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Review Paper

To the multiparameter spectral theory

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The spectral theory of operators is one of the essential directions of functional analysis. Development of physical sciences proposes new and new problems for mathematicians. Particularly, many partial differential equations and problems of mathematical physics, connecting with physical processes, demand new admission to solving of such problems.

The method of separation of variables in partial differential equations in many cases turned out to be the only acceptable because it reduces the finding a solution of a complex equation with many variables to the finding of a solution of a system of ordinary differential equations with many parameters which are much easier to study[1],[2]. For example, a multivariable problems arise in the quantum mechanics, diffraction theory, the theory of elastic shells, nuclear reactor calculations, stochastic diffusion processes, Brownian motion, boundary value problems for equations of elliptic-parabolic type, the Cauchy problem for ultraparabolic equations and etc. [1],[3].

F.V. Atkinson [4] studied the fragmentary results for multiparameter symmetric differential systems, built multiparameter spectral theory of self-adjoint systems of operators in finite-dimensional Euclidean spaces. Further, by taking the limit, Atkinson generalized these results to the case of the multiparameter system with self-adjoint completely continuous operators in infinite-dimensional Hilbert spaces.

Further, Browne [5], Sleeman [6], and other mathematicians built the spectral theory of self-adjointmultiparameter system in infinite dimensional Hilbert spaces. In spite of actuality and age-old of this problem, obtained results apply to the only self-adjointmultiparameter systems linearly depending on parameters, because the method of investigations in abovementioned works [4],[5],[6] essentially uses the self-adjointness of all operators, forming the system. The author offered the new approach to research of multiparameter problems [10],[11],[12],[13].

We consider a multiparameter system

$$A(\lambda_1,\lambda_2,...,\lambda_n)x_i = (A_{i,0}\lambda_1 + +\lambda_2 A_{i,2} + ... + \lambda_n A_{i,n})x_i = 0$$
$$i = 1,2,...,n \quad (1)$$

where $A_{i,k}$ are bounded operators acting in Hilbert space H_i .

It is necessary to give somedefinitions and notions of multiparameter system of operators.

In [4],[5] for system (1) analogs of Cramer's determinants are introduced by following manner: on decomposable tensors $x = x_1 \otimes x_2 \otimes ... \otimes x_n$ operators Δ_i are set by means of equality

$$\sum \alpha_i \Delta_i x = \otimes$$

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \\ A_{1,0}x_1 & A_{1,1}x_1 & A_{1,2}x_1 & A_{1,3} & \dots & A_{1,n}x_1 \\ A_{2,0}x_2 & A_{2,1}x_2 & A_{2,2}x_2 & A_{2,3} & \dots & A_{2,n}x_2 \\ A_{3,0}x_3 & A_{3,1}x_3 & A_{3,2}x_3 & A_{3,3} & \dots & A_{3,n}x_3 \\ . & . & . & . & \dots & . \\ A_{n,0}x_n & A_{n,1}x_n & A_{n,2}x_n & A_{n,3} & \dots & A_{n,n}x_n \end{pmatrix}. , \quad (2)$$

where $\alpha_0, \alpha_1, ..., \alpha_n$ are arbitrary complex numbers. If $\alpha_k = 1, \alpha_i = 0$ at $i \neq k$ (2) is $\Delta_k x$ where $x = x_1 \otimes x_2 \otimes ... \otimes x_n$. ,on all other elements of space $H = H_1 \otimes H_2 \otimes ... \otimes H_n$ the operator Δ_k is defined on linearity and a continuity.
The expansion of a determinant (2) is understood as its formal expansion when products of elements are tensor products of elements.

From [4] it is known, if Δ_0^{-1} exists, operators $\Gamma_i = \Delta_0^{-1}\Delta_i$ $i = (1,2,...,n)$ are pair wise commute, and also the formulae

$$A_{i,0}^{+} + A_{i,1}^{+}\Gamma_i + ... + A_{i,n}^{+}\Gamma_n = 0 \quad (3)$$

is fair .
In (3)

$$A_{i,k}^{+} = E_1 \otimes E_2 \otimes ... \otimes E_{i-1} \otimes A_{i,k} \otimes E_{i+1} \otimes ... \otimes E_n ,$$

$E_s (s = 1,2,...,n)$ are the induced operators into space H_s by operators $A_{i,k}$ and E_s correspondingly.

If for two decomposable tensors $x = x_1 \otimes x_2 \otimes ... \otimes x_n$, $y = y_1 \otimes y_2 \otimes ... \otimes y_n$ $[x, y] = (\Delta_0 x, y),$ where $(x, y) = \sum_{i=1}^{n} (x_i, y_i)$, and (x_i, y_i) is the inner productin H_i, and on all other elements of the space H the inner product $[x, y]$ is defined on linearity and a continuity, then in the space H at the condition $(\Delta_0 x, x) ,\geq \delta (x, x,) , \delta > \mathbf{0}$ all operators Γ_k are self-adjoint [4],[5],[6].

1. $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is an eigenvalue of the systrm (1) if there is n nonzero vectors x_i from H_i such that the equations in the (1) are true. $x = x_1 \otimes x_2 \otimes ... \otimes x_n$ is the corresponding eigenvector.

