Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 2 (2025) pp: 25-43
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Regular Weighted Pre-Orders in Banach Algebra

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Abstract
Many shown new pre-orders are introduced and characterized on the set of all $w_{r-1}g$-Drazin invertible elements of a Banach algebra. We follow D. Mosić [24] on his consistence to develope the generalize results for rectangular matrices and bounded linear operators between Banach spaces, correspondly improve and adding regular new characterizations.

Key words and phrases: $w_{r-1}g$–Drazin inverse; generalized Drazin pre-order; sharp order; rectangular matrices; minus partial order.

***R**eceived 03 Feb., 2025; **R**evised 11 Feb., 2025; **A**ccepted 13 Feb., 2025 © The author(s) 2025.*
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1 Introduction

Drazin inverses theory is an area which is of great theoretical interest with acceptable applications in many diverse areas, involving statistics, numerical analysis, Markov chains, differential equation, control theory and others. With the rectangular systems of differential equation arising, the weighted Drazin inverse for a rectangular matrix has been defined and studied by Cline and Greville [2]. In the dense literature Drazin inverse, the weighted Drazin inverse and their generalizations, including complex matrices, linear bounded operators on Banach or Hilbert spaces, elements in Banach algebras or rings are considered see [1, 8, 9, 15, 16, 17, 20, 21, 22, 23]. For $\mathcal{A}$ be a complex unital Banach algebra. We use $\mathcal{A}^{-1}$, $\mathcal{A}^{\cdot}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$ to denote the sets of all invertible elements, idempotents, nilpotent and quasinilpotent elements of $\mathcal{A}$, respectively.

For $w_{r-1} \in \mathcal{A}$ be a fixed nonzero element. An element $a_{r-1} \in \mathcal{A}$ is $w_{r-1}g$–Drazin invertible if there exists unique $b_{r-1} \in \mathcal{A}$ such that

$$a_{r-1}w_{r-1}b_{r-1} = b_{r-1}w_{r-1}a_{r-1}, b_{r-1}w_{r-1}a_{r-1}w_{r-1}b_{r-1} = b_{r-1} \text{ and } a_{r-1} - a_{r-1}w_{r-1}b_{r-1}w_{r-1}a_{r-1} \in \mathcal{A}^{qnil}$$

The $w_{r-1}g$-Drazin inverse b_{r-1} of a_{r-1} is denoted by $a_{r-1}^{d,w_{r-1}}$ [3]. If $a_{r-1} - a_{r-1}w_{r-1}b_{r-1}w_{r-1}a_{r-1} \in \mathcal{A}^{nil}$ in the above definition, then $a_{r-1}^{d,w_{r-1}} = a_{r-1}^{D,w_{r-1}}$ is the w_{r-1}-weighted Drazin inverse of a_{r-1}[2]. In the case that $w_{r-1} = 1$, then $a_{r-1}^{d} = a_{r-1}^{d,w_{r-1}}$ is the generalized Drazin inverse (or the Koliha-Drazin inverse) of a_{r-1} [7] and $a_{r-1}^{D} = a_{r-1}^{D,w_{r-1}}$ is the Drazin inverse of a_{r-1}[4]. The condition $a_{r-1} - a_{r-1}b_{r-1}a_{r-1} \in \mathcal{A}^{nil}$ is equivalent to $a_{r-1}^{k+1}b_{r-1} = a_{r-1}^{k}$, for some non-negative integer k. The smallest such k is called the index of a_{r-1} and it is denoted by $\text{ind}(a_{r-1})$. If $\text{ind}(a_{r-1}) \leq 1$, then a_{r-1} is group invertible and a_{r-1}^{D} is the group inverse of a_{r-1} denoted by $a_{r-1}^{\neq}$. The sets of all $w_{r-1}g$-Drazin invertible, w_{r-1}-Drazin invertible, generalized Drazin invertible, Drazin invertible and group invertible elements in $\mathcal{A}$ are denoted by $\mathcal{A}^{d,w_{r-1}}$, $\mathcal{A}^{D,w_{r-1}}$, $\mathcal{A}^{d}$, $\mathcal{A}^{D}$ and $\mathcal{A}^{\neq}$, respectively.

Lemma 1.1. [17] *Let $\mathcal{A}$ be a complex Banach algebra, and let $w_{r-1} \in \mathcal{A}\backslash\{0\}$. For $a_{r-1} \in \mathcal{A}$ the following statements are equivalent*:

(i) $a_{r-1} \in \mathcal{A}^{d,w_{r-1}}$ *and* $a_{r-1}^{d,w_{r-1}} = b_{r-1} \in \mathcal{A}$.

(ii) $a_{r-1}w_{r-1} \in \mathcal{A}^d$ *and* $(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}$.

(iii) $w_{r-1}a_{r-1} \in \mathcal{A}^d$ *and* $(w_{r-1}a_{r-1})^d = w_{r-1}b_{r-1}$.

The $w_{r-1}g$-Drazin inverse $a_{r-1}^{d,w_{r-1}}$ of a then satisfies

$$a_{r-1}^{d,w_{r-1}} = ((a_{r-1}w_{r-1})^d)^2 a_{r-1} = a_{r-1}((w_{r-1}a_{r-1})^d)^2 \quad (1)$$

For idempotents $e_r, e_{r+1}, \ldots, e_{r+n-1} \in \mathcal{A}$ such that $e_{r+i-1}e_{r+j-1} = e_{r+j-1}e_{r+i-1} = 0$, for $i \neq j$, the equality $e_r + e_{r+1} + \cdots + e_{r+n-1} = 1$ is called a decomposition of the identity of $\mathcal{A}$. Let $e_r + e_{r+1} + \cdots + e_{r+n-1} = 1$ and $f_r + f_{r+1} + \cdots + f_{r+n-1} = 1$ be two decompositions of the identity of $\mathcal{A}$. We can represent any $a_{r-1} \in \mathcal{A}$ as a matrix

$$a_{r-1} = \begin{Bmatrix} a_{rr} & \cdots & a_{r(r+n-1)} \\ \vdots & \ddots & \vdots \\ a_{(r+n-1)r} & \cdots & a_{(r+n-1)(r+n-1)} \end{Bmatrix}_{e_{r-1}\times f_{r-1}}$$

where $a_{ij} = e_{r+i-1}a_{r-1}f_{r+j-1} \in e_{r+i-1}\mathcal{A}f_{r+j-1}$. Now usual algebraic operations in $\mathcal{A}$ can be interpreted as simple operations between appropriate $(r+n-1 \times r+n-1)$ matrices over $\mathcal{A}$. Let $a_{r-1} \in \mathcal{A}$ and p, $q \in \mathcal{A}^{\cdot}$ Then we write

$$a_{r-1} = \begin{bmatrix} pa_{r-1}q & pa_{r-1}(1-q) \\ (1-p)a_{r-1}q & (1-p)a_{r-1}(1-q) \end{bmatrix}_{p,q} = \begin{bmatrix} a_{rr} & a_{r(r+1)} \\ a_{(r+1)r} & a_{(r+1)^2} \end{bmatrix}_{p,q}$$

If $p = q$, we denote $a_{r-1} = \begin{Bmatrix} a_{rr} & a_{r(r+1)} \\ a_{(r+1)r} & a_{(r+1)^2} \end{Bmatrix} p$ ' where $a_{rr} = pa_{r-1}p$, $a_{r(r+1)} = pa_{r-1}(1-p)$, $a_{(r+1)r} = (1-p)a_{r-1}p, a_{(r+1)^2} = (1-p)a_{r-1}(1-p)$.

The following result concerning the matrix form of $a_{r-1} \in \mathcal{A}^{d,w_{r-1}}$ will be needed see [24].

Lemma 1.2. [17] *Let $\mathcal{A}$ be a complex Banach algebra, and let $w_{r-1} \in \mathcal{A}\backslash\{0\}$. Then $a_{r-1} \in \mathcal{A}$ is $w_{r-1}g$-Drazin invertible if and only if there exist p, $q \in \mathcal{A}^{\cdot}$ such that p commutes with aw, q commutes with* $w_{r-1}a_{r-1}$,

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_{p,q} \text{ and } w_{r-1} = \begin{bmatrix} w_r & 0 \\ 0 & w_{r+1} \end{bmatrix}_{q,p},$$

where $a_r w_r \in (p\mathcal{A}p)^{-1}$, $w_r a_r \in (q\mathcal{A}q)^{-1}$, $a_{r+1}w_{r+1} \in \left((1-p)\mathcal{A}(1-p)\right)^{qnil}$ *and* $w_{r+1}a_{r+1} \in ((1-q)\mathcal{A}(1-q))^{qnil}$ *The $w_{r-1}g$-Drazin inverse of a_{r-1} is given by*

$$a_{r-1}^{d,w_{r-1}} = \begin{bmatrix} a_r((w_r a_r)^{-1})^2 & 0 \\ 0 & 0 \end{bmatrix}_{p,q} = \begin{bmatrix} ((a_r w_r)^{-1})^2 a_r & 0 \\ 0 & 0 \end{bmatrix}_{p,q} ((a_r w_r)^{-1})^2 (2)$$

Notice that $p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$ and $q = w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d$ in Lemma 1.2.

Let p, $q \in \mathcal{A}^{\cdot}$ such that $p \neq q$. Then $p\mathcal{A}p$ is a algebra with the unit p and we can talk about invertibility of its elements. Since $p\mathcal{A}q$ does not have a unit, we will talk about invertibility of its elements only in the following sense: let , $q \in \mathcal{A}^{\cdot}$, an element $a_{r-1} \in (p\mathcal{A}q)^{-1}$ if there exists $a'_{r-1} \in q\mathcal{A}p$ such that

$$a_{r-1} \in p\mathcal{A}q, a_{r-1}a'_{r-1} = p \text{ and } a'_{r-1}a_{r-1} = q.$$

If this inverse a'_{r-1} of a_{r-1} exists, it is unique.

Lemma 1.3 (see [24]). *Let $a_r \in p\mathcal{A}q$ and $w_r \in q\mathcal{A}p$. Then $a_r \in (p\mathcal{A}q)^{-1}$ and $w_r \in (q\mathcal{A}p)^{-1}$ if and only if $a_r w_r \in (p\mathcal{A}p)^{-1}$ and $w_r a_r \in (q\mathcal{A}q)^{-1}$*

Proof. If $a_r \in (p\mathcal{A}q)^{-1}$ and $w_r \in (q\mathcal{A}p)^{-1}$, there exist $a'_r \in q\mathcal{A}p$ and $w'_r \in p\mathcal{A}q$ such that $a_r a'_r = p$, $a'_r a_r = q$, $w_r w'_r = q$ and $w'_r w_r = p$. Then $a_r w_r w'_r a'_r = a_r q a'_r = a_r a'_r = p$ and $w'_r a'_r a_r w_r = w'_r q w_r = w'_r w_r = p$ implies $a_r w_r \in (p\mathcal{A}p)^{-1}$ and $w'_r a'_r$ is the inverse of $a_r w_r$ in the algebra $p\mathcal{A}p$. Similarly, we check that $w_r a_r \in (q\mathcal{A}q)^{-1}$.

Suppose that $a_r w_r \in (p\mathcal{A}p)^{-1}$ and $w_r a_r \in (q\mathcal{A}q)^{-1}$. Denote by $(a_r w_r)_p^{-1}$ the inverse of $a_r w_r$ in $p\mathcal{A}p$

and by $(w_r a_r)_q^{-1}$ the inverse of $w_r a_r$ in $q\mathcal{A}q$. From $a_r w_r (a_r w_r)_p^{-1} = p, (w_r a_r)_q^{-1} w_r a_r = q$ and $w_r (a_r w_r)_p^{-1} = (w_r a_r)_q^{-1} w_r \in q\mathcal{A}p$, we deduce that $a_r \in (p\mathcal{A}q)^{-1}$. In the same way, we verify that $w_r \in (q\mathcal{A}p)^{-1}$.

Recall that a reflexive and transitive binary relation on a non-empty set is called a preorder. A pre-order is a partial order if it is antisymmetric. Using various generalized inverses, various partial orders were defined. For detailed results related to pre-orders and partial orders on the set of complex matrices see [13].

The minus partial order was defined by Mitsch [12] on an arbitrary semigroup S: for $a_{r-1}, b_{r-1} \in S$, we write

$$a_{r-1} \leq^{-} b_{r-1} \text{ if } a_{r-1} = x b_{r-1} = b_{r-1} y \text{ and } x a_{r-1} = a_{r-1},$$

for some $x, y \in S^1$, where S^1 denotes S, if S has the identity, and S with the identity adjoined otherwise.

For $a_{r-1} \in \mathcal{A}^{\#}$ and $b_{r-1} \in \mathcal{A}$, we say that a_{r-1} is below b_{r-1} under the sharp order (denoted by $a_{r-1} \leq^{\#} b_{r-1}$) if $a_{r-1}^{\#} a_{r-1} = a_{r-1}^{\#} b_{r-1}$ and $a_{r-1} a_{r-1}^{\#} = b_{r-1} a_{r-1}^{\#}$. The sharp order is a partial order on the set of all group invertible elements of $\mathcal{A}$[10].

