedness of Fourier multipliers on $W^{1+\epsilon,1+2\epsilon}$.We layout the proof for Theorem 1.1. Moreover, we give sufficient conditions for the boundedness of $e^{i\mu(D)}$ for $\epsilon > 0$ (not contained in Theorem 1.1). Finally we prove Theorem 1.2.

II. Preliminaries

We denote the Schwartz class of test functions on $\mathbb{R}^{1+\epsilon}$ by $S = S(\mathbb{R}^{1+\epsilon})$ and its dual, the space of tempered distributions, by $S' = S(\mathbb{R}^{1+\epsilon})$. The Fourier transform of $f_r \in S$ is given by

$$\mathcal{F}f_r(\xi) = \hat{f}_r(\xi) = \int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{-ix\cdot\xi} f_r(x)\, dx$$

which is an isomorphism of the Schwartz space S onto itself that extends to the tempered distributions S' by duality. The inverse Fourier transform is given by $\mathcal{F}^{-1}f_r(x) = \check{f}_r(x) = \frac{1}{(2(1+\epsilon))^{1+\epsilon}}\int_{\mathbb{R}^{1+\epsilon}} \quad e^{i\xi\cdot x}f_r(\xi)\,d\xi$. Given $0 \le \epsilon \le \infty$, we denote by $\left(\frac{1+\epsilon}{\epsilon}\right)$ the conjugate exponent of $(1+\epsilon)$ (i.e. $\frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon} = 1$). We use the notation $u \lesssim v$ to denote $u \le cv$ for a positive constant c independent of u and v. Also, we use the notation $u \asymp v$ to denote $cu \le v \le Cu$ for universal positive constants c, C. The translation and modulation operators are defined by $T_x f_r(t) = f_r(t-x)$ and $M_\xi f_r(t) = e^{it\cdot\xi} f_r(t)$, respectively. The scaling operator is given by $U_\lambda f_r(t) = f_r(\lambda t)$.

Let $s \in \mathbb{R}$, we denote the weight function $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}, \xi \in \mathbb{R}^{1+\epsilon}$. For $0 \le \epsilon \le \infty, s \in \mathbb{R}$, the Wiener amalgam space $W_s^{1+\epsilon,1+2\epsilon}$ is defined as the closure of the Schwartz class with respect to the norm

$$||f_r||_{W_s^{1+\epsilon,1+2\epsilon}} = \left(\int_{\mathbb{R}^{1+\epsilon}} \left(\int_{\mathbb{R}^{1+\epsilon}} \sum_r \left|V_{g_r}f_r(y,\omega_r)\right|^{1+2\epsilon}\langle\omega_r\rangle^{s(1+2\epsilon)}\,d\omega_r\right)^{\frac{1+\epsilon}{1+2\epsilon}} dy\right)^{\frac{1}{1+\epsilon}},$$

where $V_{g_r}f_r$ is the short-time Fourier transform (STFT) of $f_r \in S'$ with respect to the window $0 \ne g_r \in S$ defined by

$$V_{g_r}f_r(y,\omega_r) = \int_{\mathbb{R}^{1+\epsilon}} \sum_r f_r(\xi)\overline{g_r(\xi - y)}e^{-i\xi\cdot\omega_r}d\xi.$$

If $s = 0$ we simply write $W^{1+\epsilon,1+2\epsilon}$ instead of $W_0^{1+\epsilon,1+2\epsilon}$. Moreover, in some event, we use the notation $W(\mathcal{F}L^{1+2\epsilon}, L^{1+\epsilon}) = W^{1+\epsilon,1+2\epsilon}$. We note that this definition is independent of the choice of window g_r. Alternatively, we could use the following equivalent norm

$$||f_r||_{W_s^{1+\epsilon,1+2\epsilon}} = \left\|\,||\{\langle k\rangle^s\varphi_r(D-k)f_r\}||_{\ell^{1+2\epsilon}}\right\|_{L^{1+\epsilon}},$$

where $\varphi_r \in S$ satisfying

$$\text{supp}\varphi_r \subset (-1,1)^{1+\epsilon}\text{and} \sum_{k\in\mathbb{Z}^{1+\epsilon}} \sum_r \varphi_r(\xi - k) = 1 \qquad \forall\xi \in \mathbb{R}^{1+\epsilon}.$$

Here we collect some properties of Wiener amalgam spaces.

Lemma 2.1 (see [27]). Let $1 \le \epsilon \le \infty$ and $s_j \in \mathbb{R}, j = 1,2$. Then

(1) $S \to W^{1+\epsilon,1+2\epsilon} \to S'$;

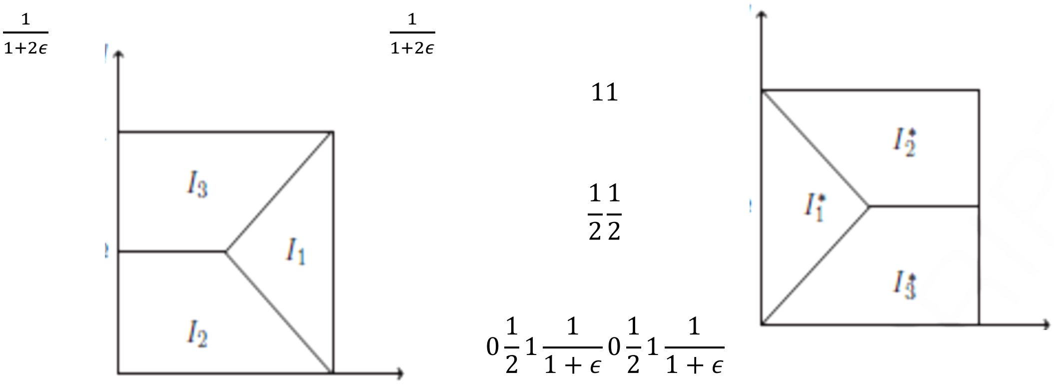


Figure 1. The index sets(see [27]).

