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Review Paper

On Weighted Differentiation Composition Operators from Mixed-Norm to Zygmund Spaces

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Abstract

We show and investigated the little Zygmund spaces, following perfectly the way of [53] the graduation steps of the boundedness and compactness of the weighted differentiation composition operator $D^n_{\varphi_j,u_j}$ assigned between the mixed-norm to Zygmund spaces are determined. Where given $H(\mathbb{D})$ the space of all sequences of analytic functions on the unit disk $\mathbb{D}$, $\varphi_j \in \mathbb{D}$ and $u_j \in H(\mathbb{D})$ be the sequences of analytic self-maps.

Keywords Boundedness; Compactness; Mixed-norm space; Weighted differentiation composition operators; Zygmund space.

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I. INTRODUCTION

For $\mathbb{D} \in \mathbb{C}$ the finite complex plane, $\partial\mathbb{D}$ the boundary , $\mathbb{N}_0$ and $\mathbb{N}$ the sets of nonnegative and positive integers respectively.

$0 \le \phi_j < 1$, sequence of positive continuous functions are called normal if there exist positive numbers $(1+\epsilon),(1+2\epsilon),\epsilon \ge 0$ and $L \in [0,1)$, so that

$$\frac{\phi_j(t)}{(1-t^2)^{1+\epsilon}} \text{ decreases for } 4 \le t < 1 \text{ and } \lim_{t\to 1^-} \sum_j \frac{\phi_j(t)}{(1-t^2)^n} = 0$$

and

$$\frac{\phi_j(t)}{(1-t^2)^6} \text{ increases for } 4 \le t < 1 \text{ and } \lim_{t\to 1^-} \sum_j \frac{\phi_j(t)}{(1-t^2)^6} = \infty$$

(see [28]).

For $0 < \epsilon < \infty$ and ϕ_j, let $H(1+\epsilon),(1+2\epsilon),(\phi_j)$ denote the space of all sequences of analytic functions f_j on $\mathbb{D}$ so that

$$\sum_j \| f_j \|_{1+\epsilon\cdot 4\phi_j} = \sum_j \left(\int_0^1 M^{1+\epsilon}_{1+2\epsilon}(f_j,r)\frac{\phi_j'(r)}{1-r} r dr\right)^{1/f_j}$$

Where $M_{1+\epsilon}(f_j,r)$ are the integral means defined by

$$M_{1+\epsilon}(f_j,r)=\sum_j\left(\frac{1}{2\pi}\int_0^{2x}\left|f_j(\pi^d)\right|^{1+\epsilon}d\theta\right)^{1/1+\epsilon},\ 0\le r<1.$$

For $0\le\epsilon<\infty, H(1+\epsilon),(1+2\epsilon),(\phi_j)$ with the norm $\|\cdot\|_{1+\epsilon,1+2\epsilon,\phi_j}$ is a Banach space. When $0\le\epsilon<1$, $\|\cdot\|_{pes}$ is a quasinorm on $H(1+\epsilon),(1+2\epsilon),(\phi_j), H(1+\epsilon),(1+2\epsilon),(\phi_j)$ is a Fréchet space but not a Banach space. If $0\le\epsilon<\infty$, then $H(1+\epsilon),(1+\epsilon),(\phi_j)$ is the Bergman-type space

$$H(1+\epsilon),(1+\epsilon),(\phi_j)=\left\{f_j\in H(\mathbf{D}):\int_{\mathbb{D}}\sum_j|f_j(z)|\frac{\phi_j^{1+\epsilon}(|z|)}{1-|z|}dA(z)<\infty\right\}$$

where $dA(z)$ denotes the normalized Lebesgue area measure on the unit disk $\mathbf{D}$ such that $A(\mathbf{D})=1$. So if $\phi_j(r)=(1-r)^{(x+1)/1\hat{+}\epsilon}$, then $H(1+\epsilon),(1+\epsilon),(\phi_j)$ is the weighted Bergman space $A_j^{1+\epsilon}$(D) defined for $0<\epsilon<\infty$ and $\alpha>-1$, as the space of all functions $f_j\in H(\mathbb{D})$ so that

$$\|f_j\|_{\Lambda_1}^{1+\epsilon}=\int_{\mathbf{D}}\sum_j|f_j(z)|^{1+\epsilon}(1-|z|^2)^x dA(z)<\infty$$

(see, [5]). For £ denote the space of all functions $f_j\in H(\mathbf{D})\cap C(\overline{\mathbb{D}})$ so that

$$\|f_j\|_I=\sup\sum_j\frac{\left|f_j\left(e^{i(\theta+h_j)}\right)+f_j\left(e^{i(\theta-k)}\right)-2f_j\left(e^{i\theta}\right)\right|}{h_j}<\infty,$$

where the supremum is taken over all $e''\in\vec{O}$D and $h_j>0$. By the Zygmund theorem (see [3, Theorem 5.3]) and the Closed Graph Theorem we see that $f_j\in\mathcal{X}$ if and only if

$$\sup_{x\in\mathbf{D}}\sum_j(1-|z|^2)\left|f_j''(z)\right|<\infty$$

So that the following asymptotic relation holds:

$$\|f_j\|_x\approx\sup_{u_j+\mathbf{D}}\sum_j(1-|z|^2)\left|f_j''(z)\right|. \tag{1}$$

Therefore, $\mathcal{A}$ is called the Zygmund class. Since the quantities in (1) are semi norms (they do not distinguish between functions differing by a linear polynomial), it is natural to add to them the quantity $|f_j(0)|+|f_j(0)|$ to obtain two equivalent norms on the Zygmund class of functions. The Zygmund class with such defined norm will be called the Zygmund space. Which again denoted by $\|\cdot\|_x$.

For some information and operation on Zygmund-type spaces on the unit disc see [1,9,13,17,21,92], for the case of the upper half-plane see [41], and information of the unit ball see [20,21,45,50 − 52].

The little Zygmund space $\mathcal{A}_0$ was introduced by L. and Stević in [13] in the following natural way:

$$f_j \in \not{f}_0 \Leftrightarrow \lim_{|z|\to 1} \sum_j (1-|z|)\left|f_j''(z)\right| = 0$$

It is easy to see that $\mathcal{J}_0$ is a closed subspace of $\not{f}$ and the set of all polynomials is dense in $\not{f}_a$.

For $D = D^1$ be the differentiation operator, that is, $Df_j = f_j'$. If $n \in \mathbb{N}_0$ then the operator D^n is defined by $D^0(\sum_j f_j) = \sum_j f_j, D^n(\sum_j f_j) = \sum_j f_j^{(n)}, f_j \in H(\mathbb{D})$.