Let $a_{r-1} \in \mathcal{A}^d$. Then $a_{r-1} = c_{a_{r-1}} + q_{a_{r-1}}$ is called the core-quasinilpotent decomposition of a_{r-1}, where $c_{a_{r-1}} = a_{r-1}^2 a_{r-1}^d$ is the core part of a_{r-1} and $q_{a_{r-1}} = (1 - a_{r-1} a_{r-1}^d) a_{r-1}$ is the quasinilpotent part of a_{r-1}[3]. Notice that $c_{a_{r-1}} q_{a_{r-1}} = q_{a_{r-1}} c_{a_{r-1}} = 0$, $c_{a_{r-1}} \in \mathcal{A}^{\#}$ and $c_{a_{r-1}}^{\#} = a_{r-1}^d$.

For $a_{r-1}, b_{r-1} \in \mathcal{A}^d$ such that $a_{r-1} = c_{a_{r-1}} + q_{a_{r-1}}$ and $b_{r-1} = c_{b_{r-1}} + q_{b_{r-1}}$ are the core-quasinilpotent decompositions of a_{r-1} and b_{r-1}, respectively, we say that a_{r-1} is related to b_{r-1} under the generalized Drazin relation (denoted by $a_{r-1} \leq^d b_{r-1}$) if $c_{a_{r-1}} \leq^{\#} c_{b_{r-1}}$[18]. The generalized Drazin relation is a pre-order on $\mathcal{A}^d$ and it is not antisymmetric (see [13, Example 4.4.5]), because it ignores the quasinilpotent parts of operators. The following result was proved in [18, Theorem 1.1] for linear bounded operators on a Banach space.

Lemma 1.4 (see [24]). *Let , $b_{r-1} \in \mathcal{A}^d$. Then the following statements are equivalent*: (i) $a_{r-1} \leq^d b_{r-1}$;
(ii) $a_{r-1}^d a_{r-1} = a_{r-1}^d b_{r-1}$ *and* $a_{r-1} a_{r-1}^d = b_{r-1} a_{r-1}^d$;
(iii) *there exists* $p \in \mathcal{A}^{\cdot}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_p \text{ and } b_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & b_{r+1} \end{bmatrix}_p,$$

where $a_r \in (p\mathcal{A}p)^{-1}$, $a_{r+1} \in ((1-p)\mathcal{A}(1-p))^{qnil}$ *and* $b_{r+1} \in ((1-p)\mathcal{A}(1-p))^d$.

Proof. (i) ⇒ (ii): Since $a_{r-1} \leq^d b_{r-1}$, then $c_{a_{r-1}} c_{a_{r-1}}^{\#} = c_{b_{r-1}} c_{a_{r-1}}^{\#} = c_{a_{r-1}}^{\#} c_{b_{r-1}}$ which give $q_{b_{r-1}} c_{a_{r-1}}^{\#} = q_{b_{r-1}} c_{a_{r-1}} (c_{a_{r-1}}^{\#})^2 = q_{b_{r-1}} c_{b_{r-1}} (c_{a_{r-1}}^{\#})^2 = 0$ and so $a_{r-1} a_{r-1}^d = c_{a_{r-1}} c_{a_{r-1}}^{\#} = c_{b_{r-1}} c_{a_{r-1}}^{\#} = b_{r-1} c_{a_{r-1}}^{\#} = b_{r-1} a_{r-1}^d$. Similarly, we check that $a_{r-1}^d a_{r-1} = a_{r-1}^d b_{r-1}$. (ii) ⇒ (iii): Recall that if $a_{r-1} \in \mathcal{A}^d$ and $p = a_{r-1} a_{r-1}^d$, then

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_p \text{ and } a_{r-1}^d = \begin{bmatrix} a_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p$$

where $a_r \in (p\mathcal{A}p)^{-1}$ and $a_{r+1} \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Let

$$b_{r-1} = \begin{bmatrix} b_r & b_{r+2} \\ b_{r+3} & b_{r+1} \end{bmatrix}_p$$

From $a_{r-1}^d a_{r-1} = a_{r-1}^d b_{r-1} = b_{r-1} a_{r-1}^d$, we get $b_r = a_r$ and $b_{r+2} = b_{r+3} = 0$. Since $b_{r-1} = \begin{Bmatrix} a_r & 0 \\ 0 & b_{r+1} \end{Bmatrix}$

p is generalized Drazin invertible, we deduce that $b_{r+1} \in ((1-p)\mathcal{A}(1-p))^d$.

(iii) $\Rightarrow$ (i) : Using $b_{r-1}^d = \begin{Bmatrix} a_r^{-1} & 0 \\ 0 & b_{r+1}^d \end{Bmatrix}$ $p'c_{a_{r-1}} = a_{r-1}^2 a_{r-1}^d$ and $c_{b_{r-1}} = b_{r-1}^2 b_{r-1}^d$, we obtain $c_{a_{r-1}} c_{a_{r-1}}^{\#} = c_{b_{r-1}} c_{a_{r-1}}^{\#} = c_{a_{r-1}}^{\#} c_{b_{r-1}}$.

Some weighted pre-orders were defined and studied on the set of rectangular matrices in [5, 6], extending the Drazin pre-order and its generalization to partial order which were introduced in [13]. These definitions were generalized to the class of $w_{r-1}g$-Drazin invertible bounded linear operators between Banach spaces in [14, 18]. The Drazin pre-order and its generalization to a partial order have been investigated for elements of a ring in [11]. Using the weighted element $w_{r-1} \in \mathcal{A}\backslash\{0\}$, the generalized Drazin pre-order of certain elements and the weighted Drazin inverse, we introduce and characterize new weighted preorders on the set of all $w_{r-1}g$-Drazin invertible elements of a Banach algebra. Comparing matrix representations of corresponding elements, notice that we generalize some results from [5, 6, 14, 18] and add new characterizations of weighted pre-orders. Considering the minus partial order, the sharp partial order, the core and the quasinilpotent parts of elements $a_{r-1}w_{r-1}$ or/and $w_{r-1}a_{r-1}$, we present new weighted pre-orders. Thus, we extend the results for bounded linear operators between Banach spaces [14] to elements of a Banach algebra without assumption that the quasinilpotent part of $a_{r-1}w_{r-1}$ (or/and $w_{r-1}a_{r-1}$) is relatively regular. As a consequence, the generalized Drazin pre-order is extended to a partial order. In the end, we define and investigate weighted pre-orders on the set of all w_{r-1}-Drazin invertible elements in a ring generalizing recent results from [11] (see also [24]).

2 Regular Weighted Pre-Orders

We establish some regular weighted pre-orders generalizing corresponding weighted pre-orders defined on the set of rectangular matrices in [6] and the set of bounded linear operators between Banach spaces in [18], to the set of all $w_{r-1}g$-Drazin invertible elements of a Banach algebra (see [24]).

Definition 2.1. *Let $a_{r-1}, b_{r-1} \in \mathcal{R}$ and $w_{r-1} \in \mathcal{A}\backslash\{0\}$. If a_{r-1} is $w_{r-1}g$-Drazin invertible, then we say that* (i) $a_{r-1} \leq^{d,w_{r-1},r_0+r-1} b_{r-1}$ *if* $a_{r-1}w_{r-1} \leq^d b_{r-1}w_{r-1}$,

(ii) $a_{r-1} \leq^{d,w_{r-1},l+r-1} b_{r-1}$ *if* $w_{r-1}a_{r-1} \leq^d w_{r-1}b_{r-1}$,

(iii) $a_{r-1} \leq^{d,w_{r-1}} b_{r-1}$ *if* $a_{r-1} \leq^{d,w_{r-1},r_0+r-1} b_{r-1}$ *and* $a_{r-1} \leq^{d,w_{r-1},l+r-1} b_{r-1}$.

Since the generalized Drazin order is a pre-order, we deduce that the relations $\leq^{d,w_{r-1},r_0+r-1}$, $\leq^{d,w_{r-1},l+r-1}$, $\leq^{d,w_{r-1}}$ are pre-orders on the set of all $w_{r-1}g$-Drazin invertible elements of $\mathcal{A}$.

The relation $\leq^{d,w_{r-1},r_0+r-1}$ will be characterized in the following result extending corresponding results from [6, 18] and adding conditions (vii) and (viii). By $a_{r-1}\{1,5\}$ we denote the set of all inner inverse of a_{r-1} which commutes with a_{r-1} , i.e. $a_{r-1}\{1,5\} = \{c_{r-1} \in \mathcal{A} : a_{r-1}c_{r-1}a_{r-1} = a_{r-1}, a_{r-1}c_{r-1} = c_{r-1}a_{r-1}\}$.

Theorem 2.1 (see [24]). *Let $w_{r-1} \in \mathcal{A}\backslash\{0\}$, and let $a_{r-1}, b_{r-1} \in \mathcal{A}$ be $w_{r-1}g$-Drazin invertible. Then the following statements are equivalent:*

(i) $a_{r-1} \leq^{d,w_{r-1},r_0+r-1} b_{r-1}$;

(ii) *aw* $(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = (a_{r-1}w_{r-1})^d b_{r-1}w_{r-1}$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}$$

where $a_r \in (e_r\mathcal{A}f_r)^{-1}$, $w_r \in (f_r\mathcal{A}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2w_r^2 + a_{r+1}^2w_{r+1}^2 + a_{r+2}^2w_r^2 + a_{r+3}^2w_{r+1}^2 \in ((1-e_r)\mathcal{A}(1-e_r))^{qnil}$, $\quad w_{r+1}a_{r+1} = w_r^2a_r^2 + w_r^2a_{r+1}^2 + w_{r+1}^2a_{r+2}^2 + w_{r+1}^2a_{r+3}^2 \in \left((1-f_r)\mathcal{A}(1-f_r)\right)^{qnil}$,

$b_{r+1}^3 w_{r+1}^2 = 0, b_r^2 \in (e_{r+1}\mathcal{A}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{A}e_{r+1})^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$ *and* $w_{r+1}^2 b_{r+1}^2 \in (f_{r+2}\mathcal{A}f_{r+2})^{qnil}$;

(iv) $(b_{r-1}w_{r-1})^2\ (b_{r-1}w_{r-1})^d\{1,5\} \subseteq (a_{r-1}w_{r-1})^2\ (a_{r-1}w_{r-1})^d\{1,5\}$;

(v) $(b_{r-1}w_{r-1})^d \in (a_{r-1}w_{r-1})^2(a_{r-1}w_{r-1})^d\{1,5\}$;

(vi) *there exist an idempotent* p *such that* $(a_{r-1}w_{r-1})^2(a_{r-1}w_{r-1})^d = p(b_{r-1}w_{r-1})^2(b_{r-1}w_{r-1})^d = (b_{r-1}w_{r-1})^2(b_{r-1}w_{r-1})^d p$;

(vii) $[(a_{r-1}w_{r-1})^d]^n = b_{r-1}w_{r-1}[(a_{r-1}w_{r-1})^d]^{n+1} = [(a_{r-1}w_{r-1})^d]^{n+1}b_{r-1}w_{r-1}$, *for all integer* $n \geq 1$;

(viii) $[(a_{r-1}w_{r-1})^d]^n = b_{r-1}w_{r-1}[(a_{r-1}w_{r-1})^d]^{n+1} = [(a_{r-1}w_{r-1})^d]^{n+1}b_{r-1}w_{r-1}$, *for some integer* $n \geq 1$.

Proof. (i) ⇔ (ii): It follows from the definition of the pre-order $\leq^{d,w_{r-1},r_0+r-1}$ and Lemma 1.4. (ii) ⇒ (iii): By Lemma 1.2, we have that, for $p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$ and $q = w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d$,

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_{p,q} \text{ and } w_{r-1} = \begin{bmatrix} w_r & 0 \\ 0 & w_{r+1} \end{bmatrix}_{p,q},$$

where $a_r w_r \in (p\mathcal{A}p)^{-1}$, $w_r a_r \in (q\mathcal{A}q)^{-1}$, $a_{r+1}w_{r+1} \in \big((1-p)\mathcal{A}(1-p)\big)^{qnil}$ and $w_{r+1}a_{r+1} \in ((1-q)\mathcal{A}(1-q))^{qnil}$. Thus,

$$a_{r-1}w_{r-1} = \begin{bmatrix} a_r w_r & 0 \\ 0 & a_{r+1}w_{r+1} \end{bmatrix}_p \text{ and } (a_{r-1}w_{r-1})^d = \begin{bmatrix} (a_r w_r)^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

By Lemma 1.3, $a_r \in (p\mathcal{A}q)^{-1}$ and $w_r \in (q\mathcal{A}p)^{-1}$.

