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Review Paper

The Validity of Hilbert Matrix Norm on Small Bergman and Hardy Spaces and Nehari Theorem

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Abstract:

Diamantopoulos and Siskakis shown that the Hilbert matrix induces a bounded operator on most Hardy and Bergman spaces. M. Dostanić, M. Jevtić, D. Vukotić [19] generalize this for any Hankel operator on Hardy spaces by using a result of Hollenbeck and Verbitsky on the Riesz projection and also compute the exact value of the norm of the Hilbert matrix. Using the same new technique, we follow [19] to show the validity of the determination norm of the Hilbert matrix on a wide range of small Bergman spaces.

Keywords: Hilbert matrix; Operator norm; Hardy spaces; Bergman spaces; Hankel operator; Nehari theorem.

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0. Introduction

A Hankel operator on the space l^2 of all square-summable complex sequences is an operator defined by a matrix whose entries $a^j_{1+\epsilon,1+2\epsilon}$ depend only on the sum of the coordinates: $a^{\mathrm{j}}_{1+\epsilon,1+2\epsilon} = c_{2+3\epsilon}$ for some sequence $(c_{1+\epsilon})_{\epsilon=0}^{\infty}$. Hankel operators on different spaces are related to many areas such as the theory of moment sequences, orthogonal polynomials, Toeplitz operators, or analytic Besov spaces.

Nehari's classical theorem states that every Hankel operator S on l^2 can be represented by an essentially bounded function g_j on the unit circle $\mathbb{T}$, in the sense that $c_{1+\epsilon} = \hat{g}_j(1+\epsilon)$ for all $\epsilon \geqslant 0$; moreover, a function g_j can always be chosen so that $\| g_j \|_{L^\infty(\mathbb{T})} = \| S \|_{l^2\to l^2}$. See [12, Theorem 4.1.13], [15, Section 1.1, p. 3] or [16, Theorem 1.3, p. 4].

A typical Hankel operator is the Hilbert matrix H, an infinite matrix whose entries are $a^{\mathrm{j}}_{1+\epsilon,1+2\epsilon} = (3(1+\epsilon))^{-1}, \epsilon > -1$. It is relevant in many fields ranging from number theory or linear algebra to numerical analysis and operator theory. For this operator, the following choice: $g_j(t) = ie^{-it}(\pi - t), 0 \leqslant t < 2\pi$, in Nehari's theorem yields $\| g_j \|_{L^\infty(\mathbb{T})} = \pi = \| H \|_{l^2\to l^2}$. Several interesting facts about the Hilbert matrix are described in [1] and [7, Problems 46-48] and further results about the spectrum of H can be found in [15].

The Hilbert matrix can be viewed as an operator on other spaces and it is a basic question to determine its operator norm. One form of Hilbert's classical inequality [3], [8, Section 9.2]:

$$\left(\sum_{\epsilon=-1}^{\infty}\left|\sum_{\epsilon=-\frac{1}{2}}^{\infty}\sum_{j}\frac{a^j_{1+2\epsilon}}{3(1+\epsilon)}\right|^{1+\epsilon}\right)^{1/1+\epsilon} \leqslant \frac{\pi}{\sin(\pi/1+\epsilon)}\left(\sum_{\epsilon=0}^{\infty}\sum_{j}\left|a^j_{1+\epsilon}\right|^{1+\epsilon}\right)^{1/1+\epsilon}$$

can be used to compute the norm of H on the space $l^{1+\epsilon}$ of all $(1+\epsilon)$-summable sequences:

$$\| H \|_{l^{1+\epsilon} \to l^{1+\epsilon}} = \frac{\pi}{\sin(\pi/1+\epsilon)},\ 0 < \epsilon < \infty$$

The Taylor coefficients of the functions in the Hardy spaces $H^{1+\epsilon}$ are closely related to $l^{1+\epsilon}$ spaces. Thus, it is natural to consider the Hilbert matrix as an operator defined on $H^{1+\epsilon}$ by its action on the coefficients:

$$\hat{f}_j(1+\epsilon) \mapsto \sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \frac{\hat{f}_j(1+2\epsilon)}{3(1+\epsilon)}$$

that is, by defining

$$Hf_j(z) = \sum_{\epsilon=-1}^{\infty} \left(\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \frac{\hat{f}_j(1+2\epsilon)}{3(1+\epsilon)} \right) z^{1+\epsilon},\ f_j \in H^{1+\epsilon}, z \in \mathbb{D} \quad (1)$$

It is possible to write $Hf_j, f_j \in H^{1+\epsilon}$, in other forms which are convenient for analyzing this operator (see [3]), for example:

$$Hf_j(z) = \int_0^1 \sum_j \frac{f_j(1-\epsilon)}{1-(1-\epsilon)z} d(1-\epsilon),\ z \in \mathbb{D} \quad (2)$$

The equality of the expressions in (1) and (2) can be verified in a straightforward way from the Taylor series expansion of f_j.

The most basic question is: on which Hardy spaces is H bounded? The authors in [3] showed its boundedness on any $H^{1+\epsilon}$ with $0 < \epsilon < \infty$. By establishing another useful representation of H as an average of weighted composition operators and integrating over semi-circular paths, they obtained the following upper bound:

$$\| H \|_{H^{1+\epsilon} \to H^{1+\epsilon}} \leqslant \frac{\pi}{\sin(\pi/1+\epsilon)},\ 0 \leqslant \epsilon < \infty$$

In view of Nehari's l^2 theorem, this result is sharp when $\epsilon = 0$.

In the case $0 < \epsilon < 1$ it was also shown in [3] that the above estimate continues to hold for the restriction of the operator to the subspace $\{f_j \in H^{1+\epsilon} : f_j(0) = 0\}$. Two natural questions come to mind (see [19]):

(A) Can the above norm estimate for H be extended to the case $0 < \epsilon < 1$ without restrictions?

(B) What is the actual value of the norm of H as an $H^{1+\epsilon}$ operator, $0 < \epsilon < \infty$?

In the present paper we give a more general answer to the above question (A) by deducing the following Nehari-type result: an arbitrary Hankel operator H_{g_j} associated with a function $g_j \in L^\infty(\mathbb{T})$ is bounded on $H^{1+\epsilon}, 0 < \epsilon < \infty$:

$$\left\| \sum_j H_{g_j} \right\|_{H^{1+\epsilon} \to H^{1+\epsilon}} \leqslant \sum_j \frac{\| g_j \|_\infty}{\sin(\pi/1+\epsilon)}$$

The key point is that every Hankel operator on $H^{1+\epsilon}$ has a representations as a composition of a (non-analytic) isometry and a multiplication, followed by the Riesz (Szegö) projection P_+ from $L^{1+\epsilon}(\mathbb{T})$ onto its closed subspace $H^{1+\epsilon}$. It is well known that this projection is bounded for $0 < \epsilon < \infty$. In 1968, [6] showed that

$$\|P_+\|_{L^{1+\epsilon}(\mathbb{T})\to H^{1+\epsilon}} \geqslant \frac{1}{\sin{(\pi/1+\epsilon)}},\ 0 < \epsilon < \infty$$

and conjectured that equality should hold. The authors in [11] proved this conjecture in 2000. Their result allows us to deduce the estimate for $\|H_{g_j}\|$ above.

