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Review Paper

Common Fixed point theorems for generalized contractions in Complete Metric Space

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Abstract:
The main objective of this paper is to present some common fixed point theorems for generalized contractions in complete metric spaceby employing the concept of P-property and generalized P-property. An example is also given to justify the result.

Keywords: *Fixed point, complete metric space, simulation function, generalized contraction, best proximity point.*
2020MSC: *47H10, 54H25.*

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I. Introduction and Preliminaries :

The metric fixed theory is most demanded and interesting in solving many problems in area of research in mathematics.It has wide applications in many field of science. In 1922, Banach [3] introduced BanachContraction Principle that is "A self-mapping in a complete metric space satisfying the contraction conditions has a unique fixed point."In state of complete metric space, fixed point theory is an important part to prove many results. Many authors defined several metric spaces and several contractions. Among all these metric spaces and contraction, generalized contraction has many applications in fixed point theory. The concept of simulation function and Z −contraction was given by Khojasteh[5]. The notion of R −function and its best proximity points was given by Aslanta M. et al.,In this paper we obtain fixed point results for generalized-contraction using P −property and generalized P −property.

In 1906, Maurice Frechet introduced concept of metric space.

Definition 1.1: Let X be a non-empty set. A metric on X is a distance function $d : X \times X \to \mathbb{R}$satisfying the following:

1. $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) + d(y,z) \geq d(x,z)$, for all$x, y, z \in X$.

The pair (X, d) is called a metric space.

Definition1.2:A sequence $\{x_n\}$ in metric space (X, d) is

(i) A Cauchy sequence if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all$n, m \geq N,\ d(x_n, x_m) < \varepsilon$.
(ii) Convergent to $x \in X$ such that for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m \geq N,\ d(x_m, x) < \varepsilon$.

Definition1.3:A metric space (X, d) is said to be complete if every Cauchy sequence is convergent in X.

Khojasteh [5] gave concept of simulation function and Z-contractions.so we give definition of Z-contraction and a theorem based on it.

Definition 1.4: Let $f : [0,\infty] \times [0,\infty] \to R$ be a function. If f satisfies the following condition, then it is called simulation function:

(1) $f(0,0) = 0$,
(2) $f(p,q) < (q - p)$for all $q, p > 0$.
(3) If $\{p_n\}, \{q_n\} \subseteq (0,\infty)$ are sequences satisfying
$\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$, then
$\lim_{n\to\infty} \sup f(p_n, q_n) < 0$.

Result 1: Let $T: X \to X$ be a mapping on a complete metric space(X, d) and $f : [0, \infty] \times [0, \infty] \to R$ be a simulation function. If the mapping $T$is a Z-contraction w.r.t.$f$satisfying$f(d(Tx, Ty), d(x, y)) \geq 0$ for each $x, y \in X$, then T has a unique fixed point u in X.Also the picard sequence $\{T^n x\}$ for any initial point $x \in X$ converges to u.

Since every Z-contraction in sense of Roldan-Lopez-de-Hierro et al. is a Meir-Keeler contraction then Roldan-Lopez-de-Hierro et al.,[9] introduced R-functions and R-contractions.

Definition 1.5: Let $\phi \neq A \subseteq R$. If a function $f: A \times A \to R$ satisfies the following conditions then it is called R-function on A:

(1) If $\{p_n\} \subseteq (0, \infty) \cap A$ is a sequence satisfying $f(p_n, p_{n+1}) > 0$ for all $n \in N \cup \{0\}$, then we have $p_n \to 0$.

(2) If $\{p_n\}$, $\{q_n\} \subseteq (0, \infty) \cap A$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = L \geq 0, L < p_n$ and satisfying and $f(p_n, q_n) > 0$ for each $n \in N$,then we have $L = 0$.

(3) If $\{p_n\}$, $\{q_n\} \subseteq (0, \infty) \cap A$ are sequences satisfying $f(p_n, q_n) > 0$ for each $n \in N$ and $q_n \to 0$ as $n \to \infty$, then we get $p_n \to 0$.

Definition 1.6: Let$T: X \to X$ be a mapping on a metric space (X, d).If there is an R-function f on A satisfying $\mathrm{ran}(d, X) = \{d(x, y): x, y \in X\} \subseteq A$ and $f\big(d(Tx, Ty), d(x, y)\big) > 0$ for each $x, y \in A$with $x \neq y$, then T is called an R-contraction with respect to f.