Let

$$b_{r-1} = \begin{Bmatrix} b_r & b_{r+2} \\ b_{r+3} & b_{r+1} \end{Bmatrix}_{p,q}$$

From

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = \begin{bmatrix} b_r w_r(a_r w_r) & 0 \\ b_{r+3}w_r(a_r w_r) & 0 \end{bmatrix}_p$$

and

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = \begin{bmatrix} b_r w_r(a_r w_r) & (a_r w_r)^{-1}b_{r+2}w_{r+1} \\ b_{r+3}w_r(a_r w_r) & 0 \end{bmatrix}_p,$$

we deduce that $b_r w_r = a_r w_r$, $b_{r+3}w_r = 0$ and $b_{r+2}w_{r+1} = 0$. So, $b_{r+3} = b_{r+3}q = b_{r+3}w_r a_r(w_r a_r)^{-1} = 0, b_r = b_r q = b_r w_r a_r (w_r a_r)^{-1} = a_r w_r a_r (w_r a_r)^{-1} = a_r q = a_r$ and

$$b_{r-1}w_{r-1} = \begin{bmatrix} a_r w_r & 0 \\ 0 & b_{r+1}w_{r+1} \end{bmatrix}_p.$$

If $w_{r+1} \neq 0$, because $b_{r-1}w_{r-1}$ is generalized Drazin invertible, then $b_{r+1}w_{r+1}$ is generalized Drazin invertible and so b_{r+1} is $w_{r+1}g$-Drazin invertible. Using Lemma 1.2, for $r_0 = b_{r+1}w_{r+1}(b_{r+1}w_{r+1})^d$ and $s = w_{r+1}b_{r+1}(w_{r+1}b_{r+1})^d$, we observe that

$$b_{r+1} = \begin{bmatrix} b_r^2 & 0 \\ 0 & b_{r+1}^2 \end{bmatrix}_p \text{ and } w_{r+1} = \begin{bmatrix} w_r^2 & 0 \\ 0 & w_{r+1}^2 \end{bmatrix}_p,$$

where $b_r^2 = r_0 b_{r+1}s$, $b_{r+1}^2 = (1-r_0)b_{r+1}(1-s)$, $w_r^2 = s w_{r+1} r_0$, $w_{r+1}^2 = (1-s)w_{r+1}(1-r_0)$, $b_r^2 w_r^2 \in (r_0\mathcal{A}r_0)^{-1}, w_r^2 b_r^2 \in (s\mathcal{A}s)^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in \big((1-r_0)\mathcal{A}(1-r_0)\big)^{qnil}$ and $w_{r+1}^2 b_{r+1}^2 \in \big((1-s)\mathcal{A}(1-s)\big)^{qnil}$. Applying again Lemma 1.3, $b_r^2 \in (r\mathcal{A}s)^{-1}$ and $w_r^2 \in (s\mathcal{A}r_0)^{-1}$. Since $b_{r+1}w_{r+1} \in (1-$

$p)\mathcal{A}(1-p)$, then $r_0 p = (b_{r+1}w_{r+1})^d b_{r+1}w_{r+1}p = 0$. In the same way, we get $pr_0 = qs = sq = 0$. Therefore,

$b_{r-1} = b_r + b_{r+2} + b_{r+1} = b_r + b_{r+2} + b_r^2 + b_{r+1}^2 = pb_{r-1}q + pb_{r-1}(1-q) + rb_{r+1}s + (1-r_0)b_{r+1}(1-s)$

$$= pb_{r-1}q + pb_{r-1}(s+1-q-s) + r_0(1-p)b_{r-1}(1-q)s + (1-r_0)(1-p)b_{r-1}(1-q)(1-s)$$
$$= pb_{r-1}q + pb_{r-1}s + pb_{r-1}(1-q-s) + rb_{r-1}s + (1-p-r)b_{r-1}(1-q-s),$$

which gives, for $b_r^3 = pb_{r-1}s$, $b_{r+1}^3 = pb_{r-1}(1-q-s)$ and by $b_r^2 = rb_{r-1}s$, $b_{r+1}^2 = (1-p-r_0)b_{r-1}(1-q-s)$,

$$b_{r-1} = \begin{bmatrix} b_r & b_r^3 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}$$

for the decompositions of identity $e_{r-1}: p + r_0 + (1-p-r_0) = 1$ and $f_{r-1}: q + s + (1-q-s) = 1$. Similarly,

$$w_{r-1} = w_r + w_r^2 + w_{r+1}^2 = qw_{r-1}p + sw_{r+1}r_0 + (1-s)w_{r+1}(1-r_0)$$
$$= qw_{r-1}p + s(1-q)w_{r-1}(1-p)r_0 + (1-s)(1-q)w_{r-1}(1-p)(1-r_0)$$
$$= qw_{r-1}p + sw_{r-1}r_0 + (1-q-s)w_{r-1}(1-p-r_0)$$

implies

$$w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}$$

From $b_{r+2} = pb_{r-1}(1-q) = pb_{r-1}s + pb_{r-1}(1-q-s) = b_r^3 + b_{r+1}^3$ and $w_{r+1} = w_r^2 + w_{r+1}^2$, we get

$$0 = b_{r+2}w_{r+1} = \begin{bmatrix} 0 & b_r^3 & b_{r+1}^3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e_{r-1}\times f_{r-1}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}} = \begin{bmatrix} 0 & b_r^3 w_r^2 & b_{r+1}^3 w_{r+1}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e_{r-1}\times e_{r-1}}$$

Hence, $b_r^3 w_r^2 = 0$ and $b_{r+1}^3 w_{r+1}^2 = 0$. Therefore, $b_r^3 = b_r^3 s = b_r^3 w_r^2 b_r^2 (w_r^2 b_r^2)^{-1} = 0$. To find the representation of a_{r-1}, notice that

$$pa_{r-1}s = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d a_{r-1}s = a_{r-1}(w_{r-1}a_{r-1})^d w_{r-1}a_{r-1}s = a_{r-1}qs = 0$$

and similarly $pa_{r-1}(1-q-s) = 0 = r_0 a_{r-1}q = (1-p-r_0)a_{r-1}q$. Thus, for $a_r^2 = r_0 a_{r-1}s$, $a_{r+1}^2 = ra_{r-1}(1-q-s)$, $a_{r+2}^2 = (1-p-r_0)a_{r-1}s$ and $a_{r+3}^2 = (1-p-r_0)a_{r-1}(1-q-s)$,

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

$a_{r+1}w_{r+1} = a_r^2 w_r^2 + a_{r+1}^2 w_{r+1}^2 + a_{r+2}^2 w_r^2 + a_{r+3}^2 w_{r+1}^2 \in \left((1-p)\mathcal{A}(1-p)\right)^{qnil}$ and $w_{r+1}a_{r+1} = w_r^2 a_r^2 + w_r^2 a_{r+1}^2 + w_{r+1}^2 a_{r+2}^2 + w_{r+1}^2 a_{r+3}^2 \in \left((1-p)\mathcal{A}(1-p)\right)^{qnil}$.

In the case that $w_{r+1} = 0$, we can write $w_r^2 = 0$ and $w_{r+1}^2 = 0$.

(iii) ⇒ (ii): The statement (iii) gives

$$a_{r-1}w_{r-1} = \begin{bmatrix} a_r w_r & 0 & 0 \\ 0 & a_r^2 w_r^2 & a_{r+1}^2 w_{r+1}^2 \\ 0 & a_{r+2}^2 w_r^2 & a_{r+3}^2 w_{r+1}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}} \qquad (a_{r-1}w_{r-1})^d = \begin{bmatrix} (a_r w_r)^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e_{r-1}\times e_{r-1}}$$

and

$$b_{r-1}w_{r-1} = \begin{bmatrix} a_r w_r & 0 & 0 \\ 0 & b_r^2 w_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 w_{r+1}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}}.$$

So,

$$a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = \begin{bmatrix} e_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e_{r-1}\times e_{r-1}} = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = (a_{r-1}w_{r-1})^d b_{r-1}w_{r-1}.$$

$[(\mathrm{iii}) \Rightarrow (\mathrm{iv}) \Rightarrow (\mathrm{v}) \Rightarrow (\mathrm{i})] \wedge [(\mathrm{vi}) \Leftrightarrow (\mathrm{i})]$: In the similar way as in the proof of [18, Theorem 2.2].

(i) ⇒ (vii) ⇒ (viii) ⇒ (i) : These implications follow by elementary computations. □

The next theorem concerning the inequality $\leq^{d,w_{r-1},l+r-1}$ can be proved as Theorem 2.1.

Theorem 2.2. *Let $w_{r-1} \in \mathcal{A}\backslash\{0\}$, and let $a_{r-1}, b_{r-1} \in \mathcal{A}$ be $w_{r-1}g$-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{d,w_{r-1},l+r-1} b_{r-1}$;

(ii) $w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d = w_{r-1}b_{r-1}(w_{r-1}a_{r-1})^d = (w_{r-1}a_{r-1})^d w_{r-1}b_{r-1}$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ b_{r+1}^4 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{A}f_r)^{-1}$, $w_r \in (f_r\mathcal{A}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2w_r^2 + a_{r+1}^2w_{r+1}^2 + a_{r+2}^2w_r^2 + a_{r+3}^2w_{r+1}^2 \in ((1-e_r)\mathcal{A}(1-e_r))^{qnil}$, $w_{r+1}a_{r+1} = w_r^2a_r^2 + w_r^2a_{r+1}^2 + w_{r+1}^2a_{r+2}^2 + w_{r+1}^2a_{r+3}^2 \in \left((1-f_r)\mathcal{A}(1-f_r)\right)^{qnil}$,

$w_{r+1}^2b_{r+1}^4 = 0, b_r^2 \in (e_{r+1}\mathcal{A}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{A}e_{r+1})^{-1}$, $b_{r+1}^2w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$ *and* $w_{r+1}^2b_{r+1}^2 \in (f_{r+2}\mathcal{A}f_{r+2})^{qnil}$;

(iv) $(w_{r-1}b_{r-1})^2\,(w_{r-1}b_{r-1})^d\{1,5\} \subseteq (w_{r-1}a_{r-1})^2(w_{r-1}a_{r-1})^d\{1,5\}$;

(v) $(w_{r-1}b_{r-1})^d \in (w_{r-1}a_{r-1})^2(w_{r-1}a_{r-1})^d\{1,5\}$;

(vi) *there exist an idempotent* p *such that* $(w_{r-1}a_{r-1})^2(w_{r-1}a_{r-1})^d = p(w_{r-1}b_{r-1})^2(w_{r-1}b_{r-1})^d = (w_{r-1}b_{r-1})^2(w_{r-1}b_{r-1})^d p$;

(vii) $[(w_{r-1}a_{r-1})^d]^n = w_{r-1}b_{r-1}[(w_{r-1}a_{r-1})^d]^{n+1} = [(w_{r-1}a_{r-1})^d]^{n+1}w_{r-1}b_{r-1}$, *for all integer* $n \geq 1$;

(viii) $[(w_{r-1}a_{r-1})^d]^n = w_{r-1}b_{r-1}[(w_{r-1}a_{r-1})^d]^{n+1} = [(w_{r-1}a_{r-1})^d]^{n+1}w_{r-1}b_{r-1}$, *for some integer* $n \geq 1$.

Combining conditions of Theorem 2.1 and Theorem 2.2, we can obtain characterizations of the pre-order $\leq^{d,w_{r-1}}$. In the following result, we present some of them and add new characterizations of $\leq^{d,w_{r-1}}$, generalizing recent results from [5, 6, 14, 18].

Theorem 2.3 (see [24]). *Let $w_{r-1} \in \mathcal{A}\backslash\{0\}$, and let $a_{r-1}, b_{r-1} \in \mathcal{A}$ be $w_{r-1}g$-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{d,w_{r-1}} b_{r-1}$;

(ii) $a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = (a_{r-1}w_{r-1})^d b_{r-1}w_{r-1}$ *and* $w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d = w_{r-1}b_{r-1}(w_{r-1}a_{r-1})^d = (w_{r-1}a_{r-1})^d w_{r-1}b_{r-1}$;

(iii) $a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$ *and* $(w_{r-1}a_{r-1})^d w_{r-1}a_{r-1} = (w_{r-1}a_{r-1})^d w_{r-1}b_{r-1}$;

(iv) $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1} = b_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1}$ *and* $w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1}a_{r-1} = w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1}b_{r-1}$;

(v) $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}} = b_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}$ *and* $a_{r-1}^{d,w_{r-1}}w_{r-1}a_{r-1} = a_{r-1}^{d,w_{r-1}}w_{r-1}b_{r-1}$;

(vi) *the elements* a_{r-1}, w_{r-1} *and* b_{r-1} *have the following matrix representations, for* $p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$ *and* $q = w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d$,

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_{p,q}, w_{r-1} = \begin{bmatrix} w_r & 0 \\ 0 & w_{r+1} \end{bmatrix}_{q,p} \; b_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & b_{r+1} \end{bmatrix}_{p,q}$$

where $a_r w_r \in (p\mathcal{A}p)^{-1}$, $w_r a_r \in (q\mathcal{A}q)^{-1}$, $a_{r+1}w_{r+1} \in \left((1-p)\mathcal{A}(1-p)\right)^{qnil}$ *and* $w_{r+1}a_{r+1} \in \left((1-q)\mathcal{A}(1-q)\right)^{qnil}$.

(vii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{A}f_r)^{-1}$, $w_r \in (f_r\mathcal{A}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2w_r^2 + a_{r+1}^2w_{r+1}^2 + a_{r+2}^2w_r^2 + a_{r+3}^2w_{r+1}^2 \in ((1-e_r)\mathcal{A}(1-e_r))^{qnil}$, $w_{r+1}a_{r+1} = w_r^2a_r^2 + w_r^2a_{r+1}^2 + w_{r+1}^2a_{r+2}^2 + w_{r+1}^2a_{r+3}^2 \in \left((1-f_r)\mathcal{A}(1-f_r)\right)^{qnil}$, $b_r^2 \in (e_{r+1}\mathcal{A}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{A}e_{r+1})^{-1}$, $b_{r+1}^2w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$ *and* $w_{r+1}^2b_{r+1}^2 \in (f_{r+2}\mathcal{A}f_{r+2})^{qnil}$.

Proof. (i) ⇔ (ii) ⇔ (vii): By Theorem 2.1 and Theorem 2.2.

(ii) ⇒ (iii): This is obvious.

