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Review Paper

Spectral decomposition of the completely continuous operators in Hilbert space.

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Spectral theory of operators took the important place in functional analysis. Many works of famous mathematicians are devoted to this topic [1],[2],[3] . For example, spectral decomposition of the completely continuous self-adjoint operators was given in all books on functional analysis. Spectral decompositionsfor the self-adjoint and normal operators are given in[1].

Let A be a bounded operator, acting in separabel Hilbert space H .

1. λ is an eigenvalue of operator A ifthere is nonzero element x such that $Ax-\lambda x=0$ Element x is called an eigenvector, corresponding to this eigenvalue λ .

2.Element x is called a root vector of height k if the following equalities

$(A-\lambda E)^k x=0$

$(A-\lambda E)^{k-1} x\neq 0$ are satisfied.

If $k=1$, then x is called an eigenvector of operator A with the eigenvalue λ .

Let now A A be a completely continuous operator in Hilbert space H .

The operator A can be represented in the form

$$A=\frac{A+A^*}{2}+i\frac{A-A^*}{2i},$$

where

$$T=\frac{A+A^*}{2}$$

and

$$S=\frac{A-A^*}{2i} \quad (1)$$

are completely continuous self-adjoint operators. It is famous that each from operators T and S has a countable set of eigenvalues and eigenvectors. Besides, eigenvectors, corresponding to the different eigenvalues of each operator T and S, are orthogonal with each other. We denote by $e_1, e_2, \dots$ the sequence of linear independent eigenvectors of operator T ,and by $g_1, g_2, , \dots$ the sequence of linear independent eigenvectors of operator S.

Further, we introduce the decomposition of the unity E_t of operator T and the decomposition of the unity F_s of operator S [1].

The following equalities are satisfied:

$E_a = 0$, $E_b = 1$

$F_c = 0$, $F_d = 1$

2. $E_m E_n = E_k$

$k = \min(m, n)$

$FpFq = Fr$

$r = \min(p, q)$ 3. $E_t - E_{t-0} = P_t$

$F_s - F_{s-0} = R_s$ (2),

where P_t is projective operator which projects into eigen subspace of operator T, corresponding to its eigenvalue t ,and R_{ss} is a projective operator that projects into eigen subspace of operator S ,corresponding to its eigenvalue s.

It is known that operator T has the spectral decomposition $Tf = \sum_{k=1}^{\infty} \lambda_k (f, e_k) e_k$ (3)

on the eigenvectors of self-adjoint completely continuous operator T .Because $g_1, g_2, \dots$ is a sequence of eigenvectors of operator S ,we have

$$f = \sum_{j=1}^{\infty} (f, g_j) g_j \quad (4).$$

Substituting the right side of (3) the decomposition of element f on eigenvectors of operator S from (4), we have

$$Tf = \sum_{k=1}^{\infty} \lambda_k (\sum_{j=1}^{\infty} \ (f, g)(g,_j \, e_k) e_k \text{ (5).}$$

Similarly, we have also for element Sf the decomposition

$$Sf = \ \sum_{j=1}^{\infty} \ \mu_j (f, g_j) g_j \text{ (6)}$$

on eigenvectors of operator S .

Further, substituted in in the rightside of (6) the decompositionof element f on eigenvectors of operator T

$$f = \sum_{\substack{jk=1 \\ 1}}^{\infty} (f, e_k) e_k \text{ (7)}$$

we have $Sf = \ \sum_{j=1}^{\infty} \mu_j \sum_{k=1}^{\infty} (f_{,,} e_k)(e_k, g_j) g_j$ (8)

Taking into account, that $Af = Tf + iSf$ and

substituting instead Tf and Sf their expressions from (5) and (8), we have

$$Aif = \sum_{k=1}^{\infty} \lambda_k (\sum_{j=1}^{\infty} \ (f, g)(g,_j \, e_k) e_k + i \sum_{j=1}^{\infty} \mu_j \sum_{k=1}^{\infty} (f_{,,} e_k)(e_k, g_j) g_j$$

or

$$Af = \sum_{k=1}^{\infty} \ \sum_{j=1}^{\infty} \ \{\lambda_k (f, g_j)(g_j, e_k) e_k + i(f, e_k)(e_k, g_j) g_j\}$$

Comments.

1. If for some two real numbers a and b we have $P_a R_b \neq 0$ (the two parameter projective operator $P_a R_b$ proijects into no-null subspace),then $a + ib$ is the eigenvalue of operator A . [4].

2.If the operator A is self-adjoint ,then $S = 0$ and decomposition of operator A coincides with the decomposition of operator T
in Hilbert space H. .

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