Definition 1.7: Let$T: X \to X$ be a mapping on a metric space (X, d).If there is a function f on A satisfying $\mathrm{ran}(d, X) = \{d(x, y): x, y \in X\} \subseteq A$ and $f\big(d(Tx, Ty), m(x, y)\big) > 0$ where $m(x, y) = \max\left\{d(x, y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$, for each $x, y \in X$with $x \neq y$, then T is called ageneralized contraction with respect to f.

Definition 1.8[1]: Let $\phi \neq P, Q$ be subsets of metric space (X, d).Then the pair (P, Q) is said to have aP-property if it satisfies:

$$d(x_1, y_1) = d(P, Q)$$
$$d(x_2, y_2) = d(P, Q)$$

gives$d(x_1, x_2) = d(y_1, y_2)$ for all $x_1, x_2 \in P$ and $y_1, y_2 \in Q$.

Definition 1.9: Let $\phi \neq P, Q$ be subsets of metric space (X, d).Then the pair (P, Q) is said to have a generalized P-property if it satisfies: $d(x_1, y_1) = d(P, Q)$ $d(x_2, y_2) = d(P, Q)$gives $d(x_1, x_2) = d(y_1, y_2)$ for all $x_1 \neq x_2 \in P$and $y_1, y_2 \in Q$.

Definition 1.10: Let $\phi \neq P, Q$ be subsets of metric space (X, d) and $T: P \to Q$ be a mapping. A point$x \in P$ is said to be a best proximity point of T if $d(x, Tx) = d(P, Q)$. It is fixed point of Tif $P = Q = X$.

II. Main Results:

In this section, we prove some common fixed point theorems using generalized contractions in complete metric space using some properties. We find the existence of a best proximity point with help of our results and definitions.

Theorem 2.1: Let $T: P \to Q$ be a mapping on a complete metric space (X, d) and P and Q are closed subsets of X. Assume that $P_0 \neq \phi$,$T(P_0) \subset Q_0$ where $P_0 = \{x \in P: d(x, y) = d(P, Q)$ for some $y \in Q\}$ and $Q_0 = \{y \in Q: d(x, y) = d(P, Q)$ for some $x \in P\}$. Also pair (P, Q) has the generalized P-property. Suppose that T is a generalized contractions with respect to f. If one of the following condition is satisfied:

(1). T is continuous,

(2). The function f satisfy the following property

If $\{p_n\}$, $\{q_n\} \subseteq (0, \infty) \cap A$ are sequences satisfying $f(p_n, q_n) > 0$ for each $n \in N$ and $q_n \to 0$ as $n \to \infty$, then we get $p_n \to 0$. Then T has a unique best proximity point in P.

Proof: Let $x_0 \in P_0$ be any arbitrary point. Since $Tx_0 \in TP_0 \subseteq Q_0$, there exists $x_1 \in P_0$ satisfying$d(x_1, Tx_0) = d(P, Q)$.

Also, since $Tx_1 \in T(P_0) \subseteq Q_0$, there exists$x_2 \in P_0$ satisfying $d(x_2, Tx_1) = d(P, Q)$.

In this way, we can have a sequence $\{x_n\}$ in P_0 such that

$$d(x_{n+1}, Tx_n) = d(P, Q) \qquad (1.1)$$

For all $n \in N \cup \{0\}$.

If $x_{n_0} = x_{n_{0+1}}$ for some $n_0 \in N \cup \{0\}$, then from (1.1), we have $d\big(x_{n_0}, Tx_{n_0}\big) = d(P, Q)$,

Then no need of proof as it is completed. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in N \cup \{0\}$. Then , since the pair (P, Q) has generalized P-property, from (1.1), we get

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \qquad (1.2)$$

For all $n \in N$. Also, since T is generalized contraction with respect to $f$we get

$f(d(Tx_{n-1}, Tx_n), m(x_{n-1}, x_n)) > 0$for all $n \in N$ and where $m(x_{n-1}, x_n) = \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},Tx_{n-1})}, \frac{d(x_n,Tx_n)}{1+d(x_n,Tx_n)}, \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\right\}$, for each $x, y \in X$ with $x \neq y$

and from (1.2) we have $f(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) > 0$ for all $n \in N$ (1.3)

therefore if we denote a sequence $\{p_n\}$ by $p_n = d(x_{n-1}, x_n)$ for all $n \in N$,then from (1.3), we have $p_n > 0$ and $f(p_{n+1}, p_n) > 0$ for all $n \in N$.