(iii) ⇒ (iv): It follows from $(a_{r-1}w_{r-1})^d = a_{r-1}^{d,w_{r-1}}w_{r-1}$ and $(w_{r-1}a_{r-1})^d = w_{r-1}a_{r-1}^{d,w_{r-1}}$.

(iv) ⇒ (v) : By the properties of the $w_{r-1}g$-Drazin inverse.

(v) ⇒ (vi): Using Lemma 1.2, for $p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$ and $q = w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d$, we get

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_{p,q} \text{ and } w_{r-1} = \begin{bmatrix} w_r & 0 \\ 0 & w_{r+1} \end{bmatrix}_{q,p},$$

where $a_r w_r \in (p\mathcal{A}p)^{-1}$, $w_r a_r \in (q\mathcal{A}q)^{-1}$, $a_{r+1}w_{r+1} \in \left((1-p)\mathcal{A}(1-p)\right)^{qnil}$ and $w_{r+1}a_{r+1} \in ((1-q)\mathcal{A}(1-q))^{qnil}$. If we suppose that

$$b_{r-1} = \begin{bmatrix} b_r & b_{r+2} \\ b_{r+3} & b_{r+1} \end{bmatrix}_{p,q},$$

by (2), $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}} = b_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}$ and $a_{r-1}^{d,w_{r-1}}w_{r-1}a_{r-1} = a_{r-1}^{d,w_{r-1}}w_{r-1}b_{r-1}$, notice that $b_r = a_r$, $b_{r+2} = 0$ and $b_{r+3} = 0$.

(vi) ⇒ (ii): We can easily check this implication.

If we suppose that $w_{r-1} = 1$ in Theorem 2.3, we obtain one more characterization of the relation $\leq^d$.

Corollary 2.1 [24]. *Let* $a_{r-1}, b_{r-1} \in \mathcal{A}$ *be generalized Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^d b_{r-1}$;

(ii) *there exist a decomposition of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}},$$

where $a_r \in (e_r\mathcal{A}e_r)^{-1}$, $a_{r+1} = a_r^2 + a_{r+1}^2 + a_{r+2}^2 + a_{r+3}^2 \in \left((1-e_r)\mathcal{A}(1-e_r)\right)^{qnil}$, $b_r^2 \in (e_{r+1}\mathcal{A}e_{r+1})^{-1}$ *and* $b_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$.

As a consequence of previous theorems, we obtain the next result (see [24]).

Corollary 2.2. *Let* $w_{r-1} \in \mathcal{A}\backslash\{0\}$, $a_{r-1} \in \mathcal{A}$ *be* $w_{r-1}g$*-Drazin invertible and let* $b_{r-1} \in \mathcal{A}$.

(i) *If* $a_{r-1} \leq^{d,w_{r-1},r_0+r-1} b_{r-1}$, *then* $(a_{r-1}w_{r-1})^n \leq^d (b_{r-1}w_{r-1})^n$, *for all integer* $n \geq 1$.

(ii) *If* $a_{r-1} \leq^{d,w_{r-1},l+r-1} b_{r-1}$, *then* $(w_{r-1}a_{r-1})^n \leq^d (w_{r-1}b_{r-1})^n$, *for all integer* $n \geq 1$.

(iii) *If* $a_{r-1} \leq^{d,w_{r-1}} b_{r-1}$, *then* $(a_{r-1}w_{r-1})^n \leq^d (b_{r-1}w_{r-1})^n$ *and* $(w_{r-1}a_{r-1})^n \leq^d (w_{r-1}b_{r-1})^n$, *for all integer* $n \geq 1$.

Proof. (i) Since $a_{r-1} \leq^{d,w_{r-1},r_0+r-1} b_{r-1}$, then, by Theorem 2.1, $a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = (a_{r-1}w_{r-1})^d b_{r-1}w_{r-1}$. Hence, for $n \geq 1$,

$$[(a_{r-1}w_{r-1})^n]^d(b_{r-1}w_{r-1})^n = [(a_{r-1}w_{r-1})^d]^n(b_{r-1}w_{r-1})^n = [(a_{r-1}w_{r-1})^d b_{r-1}w_{r-1}]^n$$
$$= [(a_{r-1}w_{r-1})^d a_{r-1}w_{r-1}]^n = [(a_{r-1}w_{r-1})^n]^d(a_{r-1}w_{r-1})^n$$

and similarly $(b_{r-1}w_{r-1})^n[(a_{r-1}w_{r-1})^n]^d = [(a_{r-1}w_{r-1})^n]^d(a_{r-1}w_{r-1})^n$, which give $(a_{r-1}w_{r-1})^n \leq^d (b_{r-1}w_{r-1})^n$.

(ii) In the same way as (i), we check this part.

(iii) It follows by (i) and (ii).

If we separate the equalities which appear in part (v) of Theorem 2.3, we define and investigate regular new pre-orders on the set $\mathcal{A}^{d,w_{r-1}}$ generalizing some results in [5, 14].

Definition 2.2. *Let* $w_{r-1} \in \mathcal{A}\backslash\{0\}$, $a_{r-1} \in \mathcal{A}$ *be* $w_{r-1}g$*-Drazin invertible and* $b_{r-1} \in \mathcal{A}$. *Then we say that*

(i) $a_{r-1} \leq^{w_{r-1}gD,r_0+r-1} b_{r-1}$ *if* $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}} = b_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}$,

(ii) $a_{r-1} \leq^{w_{r-1}gD,l+r-1} b_{r-1}$ *if* $a_{r-1}^{d,w_{r-1}}w_{r-1}a_{r-1} = a_{r-1}^{d,w_{r-1}}w_{r-1}b_{r-1}$.

Theorem 2.4 (see [24]). *Let* $w_{r-1} \in \mathcal{A}\backslash\{0\}$. *The relations* $\leq^{w_{r-1}gD,r_0+r-1}$ and $\leq^{w_{r-1}gD,l+r-1}$ *are pre-orders on the set of all* $w_{r-1}g$*-Drazin invertible elements of* $\mathcal{A}$.

Proof. Because $\leq^{w_{r-1}gD,r_0+r-1}$ and $\leq^{w_{r-1}gD,l+r-1}$ are reflexive obviously, we will only verify that $\leq^{w_{r-1}gD,r_0+r-1}$ is transitive and similarly we can check that $\leq^{w_{r-1}gD,l+r-1}$ is transitive too.

Suppose that $a_{r-1}, b_{r-1}, c_{r-1} \in \mathcal{A}$ are $w_{r-1}g$-Drazin invertible such that $a_{r-1} \leq^{w_{r-1}gD,r_0+r-1} b_{r-1}$ and $b_{r-1} \leq^{w_{r-1}gD,r_0+r-1} c_{r-1}$. Applying Lemma 1.2, for $p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$ and $q = w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d$, we have

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_{p,q} \text{ and } w_{r-1} = \begin{bmatrix} w_r & 0 \\ 0 & w_{r+1} \end{bmatrix}_{q,p},$$

where $a_r w_r \in (p\mathcal{A}p)^{-1}$, $w_r a_r \in (q\mathcal{A}q)^{-1}$, $a_{r+1}w_{r+1} \in \left((1-p)\mathcal{A}(1-p)\right)^{qnil}$ and $w_{r+1}a_{r+1} \in ((1-q)\mathcal{A}(1-q))^{qnil}$. Assume that

$$b_{r-1} = \begin{bmatrix} b_r & b_{r+2} \\ b_{r+3} & b_{r+1} \end{bmatrix}_{p,q} \text{ and } c_{r-1} = \begin{bmatrix} c_r & c_{r+2} \\ c_{r+3} & c_{r+1} \end{bmatrix}_{p,q}.$$

The equalities (2) and $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}} = b_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}$ give $b_r = a_r$ and $b_{r+3} = 0$. So,

$$b_{r-1} = \begin{bmatrix} a_r & b_{r+2} \\ 0 & b_{r+1} \end{bmatrix}_{p,q}$$

Since $b_{r-1}w_{r-1} = [a_r w_r 0\ b_{r+1}w_{r+1} b_{r+2}w_{r+1}\ p$ is generalized Drazin invertible, by [1, Theorem 2.3],

$$(b_{r-1}w_{r-1})^d = \begin{bmatrix} (a_r w_r)^{-1} & x \\ 0 & (b_{r+1}w_{r+1})^d \end{bmatrix}_p,$$

for the corresponding element x. From

$$\begin{bmatrix} p & * \\ 0 & * \end{bmatrix}_p = b_{r-1}w_{r-1}(b_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}b_{r-1}^{d,w_{r-1}}w_{r-1}$$

$$= c_{r-1}w_{r-1}b_{r-1}^{d,w_{r-1}}w_{r-1} = c_{r-1}w_{r-1}(b_{r-1}w_{r-1})^d = \begin{bmatrix} c_r w_r (a_r w_r)^{-1} & * \\ c_{r+3} w_r (a_r w_r)^{-1} & * \end{bmatrix}_p,$$

where we denoted by $*$ the entries for which we are not interested in, we get $c_r w_r = a_r w_r$ and $c_{r+3} w_r = 0$. Thus, $c_r = c_r q = c_r w_r a_r (w_r a_r)^{-1} = a_r w_r a_r (w_r a_r)^{-1} = a_r$ and $c_{r+3} = c_{r+3} q = c_{r+3} w_r a_r (w_r a_r)^{-1} = 0$. Now, we obtain

$$a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1} = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$$
$$= \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p = c_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = c_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1}.$$

Multiplying $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1} = c_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1}$ by $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}$ from the right side, we get $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}} = c_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}$, that is, $a_{r-1} \leq^{w_{r-1}gD,r_0+r-1} c_{r-1}$.
Now, we present necessary and sufficient conditions for $a_{r-1} \leq^{w_{r-1}gD,r_0+r-1} b_{r-1}$, where $a_{r-1}, b_{r-1} \in \mathcal{A}^{d,w_{r-1}}$.

Theorem 2.5 (see [24]). *Let* $w_{r-1} \in \mathcal{A}\backslash\{0\}$, *and let* $a_{r-1}, b_{r-1} \in \mathcal{A}$ *be* $w_{r-1}g$*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{w_{r-1}gD,r_0+r-1} b_{r-1}$;

(ii) $a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$;

(iii) $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1} = b_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}w_{r-1}$;

(iv) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & b_r^3 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{A}f_r)^{-1}$, $w_r \in (f_r\mathcal{A}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2 w_r^2 + a_{r+1}^2 w_{r+1}^2 + a_{r+2}^2 w_r^2 + a_{r+3}^2 w_{r+1}^2 \in ((1-e_r)\mathcal{A}(1-e_r))^{qnil}$, $w_{r+1}a_{r+1} = w_r^2 a_r^2 + w_r^2 a_{r+1}^2 + w_{r+1}^2 a_{r+2}^2 + w_{r+1}^2 a_{r+3}^2 \in \left((1-f_r)\mathcal{A}(1-f_r)\right)^{qnil}$, $b_r^2 \in (e_{r+1}\mathcal{A}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{A}e_{r+1})^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$ *and* $w_{r+1}^2 b_{r+1}^2 \in (f_{r+2}\mathcal{A}f_{r+2})^{qnil}$.

Proof. (i) ⇔ (ii) ⇔ (iii): These equivalences are clear.

(i) ⇒ (iv): Applying Lemma 1.2, for $p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^d$ and $q = w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^d$, notice that

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_{p,q} \text{ and } w_{r-1} = \begin{bmatrix} w_r & 0 \\ 0 & w_{r+1} \end{bmatrix}_{q,p},$$

where $a_r w_r \in (p\mathcal{A}p)^{-1}$, $w_r a_r \in (q\mathcal{A}q)^{-1}$, $a_{r+1}w_{r+1} \in \left((1-p)\mathcal{A}(1-p)\right)^{qnil}$ and $w_{r+1}a_{r+1} \in ((1-q)\mathcal{A}(1-q))^{qnil}$.

Assume that

$$b_{r-1} = \begin{bmatrix} b_r & b_{r+2}w_{r+1} \\ b_{r+3} & b_{r+1}w_{r+1} \end{bmatrix}_{p,q}.$$

The equalities (2) and $a_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}} = b_{r-1}w_{r-1}a_{r-1}^{d,w_{r-1}}$ give $b_r = a_r$ and $b_{r+3} = 0$. Hence,

$$b_{r-1}w_{r-1} = \begin{bmatrix} a_r w_r & b_{r+2}w_{r+1} \\ 0 & b_{r+1}w_{r+1} \end{bmatrix}_p.$$

In the case that $w_{r+1} \neq 0$, since $b_{r-1}w_{r-1}$ is generalized Drazin invertible, then $b_{r+1}w_{r+1}$ is generalized Drazin invertible which implies that b_{r+1} is $w_{r+1}g$-Drazin invertible. By Lemma 1.2, for $r_0 = b_{r+1}w_{r+1}(b_{r+1}w_{r+1})^d$ and $s = w_{r+1}b_{r+1}(w_{r+1}b_{r+1})^d$, we have that

$$b_{r+1} = \begin{bmatrix} b_r^2 & 0 \\ 0 & b_{r+1}^2 \end{bmatrix}_{r_0,s} \quad \text{and} \quad w_{r+1} = \begin{bmatrix} w_r^2 & 0 \\ 0 & w_{r+1}^2 \end{bmatrix}_{s,r_0},$$

where $b_r^2 w_r^2 \in (r_0 \mathcal{A} r_0)^{-1}$, $w_r^2 b_r^2 \in (s\mathcal{A}s)^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in \left((1-r_0)\mathcal{A}(1-r_0)\right)^{qnil}$ and $w_{r+1}^2 b_{r+1}^2 \in ((1-s)\mathcal{A}(1-s))^{qnil}$. As in the proof of Theorem 2.1, we obtain the representations for a_{r-1}, w_{r-1} and b_{r-1}.