(2) S is dense in $W^{1+\epsilon,1+2\epsilon}$ if $\epsilon < \infty$;

(3) If $\epsilon \ge 0$, then $W^{1+3\epsilon,1+4\epsilon} \to W^{1+5\epsilon,1+6\epsilon}$;

(4) $\langle D\rangle^{-s} : W^{1+\epsilon,1+2\epsilon} \to W_s^{1+\epsilon,1+2\epsilon}$ is an isomorphism.

(5) (Convolution) If $\mathcal{F}L^{1+4\epsilon} * \mathcal{F}L^{1+6\epsilon} \to \mathcal{F}L^{1+2\epsilon}$ and $L^{1+3\epsilon} * L^{1+5\epsilon} \to L^{1+\epsilon}$, then

$$W(\mathcal{F}L^{1+4\epsilon}, L^{1+3\epsilon}) * W(\mathcal{F}L^{1+6\epsilon}, L^{1+5\epsilon}) \to W(\mathcal{F}L^{1+2\epsilon}, L^{1+\epsilon}).$$

In particular, $||f_r * u||_{W(\mathcal{F}L^{1+2\epsilon},L^{1+\epsilon})} \le ||u||_{W(\mathcal{F}L^{\infty},L^{1})} ||f_r||_{W(\mathcal{F}L^{1+2\epsilon},L^{1+\epsilon})}$;

(6) (Complex interpolation) For $0 < \theta < 1$. Let $\frac{1}{1+\epsilon} = \frac{\theta}{1+3\epsilon} + \frac{1-\theta}{1+5\epsilon}, \frac{1}{1+2\epsilon} = \frac{\theta}{1+4\epsilon} + \frac{1-\theta}{1+6\epsilon}$ and $s = \theta s_1 + (1-\theta)s_2$. Then

$$[W_{s_1}^{1+3\epsilon,1+4\epsilon}, W_{s_2}^{1+5\epsilon,1+6\epsilon}][\theta] = W_s^{1+\epsilon,1+2\epsilon};$$

(7) (Duality) $\left(W_s^{1+\epsilon,1+2\epsilon}\right)' = W_{-s}^{\frac{1+\epsilon}{\epsilon},\frac{1+2\epsilon}{2\epsilon}}$.

For the proofs of these statements (see [9, 10, 11, 13, 25]).

We now recall the definition of modulation spaces $M^{1+\epsilon,1+2\epsilon}$. For $0 \le \epsilon \le \infty, s \in \mathbb{R}$, the modulation space $M_s^{1+\epsilon,1+2\epsilon}$ is defined as the closure of the Schwartz class with respect to the norm

$$||f_r||_{M_s^{1+\epsilon,1+2\epsilon}} = \left(\int_{\mathbb{R}^{1+\epsilon}} \left(\int_{\mathbb{R}^{1+\epsilon}} \sum_r \left|V_{g_r} f_r(y,\omega_r)\right|^{1+2\epsilon} dy \right)^{\frac{1+2\epsilon}{1+\epsilon}} \langle \omega_r \rangle^{s(1+2\epsilon)} \, d\omega_r \right)^{\frac{1}{1+2\epsilon}}.$$

Analogous properties to Lemma 2.1 are known for modulation spaces. We introduce the following indices

$$\mu_1(1+2\epsilon, 1+\epsilon) = \begin{cases} -\frac{1}{1+2\epsilon} \text{ if } (\frac{1}{1+\epsilon}, \frac{1}{1+2\epsilon}) \in I_1^* \\ \frac{-\epsilon}{1+\epsilon} \text{ if } (\frac{1}{1+\epsilon}, \frac{1}{1+2\epsilon}) \in I_2^* \\ -\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)} \text{ if } (\frac{1}{1+\epsilon}, \frac{1}{1+2\epsilon}) \in I_3^*. \end{cases} \quad (1)$$

and

$$\mu_2(1+2\epsilon, 1+\epsilon) = \begin{cases} -\frac{1}{1+2\epsilon} \text{ if } (\frac{1}{1+\epsilon}, \frac{1}{1+2\epsilon}) \in I_1 \\ \frac{-\epsilon}{1+\epsilon} \text{ if } (\frac{1}{1+\epsilon}, \frac{1}{1+2\epsilon}) \in I_2 \\ -\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)} \text{ if } (\frac{1}{1+\epsilon}, \frac{1}{1+2\epsilon}) \in I_3 . \end{cases} \quad (2)$$

Here we collect properties of the dilation operator in Wiener amalgam spaces by extending the result in [23].

Lemma 2.2 (see [27]). Let $0 \le \epsilon \le \infty$, and $\lambda \neq 0$. We have the following inequalities.