Since $\{p_n\} \subseteq (0, \infty) \cap A$, so there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that

$$\lim_{n\to\infty} p_{n_k} = \lim_{n\to\infty} d(x_{n_{k-1}}, x_{n_k}) = 0 \quad (1.4)$$

Now we will show that $\{p_{n_k}\}$ is a Cauchy sequence. Let us assume that $\{x_{n_k}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and two subsequences of natural numbers $\{l_r\}, \{k_r\}$ with $l_r > k_r \geq r$ such that $d\left(x_{n_{k_r}}, x_{n_{l_r}}\right) \geq \varepsilon$(1.5)

For all $r \in N$ where l_r is the least integer satisfying (1.5).

So $d\left(x_{n_{k_r}}, x_{n_{l_{r-1}}}\right) < \varepsilon$ for all $r \in N$.

Therefore $\varepsilon \leq d\left(x_{n_{k_r}}, x_{n_{l_r}}\right) \leq d\left(x_{n_{k_r}}, x_{n_{l_{r-1}}}\right) + d\left(x_{n_{l_{r-1}}}, x_{n_{l_r}}\right) < \varepsilon + d\left(x_{n_{l_{r-1}}}, x_{n_{l_r}}\right)$ for all $r \in N$. Taking the limit as $r \to \infty$, we get

$$\lim_{r\to\infty} d\left(x_{n_{k_r}}, x_{n_{l_r}}\right) = \varepsilon \quad (1.6)$$

Using (1.6), we have $\lim_{r\to\infty} d\left(x_{n_{k_{r-1}}}, x_{n_{l_{r-1}}}\right) = \varepsilon$ (1.7)

Since T is a generalized contraction with respect to f, we get

$$f\left(d\left(Tx_{n_{k_{r-1}}}, Tx_{n_{l_{r-1}}}\right), m\left(x_{n_{k_{r-1}}}, x_{n_{l_{r-1}}}\right)\right) > 0 \quad (1.8)$$

for all $r \in N$.Now since $\lim_{r\to\infty} d\left(x_{n_{k_{r-1}}}, x_{n_{l_{r-1}}}\right) = \lim_{r\to\infty} d\left(x_{n_{k_r}}, x_{n_{l_r}}\right) = \varepsilon$,

Using (1.5) and (1.8), we have $\varepsilon = 0$,

A contradiction.

Hence $\{x_{n_k}\}$ is a Cauchy sequence in P.

Also $\{Tx_{n_{k-1}}\}$is a Cauchy sequence in Q.

Since subsets P,Q are closed subsets of complete metric space(X, d), there exists$x \in P$ and $y \in Q$ such that

$$\lim_{k\to\infty} x_{n_k} = x \text{ and } \lim_{k\to\infty} T\, x_{n_{k-1}} = y \quad (1.9)$$

Using (1.1), taking the limit as $k \to \infty$, we have

$$d(x, y) = d(P, Q) \quad (1.10)$$

Also, we get

$d(x_{n_{k-1}}, x) \leq d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x)$for each $k \in N$.

Hence using (1.4) and (1.9), we get

$$\lim_{k\to\infty} x_{n_{k-1}} = x \quad (1.11)$$

Now from equation (1.9), we have $y = Tx$.

Now suppose that$x_{n_{k-1}} \neq x$ for all $k \in N$ and for some $r \in N$with $k \geq r$. Now we take following case when T is a continuous mapping. Then we get,

$\lim_{k\to\infty} T\, x_{n_{k-1}} = Tx$,

So, using (1.10)$y = Tx$.

Hence $x \in P$ is a best proximity point of$T$.

Now suppose that (2) property is satisfied. Since T is a generalized contraction mapping, we get

$f\left(d(Tx_{n_{k-1}}, Tx), m(x_{n_{k-1}}, x)\right) > 0$.

Hence using property (2) and equation (1.11), we have ,

$\lim_{k\to\infty} T\, x_{n_{k-1}} = Tx$,

So, $y = Tx$.

Now we prove uniqueness of best proximity point.