In the case that $w_{r+1} = 0$, we can write $w_r^2 = 0$ and $w_{r+1}^2 = 0$.

(iv) ⇒ (ii) We can easily check this part. □

Similarly as Theorem 2.5, we check the following characterizations for the regular pre-order $\leq^{w_{r-1}gD,l+r-1}$.

Theorem 2.6 [24]. *Let* $w_{r-1} \in \mathcal{A}\backslash\{0\}$, *and let* a_{r-1}, $b_{r-1} \in \mathcal{A}$ *be* $w_{r-1}g$*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{w_{r-1}gD,l+r-1} b_{r-1}$;

(ii) $(w_{r-1}a_{r-1})^d w_{r-1}a_{r-1} = (w_{r-1}a_{r-1})^d w_{r-1}b_{r-1}$;

(iii) $w_{r-1}a_{r-1}^{d,w_{r-1}} w_{r-1}a_{r-1} = w_{r-1}a_{r-1}^{d,w_{r-1}} w_{r-1}b_{r-1}$;

(iv) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ b_r^4 & b_r^2 & 0 \\ b_{r+1}^4 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{A}f_r)^{-1}$, $w_r \in (f_r\mathcal{A}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2 w_r^2 + a_{r+1}^2 w_{r+1}^2 + a_{r+2}^2 w_r^2 + a_{r+3}^2 w_{r+1}^2 \in ((1-e_r)\mathcal{A}(1-e_r))^{qnil}$, $w_{r+1}a_{r+1} = w_r^2 a_r^2 + w_r^2 a_{r+1}^2 + w_{r+1}^2 a_{r+2}^2 + w_{r+1}^2 a_{r+3}^2 \in \left((1-f_r)\mathcal{A}(1-f_r)\right)^{qnil}$,

$b_r^2 \in (e_{r+1}\mathcal{A}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{A}e_{r+1})^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$ *and* $w_{r+1}^2 b_{r+1}^2 \in (f_{r+2}\mathcal{A}f_{r+2})^{qnil}$.

In the case that $w_{r-1} = 1$ in Definition 2.2, we consider the following generalized Drazin pre-orders on $\mathcal{A}^d$: for $a_{r-1} \in \mathcal{A}^d$ and $b_{r-1} \in \mathcal{A}$, we say that

(i) $a_{r-1} \leq^{gD,r_0+r-1} b_{r-1}$ if $a_{r-1}a_{r-1}^d = b_{r-1}a_{r-1}^d$,

(ii) $a_{r-1} \leq^{gD,l+r-1} b_{r-1}$ if $a_{r-1}^d a_{r-1} = a_{r-1}^d b_{r-1}$.

Corollary 2.3 [24]. *Let* , $b_{r-1} \in \mathcal{A}^d$. *Then*:

(i) $a_{r-1} \leq^{gD,r_0+r-1} b_{r-1}$ *if and only if there exist a decomposition of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & b_r^3 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}},$$

where $a_r \in (e_r\mathcal{A}e_r)^{-1}$, $a_{r+1} = a_r^2 + a_{r+1}^2 + a_{r+2}^2 + a_{r+3}^2 \in \left((1-e_r)\mathcal{A}(1-e_r)\right)^{qnil}$, $b_r^2 \in (e_{r+1}\mathcal{A}e_{r+1})^{-1}$ *and* $b_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$.

(ii) $a_{r-1} \leq^{gD,l+r-1} b_{r-1}$ *if and only if there exist a decomposition of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ b_r^4 & b_r^2 & 0 \\ b_{r+1}^4 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}},$$

where $a_r \in (e_r\mathcal{A}e_r)^{-1}$, $a_{r+1} = a_r^2 + a_{r+1}^2 + a_{r+2}^2 + a_{r+3}^2 \in \left((1-e_r)\mathcal{A}(1-e_r)\right)^{qnil}$, $b_r^2 \in$

$(e_{r+1}\mathcal{A}e_{r+1})^{-1}$ *and* $b_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$.

3 Regular Weighted Pre-Orders Based on the Core-Quasinilpotent Decomposition

Using the sharp order, the minus partial order and the core-quasinilpotent decompositions of elements $a_{r-1}w_{r-1}$ and $w_{r-1}a_{r-1}$, we establish regular new weighted pre-orders on $\mathcal{A}^{d,w_{r-1}}$ extending corresponding definitions from [14] on the set of all W −Drazin invertible linear bounded operators A between two Banach spaces such that the quasinilpotent part of AW is relatively regular.

Definition 3.1. *Let* $w_{r-1} \in \mathcal{A}\backslash\{0\}$ *and* $a_{r-1}, b_{r-1} \in \mathcal{A}$ *be* $w_{r-1}g$*-Drazin invertible. Then we say that*

(i) $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$ *if* $c_{a_{r-1}w_{r-1}} \leq^{\#} c_{b_{r-1}w_{r-1}}$ *and* $q_{a_{r-1}w_{r-1}} \leq^{-} q_{b_{r-1}w_{r-1}}$,

(ii) $a_{r-1} \leq^{\neq,-,w_{r-1},l+r-1} b_{r-1}$ *if* $c_{w_{r-1}a_{r-1}} \leq^{\#} c_{w_{r-1}b_{r-1}}$ *and* $q_{w_{r-1}a_{r-1}} \leq^{-} q_{w_{r-1}a_{r-1}}$,

(iii) $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$ *if* $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$ *and* $a_{r-1} \leq^{\neq,-,w_{r-1},l+r-1} b_{r-1}$.

The relations $\leq^{\neq,-,w_{r-1},r_0+r-1}$, $\leq^{\neq,-,w_{r-1},l+r-1}$, $\leq^{\neq,-,w_{r-1}}$ are pre-orders on $\mathcal{A}^{d,w_{r-1}}$ (because the sharp order and the minus partial order are partial orders), and partial orders on $\mathcal{A}^{d,w_{r-1}}$ if $w_{r-1} \in \mathcal{A}$ is right invertible, left invertible and invertible, respectively.

We now present equivalent conditions for $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$. Notice that, for $u \in \mathcal{A}$, we denote by $u^{\circ} = \{x \in \mathcal{A} : ux = 0\}$ and $ou = \{x \in \mathcal{A} : xu = 0\}$.

Theorem 3.1 (see [24]). *Let* $w_{r-1} \in \mathcal{A}\backslash\{0\}$ *and* $a_{r-1}, b_{r-1} \in \mathcal{A}$ *be* $w_{r-1}g$*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$;

(ii) $a_{r-1} \leq^{d,w_{r-1},r_0+r-1} b_{r-1}$ *and* $a_{r-1}w_{r-1} - (a_{r-1}w_{r-1})^2(a_{r-1}w_{r-1})^d \leq^{-} b_{r-1}w_{r-1} - (b_{r-1}w_{r-1})^2(b_{r-1}w_{r-1})^{\mathrm{d}}$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & a_{r+1}^2 \\ 0 & 0 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{A}f_r)^{-1}$, $w_r \in (f_r\mathcal{A}e_r)^{-1}$, $a_{r+1}^2w_{r+1}^2 = 0$, $a_{r+3}^2w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$, $a_{r+3}^2w_{r+1}^2 \leq^{-} b_{r+1}^2w_{r+1}^2, w_{r+1}^2a_{r+3}^2 \in (f_{r+2}\mathcal{A}f_{r+2}))^{qnil}$, $b_{r+1}^3w_{r+1}^2 = 0$, $b_r^2 \in (e_{r+1}\mathcal{A}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{A}e_{r+1})^{-1}$, $b_{r+1}^2w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$ *and* $w_{r+1}^2b_{r+1}^2 \in (f_{r+2}\mathcal{A}f_{r+2})^{qnil}$.

Proof. (i) ⇔ (ii): This is obvious.

(i) ⇒ (iii): Since $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$ implies $a_{r-1} \leq^{d,w_{r-1},r_0+r-1} b_{r-1}$, then, by Theorem 2.1, there exist decompositions of the identity $1 = e_r + e_{r+1} + e_{r+2}$ and $1 = f_r + f_{r+1} + f_{r+2}$ such that

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_rw_r \in (e_r\mathcal{A}e_r)^{-1}$, $w_ra_r \in (f_r\mathcal{A}f_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2w_r^2 + a_{r+1}^2w_{r+1}^2 + a_{r+2}^2w_r^2 + a_{r+3}^2w_{r+1}^2 \in ((1-e_r)\mathcal{A}(1-e_r))^{qnil}$, $w_{r+1}a_{r+1} = w_r^2a_r^2 + w_r^2a_{r+1}^2 + w_{r+1}^2a_{r+2}^2 + w_{r+1}^2a_{r+3}^2 \in \left((1-f_r)\mathcal{A}(1-f_r)\right)^{qnil}$, $b_{r+1}^3w_{r+1}^2 = 0, b_r^2w_r^2 \in (e_{r+1}\mathcal{A}e_{r+1})^{-1}$, $w_r^2b_r^2 \in (f_{r+1}\mathcal{A}f_{r+1})^{-1}$, $b_{r+1}^2w_{r+1}^2 \in (e_{r+2}\mathcal{A}e_{r+2})^{qnil}$ and $w_{r+1}^2b_{r+1}^2 \in (f_{r+2}\mathcal{A}f_{r+2})^{qnil}$. Therefore, the core part of $a_{r-1}w_{r-1}$ is $c_{a_{r-1}w_{r-1}} = a_rw_r$ and the quasinilpotent part of aw is $q_{a_{r-1}w_{r-1}} = a_{r+1}w_{r+1}$. Also, the core and quasinilpotent part of $b_{r-1}w_{r-1}$ are $c_{b_{r-1}w_{r-1}} = a_rw_r + b_r^2w_r^2$ and $q_{b_{r-1}w_{r-1}} = b_{r+1}^2w_{r+1}^2$, respectively. Because $q_{a_{r-1}w_{r-1}} \leq^{-} q_{b_{r-1}w_{r-1}}$, that is $a_{r+1}w_{r+1} \leq^{-} b_{r+1}^2w_{r+1}^2$, there exist x, $y \in \mathcal{A}$ such that $a_{r+1}w_{r+1} = xb_{r+1}^2w_{r+1}^2 = b_{r+1}^2w_{r+1}^2y$ and $xa_{r+1}w_{r+1} = a_{r+1}w_{r+1}$. Hence, $(b_{r+1}^2w_{r+1}^2)^{\circ} \subseteq (a_{r+1}w_{r+1})^{\circ}$ and $\mathrm{o}(b_{r+1}^2w_{r+1}^2) \subseteq \circ\, (a_{r+1}w_{r+1})$. The equalities $b_{r+1}^2w_{r+1}^2 = e_{r+2}(b_{r-1}w_{r-1})e_{r+2}$ and $e_{r+1}e_{r+2} = 0 = e_{r+2}e_{r+1}$ give $e_{r+1}b_{r+1}^2w_{r+1}^2 = 0 =$

$b_{r+1}^2 w_{r+1}^2 e_{r+1}$ and so $e_{r+1} \in (b_{r+1}^2 w_{r+1}^2)^{\circ} \cap o(b_{r+1}^2 w_{r+1}^2) \subseteq (a_{r+1} w_{r+1})^{o} \cap o(a_{r+1} w_{r+1})$. From

$$a_{r+1} w_{r+1} = a_r^2 w_r^2 + a_{r+1}^2 w_{r+1}^2 + a_{r+2}^2 w_r^2 + a_{r+3}^2 w_{r+1}^2$$
$$= e_{r+1}(a_{r-1} w_{r-1}) e_{r+1} + e_{r+1}(a_{r-1} w_{r-1}) e_{r+2} + e_{r+2}(a_{r-1} w_{r-1}) e_{r+1} + e_{r+2}(a_{r-1} w_{r-1}) e_{r+2},$$

we get $0 = a_{r+1} w_{r+1} e_{r+1} = e_{r+1}(a_{r-1} w_{r-1}) e_{r+1} + e_{r+2}(a_{r-1} w_{r-1}) e_{r+1}$ and $0 = e_{r+1} a_{r+1} w_{r+1} = e_{r+1}(a_{r-1} w_{r-1}) e_{r+1} + e_{r+1}(a_{r-1} w_{r-1}) e_{r+2}$, i.e. $-e_{r+1}(a_{r-1} w_{r-1}) e_{r+1} = e_{r+2}(a_{r-1} w_{r-1}) e_{r+1} = e_{r+1}(a_{r-1} w_{r-1}) e_{r+2}$. By $e_{r+1} \mathcal{A} e_{r+2} \cap e_{r+2} \mathcal{A} e_{r+1} = \{0\}$, we observe that $e_{r+1}(a_{r-1} w_{r-1}) e_{r+1} = e_{r+2}(a_{r-1} w_{r-1}) e_{r+1} = e_{r+1}(a_{r-1} w_{r-1}) e_{r+2} = 0$, that is $a_r^2 w_r^2 = a_{r+1}^2 w_{r+1}^2 = a_{r+2}^2 w_r^2 = 0$. Thus, $a_r^2 = a_r^2 f_{r+1} = a_r^2 w_r^2 b_r^2 (w_r^2 b_r^2)^{-1} = 0$ and $a_{r+2}^2 = a_{r+2}^2 f_{r+1} = a_{r+2}^2 w_r^2 b_r^2 (w_r^2 b_r^2)^{-1} = 0$. Now, $a_{r+1} w_{r+1} = a_{r+3}^2 w_{r+1}^2$ implies $a_{r+3}^2 w_{r+1}^2 = x b_{r+1}^2 w_{r+1}^2 = b_{r+1}^2 w_{r+1}^2 y$ and $x a_{r+3}^2 w_{r+1}^2 = a_{r+3}^2 w_{r+1}^2$, i.e. $a_{r+3}^2 w_{r+1}^2 \leq^{-} b_{r+1}^2 w_{r+1}^2$. Since $a_{r+1} w_{r+1} = a_{r+3}^2 w_{r+1}^2$ is quasinilpotent, then $a_{r+3}^2 w_{r+1}^2 \in (e_{r+2} \mathcal{A} e_{r+2})^{qnil}$ and so $w_{r+1}^2 a_{r+3}^2 \in (f_{r+2} \mathcal{A} f_{r+2})^{qnil}$.