We denote the weighted differentiation composition operator by D^z_{e,u_j}, and defined as (see [44,48]) :

$$\sum_j D^n_{\varepsilon,a} f_j(z) = \sum_j u_j(z) f_j^{(\pi)}(\varphi_j(z)),\ f_j \in H(\mathbb{D})$$

where $u_j \in H(\mathbb{D})$ and φ_j are a nonconstant holomorphic self-maps of $\mathbb{D}$. If $n = 0$, then $D^x_{e,v}$ becomes the weighted composition operator $u_j C_{\varphi_j}$ and defined by

$$\sum_j u_j C_v f_j(z) = \sum_j u_j(z) f_j(\varphi_j(t)),\ z \in \mathbb{D}$$

which for $u_j(z) = 1$, is reduced to the composition operator C_{φ_j}. For weighted composition operators on some H^∞ -type spaces, see [4, 9-11, 26, 29-31,43]. If $n = 1, u_j(z) = \phi_j'(z)$, then $D^{1+\epsilon}_{c,u_j} = DC_{\varphi_j}$, which was obtained in [6,8,12,22,24,27,42,46,47]. If $n = 1, u_j(z) = 1$, then $D^n_{e,u_j} = C_\varepsilon D$, see [6,27]. If $n = 1, \varphi_j(z) = z$, then $D^{=}_{z,v} = M_a D$, give the product of differentiation operator and multiplication operator M_w defined by $\sum_j M_{u_j} f_j = \sum_j u_j f_j$. Zhu in [48] completely characterized the boundedness and compactness of linear operators which are obtained by taking products of differentiation, composition and multiplication operators and which act from Bergman type spaces to Bers spaces. Stević in [44] studied the boundedness and compactness of the weighted differentiation composition operator D^n_{e,u_j} from mixed-norm spaces to weighted-type spaces or the little weighted-type space. Zhu in [49] studied the boundedness and compactness of the generalized weighted composition operator on weighted Bergman spaces. Yang in [47] studied the boundedness and compactness of the operator $C_{\varphi_j}D$ and DC_{φ_j} from $Q_x(1+\epsilon), (1+2\epsilon)$ to $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$, spaces. Liu and Yu in [24] studied the boundedness and compactness of the operator DC_e between H^∞ to Zygmund spaces. Stevic in [32] studied the boundedness and compactness of the generalized composition operator from mixed-norm space to the Bloch-type space, the little Bloch-type space, the Zygmund space, and the little Zygmund space. For other products of operators on spaces of holomorphic functions see $[2,7,12,14-16,18,19,42,49,48]$. For the products of composition operator and integral-type operators, see [15,16,19,28,42]. By [24,92,44,48,49], we consider the boundedness and compactness of the operators D^n_{e,u_j} from $H(1+\epsilon), (1+2\epsilon), (\phi_j)$ to the Zygmund space, and the little Zygmund space.

C denotes a positive constant which may vary but it is independent of any variables. Two quantities $1+\epsilon$ and $1+2\epsilon$ are said to be comparable, denoted by $\epsilon = 0$, if $C^{-1}(1+\epsilon) \leq 1+2\epsilon \leq C(1+\epsilon)$.

We study the boundedness and compactness of $D^n_{1+2\epsilon,u_j} = H(1+\epsilon),(1+2\epsilon),(\phi_j) \rightarrow$ I. For the prove we need the following lemmas.

Lemma 1([32]). Assume that $0 \leq \epsilon < \infty, \phi_j$ is normal and $f_j \in H(1+\epsilon),(1+2\epsilon),(\phi_j)$. Then for each $n \in \mathbb{N}_0$, there is a positive canstant C independent of f_j such that

$$\left|\sum_j f_j^{(n)}(z)\right| \leq C \sum_j \frac{\| f_j \|_{(1+\epsilon),(1+2\epsilon),\phi_j}}{\phi_j(|z|)(1-|z|^2)^{1/1+2\epsilon+n}},\ z \in \mathbb{D}$$

By (see, [1] or Lemma 3 in [29]) we have.

Lemma 2. Assume that φ_j is an analylic self-map of D. Then $D^n_{1+2\epsilon,u_j} : H(1+\epsilon, 1+2\epsilon, \phi_j) \rightarrow \oint$ is compact if and only if $D^n_{1+2\epsilon,u_j} : H(1+\epsilon, 1+2\epsilon, \phi_j) \rightarrow \oint$ is bounded and for any bounded sequence $\{\boldsymbol{f_j}\}$ in $H($ compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$, we have $\sum_j \left\|D^n_{e,s} f_j\right\|_x \rightarrow 0$ as $k \rightarrow \infty$.
The following lemma was proved in [19] similar to the corresponding lemma in [25].

Lemma 3. A closed set K in $\mathcal{X}_0$ is compact if and only if K is boundnd and satisfies

$$\lim_{|z|\rightarrow 1} \sup_{f_j \in K} \sum_j (1-|z|^2)\left|f_j''(z)\right| = 0$$

Lemma 4([5]). For any real β, let

$$J_{1+\epsilon}(z) = \int_0^{z_x} \frac{d\theta}{\left|1-ze^{-\phi_j}\right|^{1+\beta}},\ z \in \mathbb{D}.$$

Then we have

$$J_F(z) \approx \begin{cases} 1, & \text{if } \beta < 0, \\ \log\dfrac{1}{1-|z|\ |}, & \text{if } \beta = 0, \qquad as|z| \rightarrow 1^- \\ \dfrac{1}{(1-|z|^2)B}, & if_j \beta > 0,. \end{cases}$$

Lemma 5([28]). For $\beta > -1$ and $\gamma > 1+\beta$ av haur

$$\int_0^1 \frac{(1-r)^s}{(1-r\rho)^r} dr \leq C(1-\rho)^{1+\beta-7},\ 0 < \rho < 1.$$

Now we characterize the boundedness of $D^e_{e,x}$:$H(1+\epsilon),(1+2\epsilon),(\phi_j) \rightarrow x$.

Theorem 1 (see [53]). Assume that φ_j is an analytic self-map of $\mathbb{D}$. Then $D^s_{e,v}\ H(1+\epsilon),(1+2\epsilon),(\phi_j) \to x$ is bounded if and anly if the following conditions are satisfied,

$$\sup_{s\in D}\sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\phi_j(z)|)\left(1-|\phi_j(z)|^2\right)^{1/\xi+n}} < \infty, \tag{2}$$

$$\sup_{z\in \mathbf{D}}\sum_j \frac{(1-|z|^2)\left|2u_j'(z)\varphi_j{}'(z)+u_j(z)\varphi_j{}''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\phi_j(z)|^2\right)^{1/2+2\epsilon+n}} < \infty \tag{3}$$

and

$$\sup_{x\in \mathbf{D}}\sum_j \frac{(1-|z|^2)\left|u_j(z)\left(\varphi_j{}'(z)\right)^2\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{\frac{1}{4+3\epsilon}}} < \infty \tag{4}$$