(1)

$$\left\| \sum_r U_\lambda f_r \right\|_{W^{1+\epsilon,1+2\epsilon}} \lesssim |\lambda^{-1}|^{1+\epsilon+(1+\epsilon)\mu_2(1+2\epsilon,1+\epsilon)} \sum_r \|f_r\|_{W^{1+\epsilon,1+2\epsilon}} \forall\, |\lambda| \ge 1, \quad \forall f_r \in W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$$

(2)

$$\left\| \sum_r U_\lambda f_r \right\|_{W^{1+\epsilon,1+2\epsilon}} \lesssim |\lambda^{-1}|^{1+\epsilon+(1+\epsilon)\mu_1(1+2\epsilon,1+\epsilon)} \sum_r \|f_r\|_{W^{1+\epsilon,1+2\epsilon}} \forall\, 0 < |\lambda| \le 1, \quad \forall f_r \in W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$$

(3)

$$\left\|\sum_r U_\lambda f_r\right\|_{W^{1+\epsilon,1+2\epsilon}} \gtrsim |\lambda^{-1}|^{1+\epsilon+(1+\epsilon)\mu_1(1+2\epsilon,1+\epsilon)}\sum_r \|f_r\|_{W^{1+\epsilon,1+2\epsilon}} \forall\ |\lambda| \geq 1, \quad \forall f_r \in W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$$

(4)

$$\left\|\sum_r U_\lambda f_r\right\|_{W^{1+\epsilon,1+2\epsilon}} \gtrsim |\lambda^{-1}|^{1+\epsilon+(1+\epsilon)\mu_2(1+2\epsilon,1+\epsilon)}\sum_r \|f_r\|_{W^{1+\epsilon,1+2\epsilon}} \forall\ 0 < |\lambda| \leq 1, \quad \forall f_r \in W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$$

Proof. We use the dilation property of Fourier transforms and thefact that $\mathcal{F}M^{1+2\epsilon,1+\epsilon} = W^{1+\epsilon,1+2\epsilon}$. We know that $\widehat{U_\lambda f_r}(\xi) = \lambda^{-(1+\epsilon)}\hat{f}_r(\lambda^{-1}\xi)$. Based on [23] we have

$$\left\|\sum_r U_\lambda f_r\right\|_{M^{1+2\epsilon,1+\epsilon}} \lesssim |\lambda|^{(1+\epsilon)\mu_1(1+2\epsilon,1+\epsilon)}\sum_r \|f_r\|_{M^{1+2\epsilon,1+\epsilon}} \qquad |\lambda| \geq 1$$

Equivalently, we have

$$|\lambda|^{-(1+\epsilon)}\left\|\sum_r \hat{f}_r(\lambda^{-1}\xi)\right\|_{W^{1+\epsilon,1+2\epsilon}} \lesssim |\lambda|^{(1+\epsilon)\mu_1(1+2\epsilon,1+\epsilon)}\sum_r \|\hat{f}_r\|_{W^{1+\epsilon,1+2\epsilon}} \qquad |\lambda| \geq 1$$

Inequality (2) follows from the change of variable $\lambda \mapsto 1/\lambda$. All the remaining estimates follows the same proof by using the appropriate inequality in [23].

In the following lemma we give sufficient conditions for a Fourier multiplier to be bounded in $W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$.

Lemma 2.3 (see [27]). A Fourier multiplier operator $\sigma(D)$ is bounded on all Wiener amalgam spaces $W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$ for $\epsilon \geq 0$ and $0 \leq \epsilon \leq \infty$ whenever $\sigma \in M^{\infty,1}$.

Proof. We use the convolution property of Wiener amalgam spaces stated in Lemma 2.1. Since $\mathcal{F}L^\infty * \mathcal{F}L^{1+2\epsilon} \to \mathcal{F}L^{1+2\epsilon}$ and $L^1 * L^{1+\epsilon} \to L^{1+\epsilon}$, we have $W(\mathcal{F}L^{1+2\epsilon}, L^{1+\epsilon}) * W(\mathcal{F}L^\infty, L^1) \hookrightarrow W(\mathcal{F}L^{1+2\epsilon}, L^{1+\epsilon})$ with

$$\left\|\sum_r \breve{\sigma} * f_r\right\|_{W^{1+\epsilon,1+2\epsilon}} \leq ||\breve{\sigma}||_{W(\mathcal{F}L^\infty,L^1)}\sum_r ||f_r||_{W^{1+\epsilon,1+2\epsilon}}.$$

By the relation $\mathcal{F}M^{1+2\epsilon,1+\epsilon} = W(\mathcal{F}L^{1+2\epsilon}, L^{1+\epsilon})$ we conclude that if $\sigma \in M^{\infty,1}, \sigma(D)f_r = \breve{\sigma} * f_r$ is bounded on $W^{1+\epsilon,1+2\epsilon}(\mathbb{R}^{1+\epsilon})$.

The next corollary follows directly from Lemma 2.3 and [3, Cor. 15].

Corollary 2.1. If $0 \leq \epsilon \leq 1$, then $e^{i|D|^{2+\epsilon}}$ is bounded on $W^{1+\epsilon,1+2\epsilon}$ for all $0 \leq \epsilon \leq \infty$.

III. Sufficient Conditions for the Boundedness of $e^{i\mu(D)}$

This section contains the proof of Theorem 1.1. In addition, we give sufficient conditions for the boundedness of $e^{i\mu(D)}$ on $W^{1+\epsilon,1+2\epsilon}$ for $0 < \epsilon < 2$ which is not covered by Theorem 1.1. Throughout, $\chi \in C_0^\infty(\mathbb{R}^{1+\epsilon})$ will denote a test function such that

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi| \geq 2 \\ 1 & \text{if} |\xi| \leq 1 \\ 0 \leq \chi(\xi) \leq 1 & \text{if } 1 \leq |\xi| \leq 2. \end{cases}$$

Moreover we let $\Phi(\xi) = (1 - \chi(\xi))$.