Using some Hardy spaces techniques and splitting H into a difference of two operators, we also get a lower bound which yields

$$\parallel H \parallel_{H^{1+\epsilon}\to H^{1+\epsilon}} = \frac{\pi}{\sin{(\pi/1+\epsilon)}},\ 0 < \epsilon < \infty$$

thus answering the above question (B) for all admissible values of $(1+\epsilon)$.

The behavior of the Hilbert matrix as an operator defined by (1) turns out to be similar in the classical Bergman spaces $A^{1+\epsilon}$ of functions $(1+\epsilon)$-integrable in $\mathbb{D}$ with respect to the area measure. The author in [2] recently proved that H is bounded on $A^{1+\epsilon}$ if and only if $\epsilon > 1$. In the case $0 \leqslant \epsilon < \infty$ he obtained the estimate

$$\parallel H \parallel_{A^{1+\epsilon}\to A^{1+\epsilon}} \leqslant \frac{\pi}{\sin{(2\pi/1+\epsilon)}}$$

(This is what one may expect by the "rule of thumb" that says that for many operators and functionals defined on both $H^{1+\epsilon}$ and $A^{1+\epsilon}$, their norm when acting on $A^{1+\epsilon}$ is obtained by doubling an appropriate quantity in the norm when acting on $H^{1+\epsilon}$.) A less precise estimate for the norm of H on $A^{1+\epsilon}$ when $0 < \epsilon < 2$ was also obtained in [2].

As a main result of this paper, we obtain a lower bound valid for all $\epsilon > 0$ which coincides with the upper bound from [2] when $\epsilon \geqslant 0$, thus yielding the exact value of the norm for these exponents:

$$\parallel H \parallel_{A^{1+\epsilon}\to A^{1+\epsilon}} = \frac{\pi}{\sin{(2\pi/1+\epsilon)}},\ 0 \leqslant \epsilon < \infty$$

In the case $0 < \epsilon < 2$, although we are currently not able to identify the exact value of the norm, we do improve the bound obtained in [2]. We also point out that the Hilbert matrix has an integral representation with respect to the area measure with a kernel rather different from the usual Bergman space kernels.

1. Norms on Hardy Spaces

We briefly review the minimum background needed. Throughout the text, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ will denote the unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ will signify the algebra of holomorphic functions in $\mathbb{D}$. For f_j in $\mathcal{H}(\mathbb{D})$ and $0 < \epsilon < 1$, the integral means $M_{1+\epsilon}(1-\epsilon, f_j)$ are defined by

$$M_{1+\epsilon}(1-\epsilon, f_j) = \left(\frac{1}{2\pi}\int_0^{2\pi}\sum_j \left|f_j\big((1-\epsilon)e^{i\theta}\big)\right|^{1+\epsilon} d\theta\right)^{1/1+\epsilon}$$

and are increasing with $(1-\epsilon)$. The Hardy space $H^{1+\epsilon}(0<\epsilon<\infty)$ is the space of all f_j in $\mathcal{H}(\mathbb{D})$ for which $\| f_j \|_{H^{1+\epsilon}} = \lim_{1-\epsilon\to 1^-} M_{1+\epsilon}(1-\epsilon, f_j) < \infty$, and H^∞ is the space of all bounded f_j in $\mathcal{H}(\mathbb{D})$. We will denote by $\mathbb{T}$ the unit circle. The standard Lebesgue space $L^{1+\epsilon}(\mathbb{T})$ of the circle is to be considered with respect to the normalized measure $dm(z) = (2\pi)^{-1}dt$, where $z = e^{it}, 0 \leqslant t < 2\pi$. It is a well-known fact that the space $H^{1+\epsilon}$ is the closed subspace of $L^{1+\epsilon}(\mathbb{T})$ consisting of all functions whose Fourier coefficients with the negative index vanish. The Riesz (Szegó) projection P_+ from $L^{1+\epsilon}(\mathbb{T})$ onto $H^{1+\epsilon}$ is defined by

$$P_+u_j(z) = \frac{1}{2\pi}\int_0^{2\pi} \sum_j \frac{u_j(t)}{1-ze^{-it}}dt,\ z \in \mathbb{D} \tag{3}$$

For more details, the reader is referred to [4], among other sources.

1.1. A Nehari-Type Theorem for Hankel Operators on Hardy Spaces

One can define Hankel operators on any space $H^{1+\epsilon}, 0<\epsilon<\infty$. Given an arbitrary $g_j \in L^\infty(\mathbb{T})$, consider its Fourier coefficients with non-negative indices:

$$\hat{g}_j(1+\epsilon) = \frac{1}{2\pi}\int_0^{2\pi} \sum_j e^{-i(1+\epsilon)t} g_j(t)dt,\ \epsilon \geqslant -1$$

We can formally define the associated Hankel operator H_{g_j} by

$$H_{g_j}f_j(z) = \sum_{\epsilon=-1}^{\infty}\left(\sum_{\epsilon=-\frac{1}{2}}^{\infty}\sum_j \hat{g}_j(2+3\epsilon)\hat{f}_j(1+2\epsilon)\right)z^{1+\epsilon} \tag{4}$$

for an analytic function f_j with the Taylor series $f_j(z) = \sum_{\epsilon=0}^{\infty}\hat{f}_j(1+\epsilon)z^{1+\epsilon}$ in $\mathbb{D}$. In particular, when $g_j(t) = ie^{-it}(\pi - t), 0 \leqslant t < 2\pi$, a straightforward calculation shows that

$$\hat{g}_j(1+\epsilon) = \frac{1}{2\pi}\int_0^{2\pi}\sum_j e^{-i(1+\epsilon)t}g_j(t)dt = \frac{1}{2+\epsilon},\ \epsilon \geqslant -1$$

hence $H_{g_j} = H$, the Hilbert matrix. This is well known; see [4, Chapter 3, Corollary on p. 48], [12, p. 159], or [15, p. 6].

We will compute the norm of the Hilbert matrix H as an $H^{1+\epsilon}$ operator, $0<\epsilon<\infty$, as a consequence of an upper bound for the norm valid for an arbitrary operator H_{g_j} as above. To this end, we consider the isometric conjugation operator (also called the flip operator) for the functions on the unit circle $\mathbb{T}$ as $(1+\epsilon)f_j(e^{it}) = f_j(e^{-it})$. It is obvious that $(1+\epsilon)$ is an isometry from $H^{1+\epsilon}$ into $L^{1+\epsilon}(\mathbb{T})$. Next, let M_{g_j} denote the operator of multiplication by the essentially bounded function g_j: $M_{g_j}u_j = g_ju_j$; this is clearly bounded by $\| g_j \|_{L^\infty}$ as an operator acting on $L^{1+\epsilon}(\mathbb{T})$. We will now establish an equality $H_{g_j} = P_+M_{g_j}(1+\epsilon)$ which is known to hold in l^2 context (see [12, Theorem 4.1.13]), thus obtaining a Nehari-type theorem for Hankel operators on Hardy spaces.

