Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 3 (2025) pp: 108-113
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Extremal Solution of First Order NQFDE

Sachin N.Kondekar
Department of Mathematics, Degloor College, Degloor

ABSTRACT: *In this paper, we discuss the Extremal solutionfor First Order Nonlinear Quadratic Functional Differential Equation in* $\mathcal{R}_+$. *For this we consider the first order nonlinear quadratic functional differential equation.*

KEYWORDS: *Extremal Solution, Banach algebras, hybridfixed point theorem, Functionaldifferential equation.*

R*eceived 19 Mar., 2025;* ***R****evised 28 Mar., 2025;* ***A****ccepted 31 Mar., 2025 © The author(s) 2025.*
P*ublished with open access at* ***www.questjournas.org***

I. INTRODUCTION:

Differential and Integral equations are most useful mathematical tools in both Applied and Pure Mathematics. Moreover the theories of differential and integral equations are rapidly developing using the tools of Topology, Functional Analysis and Fixed point theory. This is particularly true for problems in the related fields of Engineering, Mechanical Vibrations and Mathematical Physics. There are numerous applications of differential and integral equations of integer and fractional orders in Electrochemistry, Viscoelasticity, Control theory, Electromagnetism and Porous media etc. [10-15, 20-23].

Here we will study the Extremal Solutionof first order nonlinear quadratic functional differential equation.
We consider the following first order nonlinear quadratic functional differential equations:

$$\left.\begin{aligned}\mathcal{D}\left[\frac{x(t)}{f\left(t,x\left(\alpha(t)\right)\right)}\right] &= g\left[t,x\left(\mu(t)\right)\right],\ \ t\in\mathcal{R}_+\\ x(0) &= 0\end{aligned}\right\}\qquad(2.1.1)$$

Where, $f(t,x):\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}-\{0\}$, $g(t,x):\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}$ and $\alpha,\mu:\mathcal{R}_+\to\mathcal{R}$

Here the solution of nonlinear differential equations (2.1.1) we mean a function $x\in BC(\mathcal{R}_+,\mathcal{R})$ such that:

(i) The function $t\to\left[\frac{x(t)}{f\left(t,x\left(\alpha(t)\right)\right)}\right]$ is bounded and continuous for each $x\in\mathcal{R}$.

(ii) x satisfies (2.1.1)

2.2 PRELIMINARIES:

Let $X=BC(\mathcal{R}_+,\mathcal{R})$ be the space of bounded real valued continuous function on $\mathcal{R}_+$ and S be a subset of X. Let a mapping $\mathcal{A}:X\to X$ be an operator and consider the following operator equation in X, namely,

$x(t)=(\mathcal{A}x)(t)$, for all $t\in\mathcal{R}_+$(2.2.1)

Definition 2.2.1[21]: Let $f\in\mathcal{L}^1[0,\mathcal{T}]$and $\alpha>0$. The Riemann-Liouville fractional derivative of order ζ of real function f is defined as

$$\mathcal{D}^{\zeta} f(t) = \frac{1}{\Gamma(1-\zeta)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\zeta}} ds \quad , \qquad 0 < \zeta < 1$$

Such that $\mathcal{D}^{-\zeta} f(t) = I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}}\, ds$respectively.

Definition 2.2.2 [21]: The Riemann-Liouville fractional integral of order $\zeta \in (0,1)$ of the function $f \in \mathcal{L}^1[0, \mathcal{T}]$ is defined by the formula:

$$I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}} ds \, , \quad t \in [0, \mathrm{T}]$$

Where $\Gamma(\zeta)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ζ defined by

$$\mathcal{D}^{\zeta} = \frac{d^{\zeta}}{dt^{\zeta}} = \frac{d}{dt} \circ I^{1-\zeta}$$

It may be shown that the fractional integral operator I^{ζ} transforms the space $\mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ into itself and has some other properties.

Definition 2.2.3 [12]: A mapping $g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is Caratheodory if:

i) $t \to g(t, x)$ is measurable for each $x \in \mathcal{R}$ and

ii) $x \to g(t, x)$is continuous almost everywhere for $t \in \mathcal{R}_+$.

Furthermore a Caratheodory function g is $\mathcal{L}^1$ −Caratheodory if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that$|g(t, x)| \leq h_r(t)$ $a.e.$ $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \leq r$

Finally a caratheodory function g is $\mathcal{L}_X^1$ −caratheodory if:

iv) There exists a function $h \in \forall \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t, x)| \leq h(t)$, $a.e.$ $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

For convenience, the function h is referred to as a bound function for g.