Lemma 3.1 [27]. Let $J = [\frac{1+\epsilon}{2}] + 1$. Suppose that $\partial^{\gamma_r} m \in L^2(\mathbb{R}^{1+\epsilon})$ for all $\gamma_r \in N^{1+\epsilon}, |\gamma_r| \leq J$. Then

$$\|\mathcal{F}^{-1}(m)\|_{L_1} \leq C\|m\|_{L^2}^{1-\frac{1+\epsilon}{2J}} \left(\sum_{|\gamma_r|=J} \sum_r \|\partial^{\gamma_r} m\|_{L^2} \right)^{\frac{1+\epsilon}{2J}}. \quad (3)$$

The proof of this lemma can be found in [21].

Lemma 3.2 (Lemma 3.1 of [18]). Let m be a bounded function on $\mathbb{R}^{1+\epsilon}$ with compact support. Suppose that m is of class $C^{\left[\frac{1+\epsilon}{2}\right]+1}(\mathbb{R}^{1+\epsilon}\backslash\{0\})$ and suppose there exists $\epsilon > 0$ such that

$$\left| \sum_r \partial^{\gamma_r} \mu(\xi) \right| \leq \sum_r C_{\gamma_r} |\xi|^{\epsilon - |\gamma_r|}$$

for $|\gamma_r| \leq [\frac{1+\epsilon}{2}] + 1$. Then $m \in \mathcal{F}L^1$.

For the reader convenience we have the following lemmas:

Lemma 3.3 (see [27]). For $s > 0$, the function $|\xi|^s/(1 + |\xi|^2)^{\frac{s}{2}}$ is an element of $M^{\infty,1}$.

Proof. We write

$$\frac{|\xi|^s}{(1 + |\xi|^2)^{\frac{s}{2}}} = \frac{|\xi|^s}{(1 + |\xi|^2)^{\frac{s}{2}}} \chi(\xi)^2 + \frac{|\xi|^s}{(1 + |\xi|^2)^{\frac{s}{2}}} (1 - \chi(\xi)^2).$$

The second term in the sum can be distinguished as an element of $C^k \subset M^{\infty,1}$ for some k. For the first term, we split into $|\xi|^s \chi(\xi) \cdot \frac{\chi(\xi)}{(1+|\xi|^2)^{\frac{s}{2}}}$ where the second factor is again in C^k and by Lemma 3.2 the first factor belongs to $\mathcal{F}L^1 \subset M^{\infty,1}$. This ends our proof.

Lemma 3.4 (see [27]). Let $\epsilon > 0, s \in \mathbb{R}$ and let μ be a real-valued function on $\mathbb{R}^{1+\epsilon}$ which belongs to $C^{\left[\frac{1+\epsilon}{2}\right]+1}$ supported away from the origin satisfying

$$\left| \sum_r \partial^{\gamma_r} \mu(\xi) \right| \leq \sum_r C_{\gamma_r} |\xi|^{(1+\epsilon)-|\gamma_r|} \text{for } \gamma_r \in N^{1+\epsilon}, \quad |\gamma_r| \leq \left[\frac{1+\epsilon}{2}\right] + 1 \quad (4)$$

Set $m(\xi) = \Phi(\xi)|\xi|^{-s} e^{i\mu(\xi)}, \xi \in \mathbb{R}^{1+\epsilon}$. If $s > \frac{(1+\epsilon)^2}{2}$, then $m \in \mathcal{F}^{-1}L^1$.

Proof. Let $\Phi_0(\xi) = \Phi(\xi) - \Phi(\xi/2)$ and let $\Phi_\nu(\xi) = \Phi_0(2^{-\nu}\xi)$ for $\nu \in \mathbb{N}$ so that $\Phi(\xi) = \sum_{\nu\in\mathbb{N}} \Phi_\nu(\xi)$. Let $m_\nu(\xi) = \Phi_\nu(\xi)|\xi|^{-s} e^{i\mu(\xi)}$. Then from (4) and Leibniz rule we have

$$|\sum_r \partial^{\gamma_r} m_\nu(\xi)| \leq \sum_r C_{\gamma_r} 2^{\nu(-s+|\gamma_r|(\epsilon))} \text{for} \quad |\gamma_r| \leq [\frac{1+\epsilon}{2}] + 1.$$

Hence,

$$\left\| \sum_r \partial^{\gamma_r} m_\nu(\xi) \right\|_{L_2} \lesssim \sum_r 2^{\frac{(1+\epsilon)\nu}{2} - \nu s + \nu|\gamma_r|(\epsilon)}$$

and

$$\|m_\nu\|_{L_2} \lesssim 2^{\frac{\nu(1+\epsilon)}{2} - \nu s}$$

for $|\gamma_r| \leq [\frac{1+\epsilon}{2}] + 1$.Therefore, by Lemma 3.1 we have

$$\|\mathcal{F}^{-1}(m_\nu)\|_{L_1} \lesssim 2^{\frac{\nu(1+\epsilon)^2}{2} - \nu s}.$$

Finally, we see that

$$\|\mathcal{F}^{-1}(m)\|_{L_1} \le \left\|\sum_{\nu\in\mathbb{N}} \mathcal{F}^{-1}(m_\nu)\right\|_{L_1} \lesssim \sum_{\nu\in\mathbb{N}} 2^{\frac{\nu(1+\epsilon)^2}{2}-\nu s},$$

where the series converges whenever $s > \frac{(1+\epsilon)^2}{2}$.