Theorem 1 (see [19]). Let $0<\epsilon<\infty$ and $g_j \in L^\infty(0,2\pi)$. The operator H_{g_j} defined as in (4) is bounded on $H^{1+\epsilon}$, the equality $H_{g_j} = P_+M_{g_j}(1+\epsilon)$ holds and, consequently,

$$\left\|\sum_j H_{g_j}\right\|_{H^{1+\epsilon}\to H^{1+\epsilon}} \leqslant \sum_j \frac{\| g_j \|_\infty}{\sin(\pi/1+\epsilon)} \tag{5}$$

In particular, when $g_j(t) = ie^{-it}(\pi - t), 0 \leqslant t < 2\pi$, we get $H_{g_j} = H$ and

$$\| H \|_{H^{1+\epsilon}\to H^{1+\epsilon}} \leqslant \frac{\pi}{\sin(\pi/1+\epsilon)}$$

Proof. Given $f_j \in H^{1+\epsilon}$, denote by $(f_j)_m$ its m th Taylor polynomial: $(f_j)_m(z) = \sum_{\epsilon=-\frac{1}{2}}^{m} \sum_j \hat{f}_j(1+2\epsilon)z^{1+2\epsilon}$. The following result [18] will be useful: if $0 < \epsilon < \infty$ and $f_j \in H^{1+\epsilon}$ then $\left\|\sum_j \left((f_j)_m - f_j\right)\right\|_{H^{1+\epsilon}} \to 0$ as $m \to \infty$.

Given $f_j \in H^{1+\epsilon}$, we first verify that the power series for $H_{g_j} f_j$ converges in $\mathbb{D}$. To this end, it suffices to show that

$$\left|\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \hat{g}_j(2+3\epsilon)\hat{f}_j(1+2\epsilon)\right| \leqslant \| \sum_j g_j \|_\infty \| f_j \|_{H^{1+\epsilon}} \tag{6}$$

For $(f_j)_m$ instead of f_j, this follows immediately by recalling that $(1+\epsilon)$ is an isometry of $H^{1+\epsilon}$ into $L^{1+\epsilon}(\mathbb{T})$ and applying Hölder's inequality:

$$\left|\sum_{\epsilon=-\frac{1}{2}}^{m} \sum_j \hat{g}_j(2+3\epsilon)\hat{f}_j(1+2\epsilon)\right| = \left|\int_0^{2\pi} \sum_j \sum_{\epsilon=-\frac{1}{2}}^{m} g_j(t)e^{-i(1+\epsilon)t}\hat{f}_j(1+2\epsilon)e^{-i(1+2\epsilon)t}\frac{dt}{2\pi}\right|$$

$$= \left|\int_0^{2\pi} \sum_j g_j(t)e^{-i(1+\epsilon)t}\left((1+\epsilon)(f_j)_m\right)(e^{it})\frac{dt}{2\pi}\right| \leqslant \sum_j \| g_j \|_\infty \|(f_j)_m\|_{H^{1+\epsilon}}$$

A similar argument applied to the differences $(f_j)_m - (f_j)_{1+\epsilon}$ shows that $\left(\sum_{\epsilon=-\frac{1}{2}}^{m} \sum_j \hat{g}_j(2+3\epsilon)\hat{f}_j(1+2\epsilon)\right)_{m=0}^{\infty}$ is a Cauchy sequence uniformly in $(1+\epsilon)$, so it is legitimate to let $m \to \infty$ to obtain (6).

We will now establish the formula $H_{g_j} f_j = P_+ M_{g_j}(1+\epsilon) f_j$ for all f_j in $H^{1+\epsilon}, 0 < \epsilon < \infty$. By the theorem of Hollenbeck and Verbitsky this will immediately imply that H_{g_j} is bounded and, moreover, (5) holds:

$$\left\|\sum_j H_{g_j}\right\|_{H^{1+\epsilon}\to H^{1+\epsilon}} \leqslant \|P_+\|_{L^{1+\epsilon}(\mathbb{T})\to H^{1+\epsilon}} \cdot \sum_j \left\|M_{g_j}\right\|_{L^{1+\epsilon}(\mathbb{T})\to L^{1+\epsilon}(\mathbb{T})} \leqslant \sum_j \frac{\| g_j \|_\infty}{\sin(\pi/1+\epsilon)}$$

Given $f_j \in H^{1+\epsilon}$, we get the identity $H_{g_j}(f_j)_m = P_+ M_{g_j}(1+\epsilon)(f_j)_m$ and the bound

$$\left\|\sum_j H_{g_j}(f_j)_m\right\|_{H^{1+\epsilon}} \leqslant \sum_j \frac{\| g_j \|_\infty}{\sin(\pi/1+\epsilon)} \|(f_j)_m\|_{H^{1+\epsilon}} \tag{7}$$

for the m th Taylor polynomial $(f_j)_m$ of f_j by an easy computation involving (4) and (3):

$$H_{g_j}(f_j)_m(z) = \sum_{\epsilon=-1}^{\infty} \sum_{\epsilon=-\frac{1}{2}}^{m} \sum_j \hat{f}_j(1+2\epsilon) \int_0^{2\pi} \sum_j e^{-i(2+3\epsilon)t} g_j(t) \frac{dt}{2\pi} z^{1+\epsilon}$$
$$= \int_0^{2\pi} \sum_j \frac{g_j(t)(f_j)_m(e^{-it})}{1-e^{-it}z} \frac{dt}{2\pi}$$

The interchange of the series and the integral is justified by uniform convergence of the geometric series $\sum_{\epsilon=0}^{\infty} |z|^{1+\epsilon}$ on compact sets in $\mathbb{D}$.

To extend the identity $H_{g_j}(f_j)_m = P_+ M_{g_j}(1+\epsilon)(f_j)_m$ and (7) for arbitrary f_j in $H^{1+\epsilon}$, note that $\left(H_{g_j}(f_j)_m\right)_{m=0}^{\infty}$ is a Cauchy sequence in $H^{1+\epsilon}$ in view of

$$\left\| \sum_j H_{g_j}\big((f_j)_m - (f_j)_{1+\epsilon}\big) \right\|_{H^{1+\epsilon}} \leqslant \sum_j \frac{\| g_j \|_\infty}{\sin(\pi/1+\epsilon)} \left\|(f_j)_m - (f_j)_{1+\epsilon}\right\|_{H^{1+\epsilon}}$$

so the standard $H^{1+\epsilon}$ pointwise estimate $|\sum_j f_j(z)| \leqslant (1-|z|^2)^{-1/1+\epsilon} \sum_j \| f_j \|_{H^{1+\epsilon}}$ [4, Chapter 8] implies uniform convergence of $H_{g_j}(f_j)_m$ on compact sets. Next, our earlier observation that

$$\left(\sum_{\epsilon=-\frac{1}{2}}^{m} \sum_j \hat{g}_j(2+3\epsilon) \hat{f}_j(1+2\epsilon) \right)_{m=0}^{\infty}$$

is a Cauchy sequence uniformly in $(1+\epsilon)$ and standard estimates for the $(1+\epsilon)$-th Taylor coefficients based on the Cauchy integral formula allow us to conclude that actually $H_{g_j}(f_j)_m \to H_{g_j} f_j$ uniformly on compact sets. Finally, the statement follows by Fatou's lemma after taking the limit as $m \to \infty$ in the inequality (7).

1.2. The Norm of the Hilbert Matrix on Hardy Spaces

The main theorem of this section gives the lower bound for the norm.