Let $x, y \in P$ with $x \neq y$ be two proximity point such that

$d(x, Tx) = d(P, Q)$and $d(y, Ty) = d(P, Q)$.

Hence using generalized$P$-property we have,

$d(x, y) = d(Tx, Ty)$.

Since T is generalized contraction with respect to f, we get

$f(d(Tx, Ty), m(x, y)) > 0$,

A contradiction.

Therefore T has a unique best proximity point in P.

Corollary 2.2: Let $T: P \to Q$ be a mapping on a complete metric space (X, d) where P and Q are closed subsets of X. Assume that$P_0 \neq \phi$, $T(P_0) \subseteq Q_0$ and the pair (P, Q) has generalized P-property. Suppose that $m(x, y) = d(x, y)$ in above theorem for generalized contraction, we will get the same result and unique best proximity point in P.

Corollary 2.3: Let $T: P \to Q$ be a mapping on a complete metric space (X, d) where P and Q are closed subsets of X. Assume that $P_0 \neq \phi$,$T(P_0) \subseteq Q_0$and the pair (P, Q) has generalized P-property. Suppose there is a generalized R-function $f: A \times A \to R$ on A satisfying $ran(d, P \cup Q) = \{d(x, y): x, y \in P \cup Q\} \subseteq A$and $f\big(d(Tx, Ty), m(x, y)\big) > 0$, where

$m(x, y) = \max\left\{d(x, y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$,for each $x, y \in X$ with $x \neq y$. If it satisfies one of the following conditions:

(1). T is continuous,

(2). The$R$- function f satisfy the following property,

If $\{p_n\}, \{q_n\} \subseteq (0, \infty) \cap A$ are sequences satisfying $f(p_n, q_n) > 0$ for each $n \in N$ and $q_n \to 0$ as $n \to \infty$, then we get $p_n \to 0$.

Then T has a unique best proximity point in P.

Theorem 2.4: Let (X, d) be a complete metric space, P be non-empty closed subset of X and assume that $h: P \times [0,1] \to X$ is a continuous closed mapping such that

(1). $d\big(x, h(x, \lambda)\big) > 0$ for each $x \in P$and $\lambda \in [0,1]$,

(2).There exists a modified R function $f: A \times A \to$R on A such that

with$x \neq y$ and $\lambda, \mu \in [0,1]$,

(3).For all $x \in A$, $\beta, r \in [0,1]$ and $x_0 \in \overline{B(x, r)}$ there exists $x_1 \in \overline{B(x, r)}$ such that $x_1 = h(x_0, \beta)$,

If $h(.,0)$ has a fixed point in P, then $h(.,1)$ has a fixed point in P.

Proof: Assume that $h(.,0)$ has a fixed point in P.

Consider the subset $K = \{(\beta, x): d\big(x, h(x, \beta)\big) = d(P, X)\}$.

So there is a point $x \in P$ such that $d\big(x, h(x, 0)\big) = d(P, X)$.

So we have $(0, x) \in K$.

So $K \neq \phi$.

Define a partial order on K by

$(\beta, x) \leq (\mu, y)$if only if $\beta \leq \mu$ and $d(x, y) \leq (\mu - \beta)$.

Consider L be an arbitrary totally ordered subset of $k$and $\beta^* = \sup\{\beta: (\beta, x) \in L\}$.

Consider increasing sequence $\{(\beta_n, x_n)\}$ in L for all $n \in N \cup \{0\}$ and $\beta_n \to \beta^*$ as $n \to \infty$.

So we get $d(x_n, x_m) \leq (\beta_m - \beta_n)$,

For each $n, m \in N \cup \{0\}$with $m > n$.

So$\{x_n\} \subseteq P$ is a Cauchy sequence and there is $x^* \in P$such that $d(x_n, x^*) \to 0$ as $n \to \infty$.

Since $P \subseteq X$ is closed and (X, d) is a complete lattice,

So $d\big(x^*, h(x^*, \beta^*)\big) = d(P, X)$.

So $(\beta^*, x^*) \in K$.

Since L is a totally ordered, it satisfies $(\beta, x) \leq (\beta^*, x^*)$ for all $(\beta, x) \in L$.

Hence, (β^*, x^*) is an upper bound of L.

Hence K has maximal element (β_0, x_0).

Now we want to prove that $\beta_0 = 1$.