Lemma 3.5 (Lemma 3.2 of [18]). Let $\epsilon > 0$. Suppose μ is a real-valued function of class $C^{\left[\frac{1+\epsilon}{2}\right]+1}(\mathbb{R}^{1+\epsilon}\backslash\{0\})$ satisfying

$$\left|\sum_r \partial^{\gamma_r}\mu(\xi)\right| \le \sum_r C_{\gamma_r}|\xi|^{\epsilon-|\gamma_r|}$$

for $|\gamma_r| \le [\frac{1+\epsilon}{2}] + 1$. Then $\eta e^{i\mu} \in \mathcal{F}L^1$ for each $\eta \in S$ with compact support.

Proposition 3.1 (see [27]). Let $\epsilon \ge 0, s \in \mathbb{R}$ and μ be a real-valued function on $\mathbb{R}^{1+\epsilon}$ which belongs to $C^{\left[\frac{1+\epsilon}{2}\right]+1}(\mathbb{R}^{1+\epsilon}\backslash\{0\})$ satisfying

$$\left|\sum_r \partial^{\gamma_r}\mu(\xi)\right| \le \sum_r C_{\gamma_r}|\xi|^{(1+\epsilon)-|\gamma_r|} \text{for} \gamma_r \in N^{1+\epsilon}, \qquad |\gamma_r| \le \left[\frac{1+\epsilon}{2}\right] + 1. \quad (5)$$

If $s > \frac{(1+\epsilon)^2}{2}$, then $e^{i\mu(D)}$ is bounded from $W_s^{1+\epsilon,1+2\epsilon}$ to $W^{1+\epsilon,1+2\epsilon}$.

Proof. We use property 4 of Lemma 2.1.Hence, by Lemma 2.3 we have

$$\|e^{i\mu(D)}\|_{\mathcal{L}(W_s^{1+\epsilon,1+2\epsilon},W^{1+\epsilon,1+2\epsilon})}$$
$$\asymp \|D^{-s}e^{i\mu(D)}\|_{\mathcal{L}(W^{1+\epsilon,1+2\epsilon},W^{1+\epsilon,1+2\epsilon})} \lesssim \|\langle\xi\rangle^{-s}e^{-i\mu(\xi)}\|_{M^{\infty,1}}. \quad (6)$$

Now, we only need to show that $\xi^{-s}e^{i\mu(\xi)} \in M^{\infty,1}$. To do so, we rewrite

$$\xi^{-s}e^{i\mu(\xi)} = \sigma_1 + \sigma_2 \quad (7)$$

where $\sigma_1 = \chi(\xi)\xi^{-s}e^{-i\mu(\xi)}$ and $\sigma_2 = \left(1 - \chi(\xi)\right)\xi^{-s}e^{-i\mu(\xi)}$. Then by Lemma 3.5, $\sigma_1 \in \mathcal{F}L^1 \subset M^{\infty,1}$. Meanwhile, we write $\sigma_2 = \frac{|\xi|^s}{(1+|\xi|^2)^{\frac{s}{2}}} \cdot \Phi(\xi)|\xi|^{-s}e^{i\mu(\xi)}$. By Lemma 3.3 and Lemma 3.4, both factors of σ_2 are in $M^{\infty,1}$. We conclude that σ_2 is also in $M^{\infty,1}$. by using the multiplication property of modulation spaces. This ends our proof.

Remark 3.1.1. By interpolation, Proposition 3.1 implies Theorem 1.1 for $1+\epsilon \le 1+2\epsilon \le \frac{1+\epsilon}{\epsilon}$ and $\frac{1+\epsilon}{\epsilon} \le \frac{1+2\epsilon}{2\epsilon} \le 1+\epsilon$. Indeed, take $(1+\epsilon, 1+2\epsilon) = (1,\infty)$, then interpolating with [18, Theorem 1.1] for $W^{1+\epsilon,1+\epsilon}$ yields the desired threshold for s.

Remark 3.1.2. For $0 \le \epsilon \le 2$, an improvement of the estimate $s > \frac{(1+\epsilon)^2}{2}$ can be done with the use of [3, Theorem 1], where the boundedness of $e^{i\mu(D)}$ on $M^{1+\epsilon,1+\epsilon} = W^{1+\epsilon,1+\epsilon}$ is known. We interpolate with the case $(1+\epsilon, 1+2\epsilon) = (1,\infty)$ to get the improved estimate $s > \frac{(1+\epsilon)^2}{2}|\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}|$.

The proof of Theorem 1.1 relies on the following lemma that gives sufficient condition for the inclusion property of weighted Wiener amalgam spaces.

Lemma 3.6(see [27]). Let $0 \le \epsilon \le \infty$and $s_1, s_2 \in \mathbb{R}$.If $\epsilon < 0$ and $s_1 - s_2 > (1+\epsilon)\left(\frac{-\epsilon}{(1+3\epsilon)(1+2\epsilon)}\right)$,then $W_{s_1}^{1+\epsilon,1+2\epsilon} \to W_{s_2}^{1+\epsilon,1+3\epsilon}$.