2.3 EXISTENCE THEORY:

Now for the solution of (2.2.1) in the space $BC(\mathcal{R}_+, \mathcal{R})$of bounded and continuous realvalued functions defined on $\mathcal{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $BC(\mathcal{R}_+, \mathcal{R})$by,

$$\|x\| = sup\{|x(t)| : t \in \mathcal{R}_+\}, \;\; (xy)(t) = x(t)y(t), \;\; t \in \mathcal{R}_+$$

Clearly, $BC(\mathcal{R}_+, \mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ we denote the space of Lebesgue-integrable function in$\mathcal{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by

$$\|x\|_{\mathcal{L}} = \int_0^{\infty} |x(t)| dt$$

Now the FNFDE (2.1.1) is equivalent to the FNFIE

$$x(t) = \left[f\left(t, x(\alpha(t))\right)\right]\left[\int_0^t g\left(s, x(\mu(s))\right) ds\right] \tag{2.3.1}$$

Let us define the two mapping$\mathcal{A}: X \to X$ and $\mathcal{B}: B_r[0] \to X$ by

$$\mathcal{A}x(t) = f\left(t, x(\alpha(t))\right), t \in \mathcal{R}_+ \tag{2.3.2}$$

$\mathcal{B}x(t) = \int_0^t g\left(s, x(\mu(s))\right) ds \ , t \in \mathcal{R}_+$(2.3.3)

Thus from the FNDE (2.1.1), we obtain the operator equation as follows:

$x(t) = \mathcal{A}x(t)\mathcal{B}x(t), \ t \in \mathcal{R}_+$(2.3.4)

By using all above preliminaries and some hypothesiswe have already proved the operator $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditionsof fixed point theorem,so the operator equation (2.3.4) has a solution on $B_r[0]$.

2.4 EXISTENCE OF EXTREMAL SOLUTIONS

Definition 2.4.0 [10, 26]: A closed and non-empty set $\mathcal{K}$ in a Banach Algebra X is called a cone if

i. $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$

ii. $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathcal{K}, \lambda \geq 0$

iii. $\{-\mathcal{K}\} \cap \mathcal{K} = 0$, Where 0 is the zero element of$X$.

and is called positive cone if

iv. $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$

and the notation $\circ$ is a multiplication composition in X.

We introduce an order relation $\leq$ in X as follows.

Let $x, y \in X$ then $x \leq y$ if and only if $y - x \in \mathcal{K}$. A cone $\mathcal{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on$\mathcal{K}$. It is known that if the cone $\mathcal{K}$ is normal in X then every order-bounded set in X is norm-bounded set in X. The details of cone and their properties appear in Guo and Lakshikantham [10].

We equip the space $\mathcal{C}(\mathcal{R}_+, \mathcal{R})$ of continuous real valued function on $\mathcal{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$$\mathcal{K} = \{x \in \mathcal{C}(\mathcal{R}_+, \mathcal{R}) : x(t) \geq 0, \forall t \in \mathcal{R}_+\}$$

It is well known that the cone $\mathcal{K}$ is normal and positive in $\mathcal{C}(\mathcal{R}_+, \mathcal{R})$. As a result of positivity of the cone $\mathcal{K}$ we have:

Lemma 2.4.1 [13]: Let $p_1, p_2, q_1, q_2 \in \mathcal{K}$ be such that $p_1 \leq q_1$and $p_2 \leq q_2$then$p_1 p_2 \leq q_1 q_2$.

For any $p_1, p_2 \in X = \mathcal{C}(\mathcal{R}_+, \mathcal{R}), p_1 \leq p_2$ the order interval $[p_1, p_2]$ is a set in X given by,

$[p_1, p_2] = \{x \in X : p_1 \leq x \leq p_2\}$.

Definition 2.4.1 [12]: A mapping $R: [p_1, p_2] \to X$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Rx \leq Ry$ for all $x, y \in [p_1, p_2]$.

For proving the existence of Extremal solutions of the equations (2.1.1) under certain monotonicity conditions by using following fixed point theorem of Dhage [13, 14].