Proof. Assume that conditions (2), (3) and (4) hold. Then, for every $z\in\mathbb{D}$ and $f_j \in H(1+\epsilon),(1+2\epsilon),(\phi_j)$, by Lemma 1 , we have

$$\sum_j \left|(1-|z|^2)\left(D^n_{1+2\epsilon,J}f_j\right)''(z)\right|$$

$$\begin{aligned}
&= (1-|z|^2)\sum_j \left|\left(u_j(z)f_j^{(n)}\left(\varphi_j(z)\right)\right)''\right| \\
&= (1-|z|^2)\sum_j \left|\left(u_j'(z)f_j^{(n)}\left(\varphi_j(z)\right)+u_j(z)\varphi_j{}'(z)f_j^{(n+1)}\left(\varphi_j(z)\right)\right)'\right| \\
&\le (1-|z|^2)\sum_j \left|u_j''(z)f_j^{(n)}\left(\varphi_j(z)\right)\right| + (1-|z|^2)\mid\sum_j \left(2u_j'(z)\varphi_j{}'(z)\right. \\
&\quad \left.+u_j(z)\varphi_j{}''(z)\right)f_j^{(n+1)}\left(\varphi_j(z)\right)+u_j(z)\left(\varphi_j{}'(z)\right)^2 f_j^{(n+2)}\left(\varphi_j(z)\right)\mid \\
&\le C\sum_j \left(\frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j\left(\left|\varphi_j(z)\right|\right)\left(1-|\varphi_j(z)|^2\right)^{\frac{1}{1+2\epsilon}+n}}\right. \\
&\quad \left.+\frac{(1-|z|^2)\left|2u_j'(z)\varphi_j{}'(z)+u_j(z)\varphi_j{}''(z)\right|}{\phi_j\left(\left|\varphi_j(z)\right|\right)\left(1-|\varphi_j(z)|^2\right)^{\frac{1}{1+2\epsilon}+n+1}}\right)\| f_j \|_{1+\epsilon,1+2\epsilon,\phi_j} \\
&\quad +C\sum_j \frac{(1-|z|^2)\left|u_j(z)\left(\varphi_j{}'(z)\right)^2\right|}{\phi_j\left(\left|\varphi_j(z)\right|\right)\left(1-|\varphi_j(z)|^2\right)^{\frac{1}{1+2\epsilon}+n+2}}\| f_j \|_{1+\epsilon,1+2\epsilon,\phi_j}.
\end{aligned} \tag{5}$$

On the other hand, we have

$$\sum_j \left|\left(D^n_{\varphi_j,u_j}f_j\right)(0)\right| = \left|\sum_j u_j(0)f_j^{(n)}\left(\varphi_j(0)\right)\right|$$

$$\leq C\sum_j \frac{|u_j(0)|}{\phi_j(|\varphi_j(0)|)(1-|\varphi_j(0)|^2)^{\frac{1}{1+2\epsilon}+n}} \parallel f_j \parallel_{1+\epsilon,1+2\epsilon,\phi_j}, \tag{6}$$

and

$$\begin{aligned}\sum_j \left|\left(D^*_{\epsilon,\alpha}f_j\right)'(0)\right| &= \sum_j \left|u_j'(0)f_j^{(n)}\left(\varphi_j(0)\right) + u_j(0)\varphi_j{}'(0)f_j^{(n+1)}\left(\varphi_j(0)\right)\right| \\ &\leq C\sum_j \left(\frac{|u_j'(0)|}{\phi_j(|\varphi_j(0)|)(1-|\varphi_j(0)|^2)^{\frac{1}{1+2\epsilon}+x}}\right. \\ &\quad \left.+\frac{|u_j(0)\varphi_j{}'(0)|}{\phi_j(|\varphi_j(0)|)(1-|\varphi_j(0)|^2)^{\frac{1}{1+2\epsilon}+s+1}}\right) \parallel f_j \parallel_{M\phi_j}.\end{aligned} \tag{7}$$

Applying conditions (2), (3), and (4), we deduce that the operator $D^{\mathrm{n}}_{\mathrm{e.u}}:H(1+\epsilon),(1+2\epsilon),(\phi_j) \to £$ is bounded.

On the other hand, suppose that $D^n_{1+2\epsilon,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to I$ is bounded. that is there exists a constant C such that

$$\left\|\sum_j D^n_{e,\Delta}f_j\right\|_x \leq C\sum_j \parallel f_j \parallel_{1+\epsilon d\cdot\phi_j}$$

for all $f_j \in H(1+\epsilon),(1+2\epsilon),(\phi_j)$. For a fixed $w \in \mathbb{D}$, set

$$\begin{aligned}(f_j)_w(z) = &\sum_j \frac{(1-|w|^2)^{l+1}}{\phi_j(|w|)} \\ &\times\left(\frac{(\alpha+n+1)(\alpha+n+2)}{\alpha(\alpha+1)(1-\bar{w}z)^x} - \frac{2(\alpha+n+2)(1-|w|^2)}{(\alpha+1)(1-\bar{\imath}zz)^{1+\epsilon+1}} + \frac{(1-|w|^2)^2}{(1-\bar{u}_jz)^{x+2}}\right)\end{aligned} \tag{8}$$

where the constant $1+2\epsilon$ is from the definition of the normality of the function ϕ_j and $\alpha = 1/1+2\epsilon + 2+2\epsilon$.

A straightforward calculation shows that

$$\begin{aligned}(f_j)_w^{(n)}(z) = \sum_j &\frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^n}{\phi_j(|w|)} \\ &\times\left(\frac{A_{1+2\epsilon,1+2\epsilon,n}}{(\alpha+n)(1-\bar{w}z)^{\alpha+n}} - \frac{2A_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(\alpha+n+1)(1-\bar{w}z)^{\alpha+n+1}}\right) \\ &+\frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^n}{\phi_j(|w|)}\frac{A_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(\alpha+n+2)(1-\bar{w}z)^{\alpha+n+2}},\end{aligned} \tag{9}$$

$$(f_j)_w^{(n+1)}(z) = \sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+1}}{\phi_j(|w|)}\left(\frac{A_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{\alpha+n+1}} - \frac{2A_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{\alpha+n+2}}\right) + \sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+1}}{\phi_j(|w|)}\frac{A_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+n+3}} \quad (10)$$

$$(f_j)_w^{(n+2)}(z) = \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)} \times \left(\frac{(\alpha+n+1)A_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{\alpha+n+2}} - \frac{2(\alpha+n+2)A_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{\alpha+n+3}}\right) + \sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)}\frac{(\alpha+n+3)A_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+n+4}} \quad (11)$$

By Lemma 4 , we have

$$M_{1+2\epsilon}\big((f_j)_w, r\big) \le \sum_j C\frac{(1-|w|^2)^{2+2\epsilon}}{\phi_j(|w|)(1-r|w|)^{2+2\epsilon}}.$$