(iii) $\Rightarrow$ (i) : Notice that $c_{a_{r-1} w_{r-1}} = a_r w_r$, $q_{a_{r-1} w_{r-1}} = a_{r+3}^2 w_{r+1}^2$, $c_{b_{r-1} w_{r-1}} = a_r w_r + b_r^2 w_r^2$ and $q_{b_{r-1} w_{r-1}} = b_{r+1}^2 w_{r+1}^2$. The hypothesis $a_{r+3}^2 w_{r+1}^2 \leq^{-} b_{r+1}^2 w_{r+1}^2$ gives $q_{a_{r-1} w_{r-1}} \leq^{-} q_{b_{r-1} w_{r-1}}$. Using Theorem 2.1, we deduce that $a_{r-1} \leq^{d, w_{r-1}, r_0+r-1} b_{r-1}$ and so $c_{a_{r-1} w_{r-1}} \leq^{\#} c_{b_{r-1} w_{r-1}}$.

As Theorem 3.1, we prove the next characterizations of relations $\leq^{\neq,-,w_{r-1},l+r-1}$ and $\leq^{\neq,-,w_{r-1}}$.

Theorem 3.2 [24]. *Let $w_{r-1} \in \mathcal{A} \backslash \{0\}$ and a_{r-1}, $b_{r-1} \in \mathcal{A}$ be $w_{r-1} g$ -Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{\neq,-,w_{r-1},l+r-1} b_{r-1}$;

(ii) $a_{r-1} \leq^{d, w_{r-1}, l+r-1} b_{r-1}$ *and* $w_{r-1} a_{r-1} - (w_{r-1} a_{r-1})^2 (w_{r-1} a_{r-1})^d \leq^{-} w_{r-1} b_{r-1} - (w_{r-1} b_{r-1})^2 (w_{r-1} b_{r-1})^d$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1} \times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1} \times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ b_{r+1}^4 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1} \times f_{r-1}},$$

where $a_r \in (e_r \mathcal{A} f_r)^{-1}$, $w_r \in (f_r \mathcal{A} e_r)^{-1}$, $a_{r+3}^2 w_{r+1}^2 \in (e_{r+2} \mathcal{A} e_{r+2})^{qnil}$, $w_{r+1}^2 a_{r+2}^2 = 0$, $w_{r+1}^2 a_{r+3}^2 \in (f_{r+2} \mathcal{A} f_{r+2})^{qnil}$, $w_{r+1}^2 a_{r+3}^2 \leq^{-} w_{r+1}^2 b_{r+1}^2$, $w_{r+1}^2 b_{r+1}^4 = 0$, $b_r^2 \in (e_{r+1} \mathcal{A} f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1} \mathcal{A} e_{r+1})^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in (e_{r+2} \mathcal{A} e_{r+2})^{qnil}$ *and* $w_{r+1}^2 b_{r+1}^2 \in (f_{r+2} \mathcal{A} f_{r+2})^{qnil}$.

Theorem 3.3 [24]. *Let $w_{r-1} \in \mathcal{A} \backslash \{0\}$ and a_{r-1}, $b_{r-1} \in \mathcal{A}$ be $w_{r-1} g$ -Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$;

(ii) $a_{r-1} \leq^{d, w_{r-1}, l+r-1} b_{r-1}$, $a_{r-1} \leq^{d, w_{r-1}, r_0+r-1} b_{r-1}$ *and* $a_{r-1} w_{r-1} - (a_{r-1} w_{r-1})^2 (a_{r-1} w_{r-1})^d \leq^{-} b_{r-1} w_{r-1} - (b_{r-1} w_{r-1})^2 (b_{r-1} w_{r-1})^d$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{r+3}^2 \end{bmatrix}_{e_{r-1} \times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1} \times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1} \times f_{r-1}},$$

where $a_r \in (e_r \mathcal{A} f_r)^{-1}$, $w_r \in (f_r \mathcal{A} e_r)^{-1}$, $a_{r+3}^2 w_{r+1}^2 \in (e_{r+2} \mathcal{A} e_{r+2})^{qnil}$, $a_{r+3}^2 w_{r+1}^2 \leq^{-} b_{r+1}^2 w_{r+1}^2$, $w_{r+1}^2 a_{r+3}^2 \in (f_{r+2} \mathcal{A} f_{r+2}))^{qnil}$, $w_{r+1}^2 a_{r+3}^2 \leq^{-} w_{r+1}^2 b_{r+1}^2$, $b_r^2 \in (e_{r+1} \mathcal{A} f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1} \mathcal{A} e_{r+1})^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in (e_{r+2} \mathcal{A} e_{r+2})^{qnil}$ *and* $w_{r+1}^2 b_{r+1}^2 \in (f_{r+2} \mathcal{A} f_{r+2})^{qnil}$.

In the following result, we observe that $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$ implies $a_{r-1} w_{r-1} \leq^{-} b_{r-1} w_{r-1}$ and $w_{r-1} a_{r-1} \leq^{-} w_{r-1} b_{r-1}$.

Corollary 3.1 (see [24]). *Let $w_{r-1} \in \mathcal{A} \backslash \{0\}$ and a_{r-1}, $b_{r-1} \in \mathcal{A}$ be $w_{r-1} g$-Drazin invertible.*

(i) *If* $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$, *then* $a_{r-1}w_{r-1} \leq^{-} b_{r-1}w_{r-1}$.
(ii) *If* $a_{r-1} \leq^{\#,-,w_{r-1},l+r-1} b_{r-1}$, *then* $w_{r-1}a_{r-1} \leq^{-} w_{r-1}b_{r-1}$.
(iii) *If* $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$, *then* $a_{r-1}w_{r-1} \leq^{-} b_{r-1}w_{r-1}$ *and* $w_{r-1}a_{r-1} \leq^{-} w_{r-1}b_{r-1}$.
Proof. (i) If $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$, by Theorem 3.1, there exist decompositions of the identity $1 = e_{\mathrm{r}} + e_{\mathrm{r+1}} + e_{\mathrm{r+2}}$ and $1 = f_{\mathrm{r}} + f_{\mathrm{r+1}} + f_{\mathrm{r+2}}$ such that

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & a_{r+1}^2 \\ 0 & 0 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_{\mathrm{r}} \in (e_{\mathrm{r}}\mathcal{A}f_{\mathrm{r}})^{-1}$, $w_{\mathrm{r}} \in (f_{\mathrm{r}}\mathcal{A}e_{\mathrm{r}})^{-1}$, $a_{\mathrm{r+1}}^2 w_{\mathrm{r+1}}^2 = 0$, $a_{\mathrm{r+3}}^2 w_{\mathrm{r+1}}^2 \in (e_{\mathrm{r+2}}\mathcal{A}e_{\mathrm{r+2}})^{qnil}$, $a_{\mathrm{r+3}}^2 w_{\mathrm{r+1}}^2 \leq^{-} b_{\mathrm{r+1}}^2 w_{\mathrm{r+1}}^2, w_{\mathrm{r+1}}^2 a_{\mathrm{r+3}}^2 \in (f_{\mathrm{r+2}}\mathcal{A}f_{\mathrm{r+2}}))^{qnil}$, $b_{\mathrm{r+1}}^3 w_{\mathrm{r+1}}^2 = 0$, $b_{\mathrm{r}}^2 \in (e_{\mathrm{r+1}}\mathcal{A}f_{\mathrm{r+1}})^{-1}$, $w_{\mathrm{r}}^2 \in (f_{\mathrm{r+1}}\mathcal{A}e_{\mathrm{r+1}})^{-1}$, $b_{\mathrm{r+1}}^2 w_{\mathrm{r+1}}^2 \in (e_{\mathrm{r+2}}\mathcal{A}e_{\mathrm{r+2}})^{qnil}$ and $w_{\mathrm{r+1}}^2 b_{\mathrm{r+1}}^2 \in (f_{\mathrm{r+2}}\mathcal{A}f_{\mathrm{r+2}})^{qnil}$. From $a_{\mathrm{r+3}}^2 w_{\mathrm{r+1}}^2 \leq^{-} b_{\mathrm{r+1}}^2 w_{\mathrm{r+1}}^2$, we deduce that there exist $x,\ y \in \mathcal{A}$ such that $a_{\mathrm{r+3}}^2 w_{\mathrm{r+1}}^2 = x b_{\mathrm{r+1}}^2 w_{\mathrm{r+1}}^2 = b_{\mathrm{r+1}}^2 w_{\mathrm{r+1}}^2 y$ and $x a_{\mathrm{r+3}}^2 w_{\mathrm{r+1}}^2 = a_{\mathrm{r+3}}^2 w_{\mathrm{r+1}}^2$. Set

$$u = \begin{bmatrix} e_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{bmatrix}_{e_{r-1}\times e_{r-1}} \quad \text{and} \quad v = \begin{bmatrix} e_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y \end{bmatrix}_{e_{r-1}\times e_{r-1}}.$$

Now, by

$$u a_{r-1} w_{r-1} = \begin{bmatrix} a_r w_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{r+3}^2 w_{r+1}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}} = a_{r-1}w_{r-1} = u b_{r-1} w_{r-1} = b_{r-1} w_{r-1} y,$$

we conclude that $a_{r-1}w_{r-1} \leq^{-} b_{r-1}w_{r-1}$.
The parts (ii) and (iii) follow similarly.

By the previous results, we introduce and characterize the following regular partial order on $\mathcal{A}^d$:
Definition 3.2. *Let* , $b_{r-1} \in \mathcal{A}^d$. *Then we say that* $a_{r-1} \leq^{\neq,-} b_{r-1}$ *if* $c_{a_{r-1}} \leq^{\#} c_{b_{r-1}}$ *and* $q_{a_{r-1}} \leq^{-} q_{b_{r-1}}$.
Corollary 3.2 [24]. *Let* , $b_{r-1} \in \mathcal{A}^d$. *Then the following statements are equivalent*:
(i) $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$;
(ii) $a_{r-1} \leq^{d} b_{r-1}$ *and* $a — a_{r-1}^2 a_{r-1}^d \leq^{-} b_{r-1} - b_{r-1}^2 b_{r-1}^d$;
(iii) *there exist a decomposition of the identity* $1 = e_{\mathrm{r}} + e_{\mathrm{r+1}} + e_{\mathrm{r+2}}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times e_{r-1}},$$

where $a_{\mathrm{r}} \in (e_{\mathrm{r}}\mathcal{A}e_{\mathrm{r}})^{-1}$, $a_{\mathrm{r+3}}^2 \in (e_{\mathrm{r+2}}\mathcal{A}e_{\mathrm{r+2}})^{qnil}$, $a_{\mathrm{r+3}}^2 \leq^{-} b_{\mathrm{r+1}}^2$, $b_{\mathrm{r}}^2 \in (e_{\mathrm{r+1}}\mathcal{A}e_{\mathrm{r+1}})^{-1}$ *and* $b_{\mathrm{r+1}}^2 \in (e_{\mathrm{r+2}}\mathcal{A}e_{\mathrm{r+2}})^{qnil}$.
Corollary 3.3 [24]. *Let* , $b_{r-1} \in \mathcal{A}^d$. *If* $a_{r-1} \leq^{\neq,-} b_{r-1}$, *then* $a_{r-1} \leq^{-} b_{r-1}$.

4 Regular Weighted Pre-Orders in A Ring

In the same way as results of Section 3 and Section 4, we can verify the following results related to regular weighted pre-orders in a ring extending recent results from [11].
For $\mathcal{R}$ be an arbitrary ring with the unit 1. If $a_{r-1} \in \mathcal{R}^D$ with $\mathrm{ind}(a_{r-1}) = m$, then $a_{r-1} = c_{a_{r-1}} + n_{a_{r-1}}$ is the core-nilpotent decomposition of a_{r-1}, where $c_{a_{r-1}} = a_{r-1}a_{r-1}^D a_{r-1}$ is the core part of a_{r-1}, $n_{a_{r-1}} = (1 - a_{r-1}a_{r-1}^D)a_{r-1}$ is the nilpotent part of a_{r-1}, $n_{a_{r-1}}^m = 0$, $c_{a_{r-1}}n_{a_{r-1}} = n_{a_{r-1}}c_{a_{r-1}} = 0$, $c_{a_{r-1}} \in \mathcal{R}^{\neq}$ and $c_{a_{r-1}}^{\#} = a_{r-1}^D$[19].
Let a_{r-1}, $b_{r-1} \in \mathcal{R}^D$ such that $a_{r-1} = c_{a_{r-1}} + n_{a_{r-1}}$ and $b_{r-1} = c_{b_{r-1}} + n_{b_{r-1}}$ are the core-nilpotent decompositions of a_{r-1} and b_{r-1}, respectively. Then $a_{r-1} \leq^{D} b_{r-1}$ if $c_{a_{r-1}} \leq^{\#} c_{b_{r-1}}$. The relation $\leq^{D}$ is a

pre-order on $\mathcal{R}^D$. Notice that, for a_{r-1}, $b_{r-1} \in \mathcal{R}^D$, $a_{r-1} \leq^D b_{r-1}$ if and only if $a_{r-1}^D a_{r-1} = a_{r-1}^D b_{r-1}$ and $a_{r-1}a_{r-1}^D = b_{r-1}a_{r-1}^D$[11, Theorem 1]. Using the Drazin pre-order and the weighted element $w_{r-1} \in \mathcal{R}\backslash\{0\}$, we first consider the following relations on the set of all w_{r-1}-Drazin invertible elements of $\mathcal{R}$.