Theorem 2.4.1 [14]: Let $\mathcal{K}$ be a cone in Banach Algebra X and let$[p_1, p_2] \in X$. Suppose that $\mathcal{A}$,$\mathcal{B}$: $[p_1, p_2] \to \mathcal{K}$ be two operators such that

a. $\mathcal{A}$ is Lipschitz with Lipschitz constant α

b. $\mathcal{B}$ is completely continuous,

c. $\mathcal{A}x\mathcal{B}x \in [p, q]$ for each $x \in [p, q]$ and

d. $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing.

Further if the cone $\mathcal{K}$ is normal and positive then the operator equation $x = \mathcal{A}x\mathcal{B}y$ has the least and greatest positive solution in $[p_1, p_2]$ whenever $\alpha M < 1$, where $M = \|\mathcal{B}([p_1, p_2])\| = sup\{\|\mathcal{B}x\|: x \in [p_1, p_2]\}$

2.4.1 EXISTENCE THE EXTREMAL SOLUTION OF FNQFDE (2.1.1)

For existence the Extremal solution of first order nonlinear quadratic functional differential equation (FNQFDE) (2.1.1) we require following definitions.

Definition 2.4.1.1: A function $p_1 \in BC(\mathcal{R}_+, \mathcal{R})$ is called a **lower solution** of the FNQFDE (2.1.1) on $\mathcal{R}_+$ if the function $t \to \frac{p_1(t)}{f\left(t,\ p_1(\alpha(t))\right)}$ is continuous and

$$\left.\begin{array}{r} \mathcal{D}\left[\dfrac{p_1(t)}{f\left(t, p_1\left(\alpha(t)\right)\right)}\right] \le g\left[t, p_1\left(\mu(t)\right)\right], a.e., t \in \mathcal{R}_+ \\ x(0) = 0 \end{array}\right\}$$

Again a function $p_2 \in BC(\mathcal{R}_+, \mathcal{R})$ is called an **upper solution** of the FNQFDE (2.1.1) on $\mathcal{R}_+$ if the function $t \to \frac{p_2(t)}{f\left(t,\ p_2(\alpha(t))\right)}$ is continuous and

$$\left.\begin{array}{r} \mathcal{D}\left[\dfrac{p_2(t)}{f\left(t, p_2\left(\alpha(t)\right)\right)}\right] \ge g\left[t, p_2\left(\mu(t)\right)\right], a.e., t \in \mathcal{R}_+ \\ x(0) = 0 \end{array}\right\}$$

Definition 2.4.1.2: A solution x_M of the FNQFDE (2.1.1) is said to be **maximal** if for any other solution x to FNQFDE (2.1.1) one has $x(t) \le x_M(t)$ for all $\in \mathcal{R}_+$ **.** Again a solution x_M of the FNQFDE (2.1.1) is said to be minimal if $x_M(t) \le x(t)$ for all $t \in \mathcal{R}_+$ where x is any solution of the FNQFDE (2.1.1) on $\mathcal{R}_+$.

We consider the following hypothesis for existence of Extremal solution:

$\mathfrak{B}1$) g is Caratheodory.

$\mathfrak{B}2$) The functions $f\left(t, x\left(\alpha(t)\right)\right)$ and $g\left[t, x\left(\mu(t)\right)\right]$ are non-decreasing in x almost everywhere for $t \in \mathcal{R}_+$

$\mathfrak{B}3$) The FNQFDE (2.1.1) has a lower solution p_1 and an upper solution p_2 on $\mathcal{R}_+$ with $p_1 \le p_2$

$\mathfrak{B}4$) The function $k: \mathcal{R}_+ \to \mathcal{R}$ defined by

$k(t) = \left|g\left[t, p_1\left(\mu(t)\right)\right]\right| + \left|g\left[t, p_2\left(\mu(t)\right)\right]\right|$ is Lebesgue measurable

Remark 2.4.1.1: Assume that ($\mathfrak{B}2 - \mathfrak{B}4$) hold. Then

$\left|g\left[t, x\left(\mu(t)\right)\right]\right| \le k(t),\ a.e.\ t \in \mathcal{R}_+$ for all $x \in [p_1, p_2]$.

II. MAIN RESULT:

Theorem 2.4.1.1: Suppose that the assumptions required for existence of solution and$(\mathfrak{B}1-\mathfrak{B}4)$ holds and k is given in remark (2.4.1.1) further if $L\|k\|_{\mathcal{L}^1} \leq 1$ then FNQFDE (2.1.1) has a minimal and maximal positive solution on $\mathcal{R}_+$.