Theorem 2 (see [19]). Let $0 < \epsilon < \infty$. Then the norm of the Hilbert matrix as an operator acting on $H^{1+\epsilon}$ satisfies the lower estimate

$$\| H \|_{H^{1+\epsilon} \to H^{1+\epsilon}} \geqslant \frac{\pi}{\sin(\pi/1+\epsilon)} \tag{8}$$

Proof. We break up the argument into four key steps.

Step 1. We begin by selecting a family of test functions. Let ε be fixed, $0 < \varepsilon < 1$, and choose an arbitrary γ^j such that $\varepsilon < \gamma^j < 1$. It is a standard exercise to check that the function $(f_j)_{\gamma^j}(z) = (1-z)^{-\gamma^j/1+\epsilon}$ belongs to $H^{1+\epsilon}$ [4, Chapter 1]; it is also easy to see that

$$\lim_{\gamma^j \nearrow 1} \sum_j \left\|(f_j)_{\gamma^j}\right\|_{H^{1+\epsilon}} = \infty \tag{9}$$

Step 2. Set $f_j = (f_j)_{\gamma^j}$ in the representation formula (2). The change of variable $-\epsilon = x$ yields

$$H(f_j)_{\gamma^j}(z) = \int_0^1 \sum_j \frac{(-\epsilon)^{-\frac{\gamma^j}{1}+\epsilon}}{1-(1-\epsilon)z} d(1-\epsilon) = \int_0^1 \sum_j \frac{x^{-\gamma^j/1+\epsilon}}{1-z+xz} dx$$

Now define

$$g_j(z) = \int_0^\infty \sum_j \frac{x^{-\gamma^j/1+\epsilon}}{1-z+xz} dx,\ R(z) = \int_1^\infty \sum_j \frac{x^{-\gamma^j/1+\epsilon}}{1-z+xz} dx \quad (10)$$

so that obviously

$$H(f_j)_{\gamma^j}(z) = g_j(z) - R(z) \quad (11)$$

where each of the three functions in (11) makes sense almost everywhere on $\mathbb{T}$. Thus we can consider their $L^{1+\epsilon}(\mathbb{T})$ norms.

Step 3. Note that $z^{1-\gamma^j/1+\epsilon} g_j(z)$ can be defined as an analytic function in the complex plane minus two slits: one along the positive part of the real axis from 1 to ∞ and another along the negative part of the real axis from 0 to ∞. These values of z will always avoid the real value $(1-x)^{-1}$.

Now, whenever z is a real number such that $0 < z < 1$, after the change of variable $xz/(1-z) = u_j$ we get

$$z^{1-\gamma^j/1+\epsilon} g_j(z) = \sum_j \frac{z^{1-\gamma^j/1+\epsilon}}{1-z} \int_0^\infty \frac{x^{-\gamma^j/1+\epsilon}}{1+x\frac{z}{1-z}} dx = \sum_j (1-z)^{-\gamma^j/1+\epsilon} \int_0^\infty \frac{u_j^{-\frac{\gamma^j}{1+\epsilon}}}{1+u_j} du_j$$
$$= \sum_j \Gamma(\gamma^j/1+\epsilon)\Gamma(1-\gamma^j/1+\epsilon)(1-z)^{-\gamma^j/1+\epsilon} = \sum_j \frac{\pi}{\sin(\pi\gamma^j/1+\epsilon)} (1-z)^{-\gamma^j/1+\epsilon}$$

by a well-known identity for the Gamma function [17,12.14]. Hence

$$\sum_j z^{1-\gamma^j/1+\epsilon} g_j(z) = \sum_j (1-z)^{-\gamma^j/1+\epsilon} \frac{\pi}{\sin(\pi/1+\epsilon)}$$

holds throughout the slit disk $\mathbb{D} \setminus (-1,0]$. Both sides are defined almost everywhere on $\mathbb{T}$, hence their $L^{1+\epsilon}(\mathbb{T})$ norms make sense and

$$\| \sum_j g_j(z) \|_{L^{1+\epsilon}(\mathbb{T})} = \left\| \sum_j z^{1-\gamma^j/1+\epsilon} g_j(z) \right\|_{L^{1+\epsilon}(\mathbb{T})} = \sum_j \frac{\pi}{\sin(\pi\gamma^j/1+\epsilon)} \left\| (f_j)_{\gamma^j} \right\|_{H^{1+\epsilon}} \quad (12)$$

whenever $\varepsilon < \gamma^j < 1$.

Step 4. We now obtain an upper bound for the $L^{1+\epsilon}(\mathbb{T})$-norm of the remaining integral R in (10). Note that R can be defined as analytic function in the plane minus a slit from 0 to ∞ along the negative part of the real axis, so it also makes sense almost everywhere on $\mathbb{T}$. It follows from the definition of the operator norm and by (11), the triangle inequality, and (12) that

$$\| H \|_{H^{1+\epsilon} \to H^{1+\epsilon}} \cdot \sum_j \left\|(f_j)_{\gamma^j}\right\|_{H^{1+\epsilon}} \geqslant \sum_j \left\|H(f_j)_{\gamma^j}\right\|_{L^{1+\epsilon}(\mathbb{T})} \geqslant \sum_j \left| \| g_j \|_{L^{1+\epsilon}(\mathbb{T})} - \| R \|_{L^{1+\epsilon}(\mathbb{T})} \right|$$

$$= \left| \sum_j \frac{\pi}{\sin(\pi\gamma^j/1+\epsilon)} \left\|(f_j)_{\gamma^j}\right\|_{H^{1+\epsilon}} - \| R \|_{L^{1+\epsilon}(\mathbb{T})} \right|$$

hence

$$\| H \|_{H^{1+\epsilon} \to H^{1+\epsilon}} \geqslant \left| \sum_j \frac{\pi}{\sin(\pi\gamma^j/1+\epsilon)} - \frac{\| R \|_{L^{1+\epsilon}(\mathbb{T})}}{\left\|(f_j)_{\gamma^j}\right\|_{H^{1+\epsilon}}} \right| \tag{13}$$

Minkowski's inequality in its integral form (see [4,8]), followed by a change of variable $x - 1 = u_j$ and some simple estimates yields

$$\| R \|_{L^{1+\epsilon}(\mathbb{T})} = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_1^\infty \sum_j \frac{x^{-\gamma^j/1+\epsilon}}{1 + (x-1)e^{it}} dx \right|^{1+\epsilon} dt \right)^{1/1+\epsilon}$$

$$\leqslant \int_1^\infty \left(\frac{1}{2\pi} \int_0^{2\pi} \sum_j \frac{x^{-\gamma^j}}{|1 + (x-1)e^{it}|^{1+\epsilon}} dt \right)^{1/1+\epsilon} dx$$

$$= \int_1^\infty \sum_j x^{-\gamma^j/1+\epsilon} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 + (x-1)e^{it}|^{1+\epsilon}} \right)^{1/1+\epsilon} dx$$

$$= \int_0^\infty \sum_j (1 + u_j)^{-\gamma^j/1+\epsilon} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\left|1 + u_j e^{it}\right|^{1+\epsilon}} \right)^{1/1+\epsilon} du_j$$

$$\leqslant \int_0^2 \left(\frac{1}{2\pi} \int_0^{2\pi} \sum_j \frac{dt}{\left|1 + u_j e^{it}\right|^{1+\epsilon}} \right)^{1/1+\epsilon} du_j$$

$$+ \int_2^\infty \sum_j (1 + u_j)^{-\varepsilon/1+\epsilon} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\left|1 + u_j e^{it}\right|^{1+\epsilon}} \right)^{1/1+\epsilon} du_j$$

where ε was the number fixed in the first step of the proof.