As ϕ_j is normal and by applying Lemma 5 , we obtain (see [32,44])

$$\sup_{w\in\mathbb{D}}\|(f_j)_w\|_{1+\epsilon,1+2\epsilon,\phi_j} \le C \quad (12)$$

where $A_{g_j,1+2\epsilon,e} = (\alpha+2)\cdots(\alpha+n+2)$. From (9), (10) and (11), we have

$$(f_j)_w^{(n)}(w) = \sum_j \frac{B_{1+2\epsilon,1+2\epsilon,n}(\bar{w})^n}{\phi_j(|w|)(1-|w|^2)^{\frac{1}{1+2\epsilon}+n}},\ \sum_j (f_j)_w^{(n+1)}(w) = \sum_j (f_j)_w^{(n+2)}(w) = 0, \quad (13)$$

where $B_{1+2\epsilon,1+2\epsilon,n} = \frac{A_{1+2\epsilon,1+2\epsilon,n}}{\alpha+n} - \frac{2A_{1+2\epsilon,1+2\epsilon,n}}{\alpha+n+1} + \frac{A_{1+2\epsilon,1+2\epsilon,n}}{\alpha+n+2}$. Hence,

$$C \ge \left\|\sum_j D^n_{\varphi_j,u_j}(f_j)_{\varphi_j(w)}\right\|_{\mathcal{J}} \ge \sum_j (1-|w|^2)\left|u_j''(w)(f_j)^{(n)}_{\varphi_j(w)}(\varphi_j(w))\right| = \sum_j \frac{|B_{1+2\epsilon,1+2\epsilon,n}|(1-|w|^2)|u_j''(w)||\varphi(w)|^n}{\phi_j(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/1+2\epsilon+n}} \quad (14)$$

From (14) we have

$$\sup_{|\varphi_j(w)|>\frac{1}{2}}\sum_j \frac{|B_{1+2\epsilon,1+2\epsilon,n}|(1-|w|^2)|u_j''(w)|}{\phi_j(|\varphi_j(w)|)\big(1-|\varphi_j(w)|^2\big)^{1/1+2\epsilon+n}} \le \sup_{|\varphi_j(w)|>\frac{1}{2}} 2^n\sum_j \frac{|B_{1+2\epsilon,1+2\epsilon,n}|(1-|w|^2)|u_j''(w)||\varphi_j(w)|^n}{\phi_j(|\varphi_j(w)|)\big(1-|\varphi_j(w)|^2\big)^{1/1+2\epsilon+n}} \le C < \infty \quad (15)$$

Since $f_j(z) = \frac{z^n}{n!} \in H(1+\epsilon), (1+2\epsilon, \phi_j)$ it follows that

$$\sum_{j} (1-|z|^2)\left|u_j''(z)\right| \leq \left\|\sum_{j} D^n_{\varphi_j,u_j} f_j\right\|_{\mathcal{J}} \leq \sum_{j} \left\|D^n_{\varphi_j,u_j}\right\| \parallel f_j \parallel_{1+\epsilon,1+2\epsilon,\phi_j} \leq C. \tag{16}$$

From this and the fact ϕ_j is normal we obtain

$$\sup_{|\varphi_j(z)|\leq\frac{1}{2}} \sum_{j} \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}} \leq C \sup_{|\varphi_j(z)|\leq\frac{1}{2}} \sum_{j} (1-|z|^2)\left|u_j''(z)\right| \leq C < \infty \tag{17}$$

From (15) and (17) it follows that (2) holds. For a fixed $w \in \mathbb{D}$, set

$$(g_j)_w(z) = \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}}{\phi_j(|w|)} \times \left(\frac{(\alpha+n)(\alpha+n+2)}{\alpha(\alpha+1)(1-\bar{w}z)^{x}} - \frac{(2\alpha+2n+3)(1-|w|^2)}{(\alpha+1)(1-\bar{w}z)^{\alpha+1}} + \frac{(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+2}}\right). \tag{18}$$

It is easy to see that

$$(g_j)^{(n)}_w(z) = \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^n}{\phi_j(|w|)}\left(\frac{(\alpha+n+2)C_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{\alpha+n}} - \frac{(2\alpha+2n+3)C_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{\alpha+n+1}}\right) + \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^n}{\phi_j(|w|)} \frac{(\alpha+n+1)C_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+n+2}}, \tag{19}$$

$$(g_j)^{(n+1)}_w(z) = \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+1}}{\phi_j(|w|)} \frac{(\alpha+n)(\alpha+n+2)C_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{\alpha+n+1}} - \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+1}}{\phi_j(|w|)} \frac{(\alpha+n+1)(2\alpha+2n+3)C_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{\alpha+n+2}} + \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+1}}{\phi_j(|w|)} \frac{(\alpha+n+1)(\alpha+n+2)C_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+n+3}}, \tag{20}$$

$$\begin{aligned}(g_j)^{(n+2)}_\infty(z) &= \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)} \frac{(\alpha+n)(\alpha+n+1)(\alpha+n+2)C_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{x+u_j+2}} \\ &- \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)} \times \frac{(\alpha+n+1)(\alpha+n+2)(2\alpha+2n+3)C_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{x+n+3}} \\ &+ \sum_{j} \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)} \times \frac{(\alpha+n+1)(\alpha+n+2)(\alpha+n+3)C_{\varphi_j 1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{x+n+4}}\end{aligned} \tag{21}$$

By Lemmas 4 and 5 , we get (see [32,44])

$$\sup_{u_j x\in\mathbf{D}}\sum_j \|(g_j)_v\|_{1+\epsilon y\phi_j} \le C, \tag{22}$$

where $C_{\varphi_j 1+2\epsilon,n} = (\alpha+2)\cdots(\alpha+n)$. From $(19)-(21)$, we have

$$(g_j)_w^{(n+1)}(w) = \sum_j \frac{D_{1+2\epsilon,1+2\epsilon,n}(\bar{w})^{n+1}}{\phi_j(|w|)(1-|w|^2)^{\frac{1}{2+2\epsilon+n}}},\ (g_j)_w^{(n)}(w) = (g_j)_w^{(n+2)}(w) = 0, \tag{23}$$

where $D_{1+2\epsilon,i,n} = ((\alpha+n+2)(2\alpha+2n+1)-(\alpha+n+1)(2\alpha+2n+3))C_{1+2\epsilon,1+2\epsilon,\mathrm{n}}$

Hence

$$\begin{aligned} C \quad &\ge \left\|\sum_j D^n_{\epsilon,u_j}(g_j)_{\rho(w)}\right\|_x \ge (1-|w|^4)\sum_j \left|(2u_j'(w)\varphi_j{}'(w)+u_j(w)\varphi_j{}''(w)\,|)(g_j)^{(n+1)}_{\psi(u_j)}\left(\varphi_j(w)\right)\right| \\ &= \sum_j \frac{|D_{1+2\epsilon,1+2\epsilon,n}|(1-|w|^2)|2u_j'(w)\varphi_j{}'(w)+u_j(w)\varphi_j{}''(w)||\varphi_j(w)|^{n+1}}{\phi_j(|\varphi_j(w)|)(1-|\varphi_j(w)|^2)^{\frac{1}{1+2\epsilon}+n+1}} \end{aligned} \tag{24}$$

From (24) we have that

$$\sup_{|\mathfrak{f}(x)|>4}\sum_j \frac{(1-|w|^2)|2u_j'(w)\varphi_j{}'(w)+u_j(w)\varphi_j{}''(w)|}{\phi_j(|\varphi_j(w)|)(1-|\varphi_j(w)|^2)^{1/2+2\epsilon+n}}$$

$$\le \sup_{|\mathcal{F}(w)|>\frac{1}{2}}\sum_j 2^{n+1}\frac{(1-|w|^2)|2u_j'(w)\varphi_j{}'(w)+u_j(w)\varphi_j{}''(w)\,\|\,\varphi_j(w)|^{n+1}}{\phi_j(|\varphi_j(w)|)(1-|\varphi_j(w)|^2)^{\frac{1}{1+2\epsilon}+u_j+1}} \le C < \infty. \tag{25}$$