Assume that $\beta_0 < 1$.

Then there is a real number βsatisfying $\beta_0 < \beta < 1$.

Let$r_1 = \beta - \beta_0$.

Using property (2), mapping $h(., \beta): \overline{B(x_0, r_1)} \to X$ is a generalized contraction. Now considering property (3) and above theorem there exists $x_\beta \in \overline{B(x_0, r_1)}$

Such that $d(x_\beta, h\big(x_\beta, \beta\big) = d(P, X)$.

Hence $\big(x_\beta, \beta\big) \in K$,

A contradiction.

So $\beta_0 = 1$.

Hence $h(.,1)$ has a best proximity point $x_0$in P.

Example 2.5: Let $X = R^2$ be complete metric space with usual metric d.Consider the closed subset of X,

$$P = \left\{0, \frac{1}{2n}: n \in N\right\} \times \{0\}$$

And $Q = \left\{0, \frac{1}{3n}: n \in N\right\} \times \{1\}$.

Now $d(P,Q) = 1$, take $P_0 = P$ and $Q_0 = Q$.
Also the pair (P,Q) has the generalized P-property.
Now we define $T: P \to Q$and function $f: [0,\infty] \times [0,\infty] \to R$ by $T(x,0) = (0,1)$

$$\text{And } f(p,q) = \left\{ \begin{array}{c} 1, \quad p = \frac{1}{2n} \text{ and } q = 1 + \frac{1}{n}, n \geq 1, \\ or \\ p = 1 + \frac{1}{2n} \text{ and } q = \frac{1}{n}, n \geq 1 \\ 0, \quad p \notin \left\{0 \frac{1}{2n}\right\} \text{and } q = 1 + \frac{1}{3n}, n \geq 1 \\ or \\ p \notin \left\{0 \frac{1}{2n}\right\} \text{and } q = 1 + \frac{1}{3n}, n \geq 1 \\ \left(\frac{q}{3} - p\right), \quad otherwise \end{array} \right\}.$$

Now it can be seen that $T(P_0) \subseteq Q_0$ and T is continuous mapping.
Let $A = ran(d, P \cup Q)$
$A = \left\{0, \frac{1}{2n} : n \in N\right\} \cup \left\{1 + \left|\frac{1}{n} - \frac{1}{m}\right| : n, m \in N\right\} \cup \{1 + \frac{1}{3n} : n \in N\}$.
Now T is modified R-function on A.
Now for sequence $\{p_n\} \subseteq (0,\infty) \cap A$ satisfying $f(p_{n+1}, p_n) > 0$ for all $n \in N \cup \{0\}$, then there exists subsequences $\{p_{n_k}\} \to 0$ as $k \to \infty$.
So there is $n_0 \in N$ such that $p_{n_k} = \frac{1}{2n_0}$ then we have $p_{n_0+2n} \to 0$ for all $n \in N$.
Since $f(p_{n+1}, p_n) > 0$ for each $n \in N$, we have $\left(\frac{p_n}{2} - p_{n+1}\right) > 0$ for all $n \in N$ and so $\{p_n\}$ is decreasing.
Hence there exists $z \geq 0$ such that $p_n \to z$as $n \to \infty$.
Similarly $\{p_n\}, \{q_n\} \subseteq (0,\infty) \cap A$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = z \geq 0, z \leq p_n$ and $f(p_n, q_n) > 0$ for all $n \in N$then $z = 0$.
Now we prove that T is generalized contraction with respect to$f$.
Let $x = \left(\frac{1}{2n}, 0\right), y = \left(\frac{1}{m^2}, 0\right), n, m \geq 1$.
Then $Tx = (0,1)$and$Ty = (0,1)$.
So we get $d(Tx, Ty) = 0$.
$m(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$,for each $x, y \in X$ with $x \neq y$.
So $m(x,y) = \left|\frac{1}{2n} - \frac{1}{m^2}\right| = (\frac{1}{2n} - \frac{1}{m^2})$.
Hence we get,
$f\big(d(Tx,Ty), m(x,y)\big) = f\left(0, \left(\frac{1}{2n} - \frac{1}{m^2}\right)\right) = \left(\frac{1}{2n} - \frac{1}{m^2}\right) > 0$.
Thus T is generalized contraction with respect to f.
Thus T has unique best proximity point in P.

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