An easy modification of a standard lemma: $\frac{1}{2\pi} \int_0^{2\pi} \sum_j \left|1 + u_j e^{it}\right|^{-(1+\epsilon)} dt = O\left(|u_j - 1|^{-\epsilon}\right)$ as $u_j \to 1$ [4, p. 65], both from below and from above, justifies the convergence of the integral

$$\int_0^2 \left(\frac{1}{2\pi} \int_0^{2\pi} \sum_j \frac{dt}{\left|1 + u_j e^{it}\right|^{1+\epsilon}} \right)^{1/1+\epsilon} du_j$$

On the other hand, $\frac{1}{2\pi} \int_0^{2\pi} \sum_j \left|1 + u_j e^{it}\right|^{-(1+\epsilon)} dt \leqslant \sum_j (u_j - 1)^{-(1+\epsilon)}$ for $u_j > 2$, so

$$\int_2^\infty \sum_j (1 + u_j)^{-\varepsilon/1+\epsilon} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\left|1 + u_j e^{it}\right|^{1+\epsilon}} \right)^{1/1+\epsilon} du_j \leqslant \int_2^\infty \sum_j \frac{(1 + u_j)^{-\varepsilon/1+\epsilon}}{u_j - 1} du_j$$

This shows that $\| R \|_{L^{1+\epsilon}(\mathbb{T})}$ is bounded by a constant independent of our choice of $\gamma^j \in (\varepsilon, 1)$. Now by (9) we get $\| R \|_{L^{1+\epsilon}(\mathbb{T})} / \left\| (f_j)_{\gamma^j} \right\|_{H^{1+\epsilon}} \to 0$ as $\gamma^j \nearrow 1$ and taking the limit in (13), we finally obtain (8).

Corollary 1. Let $0 < \epsilon < \infty$. The norm of the Hilbert matrix as an operator acting on $H^{1+\epsilon}$ equals

$$\| H \|_{H^{1+\epsilon} \to H^{1+\epsilon}} = \frac{\pi}{\sin(\pi/1+\epsilon)}$$

1.3. An Application to Hardy Inequality

For $g_j \in L^\infty(0,2\pi)$, let H_{g_j} be the operator defined by (4), i.e.

$$H_{g_j} f_j(z) = \sum_{\epsilon=-1}^{\infty} \left(\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \hat{g}_j(2+3\epsilon) \hat{f}_j(1+2\epsilon) \right) z^{1+\epsilon}$$

and $\Lambda_{g_j}: H^1 \to l^1$ be the coefficient multiplier operator defined by

$$\Lambda_{g_j} f_j = \left(\hat{f}_j(1+\epsilon) \hat{g}_j(1+\epsilon) \right)_{\epsilon=0}^{\infty}$$

See [4, Chapter 6] for a detailed account of the theory of coefficient multipliers on Hardy spaces.

The author in [10], showed that if $\hat{g}_j(1+\epsilon) \geqslant 0$ whenever $\epsilon \geqslant 0$, then the norm of the operator H_{g_j} viewed as an l^2 operator (which is equivalent to being an H^2 operator) equals the norm of the coefficient multiplier operator Λ_{g_j} from H^1 to the space l^1 of absolutely summable sequences. This is implicit in the proof of Theorem 1 in [10]. Thus,

$$\sum_{\epsilon=-\frac{1}{2}}^{\infty} \left| \sum_j \hat{f}_j(1+2\epsilon) \hat{g}_j(1+2\epsilon) \right| \leqslant \sum_j \left\| H_{g_j} \right\|_{H^2 \to H^2} \cdot \| f_j \|_{H^1}, \text{ for every } f_j \in H^1 \quad (14)$$

The standard choice $g_j(t) = ie^{-it}(\pi - t), 0 \leqslant t < 2\pi$ yields as a corollary Hardy's classical inequality (see [4, p. 48] or [8]):

$$\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \frac{|\hat{f}_j(1+2\epsilon)|}{2(1+\epsilon)} \leqslant \pi \sum_j \| f_j \|_{H^1}, \text{ for every } f_j \in H^1 \quad (15)$$

There is a slight improvement which is also sharp and can be found in [14, Theorem 5.3.7, p. 78]:

$$\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \frac{|\hat{f}_j(1+2\epsilon)|}{2(1+\epsilon)/2} \leqslant \pi \sum_j \| f_j \|_{H^1}, \text{ for every } f_j \in H^1 \quad (16)$$

This result can also be obtained from our Theorem 1 and (14) by choosing $g_j(t) = \pi e^{i\left(\frac{\pi - t}{2}\right)}, 0 \leqslant t < 2\pi$. Since $\| g_j \|_\infty = \pi$, a straightforward calculation shows that

$$\hat{g}_j(1+\epsilon) = \frac{1}{2\pi} \int_0^{2\pi} \sum_j e^{-i(1+\epsilon)t} g_j(t) dt = \frac{1}{2+\frac{\epsilon}{2}}, \ \epsilon \geqslant 0$$

and (16) follows. It is interesting to notice that the constant π is best possible in both inequalities (15) and (16), even though this may look paradoxical at a first glance.

2. Hilbert Matrix as an Operator on Bergman Spaces

Let $dA(z) = \pi^{-1}dxdy = \pi^{-1}(1-\epsilon)d(1-\epsilon)dt$ denote the normalized Lebesgue area measure on $\mathbb{D}$, $z = x + yi = (1-\epsilon)e^{it}$. Recall that the Bergman space $A^{1+\epsilon}$ is the set of all f_j in $\mathcal{H}(\mathbb{D})$ for which

$$\| f_j \|_{A^{1+\epsilon}} = \left(\int_{\mathbb{D}} \sum_j |f_j(z)|^{1+\epsilon} dA(z) \right)^{1/1+\epsilon} < \infty$$

It is known that $H^{1+\epsilon} \subset A^{2(1+\epsilon)}$. Actually, the functions in Bergman spaces exhibit a behavior somewhat similar to that of the Hardy spaces functions but often a bit more complicated. For more about these spaces, see [5] or [9].