Since $f_j(z) = \frac{z^{n+1}}{(n+1)!} \in H(1+\epsilon),(1+2\epsilon),(\phi_j)$ it follows from (16) that

$$\sum_j (1-|w|^2)|2u_j'(w)\varphi_j{}'(w)+u_j(w)\varphi_j{}''(w)| \le (1-|w|^2)\sum_j |u_j''(w)\varphi_j(w)|$$

$$+\sum_j \|D^n_{\varphi_j,u_j}f_j\|_{\mathfrak{Z}I} \le C \tag{26}$$

Using (26) and the fact ϕ_j is normal we

$$\sup_{|\varphi_j(z)|\le\frac{1}{2}}\sum_j \frac{(1-|w|^2)|2u_j'(w)\varphi_j{}'(w)+u_j(w)\varphi_j{}''(w)|}{\phi_j(|\varphi_j(w)|)(1-|\varphi_j(w)|^2)^{1/2+2\epsilon+n}}$$

$$\le C\sup_{|\varphi_j(w)|\le\frac{1}{2}}\sum_j (1-|w|^2)|2u_j'(w)\varphi_j{}'(w)+u_j(w)\varphi_j{}''(w)| \le C < \infty \tag{27}$$

Combining (25) with (27) we get (3), as desired.

Next, we prove that (4). To see this, for a fixed $w\in\mathbb{D}$, put

$$\begin{aligned}(h_j)_w(z) = \quad &\sum_j \frac{(1-|w|^2)^{2+2\epsilon}}{\phi_j(|w|)} \\ &\times\left(\frac{(\alpha+n)(\alpha+n+1)}{\alpha(\alpha+1)(1-\bar{w}z)^x} - \frac{2(\alpha+n+1)(1-|w|^2)}{(\alpha+1)(1-\bar{w}z)^{x+1}} + \frac{(1-|w|^2)^2}{(1-\bar{w}_z)^{x+2}}\right).\end{aligned} \tag{28}$$

It is easy to see that

$$(h_j)_w^{(n)}(z) = \sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^n}{\phi_j(|w|)}$$

$$\times\left(\frac{E_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{\alpha+n}}-\frac{2E_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{\alpha+n+1}}+\frac{E_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+n+2}}\right)$$

$$(h_j)_w^{(n+1)}(z)=\sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+1}}{\phi_j(|w|)} \times\left(\frac{(\alpha+n)E_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{\alpha+n+1}}-\frac{2(\alpha+n+1)E_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{\alpha+n+2}}\right) \tag{29}$$

$$+\sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+1}}{\phi_j(|w|)}\frac{(\alpha+n+2)E_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+n+3}},$$

$$(h_j)_w^{(n+2)}(z)=\sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)}\frac{(\alpha+n)(\alpha+n+1)E_{1+2\epsilon,1+2\epsilon,n}}{(1-\bar{w}z)^{\alpha+n+2}} \tag{30}$$

$$-\sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)}\frac{2(\alpha+n+1)(\alpha+n+2)E_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)}{(1-\bar{w}z)^{\alpha+n+3}}$$

$$+\sum_j \frac{(1-|w|^2)^{2+2\epsilon}(\bar{w})^{n+2}}{\phi_j(|w|)}\frac{(\alpha+n+2)(\alpha+n+3)E_{1+2\epsilon,1+2\epsilon,n}(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+n+4}} \tag{31}$$

and (see [32,44])

$$\sup_{w\in\mathbb{D}}\sum_j \left\|(h_j)_w\right\|_{1+\epsilon,1+2\epsilon,\phi_j}\le C \tag{32}$$

where $E_{1+2\epsilon,1+2\epsilon,n}=(\alpha+2)\cdots(\alpha+n+1)$. From (29)-(31), we get

$$(h_j)_{iv}^{(n+2)}(w)=\sum_j \frac{F_{1+2\epsilon,1+2\epsilon,n}(\bar{w})^{n+2}}{\phi_j(|w|)(1-|w|^2)^{\frac{1}{1+2\epsilon}+n+2}},\ (h_j)_w^{(n)}(w)=(h_j)_{u_j}^{(n+1)}(w)=0, \tag{33}$$

where $F_{1+2\epsilon,1+2\epsilon,n}=-2(\alpha+n+1)E_{1+2\epsilon,1+2\epsilon,n}$. Hence,

$$\begin{aligned} C &\ge \left\|\sum_j D^n_{\varphi_j,u_j}(h_j)_{\varphi_j(w)}\right\|_I \ge (1-|w|^2)\left|u_j(w)\left(\varphi_j{}'(w)\right)^2(g_j)^{(n+2)}_{\psi(w)}\left(\varphi_j(w)\right)\right| \\ &=\sum_j \frac{\left|F_{1+2\epsilon,1+2\epsilon,n}\right|(1-|w|^2)\left|u_j(w)\left(\varphi_j{}'(w)\right)^2 \parallel \varphi_j(w)\right|^{n+2}}{\phi_j\left(\left|\varphi_j(w)\right|\right)\left(1-|\varphi_j(w)|^2\right)^{\frac{1}{1+2\epsilon}+n+2}} \end{aligned} \tag{34}$$

From this we have that

$$\sup_{|\varphi_j(u_j)|>\frac{1}{2}}\sum_j \frac{(1-|w|^2)\left|u_j(w)\left(\varphi_j{}'(w)\right)^2\right|}{\phi_j\left(\left|\varphi_j(w)\right|\right)\left(1-|\varphi_j(w)|^2\right)^{\frac{1}{1+2\epsilon}+n+2}}$$

$$\leq \sup_{|\varphi_j(w)|>\frac{1}{2}} \sum_j 2^{n+2} \frac{(1-|w|^2)\left|u_j(w)\left(\varphi_j'(w)\right)^2 \| \varphi_j(w)\right|^{n+2}}{\phi_j(|\varphi_j(w)|)(1-|\varphi_j(w)|^2)^{\frac{1}{1+2\epsilon}+n+2}} \quad (35)$$
$$\leq C < \infty$$

Since $f_j(z) = \frac{z^{n+2}}{(n+2)!} \in H(1+\epsilon),(1+2\epsilon),(\phi_j)$, by (16) and (26) we have

$$(1-|w|^2)\sum_j \left|u_j(w)\left(\varphi_j'(w)\right)^2\right| \leq \frac{1}{2}(1-|w|^2)\sum_j \left|u_j''(w)(\varphi_j(w))^2\right|$$
$$+\sum_j (1-|w|^2) \mid u_j(w)\varphi_j'(w)$$
$$+\sum_j u_j(w)\varphi_j''(w) \| \varphi_j(w) \mid + \sum_j \| D^n_{\varphi_j,1} f_j \|_I$$
$$\leq \frac{1}{2}\sum_j (1-|w|^2)\left|u_j''(w)\right| + \sum_j (1-|w|^2) \mid 2u_j'(w)\varphi_j'(w) \quad (36)$$
$$+\sum_j u_j(w)\varphi_j''(w) \mid + \sum_j \left\|D^n_{\varphi_j,u_j} f_j\right\|_z \leq C.$$