Proof. We write $\frac{1+3\epsilon}{1+2\epsilon} + \frac{-\epsilon}{1+2\epsilon} = 1$ and $s_2 = s_1 + (s_2 - s_1)$.By Hölder's inequality we have

$$\|f_r\|_{W_{s_2}^{1+\epsilon,1+3\epsilon}} = \left\|\sum_r \|\{\langle k\rangle^{s_2}\varphi_r(D-k)f_r\}\|_{\ell^{1+3\epsilon}}\right\|_{L^{1+\epsilon}}$$

$$\leq \sum_r \quad \| \| \|\{\langle k\rangle^{s_1}\varphi_r(D\ -\ k)f_r\}\|_{\ell^{1+2\epsilon}}\|_{L^{1+\epsilon}}\|\{\langle k\rangle^{s_1}\varphi_r(D\ -\ k)f_r\}\|_{\ell^{\frac{(1+2\epsilon)(1+3\epsilon)}{-\epsilon}}}.$$

Since $(s_1-\ s_2)\frac{(1+2\epsilon)(1+3\epsilon)}{-\epsilon} > 1+\epsilon$,we have our desired result.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1(see [27]). Using the fact that $M^{1+\epsilon,1+\epsilon}\ =\ W^{1+\epsilon,1+\epsilon}$together with the result in [18], we conclude the boundedness of the operator

$$e^{i\mu(D)}: W_{s_2}^{1+\epsilon,1+\epsilon} \rightarrow\ W^{1+\epsilon,1+\epsilon},$$

where $s_2 \geq (1+\epsilon)(\epsilon)|\frac{1-\epsilon}{2(1+\epsilon)}|$.By Lemma 2.1 $W^{1+\epsilon,1+\epsilon} \rightarrow\ W^{1+\epsilon,1+2\epsilon}$for $\epsilon > 0$. On the other hand, setting $\epsilon =$ 0and $s\ =\ s_1$in Lemma 3.6 gives us the inclusion $W_s^{1+\epsilon,1+2\epsilon} \rightarrow W_{s_2}^{1+\epsilon,1+\epsilon}$when $s-s_2 > (\frac{\epsilon}{(1+2\epsilon)})$ and $\epsilon >$ 0.Thus, the multiplier operator $e^{i\mu(D)}: W_s^{1+\epsilon,1+2\epsilon} \rightarrow W^{1+\epsilon,1+2\epsilon}$is bounded for $\epsilon > 0$whenever $s >$ $(1+\epsilon)\left[\epsilon\left|\frac{1-\epsilon}{2(1+\epsilon)}\right| + \left|\left(\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\right)\right|\right]$.The case for $\epsilon < 0$is achieved by duality. This ends our proof.

4. Necessary Condition

Here we give the proof of Theorem 1.2 which establishes the optimality of the threshold obtained in Theorem 1.1. We need the following lemmas.

Lemma 4.1. Consider the function $M_\xi 1(x)\ =\ e^{ix\xi}$. Its short-time Fourier transform is given by

$$V_{g_r}(M_\xi 1)(y,\omega_r)\ =\ e^{iy.(\xi-\omega_r)}\widehat{\overline{g_r}}(\omega_r-\ \xi)).$$

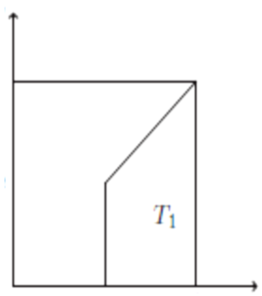
This follows easily from direct computation.

Proof of Theorem 1.2(see [27]). Our proof is an adaptation of the arguments used in [18, Section 5].

It suffices to prove Theorem 1.2 only for pairs $(1+\epsilon, 1+2\epsilon)$ satisfying $1+\epsilon \leq\ 1+2\epsilon \leq\ 2$,that is, the shaded region in Figure 2. Indeed, suppose, for contradiction, that $e^{i\mu(D)}$is bounded from $W_s^{p_0,q_0}$to $W^{p_0,q_0}$for some $(1/p_0, 1/q_0)$ in $T_1\backslash T_1'$such that $s < (1+\epsilon)(\epsilon)(1/p_0 - 1/2) + (1+\epsilon)(1/p_0 - 1, q_0)$.Then, interpolating with the estimate for point $(1,0)$ and $s\ =\ (\epsilon)(1+\epsilon)/2$ (by Theorem 1.1) would yield an improve estimate for all points of the line segment joining $(0,1)$ and $(1/p_0, 1/q_0)$ inside T_1', which is not possible.

From the assumption we have the estimate:

$$\left\|\sum_r \quad e^{i\mu(D)}\langle D\rangle^{-s}f_r\right\|_{W^{1+\epsilon,1+2\epsilon}} \leq\ C\sum_r \quad \|f_r\|_{W^{1+\epsilon,1+2\epsilon}} \forall f_r \in\ S(\mathbb{R}^{1+\epsilon}) \qquad (8)$$



$\frac{1}{1+2\epsilon}\frac{1}{1+2\epsilon}$
11

$\frac{1}{2}\frac{1}{2}$

$0\frac{1}{2}1\frac{1}{1+\epsilon}0\frac{1}{2}1\frac{1}{1+\epsilon}$

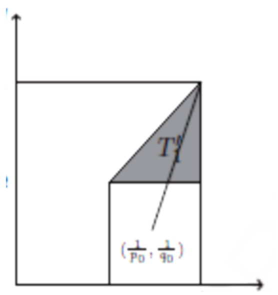


Figure 2(see [27]).