Proof: The FNQFDE (2.1.1) is equivalent to IE (2.3.4) on $\mathcal{R}_+$. Let $X=\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ and we define an order relation "$\leq$" by the cone $\mathcal{K}$ given by (2.4.1). Clearly $\mathcal{K}$ is a normal cone in X. Define two operators $\mathcal{A}$ and $\mathcal{B}$ on X by (2.3.2) and (2.3.3) respectively. Then IE (2.3.4) is transformed into an operator equation $\mathcal{A}x\mathcal{B}x=x$ in Banach algebra X. Notice that $(\mathfrak{B}1)$ implies $\mathcal{A},\mathcal{B}:[p_1,p_2]\to\mathcal{K}$ Since the cone $\mathcal{K}$ in X is normal, $[p_1,p_2]$is a norm bounded set in X. As (2.1.1) has solution, so that$\mathcal{A}$ is a Lipschitz with a Lipschitz constant L and $\mathcal{B}$ is completely continuous operator on$[p_1,p_2]$. Again the hypothesis $(\mathfrak{B}2)$ implies that $\mathcal{A}$ and $\mathcal{B}$ are non-decreasing on $[p_1,p_2]$. To see this, let $x,y\in[p_1,p_2]$ be such that $x\leq y$. Then by $(\mathfrak{B}2)$

$$\mathcal{A}x(t)=f\left(t,x\big(\alpha(t)\big)\right)\leq f\left(t,y\big(\alpha(t)\big)\right)\leq \mathcal{A}y(t),\forall t\in\mathcal{R}_+$$

Similarly,

$$\mathcal{B}x(t)=\int_0^t g\left(s,x\big(\mu(s)\big)\right)ds\leq\int_0^t g\left(s,y\big(\mu(s)\big)\right)ds$$

$$\leq\mathcal{B}y(t),\forall t\in\mathcal{R}_+$$

Implies that $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on$[p_1,p_2]$. Again definition (2.4.1.1) and hypothesis $(\mathfrak{B}3)$ implies that

$$p_1(t)\leq f\left(t,p_1\big(\alpha(t)\big)\right)\int_0^t g\big[s,p_1\big(\mu(s)\big)\big]ds$$

$$\leq f\left(t,x\big(\alpha(t)\big)\right)\int_0^t g\left(s,x\big(\mu(s)\big)\right)ds$$

$$\leq f\left(t,p_2\big(\alpha(t)\big)\right)\int_0^t g\left(s,p_2\big(\mu(s)\big)\right)ds$$

$$\leq p_2(t),\forall t\in\mathcal{R}_+\text{ and}x\in[p_1,p_2]$$

As a result $p_1(t)\leq\mathcal{A}x(t)\mathcal{B}x(t)\leq p_2(t),\forall t\in\mathcal{R}_+$ and $x\in[p_1,p_2]$

Hence$\mathcal{A}x\mathcal{B}x\in[p_1,p_2],\forall x\in[p_1,p_2]$

Again $M=\|\mathcal{B}([p_1,p_2])\|=sup\{\|\mathcal{B}x\|:x\in[p_1,p_2]\}$

$$\leq sup\left\{sup_{t\in\mathcal{R}_+}\int_0^t\left|g\left(s,x\big(\mu(s)\big)\right)ds\right|:x\in[p_1,p_2]\right\}$$

$$\leq \int_0^t k(s)\, ds \leq \|k\|_{\mathcal{L}^1}$$

Since $LM \leq L\|k\|_{\mathcal{L}^1} \leq 1$

We apply theorem (2.4.1) to the operator equation $\mathcal{A}x\mathcal{B}x = x$ to yield that the FNQFDE (2.1.1) has minimum and maximum positive solution on $\mathcal{R}_+$.

III. Conclusion:

In this Research Paper we have studied the Extremal solutions to the first order nonlinear quadratic functional differential equations in Banach Space by Hybrid Fixed Point Theory.

REFERENCES

[1]. A.A.Kilbas, Hari M. Srivastava and Juan J.Trujillo, Theory and Applications Fractional Differential equations , North-Holland Mathematics Studies,204, Elsevier Science B.V., Amsterdam ,2006, MR2218073 (2007a:34002).Zbl 1092.45003.

[2]. A.A.Kilbas, J.J.Trujillo, Differential equations of fractional order: Methods, results, Problems, I.Appl.Anal. Vol.78 (2001), pp.153-192.