Using (36) and the fact ϕ_j is normal we obtain

$$\sup_{|\varphi_j(z)|\leq\frac{1}{2}} \sum_j \frac{(1-|w|^2)\left|u_j(w)\left(\varphi_j'(w)\right)^2\right|}{\phi_j(|\varphi_j(w)|)(1-|\varphi_j(w)|^2)^{\frac{1}{1+2\epsilon}+n+2}}$$
$$\leq C\sum_j \sup_{|\varphi_j(x)|\leq\frac{1}{2}} (1-|w|^2)\left|u_j(w)\left(\varphi_j'(w)\right)^2\right| \quad (37)$$
$$\leq C < \infty$$

From (35) and (37) it follows that (4) holds, finishing the proof of the theorem.

Theorem 2 (see [53]). Assume that φ_j is an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi_j,u_j}$: $H(1+\epsilon),(1+2\epsilon),(\phi_j) \rightarrow \pm$ is compact if and only if $D^n_{\varphi_j,u_j}$: $H(1+\epsilon),(1+2\epsilon),(\phi_j) \rightarrow \mathcal{I}$ is bounded, and

$$\lim_{|\varphi_j(z)|\rightarrow 1} \sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}} = 0 \quad (38)$$

$$\lim_{|\varphi_j(z)|\rightarrow 1} \sum_j \frac{(1-|z|^2)\left|2u_j'(z)\varphi_j'(z)+u_j(z)\varphi_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/2+2\epsilon+n}} = 0 \quad (39)$$

and

$$\lim_{|\varphi_j(z)|\rightarrow 1} \sum_j \frac{(1-|z|^2)\left|u_j(z)\left(\varphi_j'(z)\right)^2\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/3+2\epsilon+n}} = 0 \quad (40)$$

Proof. Assume that $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to \mathcal{J}$ is bounded and that conditions (38), (39) and (40) hold. For any bounded sequence $\{(f_j)_k\}$ in $H(1+\epsilon),(1+2\epsilon),(\phi_j)$ with $(f_j)_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. To establish the assertion, it suffices, in view of Lemma 2 , to show that

$$\left\|\sum_j D^n_{\varphi_j,1}(f_j)_k\right\|_I \to 0 \text{ as } k \to \infty.$$

We assume that $\|\sum_j (f_j)_k\|_{1+\epsilon,1+2\epsilon,\phi_j} \le 1$. From $(38)-(40)$ given $\epsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |\varphi_j(z)| < 1$, we have

$$\sum_j \frac{(1-|z|^2)}{\phi_j(|\varphi_j(z)|)}\left(\frac{|u_j''(z)|}{(1-|\varphi_j(z)|^2)^{\frac{1}{1+2\epsilon}+n}} + \frac{|2u_j'(z)\varphi_j'(z)+u_j(z)\varphi_j''(z)|}{(1-|\varphi_j(z)|^2)^{\frac{1}{1+2\epsilon}+n+1}} + \frac{\left|u_j(z)\left(\varphi_j'(z)\right)^2\right|}{(1-|\varphi_j(z)|^2)^{\frac{1}{1+2\epsilon}+n+2}}\right) < \epsilon \tag{41}$$

From the boundedness of $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to \pm$ by Theorem 1 , we see that (2)-(4) hold. Since $\sum_j (f_j)_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $(f_j)_k^{(n)}, (f_j)_k^{(n+1)}$ and $(f_j)_k^{(n+2)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$\sum_j \left|u_j(0)(f_j)_k^{(n)}(\varphi_j(0))\right| + \sum_j \left|u_j'(0)(f_j)_k^{(n)}(\varphi_j(0))\right| + \sum_j \left|u_j(0)\varphi_j'(0)(f_j)_k^{(n+1)}(\varphi_j(0))\right| + \sup_{|\varphi_j(z)|\le\delta}\sum_j (1-|z|^2)\left|u_j''(z)(f_j)_k^{(n)}(\varphi_j(z))\right| + \sup_{|\varphi_j(z)|\le\delta}\sum_j (1-|z|^2) \mid \left(2u_j'(z)\varphi_j'(z) + \sum_j u_j(z)\varphi_j''(z)\right)(f_j)_k^{(n+1)}(\varphi_j(z)) + \sum_j u_j(z)\left(\varphi_j'(z)\right)^2 (f_j)_k^{(n+2)}(\varphi_j(z)) \mid < C\epsilon \tag{42}$$

From (41) and (42) we have

$$\sum_j \left\|D^n_{\varphi_j,1}(f_j)_k\right\|_I = \sum_j \left|\left(D^n_{\varphi_j,u_j}(f_j)_k\right)(0)\right| + \sum_j \left|\left(D^n_{\varphi_j,1}(f_j)_k\right)'(0)\right| + \sup_{z\in\mathbf{D}}\sum_j (1-|z|^2)\left|\left(D^n_{\varphi_j,1,}(f_j)_k\right)''(z)\right|$$

$$\left|u_j(0)(f_j)_k^{(n)}(\varphi_j(0))\right| + \sum_j \left|u_j'(0)(f_j)_k^{(n)}(\varphi_j(0))\right| + \sum_j \left|u_j(0)\varphi_j{}'(0)(f_j)_k^{(n+1)}(\varphi_j(0))\right|$$
$$+ \sup_{z\in\mathbb{D}} \sum_j (1-|z|^2)\left|u_j''(z)(f_j)_k^{(n)}(\varphi_j(z))\right| + \sup_{z\in\mathbf{D}} \sum_j (1-|z|^2)$$
$$|\left(2u_j'(z)\varphi_j{}'(z)\right.$$

$$\left.+u_j(z)\varphi_j{}''(z)\right)(f_j)_k^{(n+1)}(\varphi_j(z)) + u_j(z)\left(\varphi_j{}'(z)\right)^2(f_j)_k^{(n+2)}(\varphi_j(z))\ |$$
$$\le \sum_j \left|u_j(0)(f_j)_k^{(n)}(\varphi_j(0))\right| + \sum_j \left|u_j'(0)(f_j)_k^{(n)}(\varphi_j(0))\right|$$
$$+ \sum_j \left|u_j(0)\varphi_j{}'(0)(f_j)_k^{(n+1)}(\varphi_j(0))\right|$$

$$+\sup_{|\varphi_j(z)|\le\delta} \sum_j (1-|z|^2)\left|u_j''(z)(f_j)_k^{(n)}(\varphi_j(z))\right| + \sup_{|\varphi_j(z)|\le\delta} \sum_j (1-|z|^2)$$
$$|\left(2u_j'(z)\varphi_j{}'(z) + u_j(z)\varphi_j{}''(z)\right)(f_j)_k^{(n+1)}(\varphi_j(z))$$