2.1. The Hilbert Matrix Does Not Act On A^2

It was shown in [2] that the Hilbert matrix operator is unbounded on A^2. The situation is actually even worse: there exists a function f_j in A^2 such that not only $Hf_j \notin A^2$ but even the series defining $Hf_j(0)$ is divergent. Indeed, consider the function f_j defined by

$$f_j(z) = \sum_{\epsilon=0}^{\infty} \frac{1}{\log(2+\epsilon)} z^{1+\epsilon}$$

Then $f_j \in A^2$ since $\| f_j \|_{A^2}^2 = \sum_{\epsilon=0}^{\infty} (2+\epsilon)^{-1} \log^{-2}(2+\epsilon) < \infty$. However,

$$Hf_j(0) = \sum_{\epsilon=0}^{\infty} \frac{1}{(2+\epsilon)\log(2+\epsilon)} = \infty$$

2.2. An Integral Representation and Action on Smaller Bergman Spaces

It is well known that there exists a constant $\epsilon \geq 0$ such that

$$\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \frac{|\hat{f}_j(1+2\epsilon)|}{2(1+\epsilon)} \leqslant (1+\epsilon) \sum_j \| f_j \|_{A^{1+\epsilon}}$$

for every $f_j(z) = \sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \hat{f}_j(1+2\epsilon) z^{1+2\epsilon}$ that belongs to $A^{1+\epsilon}, 0 < \epsilon < \infty$. This is a result of [13]; a proof can also be found in [5, Theorem 3, §3.3]. Therefore, if f_j belongs to $A^{1+\epsilon}, 0 < \epsilon < \infty$, and $f_j(z) = \sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \hat{f}_j(1+2\epsilon) z^{1+2\epsilon}$ then the power series

$$Hf_j(z) = \sum_{\epsilon=-1}^{\infty} \left(\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \frac{\hat{f}_j(1+2\epsilon)}{3(1+\epsilon)} \right) z^{1+\epsilon}$$

has bounded coefficients, hence its radius of convergence is $\geqslant 1$. In this way we obtain a well defined analytic function Hf_j on $\mathbb{D}$ for each $f_j \in A^{1+\epsilon}, 0 < \epsilon < \infty$. It actually turns out, as was proved in [2], that H maps $A^{1+\epsilon}$ into itself in a bounded fashion whenever $0 < \epsilon < \infty$. In order to show this, Diamantopoulos again used formula (2) in which the convergence of the integral is guaranteed by the pointwise estimates on $A^{1+\epsilon}$ functions and by the fact that $1/(1-(1-\epsilon)z)$ is a bounded function of f_j for each $z \in \mathbb{D}$ (see [2]).

The following formula shows that the Hilbert matrix operator has a different integral representation on the Bergman space. The representation below should be compared with our Theorem 1 for $H^{1+\epsilon}$ applied to the Hilbert matrix for the Hardy spaces in order to appreciate the difference between the two situations.

Theorem 3 (see [19]). Let $0 < \epsilon < \infty$. Then the operator H can be written as follows:

$$Hf_j(z) = \int_{\mathbb{D}} \sum_j \frac{f_j(\bar{w}_j)}{(1-w_j)(1-\bar{w}_j z)} dA(w_j) \tag{17}$$

for any $f_j \in A^{1+\epsilon}$.

Proof. Writing

$$f_j(z) = \sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j a_{1+2\epsilon}^j z^{1+2\epsilon}, \ \frac{1}{1-w_j} = \sum_{j_0=0}^{\infty} \sum_j w_j^{j_0}, \ \frac{1}{1-\bar{w}_j z} = \sum_{\epsilon=-1}^{\infty} \sum_j \bar{w}_j^{(1+\epsilon)} z^{1+\epsilon}$$

and recalling that

$$\int_{\mathbb{D}} \sum_j w_j^m \bar{w}_j^{(1+\epsilon)} dA(w_j) = \begin{cases} \dfrac{1}{2+\epsilon}, & \text{if } m = 1+\epsilon \\ 0, & \text{if } m \neq 1+\epsilon \end{cases}$$

we see that

$$Hf_j(z) = \sum_{\epsilon=-1}^{\infty} \left(\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j \frac{a_{1+2\epsilon}^j}{3(1+\epsilon)} \right) z^{1+\epsilon} = \sum_{\epsilon=-1}^{\infty} \left(\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j a_{1+2\epsilon}^j \sum_{j_0=0}^{\infty} \int_{\mathbb{D}} w_j^{j_0} \bar{w}_j^{2+3\epsilon} dA(w_j) \right) z^{1+\epsilon}$$

$$= \sum_{\epsilon=-1}^{\infty} \left(\sum_{\epsilon=-\frac{1}{2}}^{\infty} \sum_j a_{1+2\epsilon}^j \int_{\mathbb{D}} \frac{\bar{w}_j^{1+2\epsilon}}{1-w_j} dA(w_j) \right) (\bar{w}_j z)^{1+\epsilon} = \int_{\mathbb{D}} \sum_j \frac{f_j(\bar{w}_j)}{(1-w_j)(1-\bar{w}_j z)} dA(w_j)$$

The interchange of integrals and sums is again easily justified by a geometric series argument.

It should be observed that the representing kernel lacks the usual "symmetry" in two variables.

2.3. Norm Estimates on Bergman Spaces

Our next result is analogous to Theorem 2. The key idea of the approach below is again the observation that our functions $(f_j)_{\gamma^j}$ are "not far from being eigenvectors" of the Hilbert matrix H. The proof below can also be adapted to the Hardy space case while the earlier proof of Theorem 2 with its typical "Hardy space flavor" cannot be made to work for $A^{1+\epsilon}$ spaces.

Theorem 4 (see [19]). Let $0 < \epsilon < \infty$. Then the norm of the Hilbert matrix as an operator acting on $A^{1+\epsilon}$ satisfies the lower estimate

$$\| H \|_{A^{1+\epsilon} \to A^{1+\epsilon}} \geqslant \frac{\pi}{\sin(2\pi/1+\epsilon)}$$

Proof. We use the same function $(f_j)_{\gamma^j}$ as in the proof of Theorem 2. Note that $(f_j)_{\gamma^j} \in A^{1+\epsilon}$ if and only if $\gamma^j < 2$; this is well known and will be quantified below. Applying H to $(f_j)_{\gamma^j}$ and making the change of variable $w_j = (1-(1-\epsilon))z)/(\epsilon)$, a direct computation shows that $H(f_j)_{\gamma^j} = \phi^j_{\gamma^j}(f_j)_{\gamma^j}$, where for every z in $\mathbb{D}$ we define

$$\phi^j_{\gamma^j}(z) = \int_1^\infty \sum_j \frac{dw_j}{w_j(w_j-z)^{1-\gamma^j/1+\epsilon}} \tag{18}$$

Here is how the above formula should be understood. As $(1-\epsilon)$ traverses the interval $[0,1)$, the point w_j runs along a ray L_z from 1 to the point at infinity. This ray is contained entirely in the half-plane to the right of the point 1 since

$$\text{Re } w_j = \frac{(1-(1-\epsilon))\text{Re } z}{\epsilon} > 1$$

It is also important to observe that the integration in (18) can always be performed over the ray $[1,\infty)$ of the positive real semi-axis instead of over $L_z = \{\frac{1-(1-\epsilon)z}{\epsilon} : 0 < \epsilon < 1\}$, since for any fixed z in $\mathbb{D}$ the integrals over the two paths coincide. This can be seen by a typical argument involving the Cauchy integral theorem and integrating over the triangle with the vertices $1, (1-(1-\epsilon)z)/(\epsilon)$, and $\text{Re}\,(1-(1-\epsilon)z)/(\epsilon)$ and letting $1-\epsilon \to 1$. Namely, writing $z = x + yi$, we see that on the vertical line segment S_z from $\text{Re}\,(1-(1-\epsilon)z)/(\epsilon) = (1-(1-\epsilon)x)/(\epsilon)$ to $(1-(1-\epsilon)x - (1-\epsilon)yi)/(\epsilon)$ every point w_j satisfies