Let $f_r$be a fixed Schwartz function whose Fourier transform is supported in a small neighborhood $\mathcal{U} \subset \mathbb{R}^{1+\epsilon}\backslash\{0\}$of the point $\xi_0$in the assumption and is equal to 1 in some neighborhood of ξ_0.Using the dilated $U_\lambda f_r, \lambda\ \geq\ 1$,in the above estimate gives us

$$\left\|\sum_r \quad U_\lambda e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}f_r\right\|_{W^{1+\epsilon,1+2\epsilon}} \leq\ C\sum_r \quad \|U_\lambda f_r\|_{W^{1+\epsilon,1+2\epsilon}},$$

since $e^{i\mu(D)}\langle D\rangle^{-s}U_\lambda =\ U_\lambda e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}$.Using the dilation properties of Wiener amalgam spaces given in Lemma 2.2 we have

$$|\lambda|^{1+\epsilon+(1+\epsilon)\mu_2(1+2\epsilon,1+\epsilon)}\left\|\sum_r \quad e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}f_r\right\|_{W^{1+\epsilon,1+2\epsilon}}$$

$$\leq C|\lambda|^{1+\epsilon+(1+\epsilon)\mu_1(1+2\epsilon,1+\epsilon)} \sum_r \|f_r\|_{W^{1+\epsilon,1+2\epsilon}}. \quad (9)$$

We now compute for a convenient lower bound of (9). Applying Lemma 4.1 and by interchanging the integrals we see that

$$\left|\sum_r V_{g_r}\left(e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}f_r\right)(y,w)\right|$$
$$= (2(1+\epsilon))^{-(1+\epsilon)}\left|\int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{iy\xi+i\lambda^{2+\epsilon}\mu(\xi)}\widehat{\overline{g_r}}(\omega_r-\xi)\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right|.$$

Taking the integral over $|\omega_r| \leq 1$ we have

$$\sum_r \left\|V_{g_r}\left(e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}f_r\right)(y,\cdot)\right\|_{L^{1+2\epsilon}}$$
$$\geq (2(1+\epsilon))^{-(1+\epsilon)}\left(\int_{|\omega_r|\leq 1}\left|\int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{iy\xi+i\lambda^{2+\epsilon}\mu(\xi)}\widehat{\overline{g_r}}(\omega_r-\xi)\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right|^{1+2\epsilon} d\omega_r\right)^{\frac{1}{1+2\epsilon}}$$
$$\gtrsim \left|\int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{iy\xi+i\lambda^{2+\epsilon}\mu(\xi)}\widehat{\overline{g_r}}(\omega_r-\xi)\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right|. \quad (10)$$

Hence by a change of variables $y \mapsto \lambda^{2+\epsilon}y$we get

$$\left(\int_{\mathbb{R}^{1+\epsilon}} \sum_r \left|\int_{\mathbb{R}^{1+\epsilon}} e^{iy\xi+i\lambda^{2+\epsilon}\mu(\xi)}\widehat{\overline{g_r}}(\xi)\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right|^{1+\epsilon} d\omega_r\right)^{\frac{1}{1+\epsilon}}$$
$$= \lambda^{2+\epsilon}\left(\int_{\mathbb{R}^{1+\epsilon}} \sum_r \left|\int_{\mathbb{R}^{1+\epsilon}} e^{iy\xi+i\lambda^{2+\epsilon}\mu(\xi)}\widehat{\overline{g_r}}(\xi)\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right|^{1+\epsilon} d\omega_r\right)^{\frac{1}{1+\epsilon}} \quad (11)$$

Let $y_0 = -\nabla\mu(\xi_0)$.Since the Hessian matrix $d^2\mu(\xi_0)$ of μat ξ_0 is invertible, it follows from the implicit function theorem that we can find a neighborhood $\mathcal{V}$of y_0 such that for a small $\mathcal{U}$, the phase $\Phi(\xi) = y\xi + \mu(\xi)$ has a unique non-degenerate critical point $\xi = \xi(y) \in \mathcal{U}$, for every $y \in \mathcal{V}$.Also, choose $g_r \in S(\mathbb{R}^{1+\epsilon})$, with $\widehat{g_r}(\xi) = 1$ on the support of $\widehat{f_r}$.Now it follows from the stationary phase theorem that for $y \in \mathcal{V}$,

$$\left|\int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{i\lambda^{2+\epsilon}(y\xi+\mu(\xi))}\widehat{\overline{g_r}}(\xi)\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right|$$
$$= \lambda^{-s}\left|\int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{i\lambda^{2+\epsilon}(y\xi+\mu(\xi))}\widehat{\overline{g_r}}(\xi)\lambda^s\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right|$$
$$\asymp \left|\det\left(d^2\mu(\xi(y))\right)\right|^{-\frac{1}{2}}\langle\lambda\xi(y)\rangle^{-s}\lambda^{-\frac{(2+\epsilon)(1+\epsilon)}{2}} + O(\lambda^{-\frac{(2+\epsilon)(3+\epsilon)}{2}-s}),$$

where $\det(d^2\mu(\xi(y)))$ is the Hessian matrix of μat the critical point $\xi(y)$. Indeed, here we use the uniform estimates $|\lambda^s\langle\lambda\xi(y)\rangle^{-s}| \geq C_\lambda$ on the support of $\widehat{f_r}$and the fact that all derivatives of the phase are uniformly bounded with respect to $y \in \mathcal{V}$.Hence, for some $C > 0$,

$$\left|\int_{\mathbb{R}^{1+\epsilon}} \sum_r e^{i\lambda^{2+\epsilon}(y\xi+\mu(\xi))}\widehat{\overline{g_r}}(\xi)\langle\lambda\xi\rangle^{-s}\widehat{f_r}(\xi)d\xi\right| \geq C\lambda^{-\frac{(2+\epsilon)(1+\epsilon)}{2}-s} y \in \mathcal{V}$$

Combining this estimate with equations (10) and (11) we arrive to the following estimate

$$\sum_r \left\|e^{i\mu(\lambda D)}\langle\lambda D\rangle^{-s}f_r\right\|_{W^{1+\epsilon,1+2\epsilon}} \gtrsim \lambda^{\left(\frac{(2+\epsilon)(1-\epsilon)}{2}\right)-s}. \quad (12)$$