$$+u_j(z)\left(\varphi_j{}'(z)\right)^2(f_j)_k^{(n+2)}(\varphi_j(z))$$
$$|\ \ +\sup_{\delta<|\varphi_j(z)|<1} \sum_j (1-|z|^2)\left|u_j''(z)(f_j)_k^{(n)}(\varphi_j(z))\right| + \sup_{\delta<|1+2\epsilon(z)|<1}(1-|z|^2)$$
$$|\left(2u_j'(z)\varphi_j{}'(z) + u_j(z)\varphi_j{}''(z)\right)(f_j)_k^{(n+1)}(\varphi_j(z)) + u_j(z)\left(\varphi_j{}'(z)\right)^2((f_j))_k^{(n+2)}(\varphi_j(z))\ |$$

$$\le C\epsilon + C\sup_{\delta<|1+2\epsilon(z)|<1} \sum_j (1-|z|^2)\left|u_j''(z)(f_j)_k^{(n)}(\varphi_j(z))\right| + C\sup_{\delta<|\varphi_j(z)|<1} \sum_j (1-|z|^2)$$
$$|\left(2u_j'(z)\varphi_j{}'(z) + u_j(z)\varphi_j{}''(z)\right)(f_j)_k^{(n+1)}(\varphi_j(z)) + u_j(z)\left(\varphi_j{}'(z)\right)^2(f_j)_k^{(n+2)}(\varphi_j(z))\ |$$
$$\le C\epsilon + C\sup_{\delta<|1+2\epsilon(z)|<1} \sum_j \frac{(1-|z|^2)}{\phi_j(|\varphi_j(z)|)}\left(\frac{\left|u_j''(z)\right|}{\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}}\right.$$
$$\left.+\frac{\left|2u_j'(z)\varphi_j{}'(z) + u_j(z)\varphi_j{}''(z)\right|}{\left(1-|\varphi_j(z)|^2\right)^{1/2+2\epsilon+n}} + \frac{\left|u_j(z)\left(\varphi_j{}'(z)\right)^2\right|}{\left(1-|\varphi_j(z)|^2\right)^{1/3+2\epsilon+n}}\right) < 2C\epsilon,$$

when $k > K_0$. It follows that the operator $D^n_{\varphi_j,u_j}: H(1+\epsilon), (1+2\epsilon), (\phi_j) \to I\pounds$ is compact. Conversely, assume that $D^n_{\varphi_j,u_j}: H(1+\epsilon), (1+2\epsilon), (\phi_j) \to Z$ is compact. Then it is clear that $D^n_{\varphi_j,u_j}: H(1+\epsilon), (1+2\epsilon), (\phi_j) \to \pm$ is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $\sum_j \left|\varphi_j(z_k)\right| \to 1$ as $k \to \infty$. We can use the test functions

$$\sum_j (f_j)_k(z) = \sum_j (f_j)_{\varphi_j(z_k)}(z), \tag{43}$$

From (12) and (13) we have

$$\sup_{k\in\mathbb{N}} \left\|\sum_j (f_j)_k\right\|_{1+\epsilon,1+2\epsilon,\phi_j} \le C$$

and

$$\sum_j (f_j)_k^{(n)}\left(\varphi_j(z_k)\right) = \sum_j \frac{B_{1+2\epsilon,1+2\epsilon,n}\left(\overline{\varphi_j(z_k)}\right)^n}{\phi_j(|\varphi_j(z_k)|)\left(1-|\varphi_j(z_k)|^2\right)^{1/1+2\epsilon+n}}, \sum_j (f_j)_k^{(n+1)}\left(\varphi_j(z_k)\right)$$
$$= \sum_j (f_j)_k^{(n+2)}\left(\varphi_j(z_k)\right) = 0$$

For $|z| = r < 1$, using the fact that ϕ_j is normal, we have

$$\left|\sum_j (f_j)_k(z)\right| \le \frac{C}{(1-r)^{1/2+2\epsilon}} \sum_j \left(1-|\varphi_j(z_k)|\right) \to 0 \text{ (as } k\to\infty)$$

that is, $(f_j)_k$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, using the compactness of $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to I$ we obtain

$$\sum_j \frac{\left|B_{1+2\epsilon,1+2\epsilon,n}\right|(1-|z_k|^2)\left|u_j''(z_k)\right|\left|\varphi_j(z_k)\right|^n}{\phi_j(|\varphi_j(z_k)|)\left(1-|\varphi_j(z_k)|^2\right)^{1/1+2\epsilon+n}} \le \sum_j \left\|D^n_{\varphi_j,u_j}(f_j)_k\right\|_I \to 0 \text{ as } k\to\infty.$$

From this, and $\sum_j |\varphi_j(z_k)| \to 1$, it follows that

$$\lim_{k\to\infty} \sum_j \frac{|(1-|z_k|^2)|u_j''(z_k)\,|}{\phi_j(|\varphi_j(z_k)|)\left(1-|\varphi_j(z_k)|^2\right)^{1/1+2\epsilon+n}} = 0$$

and consequently (38) holds.

In order to prove (39), choose

$$(g_j)_k(z) = \sum_j (g_j)_{\varphi_j(z_k)}(z). \tag{44}$$

It follows from (22) and (23) that

$$\sup_{k\in\mathbb{N}} \sum_j \|(g_j)_k\|_{1+\epsilon,1+2\epsilon,\phi_j} \le C$$

and

$$\sum_j (g_j)_k^{(n+1)}\left(\varphi_j(z_k)\right) = \sum_j \frac{D_{1+2\epsilon,1+2\epsilon,n}\left(\overline{\varphi_j(z_k)}\right)^{n+1}}{\phi_j(|\varphi_j(z_k)|)\left(1-|\varphi_j(z_k)|^2\right)^{1/2+2\epsilon+n}}$$
$$\sum_j (g_j)_k^{(n)}\left(\varphi_j(z_k)\right) = \sum_j (g_j)_k^{(n+2)}\left(\varphi_j(z_k)\right) = 0$$

and $(g_j)_k$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. The compactness of $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to \mathcal{I}$ implies that

$$\lim_{k\to\infty} \sum_j \left\| D^n_{\varphi_j,u_j}(g_j)_k \right\|_{\mathcal{I}} = 0$$

It follows that

$$(1 \quad -|z_k|^2)\sum_j \left| \frac{D_{1+2\epsilon,1+2\epsilon,n}\left(2u_j'(z_k)\varphi_j{}'(z_k) + u_j(z_k)\varphi_j{}''(z_k)\right)\left(\overline{\varphi_j(z_k)}\right)^{n+1}}{\phi_j\left(\left|\varphi_j(z_k)\right|\right)\left(1-\left|\varphi_j(z_k)\right|^2\right)^{1/2+2\epsilon+n}} \right| \\ \le \sum_j C\left\| D^n_{\varphi_j,u_j}(g_j)_k \right\|_{\mathcal{I}} \to 0 \text{ as } k\to\infty \tag{45}$$

$\sum_j \left|\varphi_j(z_k)\right| \to 1$ implies that

$$\lim_{k\to\infty} \sum_j \frac{(1-|z_k|^2)\left|2u_j'(z_k)\varphi_j{}'(z_k) + u_j(z_k)\varphi_j{}''(z_k)\right|}{\phi_j\left(\left|\varphi_j(z_k)\right|\right)\left(1-\left|\varphi_j(z_k)\right|^2\right)^{1/2+2\epsilon+n}} = 0$$

(39) holds. (40) can be proved in a similar manner by choosing the test function $(h_j)_k(z) = \sum_j (h_j)_{\varphi_j(z_k)}(z)$.