Definition 4.1. *Let a_{r-1}, $b_{r-1} \in \mathcal{R}$ and $w_{r-1} \in \mathcal{R}\backslash\{0\}$. If a_{r-1} is w_{r-1}-Drazin invertible, then we say that*
(i) $a_{r-1} \leq^{D,w_{r-1},r_0+r-1} b_{r-1}$ *if* $a_{r-1}w_{r-1} \leq^D b_{r-1}w_{r-1}$,
(ii) $a_{r-1} \leq^{D,w_{r-1},l+r-1} b_{r-1}$ *if* $w_{r-1}a_{r-1} \leq^D w_{r-1}b_{r-1}$,
(iii) $a_{r-1} \leq^{D,w_{r-1}} b_{r-1}$ *if* $a_{r-1} \leq^{D,w_{r-1},r_0+r-1} b_{r-1}$ *and* $a_{r-1} \leq^{D,w_{r-1},l+r-1} b_{r-1}$.

The relations $\leq^{D,w_{r-1},r_0+r-1}$, $\leq^{D,w_{r-1},l+r-1}$, $\leq^{D,w_{r-1}}$ are pre-orders on the set of all w_{r-1}-Drazin invertible elements of $\mathcal{R}$.

Now we have new characterizations of regular pre-orders $\leq^{D,w_{r-1},r_0+r-1}$ and $\leq^{D,w_{r-1},l+r-1}$ in the next results.

Corollary 4.1 [24]. *Let $w_{r-1} \in \mathcal{R}\backslash\{0\}$, and let a_{r-1}, $b_{r-1} \in \mathcal{R}$ be w_{r-1}-Drazin invertible. Then the following statements are equivalent*:
(i) $a_{r-1} \leq^{D,w_{r-1},r_0+r-1} b_{r-1}$;
(ii) *aw* $(a_{r-1}w_{r-1})^D = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D = (a_{r-1}w_{r-1})^D b_{r-1}w_{r-1}$;
(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{R}f_r)^{-1}$, $w_r \in (f_r\mathcal{R}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2w_r^2 + a_{r+1}^2w_{r+1}^2 + a_{r+2}^2w_r^2 + a_{r+3}^2w_{r+1}^2 \in ((1 - e_r)\mathcal{R}(1-e_r))^{nil}$, $w_{r+1}a_{r+1} = w_r^2a_r^2 + w_r^2a_{r+1}^2 + w_{r+1}^2a_{r+2}^2 + w_{r+1}^2a_{r+3}^2 \in \left((1-f_r)\mathcal{R}(1-f_r)\right)^{nil}$,

$b_{r+1}^3w_{r+1}^2 = 0, b_r^2 \in (e_{r+1}\mathcal{R}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{R}e_{r+1})^{-1}$, $b_{r+1}^2w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$ *and* $w_{r+1}^2b_{r+1}^2 \in (f_{r+2}\mathcal{R}f_{r+2})^{nil}$;
(iv) $(b_{r-1}w_{r-1})^2\ (b_{r-1}w_{r-1})^D\{1,5\} \subseteq (a_{r-1}w_{r-1})^2\ (a_{r-1}w_{r-1})^D\{1,5\}$;
(v) $(b_{r-1}w_{r-1})^D \in (a_{r-1}w_{r-1})^2(a_{r-1}w_{r-1})^D\{1,5\}$;
(vi) *there exist an idempotent* p *such that* $(a_{r-1}w_{r-1})^2(a_{r-1}w_{r-1})^D = p(b_{r-1}w_{r-1})^2(b_{r-1}w_{r-1})^D = (b_{r-1}w_{r-1})^2(b_{r-1}w_{r-1})^D p$;
(vii) $(a_{r-1}w_{r-1})^{k+1} = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^k = (a_{r-1}w_{r-1})^k b_{r-1}w_{r-1}$, *for* $k = \mathrm{ind}(a_{r-1}w_{r-1})$;
(viii) $(a_{r-1}w_{r-1})^{k+1} = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^k = (a_{r-1}w_{r-1})^k b_{r-1}w_{r-1}$, *for some integer* $k \geq 0$;
(ix) $[(a_{r-1}w_{r-1})^D]^n = b_{r-1}w_{r-1}[(a_{r-1}w_{r-1})^D]^{n+1} = [(a_{r-1}w_{r-1})^D]^{n+1}b_{r-1}w_{r-1}$, *for all integer* $n \geq 1$;
(x) $[(a_{r-1}w_{r-1})^D]^n = b_{r-1}w_{r-1}[(a_{r-1}w_{r-1})^D]^{n+1} = [(a_{r-1}w_{r-1})^D]^{n+1}b_{r-1}w_{r-1}$, *for some integer* $n \geq 1$.

Corollary 4.2 [24]. *Let $w_{r-1} \in \mathcal{R}\backslash\{0\}$, and let a_{r-1}, $b_{r-1} \in \mathcal{R}$ be w_{r-1}-Drazin invertible. Then the following statements are equivalent*:
(i) $a_{r-1} \leq^{D,w_{r-1},l+r-1} b_{r-1}$;
(ii) $w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^D = w_{r-1}b_{r-1}(w_{r-1}a_{r-1})^D = (w_{r-1}a_{r-1})^D w_{r-1}b_{r-1}$;
(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ b_{r+1}^4 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{R}f_r)^{-1}$, $w_r \in (f_r\mathcal{R}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2w_r^2 + a_{r+1}^2w_{r+1}^2 + a_{r+2}^2w_r^2 + a_{r+3}^2w_{r+1}^2 \in ((1 -$

$e_r)\mathcal{R}(1-e_r))^{nil}$, $w_{r+1}a_{r+1} = w_r^2a_r^2 + w_r^2a_{r+1}^2 + w_{r+1}^2a_{r+2}^2 + w_{r+1}^2a_{r+3}^2 \in \left((1-f_r)\mathcal{R}(1-f_r)\right)^{nil}$, $w_{r+1}^2b_{r+1}^4 = 0, b_r^2 \in (e_{r+1}\mathcal{R}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{R}e_{r+1})^{-1}$, $b_{r+1}^2w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$ *and* $w_{r+1}^2b_{r+1}^2 \in (f_{r+2}\mathcal{R}f_{r+2})^{nil}$;

(iv) $(w_{r-1}b_{r-1})^2(w_{r-1}b_{r-1})^D\{1,5\} \subseteq (w_{r-1}a_{r-1})^2(w_{r-1}a_{r-1})^D\{1,5\}$;

(v) $(w_{r-1}b_{r-1})^D \in (w_{r-1}a_{r-1})^2\,(w_{r-1}a_{r-1})^D\{1,5\}$;

(vi) *there exist an idempotent* p *such that* $(w_{r-1}a_{r-1})^2(w_{r-1}a_{r-1})^D = p(w_{r-1}b_{r-1})^2(w_{r-1}b_{r-1})^D = (w_{r-1}b_{r-1})^2(w_{r-1}b_{r-1})^Dp$;

(vii) $(w_{r-1}a_{r-1})^{k+1} = w_{r-1}b_{r-1}(w_{r-1}a_{r-1})^k = (w_{r-1}a_{r-1})^kw_{r-1}b_{r-1}$, *for* $k = \mathrm{ind}(w_{r-1}a_{r-1})$;

(viii) $(w_{r-1}a_{r-1})^{k+1} = w_{r-1}b_{r-1}(w_{r-1}a_{r-1})^k = (w_{r-1}a_{r-1})^kw_{r-1}b_{r-1}$, *for some integer* $k \geq 0$;

(ix) $[(w_{r-1}a_{r-1})^D]^n = w_{r-1}b_{r-1}[(w_{r-1}a_{r-1})^D]^{n+1} = [(w_{r-1}a_{r-1})^D]^{n+1}w_{r-1}b_{r-1}$, *for all integer* $n \geq 1$; (x) $[(w_{r-1}a_{r-1})^D]^n = w_{r-1}b_{r-1}[(w_{r-1}a_{r-1})^D]^{n+1} = [(w_{r-1}a_{r-1})^D]^{n+1}w_{r-1}b_{r-1}$, *for some integer* $n \geq 1$.

Corollary 4.3 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$, *and let* a_{r-1}, $b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{D,w_{r-1}} b_{r-1}$;

(ii) $a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D = (a_{r-1}w_{r-1})^Db_{r-1}w_{r-1}$ *and* $w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^D = w_{r-1}b_{r-1}(w_{r-1}a_{r-1})^D = (w_{r-1}a_{r-1})^Dw_{r-1}b_{r-1}$;

(iii) $a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D$ *and* $(w_{r-1}a_{r-1})^Dw_{r-1}a_{r-1} = (w_{r-1}a_{r-1})^Dw_{r-1}b_{r-1}$;

(iv) $a_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1} = b_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1}$ *and* $w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1}a_{r-1} = w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1}b_{r-1}$;

(v) $a_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}} = b_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}}$ *and* $a_{r-1}^{D,w_{r-1}}w_{r-1}a_{r-1} = a_{r-1}^{D,w_{r-1}}w_{r-1}b_{r-1}$;

(vi) *the elements* a_{r-1}, w_{r-1} *and* b_{r-1} *have the following matrix representations, for* $p = a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D$ *and* $q = w_{r-1}a_{r-1}(w_{r-1}a_{r-1})^D$,

$$a_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & a_{r+1} \end{bmatrix}_{p,q}, w_{r-1} = \begin{bmatrix} w_r & 0 \\ 0 & w_{r+1} \end{bmatrix}_p, b_{r-1} = \begin{bmatrix} a_r & 0 \\ 0 & b_{r+1} \end{bmatrix}_{p,q}$$

where $a_rw_r \in (p\mathcal{R}p)^{-1}$, $w_ra_r \in (q\mathcal{R}q)^{-1}$, $a_{r+1}w_{r+1} \in \left((1-p)\mathcal{R}(1-p)\right)^{nil}$ *and* $w_{r+1}a_{r+1} \in \left((1-q)\mathcal{R}(1-q)\right)^{nil}$

(vii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{R}f_r)^{-1}$, $w_r \in (f_r\mathcal{R}e_r)^{-1}$, $a_{r+1}w_{r+1} = a_r^2w_r^2 + a_{r+1}^2w_{r+1}^2 + a_{r+2}^2w_r^2 + a_{r+3}^2w_{r+1}^2 \in ((1-e_r)\mathcal{R}(1-e_r))^{nil}$, $w_{r+1}a_{r+1} = w_r^2a_r^2 + w_r^2a_{r+1}^2 + w_{r+1}^2a_{r+2}^2 + w_{r+1}^2a_{r+3}^2 \in \left((1-f_r)\mathcal{R}(1-f_r)\right)^{nil}$, $b_r^2 \in (e_{r+1}\mathcal{R}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{R}e_{r+1})^{-1}$, $b_{r+1}^2w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$ *and* $w_{r+1}^2b_{r+1}^2 \in (f_{r+2}\mathcal{R}f_{r+2})^{nil}$.

Corollary 4.4 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$, $a_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible and let* $b_{r-1} \in \mathcal{R}$.

(i) *If* $a_{r-1} \leq^{D,w_{r-1},r_0+r-1} b_{r-1}$, *then* $(a_{r-1}w_{r-1})^n \leq^D (b_{r-1}w_{r-1})^n$, *for all integer* $n \geq 1$.

(ii) *If* $a_{r-1} \leq^{D,w_{r-1},l+r-1} b_{r-1}$, *then* $(w_{r-1}a_{r-1})^n \leq^D (w_{r-1}b_{r-1})^n$, *for all integer* $n \geq 1$.

(iii) *If* $a_{r-1} \leq^{D,w_{r-1}} b_{r-1}$, *then* $(a_{r-1}w_{r-1})^n \leq^D (b_{r-1}w_{r-1})^n$ *and* $(w_{r-1}a_{r-1})^n \leq^D (w_{r-1}b_{r-1})^n$, *for all* *integer* $n \geq 1$.

Using the w_{r-1}-Drazin inverse, we define and investigate more regular weighted pre-orders on $\mathcal{R}^{D,w_{r-1}}$.

Definition 4.2. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$, $a_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible and* $b_{r-1} \in \mathcal{R}$. *Then we say that*

(i) $a_{r-1} \leq^{w_{r-1}D,r_0+r-1} b_{r-1}$ *if* $a_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}} = b_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}}$,

(ii) $a_{r-1} \leq^{w_{r-1}D,l+r-1} b_{r-1}$ *if* $a_{r-1}^{D,w_{r-1}}w_{r-1}a_{r-1} = a_{r-1}^{D,w_{r-1}}w_{r-1}b_{r-1}$.