2.Let $m_1, m_2, \ldots, m_n$ is n the whole non-negative numbers. The element $z_{m_1, m_2, \ldots, m_n} \in H$ is called the ($m_1, m_2, \ldots, m_n$)-th associated vector to an eigenvector $z_{0,\ldots,0}$, corresponding to an eigenvalue λ^0 of system (1) if there is a set
$(z_{i_1, i_2, \ldots, i_n}) \subset H_1 \otimes \ldots \otimes H_n, 0 \le i_k \le m_k, k = 1,2,\ldots,n$ such, that following equalities are fulfilled

$$A_k^+(\lambda^0) z_{i_1, i_2, \ldots, i_n} + A_{1,k}^+ z_{i_1 - 1, i_2, \ldots, i_n} + \ldots + A_{n,k}^+ z_{i_1, \ldots, i_{n-1}, i_n - 1} = 0, \qquad k = 1,2,\ldots,n$$

;

elements $z_{i_1, i_2, \ldots, i_n} = 0$ if $i_k < 0$.

Let's designate through $M(\lambda^0)$ a subspace tense on eigenvectors and associate vectors of system (`1), corresponding to an eigenvalue λ^0.

Linearly-independent vectors form a line-up of aset $(z_{i_1, \ldots, i_n}) \subset H$ of eigen and associated(e.a) vectors . The multiplicity of eigenvalue $z_{0,\ldots,0}$ designates the greatest number of eigenvectors and correspondingto them associated vectors $z_{0,\ldots,0}$ a plus 1. [10],[11],[12]

We give the sufficient condition for the existenceof eigenvalue
of the not self-adjointmultiparameter system (1.For this we use the results from [10],[11].
We have

$$\Gamma_k = \Gamma_{k,1} + i\Gamma_{k,2}$$

where

$$\Gamma_{i,1} = \frac{\Gamma_i + \Gamma_i^*}{2} \text{ and } \Gamma_{i,2} = \frac{\Gamma_i - \Gamma_i^*}{2i}$$

$$i = 1, 2, \ldots, n$$

are bounded self-adjointoperators acting in tensor product space $H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$, the inner product of H is $(x, y) = \sum_{i=1}^{n} (x_i, y_i)$ for two arbitrary decomposable tensors $x = x_1 \otimes x_2 \otimes \ldots \otimes x_n$ and $y = y_1 \otimes y_2 \otimes \ldots \otimes y_n$, (x_i, y_i) is the inner product in H_i, and on all other elements of the space H the inner product (x, y) is defined on linearity and a continuity.

Let be E_k the expansion of unity of operator $\Gamma_{k,1}$,and F_k is the expansion of unity of operator $\Gamma_{k,2}$ [7],[8].

Projective operators E_k and F_k satisfy the following conditions:

1. $E_{m,a} = 0$, $E_{m,b} = 1$

$F_{m,c} = 0$, $F_{m,,d} = 1$

2. $E_{m,l} E_{m,n} = E_{m,k}$

$k = \min(l, n)$

$F_{,mr} F_{m,s} = F_{m,q}$

$q = \min(r, s)$

3. $E_{m,t} - E_{m,t-0} = P_{m,t}$

$F_{sm,m,} - F_{sm,-0} = R_{m,s}$

where $P_{tm,t}$ is projective operator which projects onto eigen subspace of operator $_m \Gamma_{m,1}$ corresponding to its eigenvalue t ,and $R_{m,s}$ is a projective operator that projects onto subspace of operator $\Gamma_{k,2}$ corresponding to its eigenvalue S [7],[8].

The Theorem.

Let the following conditions are satisfied:

1. $Ker\Delta_0 = \{0\}$

$KerA_{i,n} = \{0\}(i = 1, 2, ..k., n)$

2. Operators $\Gamma_k (k = 1, 2, ..., n)$ are completely continuous in Hilbert space $H = H_1 \otimes H_2 \otimes ... \otimes H_n$

3.For $2n$ complex numbers $a1i, b_1, ..., , a, ibn$

$$P_{1,a_1} R_{1\ b_1} ... P_{n,a_n} R_{n\ b_n} \neq 0 \quad (4)$$

Then the multiparameter system (1) has an eigenvalue $(a_1 + ib_1, ..., a_n + ib_n)$.

Proof. The condition (4) means that all projective operators $P_{i,ai} R_{i,b_i} \neq 0$ .The last means thateach operator Γ_k has an eigenvalue $a_k + ib_k$ [14],[15].

From (4) we have that all operators $\Gamma_k = \Gamma_{k,1} + i\Gamma_{k,2}$ have a common eigenvector[15],[16]. Eigen subspace of operator Γ_k contains elements of the form x_1

$$x_{1,i_1} \otimes x_{2,i_2} \otimes ... \otimes x_{k-1,i_{k-1}} \otimes x_{k,0} \otimes ... \otimes x_{n,i_n}$$

$i_1 + i_2 + ... + i_n = 0;1;2;...;i_{k=0}$ (5)

and its linear combinations
[10],[18].

It is clear that the common eigenvector of all operators Γ_k is the vector for which all indices $i_k (k = 1, 2, ..., n)$ in (5)are equal to zero. We denote it as $x = x_1 \otimes x_2 \otimes ... \otimes x_n$.

Thus $\Gamma_k x = (a_k + ib_k)x, (k = 1, 2, ..., n)$ (6)

Substituting the expressions from (6) into (3) and the taking in account the definitions of eigenvalues and eigenvectors of the multiparameter system we have that $(a_1 + ib_1, ..., a_n + ib_n)$ is the eigenvalue of the system (1) and the common eigenvector of operators Γ_k is the corresponding to this eigenvalue $(a_1 + ib_1, ..., a_n + ib_n)$ eigenvector $x = x_1 \otimes x_2 \otimes ... \otimes x_n$ of the system (1). The Theorem is proven.

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