The following result is proved similar to Theorem in [32], hence we omit it.

Theorem 3. Assume that φ_j is an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to \mathcal{I}_0$ is bounded if and only if $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to \pm$ is bounded,

$$\lim_{|z|\to 1} \sum_j (1-|z|^2)\left|u_j''(z)\right| = 0, \tag{46}$$

$$\lim_{|z|\to 1} \sum_j (1-|z|^2)\left|2u_j'(z)\varphi_j{}'(z) + u_j(z)\varphi_j{}''(z)\right| = 0 \tag{47}$$

and

$$\lim_{|z|\to 1} \sum_j (1-|z|^2)\left|u_j(z)\left(\varphi_j{}'(z)\right)^2\right| = 0 \tag{48}$$

In the next theorem, we characterize the compactness of $D^n_{\varphi_j,u_j}$: $H(1+\epsilon),(1+2\epsilon),(\phi_j) \to \mathcal{I}_0$

Theorem 4 (see [53]). Assume that φ_j is an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j) \to \mathcal{I}_0$ is compact if and only if

$$\lim_{|z|\to 1}\sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}}=0 \qquad (49)$$

$$\lim_{|z|\to 1}\sum_j \frac{(1-|z|^2)\left|2u_j'(z)\varphi_j'(z)+u_j(z)\varphi_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/2+2\epsilon+n}}=0 \qquad (50)$$

and

$$\lim_{|z|\to 1}\sum_j \frac{(1-|z|^2)\left|u_j(z)\left(\varphi_j'(z)\right)^2\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/3+2\epsilon+n}}=0 \qquad (51)$$

Proof. Assume that conditions $(49)-(51)$ hold. Then it is clear that (2)-(4) hold. Hence $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j)\to I$ is bounded by theorem 1 . From inequality (5) we see that $D^n_{\varphi_j,u_j}f_j\in \mathcal{J}_0$ for each $f_j\in H(1+\epsilon),(1+2\epsilon),(\phi_j)$, it follows that $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j)\to \mathcal{J}_0$ is bounded. Taking the supremum in inequality (5) over all $f_j\in H(1+\epsilon),(1+2\epsilon),(\phi_j)$ such that $\|\sum_j f_j\|_{1+\epsilon,1+2\epsilon,\phi_j}\le 1$ and letting $|z|\to 1$, yields

$$\lim_{|z|\to 1}\sup_{\|/1_{1+\epsilon,1+2\epsilon,\phi_j}\le 1}\sum_j (1-|z|^2)\left|\left(D^n_{\varphi_j,n}f_j\right)''(z)\right|=0$$

Hence, by Lemma 3 we see that the operator $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j)\to \mathcal{J}_0$ is compact.

Now assume that $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),(\phi_j)\to \mathcal{J}_0$ is compact. Then $D^n_{\varphi_j,u_j}: H(1+\epsilon),(1+2\epsilon),$

$(\phi_j)\to \mathcal{J}_0$ is bounded, and by taking the function $f_j(z)=z^n$ it follows that

$$\lim_{|z|\to 1}\sum_j (1-|z|^2)\left|u_j''(z)\right|=0 \qquad (52)$$

By taking the function $f_j(z)=z^{n+1}$, we have

$$\lim_{|z|\to 1}\sum_j (1-|z|^2)\left|u_j''(z)\varphi_j(z)+2u_j'(z)\varphi_j'(z)+u_j(z)\varphi_j''(z)\right|=0 \qquad (53)$$

from (52), (53), and the fact that $\|\sum_j \varphi_j\|_\infty\le 1$, we get

$$\lim_{|z|\to 1}\sum_j (1-|z|^2)\left|u_j'(z)\varphi_j'(z)+u_j(z)\varphi_j''(z)\right|=0 \qquad (54)$$

By taking the function $f_j(z)=z^{n+2}$, from (52), (54) and the fact that $\|\sum_j \varphi_j\|_\infty\le 1$, we have

$$\lim_{|z|\to 1}\sum_j (1-|z|^2)\left|u_j(z)\left(\varphi_j'(z)\right)^2\right|=0 \qquad (55)$$

By (14), (24), (34), and observe that $D^n_{\varphi_j,1}(f_j)_{\varphi_j(w)}, D^n_{\varphi_j,u_j}(g_j)_{\varphi_j(w)}, D^n_{\varphi_j,u_j}(h_j)_{\varphi_j(w)} \in \mathcal{I}_0$ we know that

$$\lim_{|\varphi_j(z)|\to 1}\sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}} = 0 \tag{56}$$

$$\lim_{|\varphi_j(z)|\to 1}\sum_j \frac{(1-|z|^2)\left|2u_j'(z)\varphi_j'(z)+u_j(z)\varphi_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/2+2\epsilon+n}} \tag{57}$$

and

$$\lim_{|\varphi_j(z)|\to 1}\sum_j \frac{(1-|z|^2)\left|u_j(z)\left(\varphi_j'(z)\right)^2\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/3+2\epsilon+n}} = 0 \tag{58}$$

We prove that (52) and (56) imply (49). The proof of (50) and (51) is similar, hence it will be omitted.

From (56), it follows that for every $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}} < \epsilon \tag{59}$$

when $\delta < \sum_j |\varphi_j(z)| < 1$. Using (52) we see that there exists $\tau \in (0,1)$ such that

$$\sum_j (1-|z|^2)\left|u_j''(z)\right| < \epsilon \inf_{t\in[0,\delta]}\sum_j \phi_j(t)(1-t^2)^{\frac{1}{1+2\epsilon}+n}, \tag{60}$$

when $\tau < |z| < 1$.

Therefore, when $\tau < |z| < 1$ and $\delta < \sum_j |\varphi_j(z)| < 1$, by (59) we have

$$\sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\varphi_j(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}} < \epsilon \tag{61}$$

On the other hand, when $\tau < |z| < 1$ and $|\varphi_j(z)| \le \delta$, by (60) we obtain

$$\sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\phi_j(|\varphi_{j_j}(z)|)\left(1-|\varphi_j(z)|^2\right)^{1/1+2\epsilon+n}} \le \sum_j \frac{(1-|z|^2)\left|u_j''(z)\right|}{\inf_{t\in[0,1+2\epsilon]}\phi_j(t)(1-t^2)^{\frac{1}{1+2\epsilon}+n}} < \epsilon \tag{62}$$

From (61) and (62), we obtain (49), as desired. The proof is completed.

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