[3]. A.Babakhani, V.Daftardar-Gejii, Existence of positive solutions of nonlinear fractional differential equations, J.Math.Appl. Vol.278 (2003), pp.434-442.

[4]. Ahmad B., Ntouyas S.K., Alsaedi A., Existence result for a system of coupled hybrid fractional differential equations: Sci. World J. 2013, Article ID 426438(2013).

[5]. B.C. Dhage, A Fixed point theorem in Banach algebras involving three operators with applications, Kyungpook Math J. Vol.44 (2004), pp.145-155.

[6]. B.C.Dhage , On Existence of Extremal solutions of nonlinear functional integral equations in Banach Algebras, Journal of applied mathematics and stochastic Analysis 2004:3(2004),pp.271-282

[7]. B.D.Karande, Existence of uniform global locally attractive solutions for fractional order nonlinear random integral equation, Journal of Global Research in Mathematical Archives, Vol.1 (8) (2013), pp.34-43.

[8]. B.D.Karande, Fractional Order Functional Integro-Differential Equation in Banach Algebras, Malaysian Journal of Mathematical Sciences, Volume 8(S), (2014), pp. 1-16.

[9]. B.D.Karande, Global attractively of solutions for a nonlinear functional integral equation of fractional order in Banach Space, AIP Conf.Proc. "10th international Conference on Mathematical Problems in Engineering, Aerospace and Sciences"1637 (2014), pp.469-478.

[10]. D.J.Guo and V. Lakshmikantham, Nonlinear problems in Abstract cones, Notes and Reports in Mathematics in Science and engineering, vol.5, Academic press, Massachusetts, 1988.

[11]. Das S. Functional Fractional Calculus for System Identification and Controls, Berlin, Heidelberg: Springer-Verlag, 2008.

[12]. Das S., Functional Fractional Calculus. Berlin, Heidelberg: Springer-Verlag, 2011.

[13]. Dhage B.C., A Nonlinear alternative in Banach Algebras with applications to functional differential equations, Non-linear functional Analysis Appl 8(2004),pp.563-575.

[14]. Dhage B.C. , Fixed Point theorems in ordered Banach Algebras and applications, Panam Math J 9(1999),pp. 93-102.

[15]. Dhage B.C., Basic results in the theory of hybrid differential equations with mixed perturbations of second type, Funct. Differ. Equ. 19(2012(2012).), pp.1-20.

[16]. Dhage B.C., Periodic boundary value problems of first order Caratheodory and discontinuous differential equation: Nonlinear, *Funct. Anal. Appl., 13(2), 323-352,*

[17]. Dhage B.C., Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, *Differ. Equ. Appl. 2, 465-4869,*(2010).

[18]. Dugungi, A.Granas, Fixed point Theory, Monographic Math., Warsaw, 1982.

[19]. H.M.Srivastava, R.K.Saxena, Operators of fractional integration and applications, Appl.Math.Computer Vol.118 (2006), pp.147-156.

[20]. I.Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1993.

[21]. I.Podlubny, Fractional differential equations, Mathematics in science and engineering, volume 198.

[22]. J.Banas, B.C. Dhage, Globally Asymptotic Stability of solutions of a functional integral equations, Non-linear functional Analysis 69 (7)(2008) ,pp.1945-1952.

[23]. K.S.Miller, B.Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.

[24]. Lakshmikantham and A.S.Vatsala, Basic theory of fractional differential equations, Nonlinear Analysis, 69(2008), pp.2677-2682.

[25]. Lakshmikantham V, Leela.S, VasundharaDevi,Theory of fractional dynamic systems,Cambridge Academic publishers, Cambridge (2009).

[26]. S.Samko, A.A.Kilbas, O.Marichev, *Fractional Integrals and Derivatives: Theory and Applications,* Gordon and Breach, Amsterdam, (1993).

[27]. Sabatier J., Nguyen H. C., Farges C., Deletage J. Y., Moreau X., Guillemard F., Bavoux B., *Fractional models for Thermal modelling and temperature estimation of a transistor junction,* Adv. Difference Equations, Article 687363, Pp.1-12,(2011).

[28]. Sabatier J, Agrawal. O.P, Machado, *Advances in fractional calculus: Theoretical development and applications in Physics and Engineering,* (Springer, Dordrecht) (2007).

[29]. Samko S.G., *Hyper singular integrals and their applications,* CRC press, (2001).