Using this last estimate with equation (9) and letting $\lambda \to +\infty$gives $s \geq \frac{(2+\epsilon)(1-\epsilon)}{2} + (1+\epsilon)(\mu_2(1+2\epsilon,1+\epsilon) - \mu_1(1+2\epsilon,1+\epsilon))$Substituting appropriate values of μ_1 and μ_2 for $2(1-\epsilon) \leq 2-\epsilon \leq 2$ we have $\mu_2(2-\epsilon,2(1-\epsilon)) - \mu_1(2-\epsilon,2(1-\epsilon)) = 1 - \frac{1}{2(1-\epsilon)} - \frac{1}{2-\epsilon}$.Thus, $s \geq \frac{\epsilon(1+\epsilon)(1+2\epsilon-\epsilon^2)}{2(1-\epsilon)(2-\epsilon)}$ as desired.

References

[1]. Á. Bényi, T. Oh, Modulation spaces, Wiener amalgam spaces and Brownian motions, Adv. Math. 228 (2011) 2943-2981.

[2]. Á. Bényi, L. Grafakos, K. Gröchenig and K. Okoudjou, class of Fourier multipliers for modulation spaces, Appl. Comput. Harmon. Anal. 19 (1) (2005) 131-139.

[3]. Á. Bényi, K. Gröchenig, K. Okoudjou and L.G. Rogers, Unimodular Fourier multipliers for modulationspaes, J. Funct. Anal. 246 (2007) 366-384.
[4]. E. Cordero and F. Nicola, Boundedness of Schrödinger type propagators on modulation spaces, J. Fourier Anal. Appl. 16 (3) (2010) 311-339.
[5]. E. Cordero and F. Nicola, Remarks on Fourier multipliers and applications to wave equation, J. Math. Anal. Appl. 353 (2009) 583-591.
[6]. E. Cordero and F. Nicola, Strichartz estimates in Wiener amalgam spaces for the Schr¨odinger equation, Math. Nachr 281 (1) (2008) 25-41.
[7]. E. Cordero and F. Nicola and L. Rodino, Time-frequency analysis of Fourier integral operators, Comm. on Pure and Appl. 16 (3) (2010) 1-21.
[8]. E. Cordero and F. Nicola and P. Wahlberg, Schrödinger type propagators, pseudodifferential operators and modulation spaces, J. London Math. Soc., to appear (available at ArXiv:1207.2099[math.FA]).
[9]. H. G. Feichtinger, Banach convolution algebras of Wiener's type. In Proc. Conf. "Function, Series, Operators", Budapest August 1980, Colloq. Math. Soc. J´anos Bolyai, (35) 509-524, North-Holland, Amsterdam, 1983.
[10]. H. G. Feichtinger, Banach spaces of distributions of Wiener's type and interpolation. In Proc. Conf. Functional Analysis and Approximation, Oberwolfach August 1980,Internat. Ser. Numer. Math. 69 (1981) 153-165.
[11]. H. G. Feichtinger, Generalized amalgams, with applications to Fourier transform, Canad. J. Math. 42 (3) (1990) 395-409.
[12]. H. G. Feichtinger, Modulation spaces: looking back and ahead, Sampl. Theory Signal Image Process. 5 (2006) 109-140.
[13]. H. G. Feichtinger, Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, 1983.
[14]. H. G. Feichtinger, G. Narimani, Fourier multipliers of classical modulation spaces, Appl. Comput. Harmon. Anal. 21 (2006) 349-359.
[15]. L. Grafakos, Classical Fourier Analysis, Springer; 2nd ed. 2008.
[16]. L. H¨ormander, Estimates for translation invariant operators in L^p spaces, Acta Math. 104 (1960) 93-140.
[17]. H. Ishii, On some Fourier multipliers and partial differential equations, Math. Japon. (1974) 139-163.
[18]. A. Miyachi, F. Nicola, S. Rivetti, A. Tabacco and N. Tomita, Estimates for unimodular Fourier multipliers on modulation spaces, Proc. Amer. Math. Soc. 137 (11) (2009) 3869-3883.
[19]. M. Ruzhansky, M. Sugimoto, J. Toft and N. Tomita, Changes of variables in modulation and Wiener amalgam spaces, Math. Nachr. 284 (2011) 2078-2092.
[20]. M. Ruzhansky, M. Sugimoto and B. Wang, Modulation spaces and nonlinear evolution equations, Evolution Equations of Hyperbolic and Schrdinger Type, Progress in Mathematics 301 (2012) 267-283.
[21]. S. Sj¨ostrand, On Riesz means of the solutions of the Schr¨odinger equation, Ann. Scuola Nor. Sup. Pisa 24 (1970) 331-348.
[22]. E. M. Stein, Harmonic Analysis. Princeton University Press, Princeton, 1993.
[23]. M. Sugimoto and N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, J. Funct. Anal. 248 (2007) 79-106.
[24]. J.Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus-I, J. Funct. Anal. 207 (2004) 399-429.
[25]. J.Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus-II, Ann. Global Anal. Geom. 26 (2004) 73-106.
[26]. B. Wang and C. Huang, Frequency-uniform decomposition method for generalized BO, KdV and NLS equations, J. Differential Equations 239 (2007) 213-250.
[27]. J. Cunanan, M. Sugimoto, Unimodular Fourier multipliers on Wiener Amalgam Spaces, J. Math. Anal. Appl. 419 (2) (2014) 738-747.