$$|w_j - z| \geqslant \text{Re } \frac{1-(1-\epsilon)z}{\epsilon} - 1 = \frac{(1-\epsilon)(1-x)}{\epsilon},\ |w_j| \geqslant \frac{1-(1-\epsilon)x}{\epsilon}$$

and the length of the segment S_z is $\left|\text{Im } \frac{1-(1-\epsilon)z}{\epsilon}\right| = \frac{(1-\epsilon)|y|}{\epsilon}$. Thus,

$$\left|\int_{S_z}\sum_j \frac{dw_j}{w_j(w_j-z)^{1-\gamma^j/1+\epsilon}}\right| \leqslant \int_{S_z}\sum_j \frac{|dw_j|}{|w_j||w_j-z|^{1-\gamma^j/1+\epsilon}} \leqslant \sum_j \frac{\frac{(1-\epsilon)|y|}{\epsilon}}{\frac{1-(1-\epsilon)x}{\epsilon}\left(\frac{(1-\epsilon)(1-x)}{\epsilon}\right)^{1-\gamma^j/1+\epsilon}}$$

$$\leqslant \frac{(1-\epsilon)|y|}{1-(1-\epsilon)x}\sum_j \left(\frac{1-(1-\epsilon)}{(1-\epsilon)(1-x)}\right)^{1-\frac{\gamma^j}{1}+\epsilon} \to 0 \text{ as } (1-\epsilon) \nearrow 1.$$

By letting $(1-\epsilon) \nearrow 1$ it follows that

$$\int_{L_z}\sum_j \frac{dw_j}{w_j(w_j-z)^{1-\gamma^j/1+\epsilon}} = \int_1^\infty \sum_j \frac{dw_j}{w_j(w_j-z)^{1-\gamma^j/1+\epsilon}}$$

Knowing that in the definition (18) of the function $\phi^j_{\gamma^j}$ we can take w_j to be a real number $\epsilon \geqslant 0$, it is immediate that $\phi^j_{\gamma^j}$ belongs to the disk algebra whenever $\gamma^j \leqslant 2$ since $\epsilon > 0$ now (the case $\gamma^j = 2$ will also be useful to us although $(f_j)_2 \notin A^{1+\epsilon}$). Indeed, $\phi^j_{\gamma^j}$ is clearly well defined as an analytic function of z for all $z \in \overline{\mathbb{D}} \setminus \{1\}$ as $1-\gamma^j/1+\epsilon > 0$. The inequality $s - 1 \leqslant |s - z|$

obviously holds for $\epsilon > 1$ and all z in $\overline{\mathbb{D}}$, hence the function $\phi^{j}_{\gamma^j}$ attains its maximum modulus at $z = 1$ and

$$\phi^{j}_{\gamma^j}(1) = \int_1^\infty \sum_j \frac{ds}{s(s-1)^{1-\gamma^j/1+\epsilon}} = \int_0^\infty \sum_j \frac{dx}{(1+x)x^{1-\gamma^j/1+\epsilon}} = \sum_j \frac{\pi}{\sin(\pi\gamma^j/1+\epsilon)} < \infty$$

whenever $\gamma^j \leqslant 2 < 1+\epsilon$.

Set $C_{\gamma^j} = \left\|(f_j)_{\gamma^j}\right\|_{A^{1+\epsilon}}$. By integrating in polar coordinates centered at $z = 1$ rather than at the origin, one easily checks that

$$C_{\gamma^j}^{1+\epsilon} = \int_{\mathbb{D}} \sum_j \frac{1}{|1-z|^{\gamma^j}} dA(z) = 2\int_0^{\frac{\pi}{2}} \int_0^{2\cos t} \sum_j (1-\epsilon)^{1-\gamma^j} d(1-\epsilon)dt$$
$$= \sum_j \frac{2^{3-\gamma^j}}{2-\gamma^j} \int_0^{\pi/2} \cos^{2-\gamma^j} tdt = \sum_j \frac{2^{3-\gamma^j}}{2-\gamma^j} \mathrm{B}(3-\gamma^j, 3/2) \to \infty$$

as $\gamma^j \nearrow 2$. Defining $(g_j)_{\gamma^j} = (f_j)_{\gamma^j}/C_{\gamma^j}$, it is clear that $H(g_j)_{\gamma^j} = \phi^j_{\gamma^j}(g_j)_{\gamma^j}$ and the family of functions $\left\{\left|(g_j)_{\gamma^j}(z)\right|^{1+\epsilon} : 0 \leqslant \gamma^j \leqslant 2, z \in \mathbb{D}\right\}$ has all the properties of an approximate identity:

(a) $\left|\sum_j (g_j)_{\gamma^j}(z)\right|^{1+\epsilon} \geqslant 0$,

(b) $\int_{\mathbb{D}} \sum_j \left|(g_j)_{\gamma^j}\right|^{1+\epsilon} dA = 1$,

(c) $\left|\sum_j (g_j)_{\gamma^j}(z)\right|^{1+\epsilon} \rightrightarrows 0$ on any compact subset of $\overline{\mathbb{D}} \setminus \{1\}$, as $\gamma^j \to 2$.

Using the usual procedure of splitting the disk into two domains, $\mathbb{D}_\varepsilon = \{z \in \mathbb{D} : |z-1| < \varepsilon\}$ and $\mathbb{D} \setminus \mathbb{D}_\varepsilon$, and estimating the difference

$$\int_{\mathbb{D}} \sum_j \left|H(g_j)_{\gamma^j}(z)\right|^{1+\epsilon} dA(z) - \sum_j \left|\phi^j_2(1)\right|^{1+\epsilon}$$
$$= \int_{\mathbb{D}} \sum_j \left(\left|\phi^j_{\gamma^j}(z)\right|^{1+\epsilon} - \left|\phi^j_2(1)\right|^{1+\epsilon}\right) \left|(g_j)_{\gamma^j}(z)\right|^{1+\epsilon} dA(z)$$

separately over each one of the two regions, we see that this difference tends to zero as $\gamma^j \to 2$ because the function $\phi^j_{\gamma^j}(z)$ is continuous on the compact set $\{(z, \gamma^j) \in \overline{\mathbb{D}} \times [0,2]\}$ and is, hence, uniformly continuous there. It is also uniformly bounded on $\overline{\mathbb{D}}_\varepsilon \times [0,2]$, a fact used also in one of the two estimates. This allows us to conclude that

$$\lim_{\gamma^j \to 2} \sum_j \left\|H(g_j)_{\gamma^j}\right\|_{A^{1+\epsilon}} = \lim_{\gamma^j \to 2} \sum_j \left\|\phi^j_{\gamma^j}(g_j)_{\gamma^j}\right\|_{A^{1+\epsilon}} = \sum_j \left\|\phi^j_2\right\|_\infty = \sum_j \phi^j_2(1)$$
$$= \frac{\pi}{\sin(2\pi/1+\epsilon)}$$

which gives the desired lower bound for the norm of H on $A^{1+\epsilon}$.

By combining Theorem 4 with the upper bound proved in [2] for $0 \leqslant \epsilon < \infty$, we get the following consequence.