The relations $\leq^{w_{r-1}D,r_0+r-1}$ and $\leq^{w_{r-1}D,l+r-1}$ are pre-orders on the set of all w_{r-1}-Drazin invertible elements of $\mathcal{R}$.

Corollary 4.5 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$, *and let* a_{r-1}, $b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{w_{r-1}D,r_0+r-1} b_{r-1}$;

(ii) $a_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D = b_{r-1}w_{r-1}(a_{r-1}w_{r-1})^D$;

(iii) $a_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1} = b_{r-1}w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1}$;

(iv) *there exist decompositions of the identity* $1 = e_{\mathrm{r}} + e_{\mathrm{r+1}} + e_{\mathrm{r+2}}$ *and* $1 = f_{\mathrm{r}} + f_{\mathrm{r+1}} + f_{\mathrm{r+2}}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & b_r^3 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_{\mathrm{r}} \in (e_{\mathrm{r}}\mathcal{R}f_{\mathrm{r}})^{-1}$, $w_{\mathrm{r}} \in (f_{\mathrm{r}}\mathcal{R}e_{\mathrm{r}})^{-1}$, $a_{\mathrm{r+1}}w_{\mathrm{r+1}} = a_{\mathrm{r}}^2w_{\mathrm{r}}^2 + a_{\mathrm{r+1}}^2w_{\mathrm{r+1}}^2 + a_{\mathrm{r+2}}^2w_{\mathrm{r}}^2 + a_{\mathrm{r+3}}^2w_{\mathrm{r+1}}^2 \in ((1 - e_{\mathrm{r}})\mathcal{R}(1 - e_{\mathrm{r}}))^{nil}$, $w_{\mathrm{r+1}}a_{\mathrm{r+1}} = w_{\mathrm{r}}^2a_{\mathrm{r}}^2 + w_{\mathrm{r}}^2a_{\mathrm{r+1}}^2 + w_{\mathrm{r+1}}^2a_{\mathrm{r+2}}^2 + w_{\mathrm{r+1}}^2a_{\mathrm{r+3}}^2 \in \left((1 - f_{\mathrm{r}})\mathcal{R}(1 - f_{\mathrm{r}})\right)^{nil}$, $b_{\mathrm{r}}^2 \in (e_{\mathrm{r+1}}\mathcal{R}f_{\mathrm{r+1}})^{-1}$, $w_{\mathrm{r}}^2 \in (f_{\mathrm{r+1}}\mathcal{R}e_{\mathrm{r+1}})^{-1}$, $b_{\mathrm{r+1}}^2w_{\mathrm{r+1}}^2 \in (e_{\mathrm{r+2}}\mathcal{R}e_{\mathrm{r+2}})^{nil}$ *and* $w_{\mathrm{r+1}}^2b_{\mathrm{r+1}}^2 \in (f_{\mathrm{r+2}}\mathcal{R}f_{\mathrm{r+2}})^{nil}$.

Corollary 4.6 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$, *and let* a_{r-1}, $b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{w_{r-1}D,l+r-1} b_{r-1}$;

(ii) $(w_{r-1}a_{r-1})^Dw_{r-1}a_{r-1} = (w_{r-1}a_{r-1})^Dw_{r-1}b_{r-1}$;

(iii) $w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1}a_{r-1} = w_{r-1}a_{r-1}^{D,w_{r-1}}w_{r-1}b_{r-1}$;

(iv) *there exist decompositions of the identity* $1 = e_{\mathrm{r}} + e_{\mathrm{r+1}} + e_{\mathrm{r+2}}$ *and* $1 = f_{\mathrm{r}} + f_{\mathrm{r+1}} + f_{\mathrm{r+2}}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & a_r^2 & a_{r+1}^2 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ b_r^4 & b_r^2 & 0 \\ b_{r+1}^4 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_{\mathrm{r}} \in (e_{\mathrm{r}}\mathcal{R}f_{\mathrm{r}})^{-1}$, $w_{\mathrm{r}} \in (f_{\mathrm{r}}\mathcal{R}e_{\mathrm{r}})^{-1}$, $a_{\mathrm{r+1}}w_{\mathrm{r+1}} = a_{\mathrm{r}}^2w_{\mathrm{r}}^2 + a_{\mathrm{r+1}}^2w_{\mathrm{r+1}}^2 + a_{\mathrm{r+2}}^2w_{\mathrm{r}}^2 + a_{\mathrm{r+3}}^2w_{\mathrm{r+1}}^2 \in ((1 - e_{\mathrm{r}})\mathcal{R}(1 - e_{\mathrm{r}}))^{nil}$, $w_{\mathrm{r+1}}a_{\mathrm{r+1}} = w_{\mathrm{r}}^2a_{\mathrm{r}}^2 + w_{\mathrm{r}}^2a_{\mathrm{r+1}}^2 + w_{\mathrm{r+1}}^2a_{\mathrm{r+2}}^2 + w_{\mathrm{r+1}}^2a_{\mathrm{r+3}}^2 \in \left((1 - f_{\mathrm{r}})\mathcal{R}(1 - f_{\mathrm{r}})\right)^{nil}$, $b_{\mathrm{r}}^2 \in (e_{\mathrm{r+1}}\mathcal{R}f_{\mathrm{r+1}})^{-1}$, $w_{\mathrm{r}}^2 \in (f_{\mathrm{r+1}}\mathcal{R}e_{\mathrm{r+1}})^{-1}$, $b_{\mathrm{r+1}}^2w_{\mathrm{r+1}}^2 \in (e_{\mathrm{r+2}}\mathcal{R}e_{\mathrm{r+2}})^{nil}$ *and* $w_{\mathrm{r+1}}^2b_{\mathrm{r+1}}^2 \in (f_{\mathrm{r+2}}\mathcal{R}f_{\mathrm{r+2}})^{nil}$.

We now introduce regular pre-orders on the set of all w_{r-1}-Drazin invertible elements of $\mathcal{R}$, applying the minus and sharp partial orders.

Definition 4.3. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$ *and* a_{r-1}, $b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible. Then we say that*

(i) $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$ *if* $c_{a_{r-1}w_{r-1}} \leq^{\#} c_{b_{r-1}w_{r-1}}$ *and* $n_{a_{r-1}w_{r-1}} \leq^{-} n_{b_{r-1}w_{r-1}}$,

(ii) $a_{r-1} \leq^{\#,-,w_{r-1},l+r-1} b_{r-1}$ *if* $c_{w_{r-1}a_{r-1}} \leq^{\#} c_{w_{r-1}b_{r-1}}$ *and* $n_{w_{r-1}a_{r-1}} \leq^{-} n_{w_{r-1}a_{r-1}}$,

(iii) $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$ *if* $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$ *and* $a_{r-1} \leq^{\neq,-,w_{r-1},l+r-1} b_{r-1}$.

The relations $\leq^{\#,-,w_{r-1},r_0+r-1}$, $\leq^{\#,-,w_{r-1},l+r-1}$ and $\leq^{\#,-,w_{r-1}}$ are partial orders on $\mathcal{R}^{D,w_{r-1}}$ if $w_{r-1} \in \mathcal{R}$ is right invertible, left invertible and invertible, respectively.

Corollary 4.7 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$ *and* a_{r-1}, $b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$;

(ii) $a_{r-1} \leq^{D,w_{r-1},r_0+r-1} b_{r-1}$ *and* $a_{r-1}w_{r-1} - (a_{r-1}w_{r-1})^2(a_{r-1}w_{r-1})^D \leq^- b_{r-1}w_{r-1} - (b_{r-1}w_{r-1})^2(b_{r-1}w_{r-1})^D$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & a_{r+1}^2 \\ 0 & 0 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & b_{r+1}^3 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{R}f_r)^{-1}$, $w_r \in (f_r\mathcal{R}e_r)^{-1}$, $a_{r+1}^2 w_{r+1}^2 = 0$, $a_{r+3}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$, $a_{r+3}^2 w_{r+1}^2 \leq^- b_{r+1}^2 w_{r+1}^2$, $w_{r+1}^2 a_{r+3}^2 \in (f_{r+2}\mathcal{R}f_{r+2}))^{nil}$, $b_{r+1}^3 w_{r+1}^2 = 0$, $b_r^2 \in (e_{r+1}\mathcal{R}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{R}e_{r+1})^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$ *and*

$$w_{r+1}^2 b_{r+1}^2 \in (f_{r+2}\mathcal{R}f_{r+2})^{nil}.$$

Corollary 4.8 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$ *and* $a_{r-1}, b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{\neq,-,w_{r-1},l+r-1} b_{r-1}$;

(ii) $a_{r-1} \leq^{D,w_{r-1},}\iota b_{r-1}$ *and* $w_{r-1}a_{r-1} - (w_{r-1}a_{r-1})^2(w_{r-1}a_{r-1})^D \leq^- w_{r-1}b_{r-1} - (w_{r-1}b_{r-1})^2(w_{r-1}b_{r-1})^D$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{r+2}^2 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ b_{r+1}^4 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{R}f_r)^{-1}$, $w_r \in (f_r\mathcal{R}e_r)^{-1}$, $a_{r+3}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$, $w_{r+1}^2 a_{r+2}^2 = 0$, $w_{r+1}^2 a_{r+3}^2 \in (f_{r+2}\mathcal{R}f_{r+2})^{nil}$, $w_{r+1}^2 a_{r+3}^2 \leq^- w_{r+1}^2 b_{r+1}^2$, $w_{r+1}^2 b_{r+1}^4 = 0$, $b_r^2 \in (e_{r+1}\mathcal{R}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{R}e_{r+1})^{-1}$,

$b_{r+1}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$ *and* $w_{r+1}^2 b_{r+1}^2 \in (f_{r+2}\mathcal{R}f_{R+2})^{nil}$.

Corollary 4.9 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$ *and* $a_{r-1}, b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible. Then the following statements are equivalent*:

(i) $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$;

(ii) $a_{r-1} \leq^{D,w_{r-1},l+r-1} b_{r-1}$, $a_{r-1} \leq^{D,w_{r-1},r_0+r-1} b_{r-1}$ *and* $a_{r-1}w_{r-1} - (a_{r-1}w_{r-1})^2(a_{r-1}w_{r-1})^D \leq^- b_{r-1}w_{r-1} - (b_{r-1}w_{r-1})^2(b_{r-1}w_{r-1})^D$;

(iii) *there exist decompositions of the identity* $1 = e_r + e_{r+1} + e_{r+2}$ *and* $1 = f_r + f_{r+1} + f_{r+2}$ *such that*

$$a_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{r+3}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}}, w_{r-1} = \begin{bmatrix} w_r & 0 & 0 \\ 0 & w_r^2 & 0 \\ 0 & 0 & w_{r+1}^2 \end{bmatrix}_{f_{r-1}\times e_{r-1}}, b_{r-1} = \begin{bmatrix} a_r & 0 & 0 \\ 0 & b_r^2 & 0 \\ 0 & 0 & b_{r+1}^2 \end{bmatrix}_{e_{r-1}\times f_{r-1}},$$

where $a_r \in (e_r\mathcal{R}f_r)^{-1}$, $w_r \in (f_r\mathcal{R}e_r)^{-1}$, $a_{r+3}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$, $a_{r+3}^2 w_{r+1}^2 \leq^- b_{r+1}^2 w_{r+1}^2$, $w_{r+1}^2 a_{r+3}^2 \in (f_{r+2}\mathcal{R}f_{r+2}))^{nil}$, $w_{r+1}^2 a_{r+3}^2 \leq^- w_{r+1}^2 b_{r+1}^2$, $b_r^2 \in (e_{r+1}\mathcal{R}f_{r+1})^{-1}$, $w_r^2 \in (f_{r+1}\mathcal{R}e_{r+1})^{-1}$, $b_{r+1}^2 w_{r+1}^2 \in (e_{r+2}\mathcal{R}e_{r+2})^{nil}$ *and*

$$w_{r+1}^2 b_{r+1}^2 \in (f_{r+2}\mathcal{R}f_{r+2})^{nil}.$$

Corollary 4.10 [24]. *Let* $w_{r-1} \in \mathcal{R}\backslash\{0\}$ *and* $a_{r-1}, b_{r-1} \in \mathcal{R}$ *be* w_{r-1}*-Drazin invertible.*

(i) *If* $a_{r-1} \leq^{\neq,-,w_{r-1},r_0+r-1} b_{r-1}$, *then* $a_{r-1}w_{r-1} \leq^- b_{r-1}w_{r-1}$.

(ii) *If* $a_{r-1} \leq^{\neq,-,w_{r-1},l+r-1} b_{r-1}$, *then* $w_{r-1}a_{r-1} \leq^- w_{r-1}b_{r-1}$.

(iii) *If* $a_{r-1} \leq^{\neq,-,w_{r-1}} b_{r-1}$, *then* $a_{r-1}w_{r-1} \leq^- b_{r-1}w_{r-1}$ *and* $w_{r-1}a_{r-1} \leq^- w_{r-1}b_{r-1}$.

For $w_{r-1} = 1$ in corresponding results of this section, we obtain results from [11]. It is interesting to consider the previous results related to regular weighted pre-orders on the set of all $w_{r-1}g$-Drazin invertible elements of a ring.

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