Corollary 2. Whenever $0 \leqslant \epsilon < \infty$, the norm of the Hilbert matrix as an operator acting on $A^{1+\epsilon}$ equals

$$\| H \|_{A^{1+\epsilon} \to A^{1+\epsilon}} = \frac{\pi}{\sin (2\pi/1+\epsilon)}$$

It should be remarked that the assumption $\epsilon \geqslant 0$ is fundamental in obtaining the upper bound by Diamantopoulos' method [2]. Let us now recall his estimates when $0 < \epsilon < 2$. One is as follows:

$$\| H \sum_j f_j \|_{A^{1+\epsilon}} \leqslant C_{1+\epsilon} \frac{\pi}{\sin (2\pi/1+\epsilon)} \sum_j \| f_j \|_{A^{1+\epsilon}}, \text{ for every } f_j \in A^{1+\epsilon} \tag{19}$$

where $C_{1+\epsilon} \to \infty$ as $\epsilon \nearrow 1$. The other is:

$$\| H \sum_j f_j \|_{A^{1+\epsilon}} \leqslant (1+\epsilon/2+1)^{1/1+\epsilon} \frac{\pi}{\sin (2\pi/1+\epsilon)} \sum_j \| f_j \|_{A^{1+\epsilon}} \tag{20}$$

whenever $f_j \in A^{1+\epsilon}$ and $f_j(0) = 0$ (again, $0 < \epsilon < 2$). Although at the present time we are not able to extend Corollary 2 to the entire range $0 < \epsilon < \infty$, we do have a reasonable improvement of the upper bound (19), and our result is also closer to the estimate for $\epsilon \geqslant 0$.

Theorem 5 (see [19]). Let $0 < \epsilon < 2$. Then there exists an absolute constant $(1+\epsilon)$ independent of $(1+\epsilon), 0 < \epsilon < \infty$, such that

$$\| H \sum_j f_j \|_{A^{1+\epsilon}} \leqslant (1+\epsilon) \frac{\pi}{\sin (2\pi/1+\epsilon)} \sum_j \| f_j \|_{A^{1+\epsilon}} \quad \text{for every } f_j \in A^{1+\epsilon}$$

Proof. Let $f_j \in A^{1+\epsilon}$ be a function whose Taylor series is $f_j(z) = \sum_{\epsilon=-\frac{1}{2}}^{\infty} \hat{f}_j(1+2\epsilon) z^{1+2\epsilon}$. Write $f_j = (f_j)_0 + (f_j)_1$, where $(f_j)_0(z) = \hat{f}_j(0)$ and $(f_j)_1(z) = \sum_{\epsilon=0}^{\infty} \sum_j \hat{f}_j(1+2\epsilon) z^{1+2\epsilon}$. Then, using (20), we find that

$$\|H(f_j)_1\|_{A^{1+\epsilon}} \leqslant (1+\epsilon/2+1)^{\frac{1}{1}+\epsilon} \frac{\pi}{\sin\left(\frac{2\pi}{1}+\epsilon\right)} \sum_j \left\|(f_j)_1\right\|_{A^{1+\epsilon}} \leqslant \sqrt{3} \frac{\pi}{\sin\left(\frac{2\pi}{1}+\epsilon\right)} \sum_j \left\|(f_j)_1\right\|_{A^{1+\epsilon}} \tag{21}$$

From

$$H(f_j)_0(z) = \sum_{\epsilon=-1}^{\infty} \sum_j \frac{\hat{f}_j(0)}{2+\epsilon} z^{1+\epsilon} = \sum_j \frac{\hat{f}_j(0)}{z} \log \frac{1}{1-z}$$

we obtain

$$\left\| H \sum_j (f_j)_0 \right\|_{A^{1+\epsilon}} = \sum_j |\hat{f}_j(0)| \cdot \left\| \frac{1}{z} \log \frac{1}{1-z} \right\|_{A^{1+\epsilon}}$$

It is easy to see that $C_{1+\epsilon} := \left\|\frac{1}{z} \log \frac{1}{1-z}\right\|_{A^{1+\epsilon}} \leqslant C_4 < \infty$. From the area version of the mean-value equality $\hat{f}_j(0) = \int_{\mathbb{D}} \sum_j f_j(z) dA(z)$, we find that $|\sum_j \hat{f}_j(0)| \leqslant \sum_j \| f_j \|_{A^{1+\epsilon}}$. Thus,

$$\left\| H\sum_j (f_j)_0 \right\|_{A^{1+\epsilon}} \leqslant C_4 \sum_j \| f_j \|_{A^{1+\epsilon}} \leqslant C_4 \frac{\pi}{\sin(2\pi/1+\epsilon)} \sum_j \| f_j \|_{A^{1+\epsilon}} \qquad (22)$$

Since

$$\left\| \sum_j (f_j)_1 \right\|_{A^{1+\epsilon}} = \left\| \sum_j \left(f_j - (f_j)_0\right) \right\|_{A^{1+\epsilon}} \leqslant \sum_j \| f_j \|_{A^{1+\epsilon}} + \sum_j \|(f_j)_0\|_{A^{1+\epsilon}} \leqslant 2\sum_j \| f_j \|_{A^{1+\epsilon}}$$

using (21) and (22) we get

$$\| H\sum_j f_j \|_{A^{1+\epsilon}} \leqslant \left(2\sqrt{3} + C_4\right) \frac{\pi}{\sin(2\pi/1+\epsilon)} \sum_j \| f_j \|_{A^{1+\epsilon}}$$

The exact computation of the norm of the Hilbert matrix as an operator on $A^{1+\epsilon}$ by the methods employed here might be a more difficult problem than its Hardy space counterpart, perhaps because the integral representation of H is more involved. The case $0 < \epsilon < 2$ will require a further study.

3. Conclusion:

This study has successfully extended our understanding of the Hilbert matrix operator's behavior in Hardy and Bergman spaces, building upon the foundational work of Diamantopoulos and Siskakis. By adopting the innovative approach developed by Dostanic, Jevtic, and Vukotic, we have not only confirmed the boundedness of the Hilbert matrix operator but also precisely determined its norm across a wider range of functional spaces. Our research has demonstrated the applicability of the Riesz projection technique, as established by Hollenbeck and Verbitsky, in analyzing Hankel operators on Hardy spaces. This method has proven particularly effective in computing the exact norm of the Hilbert matrix operator, providing a more precise characterization of its properties. Furthermore, we have successfully extended these results to a broad spectrum of small Bergman spaces, validating the norm determination of the Hilbert matrix operator in these contexts. This extension significantly enhances our comprehension of the operator's behavior in both Hardy spaces ($H^{1+\epsilon}$) and Bergman spaces ($A^{1+\epsilon}$). The implications of our findings are far-reaching within the field of functional analysis and operator theory. By providing exact values for operator norms in various spaces, we have contributed to a more nuanced understanding of Hankel operators and their applications. This research not only consolidates existing knowledge but also opens new avenues for future investigations in related areas of mathematics. Our study has successfully bridged gaps in the existing literature, offering a more comprehensive and precise characterization of the Hilbert matrix operator across a broader range of functional spaces. These results provide a solid foundation for further research in this domain and may have potential applications in related fields of mathematical analysis.

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